Geometry and Dynamics of Singular Symplectic manifolds

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Hénan University, Day 2
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The restricted 3-body problem

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem.
- After doing a change to Mc Gehee coordinates ($r = \frac{2}{x^2}, \quad x \in \mathbf{R}^+,$) the symplectic structure becomes a singular object $\omega = -\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG.$ for $x > 0$

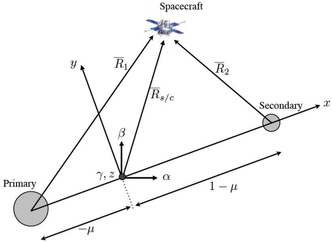


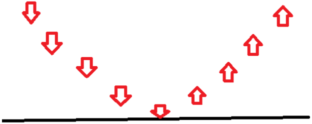
Figure: Circular 3-body problem

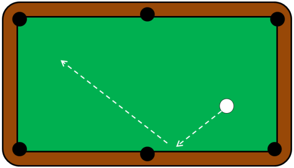
Model of these systems

$$\omega = \frac{\mathbf{1}}{\mathbf{x_1^m}} \mathbf{dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i} \geq \mathbf{2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

Close to $x_1 = 0$, the systems behave like,

and not like,





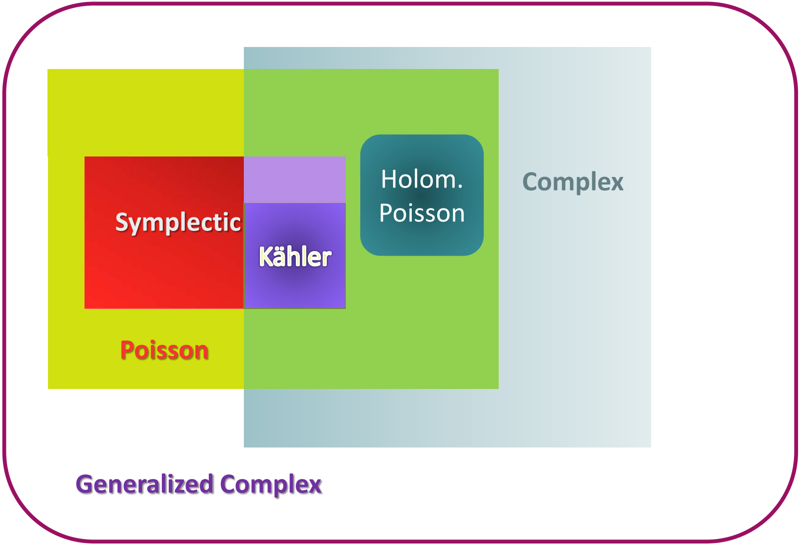
Symplectic surfaces with singularities (Radko's surfaces)

We want to modify the volume form on S by making it "explode" when we get close to a union of curves Z. We want this "blow up" process to be controlled.


Figure: A Radko surface and Olga Radko

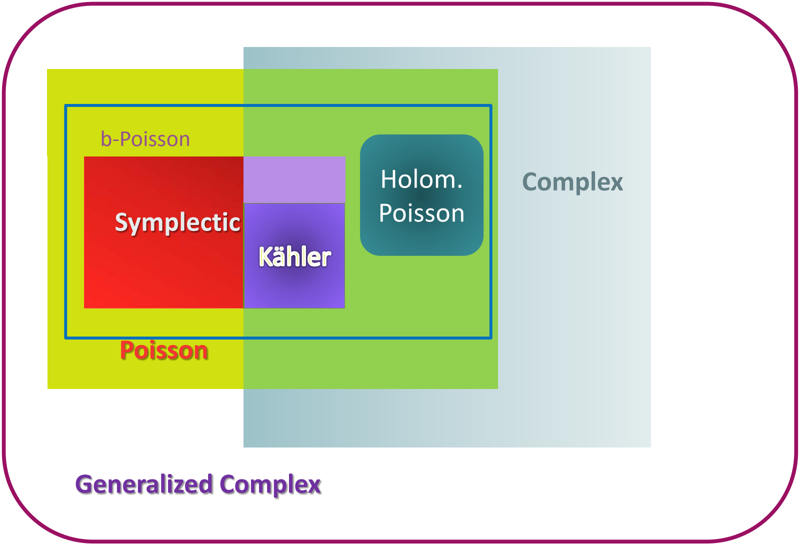
What does "controlled" mean here? We want that the 2-form looks locally $\omega = \frac{c}{x} dx \wedge dy$ (for points in Z).

Geometries involved



Zooming in...

b-Poisson close to symplectic



But sometimes it is good to zoom out...

Zooming out...to gain perspective

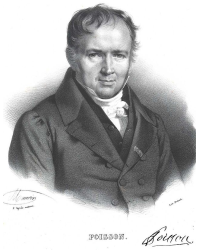
Poisson, 1809

MÉMOIRE

Sur la Variation des Constantes arbitraires dans les questions de Mécanique,

Lu à l'Institut le 16 Octobre 1809;

Par M. POISSON.



constante a ni la constante b; dans d'autres cas elle ne contiendra aucune constante arbitraire, et se réduira à une constante déterminée; mais, afin de rappeler l'origine de cette quantité, qui représente une certaine combinaison des différences partielles des valeurs de a et b, nous ferons usage de cette notation (b, a), pour la désigner; de manière que nous aurons généralement

$$\frac{db}{ds}\cdot\frac{da}{d\varphi} - \frac{da}{ds}\cdot\frac{db}{d\varphi} + \frac{db}{du}\cdot\frac{da}{d\psi} - \frac{da}{du}\cdot\frac{db}{d\psi} + \frac{db}{dv}\cdot\frac{da}{d\theta} - \frac{da}{dv}\cdot\frac{db}{d\theta} = (b, a).$$

Figure: Poisson bracket

Singular symplectic manifolds as Poisson manifolds

The local models

$$\omega = \frac{\mathbf{1}}{\mathbf{x_1^m}}\mathbf{dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

are formally not a smooth form but their dual defines a smooth Poisson structure! as their dual

$$\mathbf{\Pi} = \mathbf{x_1^m}\frac{\partial}{\partial \mathbf{x_1}} \wedge \frac{\partial}{\partial \mathbf{y_1}} + \sum_{\mathbf{i \geq 2}}^{\mathbf{n}} \frac{\partial}{\partial \mathbf{x_i}} \wedge \frac{\partial}{\partial \mathbf{y_i}}$$

is well-defined. The structure Π is a bivector field which satisfies the integrability equation $[\Pi, \Pi] = 0$. The Poisson bracket associated to Π is given by the equation

$$\{f, g\} := \Pi(df, dg)$$

Poisson structures as brackets

A Poisson bracket on a manifold is given by $\mathbb{R}$-bilinear operation

$$\begin{array}{rccc} \{\cdot,\cdot\}: & C^\infty(M) & \longrightarrow & C^\infty(M) \\ & (f,g) & \longmapsto & \{f,g\} \end{array}$$

which satisfies:

1. Anti-symmetry, $\{f,g\} = -\{g,f\}$ for any $f,g \in C^\infty(M)$
2. Leibnitz rule, $\{f, g\cdot h\} = g\cdot\{f,h\} + \{f,g\}\cdot h$
3. Jacobi identity, $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

Space for proofs

Space for proofs

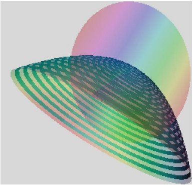
Poisson structures as bivector fields

Poisson structures

A Poisson structure is a bivector field Π with $[\Pi, \Pi] = 0$.

The Poisson manifold is locally a product of a symplectic manifold with a Poisson manifold with vanishing Poisson structure at the point (Weinstein's splitting theorem).

$$(P^n, \Pi, p) \approx (M^{2k}, \omega, p_1) \times (P_0^{n-2k}, \Pi_0, p_2)$$



This defines a symplectic foliation.

b-Poisson structures

Definition

Let (M^{2n}, Π) be an (oriented) Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface called *the critical hypersurface* and we say that Π is a b-**Poisson structure** on (M, Z).

Other singularities

It is possible to generalize this definition (M.-Planas-Scott) to consider more general Poisson structures.

Symplectic foliation of a b-Poisson manifold

The symplectic foliation has dense symplectic leaves and codimension 2 symplectic leaves whose union is Z.

b-Darboux theorem

Theorem

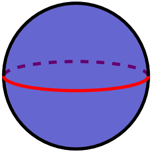
For all $p \in Z$, there exists a Darboux coordinate system $x_1, y_1, \ldots, x_n, y_n$ centered at p such that Z is defined by $x_1 = 0$ and

$$\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

Space for notes

Examples

- A Radko surface.



- The product of (R, π_R) a Radko compact surface with a compact symplectic manifold (S, ω) is a b-Poisson manifold.
- corank 1 Poisson manifold (N, π) and X Poisson vector field $\Rightarrow$ $(S^1 \times N, f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi)$ is a b-Poisson manifold if,
  1. f vanishes linearly.
  2. X is transverse to the symplectic leaves of N.

  We then have as many copies of N as zeroes of f.

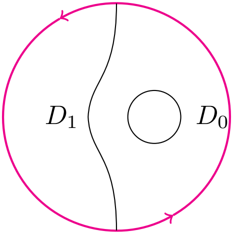
Space for notes

Another example (exercise in the list)

The cubic polynomial $g(x) = x(x-1)(x-t)$, $0 < t < 1$, defines a Poisson structure on $\mathbb{R}^2$ given by

$$\pi = (g(x) - y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},$$

which extends smoothly to a b-symplectic structure on $\mathbb{R}P^2$ with critical set Z given by the real elliptic curve $y^2 = g(x)$.



The critical set has two connected components: D_0, containing $\{(0,0),(t,0)\}$ and with trivial normal bundle, and D_1, containing $\{(1,0),(\infty,0)\}$ and with nontrivial normal bundle.

Modular vector field

Definition

Given a Poisson manifold (M, Π) and a volume form Ω, the modular vector field X_Π^Ω associated to the pair (Π, Ω) is the derivation given by the mapping

$$f \mapsto \frac{L_{X_f}\Omega}{\Omega}$$

1. $L_{X_\Pi^\Omega}(\Pi) = 0$ and $L_{X_\Pi^\Omega}(\Omega) = 0$.
2. $X^{H\Omega} = X^\Omega - X_{log(H)}$. $\rightsquigarrow$ its first cohomology class in Poisson cohomology does not depend on Ω.
3. Examples of unimodular (vanishing modular class) Poisson manifolds: symplectic manifolds.
4. In the case of b-Poisson manifolds in dimension 2, $\{x, y\} = y$ and the modular vector field is $\frac{\partial}{\partial x}$.

Modular vector fields

Modular vector field for Darboux form

The modular vector field of a local b-Poisson manifold with local normal form,

$$\Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

with respect to the volume form $\Omega = \sum_i dx_i \wedge dy_i$ is,

$$X^\Omega = \frac{\partial}{\partial x_1}.$$

The modular vector field of a b-Poisson manifold is tangent to the critical set Z and is transverse to the symplectic leaves of the induced symplectic foliation on Z.

Induced Poisson structures

Induced Poisson structures

a b-Poisson structure Π on M^{2n} induces a regular corank 1 Poisson structure on Z.

Given a Poisson manifold Z with codimension 1 symplectic foliation $\mathcal{L}$,

1. Does $(Z, \Pi_{\mathcal{L}})$ extend to a b-Poisson structure on a neighbourhood of Z in M?
2. If so to what extent is this structure unique?

The $\mathcal{L}$-De Rham complex

Choose $\alpha \in \Omega^1(Z)$ and $\omega \in \Omega^2(Z)$ such that for all $L \in \mathcal{L}$ (symplectic foliation) such that for all $L \in \mathcal{L}$, $i_L^*\alpha = 0$ and $i_L^*\omega = \omega_L$.

$$d\alpha = \alpha \wedge \beta, \beta \in \Omega^1(Z) \tag{1}$$

Therefore we can consider the complex

$$\Omega_{\mathcal{L}}^k = \Omega^K / \alpha\Omega^{k-1}$$

Consider $\Omega_0 = \alpha \wedge \Omega$ we get a short exact sequence of complexes

$$0 \longrightarrow \Omega_0 \xrightarrow{i} \Omega \xrightarrow{j} \Omega_{\mathcal{L}} \longrightarrow 0$$

By differentiation of 1 we get $0 = d(d\alpha) = d\beta \wedge \alpha - \beta \wedge \beta \wedge \alpha = d\beta \wedge \alpha$, so $d\beta$ is in Ω_0, i.e., $d(j\beta) = 0$.

First obstruction class

We define the **obstruction class** $c_1(\Pi_{\mathcal{L}}) \in H^1(\Omega_{\mathcal{L}})$ to be $c_1(\Pi_{\mathcal{L}}) = [j\beta]$

Notice that $c_1(\Pi_{\mathcal{L}}) = 0$ iff we can find a closed one form for the foliation.

The $\mathcal{L}$-De Rham complex

Assume now $c_1(\Pi_{\mathcal{L}}) = 0$ then, we obtain $d\omega = \alpha \wedge \beta_2$.

Second obstruction class

We define the **obstruction class** $c_2(\Pi_{\mathcal{L}}) \in H^2(\Omega_{\mathcal{L}})$ to be

$$c_2(\Pi_{\mathcal{L}}) = [j\beta_2]$$

Main property

$c_2(\Pi_{\mathcal{L}}) = 0 \Leftrightarrow$ there exists a **closed** 2-form, ω, such that $i_L^*(\omega) = \omega_L$.

The role of these invariants

The role of these invariants

$c_1(\Pi_{\mathcal{L}}) = c_2(\Pi_{\mathcal{L}}) = 0 \Leftrightarrow$ there exists a Poisson vector field v transversal to L.

Relation of v, ω and α:

1. $\iota_v \alpha = 1$.
2. $\iota_v \omega = 0$.

The fibration is a symplectic fibration and v defines an Ehresmann connection.

Dynamics of codimension-1 foliations on Poisson manifolds with vanishing invariants

Let β satisfy $d\alpha = \beta \wedge \alpha$. With respect to the volume form $\alpha \wedge \omega^n$
$\iota(v_{\text{mod}})\omega_L = \beta_L$.

Theorem

A regular corank 1 Poisson manifold is unimodular iff we can choose closed defining one-form α for the symplectic foliation (i.e. if and only if $c_1(\Pi_{\mathcal{L}}) = 0$).

The b-Poisson case

The Poisson structure induced on the critical hypersurface of a b-Poisson structure manifold has vanishing invariants $c_1(\Pi_{\mathcal{L}})$ and $c_2(\Pi_{\mathcal{L}})$.

Summing up,

The foliation induced by a b-Poisson structure on its critical hypersurface satisfies,

- we can choose the defining one-form α to be closed.
- symplectic structure on leaves which extends to a closed 2-form ω on M

Given a symplectic foliation on a corank 1 regular Poisson manifold α and ω exist if and only if the invariants $c_1(\Pi_{\mathcal{L}})$ and $c_2(\Pi_{\mathcal{L}})$ vanish.

Question

Is every codimension one regular Poisson manifold with vanishing invariants the critical hypersurface of a b-Poisson manifold?
We will answer this question next week.

A theorem of Tischler: Foliations given by closed forms

Theorem

Let M be a compact manifold without boundary that admits a non-vanishing closed 1-form. Then M is a fibration over S^1.

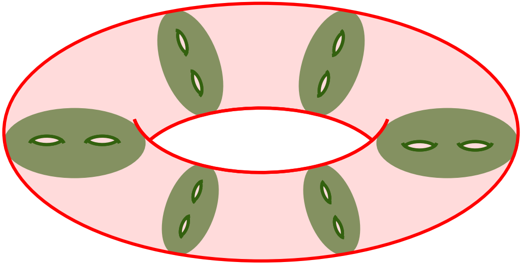
The irrational flow

Observe that this is NOT telling us that the foliation given by α itself IS a fibration.

The singular hypersurface of a b-Poisson manifold

Theorem (Guillemin-M.-Pires)

If $\mathcal{L}$ contains a compact leaf L, then Z is the mapping torus of the symplectomorphism $\phi : L \to L$ determined by the flow of a Poisson vector field v transverse to the symplectic foliation.



This description also works for b^m-Poisson structures.

Invariants: Dimension 2

Radko classified b-Poisson structures on compact oriented surfaces giving a list of invariants:

- Geometrical: The topology of S and the curves γ_i where Π vanishes.
- Dynamical: The periods of the "modular vector field" along γ_i.
- Measure: The regularized Liouville volume of S, $V_h^{\epsilon}(\Pi) = \int_{|h|>\epsilon} \omega_{\Pi}$ for h a function vanishing linearly on the curves $\gamma_1, \dots, \gamma_n$.

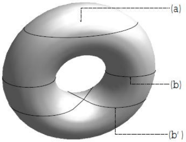


Figure: Two admissible vanishing curves (a) and (b) for Π; the ones in (b') is not admissible.

Singular forms

- A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The **b-tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$

- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

- A **b-symplectic form** is a closed, nondegenerate, b-form of degree 2.
- This dual point of view, allows to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path method because we can play the same tricks as in the symplectic case.

Space for notes

Example

$M = \mathbb{T}^4$ and $Z = \mathbb{T}^3 \times \{0\}$. Consider on Z the codimension 1 foliation given by $\theta_3 = a\theta_1 + b\theta_2 + k$, with rationally independent $a, b \in \mathbb{R}$. Then take

$$
\begin{aligned}
h &= \log(\sin\theta_4), \\
\alpha &= \frac{a}{a^2+b^2+1}\, d\theta_1 + \frac{b}{a^2+b^2+1}\, d\theta_2 - \frac{1}{a^2+b^2+1}\, d\theta_3, \\
\omega &= d\theta_1 \wedge d\theta_2 + b\, d\theta_1 \wedge d\theta_3 - a\, d\theta_2 \wedge d\theta_3,
\end{aligned}
$$

The 2-form $\omega_\Pi = dh \wedge \alpha + \omega$ defines a b-symplectic form in a neighbourhood of Z, which can be extended to M.

Geometrical invariants

Theorem (Mazzeo-Melrose)

The b-cohomology groups of a compact M are computable by

$$ {}^{b}H^*(M) \cong H^*(M) \oplus H^{*-1}(Z). $$

Corollary (Classification of b-symplectic surfaces à la Moser, Guillemin-M.-Pires)

Two b-symplectic forms ω_0 and ω_1 on an orientable compact surface are b-symplectomorphic if and only if $[\omega_0] = [\omega_1]$.

Indeed,

$$ {}^{b}H^*(M) \cong H^*_{\Pi}(M) $$

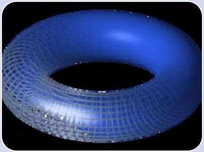
(Singular) symplectic manifolds

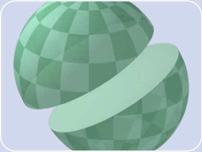
b^m -Symplectic

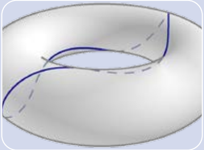
Symplectic

Folded symplectic

Déjà-vu...







Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle coordinates

b-Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle theorem

Folded symplectic manifolds

- Darboux theorem (Martinet)
- Delzant-type theorems (Cannas da Silva-Guillemin-Pires)
- Action-agle theorem (M-Cardona)

Examples and counterexamples

Orientable Surface	CP^2	S^4
• Is symplectic • Is folded symplectic • (orientable or not) is b-symplectic	• Is symplectic • Is folded symplectic • Is not b-symplectic	• Is not symplectic • Is not b-symplectic • Is folded-symplectic

Desingularizing b^m-symplectic structures

Theorem (Guillemin-M.-Weitsman)

Given a b^m-symplectic structure ω on a compact manifold (M^{2n}, Z):

- *If $m = 2k$, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*
- *If $m = 2k + 1$, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z.*

In particular:

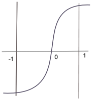
- Any b^{2k}-symplectic manifold admits a symplectic structure.
- Any b^{2k+1}-symplectic manifold admits a folded symplectic structure.
- The converse is not true: S^4 admits a folded symplectic structure but no b-symplectic structure.

Sketch of the proof: $m = 2k$

General principle: If you do not like something, just change it!

$$\omega = \frac{dx}{x^{2k}} \wedge \left(\sum_{i=0}^{2k-1} \alpha_i x^i\right) + \beta \qquad (2)$$

- $f \in \mathcal{C}^\infty(\mathbb{R})$ odd function s.t. $f'(x) > 0$ for $x \in [-1,1]$,



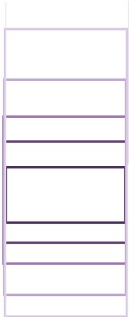
and such that outside $[-1,1]$,

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \text{for} \quad x < -1 \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \text{for} \quad x > 1 \end{cases}$$

- Re-scale on ϵ.
- Replace $\frac{dx}{x^{2k}}$ by df_ϵ to obtain $\omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k-1} \alpha_i x^i) + \beta$ which is symplectic.

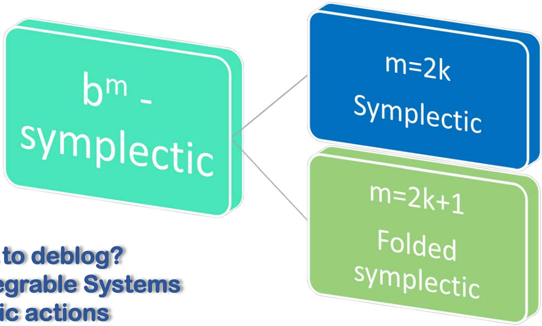
Applications of desingularization

- Convexity for $\mathbb{T}^k$-actions.



- Delzant theorem and Delzant-type theorem for semitoric systems (bolytopes).
- Applications to KAM.
- Periodic orbits of problems in celestial mechanics and applications to stability.

Desingularizing everything...



What to deblog?

- **Integrable Systems**
- **Toric actions**
- **Forms..**