

Smooth structures on stratified symplectic spaces

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Outline

- 1 Stratified spaces and their smooth structures
 - Stratified spaces
 - Smooth structures on stratified spaces
 - Structures theorems
- 2 Compatible smooth structures on symplectic stratified spaces
 - Symplectic stratified spaces
 - Compatible smooth structures
- 3 Properties of compatible smooth structures on stratified symplectic spaces
 - Brylinski-Poisson homology
 - Existence of Hamiltonian flow
 - Leftschetz decomposition
- 4 End

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Decomposed spaces

- Let $(X, \mathcal{S})$ be a Hausdorff and paracompact topological space of finite dimension with **ordered relation**.
- A **$\mathcal{S}$ -decomposition** of X is

$$X = \cup_{i \in \mathcal{S}} S_i,$$

where S_i are locally closed smooth manifolds

- such that

$$S_i \cap \bar{S}_j \neq \emptyset \iff S_i \subset \bar{S}_j \iff i \leq j.$$

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Stratified spaces

A **cone** $cL := L \times [0, \infty] / L \times \{0\}$.

A decomposed space X is called a **stratified space** if for any x in a **stratum** S there exists an open neighborhood $U(x) \subset X$, an open ball $B(x) \subset S$, a compact stratified space L , called the **link** of x , and a **stratified diffeomorphism** $\phi : B \times cL \rightarrow U$ that **preserves the decomposition**.

$depth_X S := \sup\{n \mid \text{there exist pieces } S = S^0 < S^1 < \dots < S^n\}$.

$$depth X := \sup_{i \in S} depth S_i.$$

X^{reg} is assumed to be connected.

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Examples of pseudomanifolds w.i.c.s.

- Let M be a smooth manifold with boundary $\partial M = L = \cup L_i$.
 $M_1 := M \cup cL := L \times [0, 1]/L \times \{0\}$,
 $M_2 : M \cup_i cL_i$.
- The quadric $Q_m = \{z \in \mathbb{C}^{m+1} \mid \sum_{i=1}^{m+1} z_i^2 = 0\}$ with isolated singularity at 0 is a pseudomanifold w.i.c.s..
 $Q_m = cL$, where $L = V_{2,m+1} = SO(m+1)/SO(m-1)$.
- Any smooth manifold with k marked points is a pseudomanifold w.i.c.s.

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Smooth structures on a pseudomanifold w.i.c.s.

A **smooth structure** on a pseudomanifold w.i.c.s. M is a choice of a $\mathbb{R}$ -subalgebra $C^\infty(M) \subset C^0(M)$ satisfying the following three properties.

1. $C^\infty(M)$ is a **germ-defined C^∞ -ring**, i.e. it is the C^∞ -ring of all sections of a sheaf $SC^\infty(M)$ of continuous real-valued functions.
2. $C^\infty(M)|_{M^{reg}} \subset C^\infty(M^{reg})$.
3. $j_*(C_0^\infty(M^{reg})) \subset C^\infty(M)$.

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Product smooth structures

- Assume that L is a compact manifold, $B \subset \mathbb{R}^k$ and $C^\infty(cL(1))$ is a smooth structure on $cL(1)$. $X := B \times cL(1)$.
- A **product smooth structure** $C^\infty(X)$ is the germ-defined C^∞ -ring whose sheaf $SC^\infty(X)$ is generated by $\pi_1^*(SC^\infty(B))$ and $\pi_2^*(SC^\infty(cL(1)))$.

Lemma

$C^\infty(X)$ satisfies the following properties:

- $C^\infty(X)|_S \subset C^\infty(S)$ for each stratum $S \subset X$.
- $j_*(C_0^\infty(X^{reg})) \subset C^\infty(X)$.
- If $f \in C^\infty(X)$ is nowhere vanishing, then $1/f \in C^\infty(X)$.

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Smooth structures on stratified spaces

Definition

Assume that $\text{depth}(L) = k - 1$, $C^\infty(L)$ be a smooth structure. Let $X := cL(1)$. Let $C^\infty(X^{\text{reg}})$ be the product smooth structure on X^{reg} . Then a **smooth structure** on X is a g.d. C^∞ -ring with

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A **smooth structure** on a stratified space X is a choice of a g.d. C^∞ -ring $C^\infty(X) \subset C^0(X)$ with

1. $j_*(C_0^\infty(X^{\text{reg}})) \subset C^\infty(X)$.
2. $\forall x \in X$ there exists $\phi_x : U(x) \rightarrow B_x \times cL(1)$ which is a local diffeomorphism of stratified spaces.

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Examples of smooth structures

- $i : X \rightarrow \mathbb{R}^n$ s.t. $i(X)$ is locally closed. $C^\infty(X) := i^*(C^\infty(\mathbb{R}^n))$.
- M/G , where $G \subset \text{Diff}(M)$ is compact.
 $C^\infty(M/G) := C^\infty(M)^G$.
- **Resolvable smooth structure** Assume that we have a continuous projection $M \xrightarrow{\pi} X$ from a smooth manifold M with corner to a stratified space X such that for each stratum $S_i \subset X$ the triple $(\pi^{-1}(S_i), \pi_i, S_i)$ is a differentiable fibration. Then $C^\infty(X) := \{f \in C^0(N) \mid \pi^*f \in C^\infty(M)\}$ is a smooth structure.
- Every pseudomanifold X with edges has a resolvable smooth structure, which is **locally smoothly contractible**.

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 $C^\infty(M/G) := C^\infty(M)^G$.
- **Resolvable smooth structure** Assume that we have a continuous projection $M \xrightarrow{\pi} X$ from a smooth manifold M with corner to a stratified space X such that for each stratum $S_i \subset X$ the triple $(\pi^{-1}(S_i), \pi|_{S_i}, S_i)$ is a differentiable fibration. Then $C^\infty(X) := \{f \in C^0(X) \mid \pi^*f \in C^\infty(M)\}$ is a smooth structure.
- Every pseudomanifold X with edges has a resolvable smooth structure, which is **locally smoothly contractible**.

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 - Smooth structures on stratified spaces
 - **Structures theorems**
- 2 Compatible smooth structures on symplectic stratified spaces
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- 3 Properties of compatible smooth structures on stratified symplectic spaces
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Two structure theorems

Lemma

For any locally finite open covering $\{U_i\}$ of M there exists a smooth partition of unity subordinate to U_i (i.e. there are nonnegative smooth functions $f_i \in C^\infty(X)$ with support in U_i satisfying $\sum f_i = 1$).

Lemma

*Any resolvable smooth structure on X obtained from a smooth manifold M is not **finitely generated**, if there exists $x \in X$ such that $\dim \pi^{-1}(x) \geq 1$, where $\pi : M \rightarrow X$ is the associated projection.*

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Germes of 1-forms

- $C_x^\infty(X)$ is a local $\mathbb{R}$ -algebra with the unique maximal ideal $\mathfrak{m}_x$ consisting of functions vanishing at x . Set $T_x^*(X) := \mathfrak{m}_x / \mathfrak{m}_x^2$.
- Using $0 \rightarrow \mathfrak{m}_x \rightarrow C_x^\infty \xrightarrow{j} \mathbb{R} \rightarrow 0$ we define the Kähler derivation $d : C_x^\infty(X) \rightarrow T_x^*X$ by: $d(f_x) = (f_x - j^{-1}(f_x(x))) + \mathfrak{m}_x^2$.
- We call $\Omega_x^1(X) := C_x^\infty(X) \otimes_{\mathbb{R}} \mathfrak{m}_x / \mathfrak{m}_x^2$ the germs of 1-forms at x .

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Derivation of a germ of k -forms

Set $\Omega_X^k(X) := C_X^\infty(X) \otimes_{\mathbb{R}} \Lambda^k(\mathfrak{m}_X/\mathfrak{m}_X^2)$. Then $\oplus_k \Omega_X^k(X)$ is an exterior algebra with the following wedge product

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where $f, f' \in C_X^\infty$ and $dg_i \in T_X^*M$.

Note that the Kähler derivation $d : C_X^\infty(X) := \Omega_X^0(X) \rightarrow \Omega_X^1(X)$ extends to the unique derivation $d : \Omega_X^k(X) \rightarrow \Omega_X^{k+1}(X)$ satisfying the Leibniz property. Namely we set

$$d(f \otimes 1) = 1 \otimes df,$$

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Differential k -forms

- A section $\alpha : X \rightarrow \Lambda^k T^*(X)$ is called a **smooth differential k -form**, if $\forall x \in X$ there exists $U(x) \subset X$ such that $\alpha(x) = \sum_{i_0 i_1 \dots i_k} f_{i_0} df_{i_1} \wedge \dots \wedge df_{i_k}$ for some $f_{i_0}, \dots, f_{i_k} \in C^\infty(X)$.
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- Then $d : \Omega_x^k(X) \rightarrow \Omega_x^{k+1}(X)$ extends to a map also denoted by d mapping $\Omega(X)$ to $\Omega(X)$.
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Symplectic stratified spaces

- A stratified space X is called **symplectic**, if every stratum S_i is provided with a symplectic form ω_i . The collection $\{\omega_i\}$ is called a *stratified symplectic form* or simply a *symplectic form*.
- **Examples.** - $S^2 \times S^2 = D^4 \cup D^2 \cup D^2 \cup \{pt\}$.
- Any contact 3-manifold (M^3, α) is a concave boundary of some symplectic manifold (M^4, ω) . Attaching a symplectic cone $cM^3(1)$

$$\bar{\omega}([z, t]) = \frac{1}{2}t^2 d\alpha(z) + t dt \wedge \alpha(z),$$

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Smooth structures on stratified symplectic spaces

Let (X, ω) be a symplectic stratified space and $C^\infty(X)$ be a smooth structure on X .

1. A smooth structure $C^\infty(X)$ is said to be **weakly symplectic**, if there is a smooth 2-form $\tilde{\omega} \in \Omega^2(X)$ such that the restriction of $\tilde{\omega}$ to each stratum S_i coincides with ω_i . In this case we also say that ω is **compatible with $C^\infty(X)$** , and $C^\infty(X)$ is **compatible with ω** .

2. A smooth structure $C^\infty(X)$ is called **Poisson**, if there is a Poisson structure $\{, \}_\omega$ on $C^\infty(X)$ such that $(\{f, g\}_\omega)|_{S_i} = \{f|_{S_i}, g|_{S_i}\}_{\omega_i}$ for any stratum $S_i \subset X$.

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Example of compatible smooth structures I

We assume that a compact Lie group G acts on a symplectic manifold (M, ω) with proper moment map $J : M \rightarrow \mathfrak{g}^*$. Let $Z = J^{-1}(0)$. The quotient space $M_0 = Z/G$ is called a symplectic reduction of M . $C^\infty(M_0)_{can} := C^\infty(M)^G / I^G$, where I^G is the ideal of G -invariant functions vanishing on Z .

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$$C_1^\infty(\overline{G(x)}) := \{f \in C^0(\overline{G(x)}) \mid f = \tilde{f}|_{\overline{G(x)}} \text{ for some } \tilde{f} \in C^\infty(\mathfrak{g})\}.$$

- $\overline{G(x)}$ possesses an algebraic resolution of the singularity at $(x_s, 0) \in \overline{G(x)} \subset \mathfrak{g}$. If $x_s = 0$ the resolvable smooth structure is weakly symplectic and Poisson.

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$$C_1^\infty(\overline{G(x)}) := \{f \in C^0(\overline{G(x)}) \mid f = \tilde{f}|_{\overline{G(x)}} \text{ for some } \tilde{f} \in C^\infty(\mathfrak{g})\}.$$

- $\overline{G(x)}$ possesses an algebraic resolution of the singularity at $(x_s, 0) \in \overline{G(x)} \subset \mathfrak{g}$. If $x_s = 0$ the resolvable smooth structure is weakly symplectic and Poisson.

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 - Structures theorems
- 2 Compatible smooth structures on symplectic stratified spaces
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Symplectic homology

Assume that $C^\infty(M)$ is Poisson.

- $\rightarrow \Omega^{n+1} \xrightarrow{\delta} \Omega^n(M) \rightarrow \dots$
- $\delta(f_0 df_1 \wedge \dots \wedge df_n) := \sum_{i=1}^n (-1)^{i+1} \{f_0, f_i\}_\omega df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n$
 $+ \sum_{1 \leq i < j \leq n} f_0 d\{f_i, f_j\}_\omega \wedge df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge \widehat{df_j} \wedge \dots \wedge df_n.$
- $\delta^2 = 0.$

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Brylinski-Poisson homology of compatible Poisson smooth structures

Theorem

Suppose (X, ω) is a stratified symplectic space provided with a Poisson smooth structure $C^\infty(M)$ which is compatible with symplectic from ω . Then the symplectic homology of the complex $(\Omega(X), \delta)$ is isomorphic to the deRahm cohomology with inverse grading : $H_k(\Omega(X), \delta) = H^{m-k}(\Omega, d)$. It is equal to the singular cohomology $H^{m-k}(X, \mathbb{R})$, if the smooth structure is locally contractible.

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Existence of Hamiltonian flow I

Let (X, ω) be a stratified symplectic space and $C^\infty(X)$ a Poisson smooth structure on X .

Lemma

For any $H \in C^\infty(X)$ the associated Hamiltonian vector field X_H defined on X by setting

$$X_H(f) := \{H, f\}_\omega \text{ for any } f \in C^\infty(X)$$

is a smooth Zariski vector field on X . If x is a point in a stratum S , then $X_H(x) \in T_x S$.

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Existence of Hamiltonian flow II

Theorem

Given a Hamiltonian function $H \in C^\infty(X)$ and a point $x \in X$ there exists a unique smooth curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow X$ such that for any $f \in C^\infty(X)$ we have

$$\frac{d}{dt}f(\gamma(t)) = \{H, f\}.$$

The decomposition of X into strata of equal dimension can be defined by the Poisson algebra of smooth functions.

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Leftschetz decomposition of $\Omega(X)$

Let L be the wedge multiplication by ω .

$$\mathcal{P}_{n-k}(X^{2n}) := \{\alpha \in \Omega^{n-k}(X^{2n}) \mid L^{k+1}\omega = 0\}.$$

Theorem

We have the following decomposition for $k \geq 0$

$$\Omega^{n-k}(X^{2n}) = \mathcal{P}_{n-k}(X^{2n}) \oplus L(\mathcal{P}_{n-k-2}(X^{2n})) \oplus \dots$$

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THANK YOU!

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