Second list of exercises of the course "Geometry and Dynamics of Singular Symplectic manifolds"

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The problems of this course mainly follow the articles by Victor Guillemin, Eva Miranda, Ana Rita Pires, Geoffrey Scott, Jonathan Weitsman and others. We would refer the reader to [GMP11], [GMP14], [GMPS15], [GMPS17], [GMW18b], [GMW18a], [GMW19], [GMW21] and [KMS16] to cite a few places to check the basics and not-so-basics of b-symplectic geometry.

Problem session 2. Classical problems for b^m-symplectic and b^m-contact manifolds. Toric actions, action-angle coordinates and integrable systems on b^m-symplectic manifolds. Perturbations of integrable systems and KAM theory

Exercise 0.1. *Let (M^n, Π) an orientable, connected Poisson manifold. Then, we know that $\Omega^n(M) \cong \mathcal{C}^\infty(M)$. We define the* **modular vector field** *as*

$$X_\Pi^\Omega : \begin{array}{ccc} \mathcal{C}^\infty(M) & \longrightarrow & \mathcal{C}^\infty(M) \\ f & \longmapsto & \frac{\mathcal{L}_{u_f}\Omega}{\Omega} \end{array},$$

or, more formally, $X_\Pi^\Omega(f)$ is the only function such that $\mathcal{L}_{u_f}\omega = X_\Pi^\Omega(f)\Omega$.

u_f, here, denotes the Hamiltonian vector field of f, this means, such that $u_f(g) = \{f, g\}$.

a) Show that it is a well defined derivation.

b) Show that, for any $H \in \mathcal{C}^\infty(M)$ that doesn't vanish anywhere,

$$X_\Pi^{H\Omega} = X_\Pi^\Omega - u_{\log|H|}.$$

c) Let (M^{2m}, ω) a symplectic manifold. Show that the modular vector field $X_{\omega^{-1}}^\Omega$ is a Hamiltonian vector field.

(Hint: Compute the modular vector field in local Darboux coordinates and use the last part of the exercise to get the global result)

d) Compute the modular vector field for the b-Poisson manifold $(\mathbb{R}^2, \{\cdot,\cdot\})$, where $\{x, y\} = y$.

Exercise 0.2. *Consider $\Lambda^*(M)$ the algebra of multivector fields. Recall that the Schouten-Nijenhuis bracket is a bilinear map*

$$[\cdot,\cdot] : \Lambda^k(M) \times \Lambda^l(M) \longrightarrow \Lambda^{(k-1)(l-1)}(M)$$

such that

(i) $[a, b] = (-1)^{(|a|-1)(|b|-1)}[b, a]$

(ii) $[a, [b, c]] = [[a, b], c] + (-1)^{(|a|-1)(|b|-1)}[b, [a, c]]$

(iii) If $|a| = 1$, then $[a, b] = \mathcal{L}_a b$

(iv) If $f \in \mathcal{C}^\infty(M)$, then $[f, a] = -i_{df}(a)$.

Let $\Pi \in \Lambda^2(M)$ a Poisson structure, and let

$$d_\Pi^k : \begin{array}{ccc} \Lambda^k(M) & \longrightarrow & \Lambda^{k+1}(M) \\ V & \longmapsto & [\Pi, V] \end{array}.$$

a) Show that d_Π^* is a cochain differential. This means, prove that $d_\Pi^{k+1} \circ d_\Pi^k = 0$.

The resulting cohomology, denoted by $H_\Pi^*(M)$, is called **Poisson cohomology**.

b) What do the classes of $H_\Pi^1(M)$ represent?

Exercise 0.3. *Consider the b-symplectic manifold $(S^2, Z = \{h = 0\}, \omega = \frac{dh}{h} \wedge d\theta)$, where the coordinates on the sphere are $h \in [-1, 1]$ and $\theta \in [0, 2\pi]$. Compute a moment map of the S^1-action given by the flow of $-\frac{\partial}{\partial\theta}$ and draw its image.*

Exercise 0.4. *Consider the b-symplectic manifold*

$$(\mathbb{T}^2, Z = \{\theta_1 \in \{0, \pi\}\}, \omega = \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2),$$

where the coordinates on the torus are $\theta_1, \theta_2 \in [0, 2\pi]$. Find the ${}^bC^\infty$ Hamiltonian function of the circle action of rotation on the θ_2 coordinate and draw it

Exercise 0.5. *The moment image of a $2n$-dimensional b-symplectic toric manifold is represented by an n-dimensional polytope P, and the corresponding extremal polytope Δ_P is an $(n-1)$-dimensional Delzant polytope. Describe the extremal polytope for $n = 1$ and $n = 2$.*

Exercise 0.6. *Compute the moment map of the toric action $\mathbb{T}^2$ on $\mathbb{C}P^2$ given by $((\theta_1, \theta_2), [z_0 : z_1 : z_2]) \mapsto ([z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2])$. Then, construct a b-toric manifold applying symplectic blow-up and the Gompf sum on $\mathbb{C}P^2$ such that:*

- *it has 6 fixed points, or*
- *it has 12 fixed points.*

What you will obtain is a Hirzebruch surface.

The following exercises are related with the cotangent lift, which is an essential tool since Arnold-Liouville-Mineur Theorem can be restated in a cotangent lift version ([KM17]). These problems also show the connection of singularities with physical systems.

Exercise 0.7. *The coupling of two harmonic oscillators gives can be modelled in $T^*(\mathbb{R}^2)$. Check that, in this system, the energy function*

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(x_1^2 + x_2^2)$$

and the angular momentum function

$$L = x_1 y_2 - x_2 y_1$$

Poisson commute. I.e. check that $\{H, L\} = 0$.

Exercise 0.8. *Prove that the singularity at the top of the spherical pendulum is of focus-focus type. Hint: use local coordinates $(x, y, z) = (x, y, \sqrt{l^2 - x^2 - y^2})$.*

Exercise 0.9. *Compute the infinitesimal generator of the cotangent lift of the action given by:*

$$\begin{array}{rccl} \rho: & (S^1 \times \mathbb{R}) \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & ((\theta, t), \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) & \longmapsto & \rho_{\theta,t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \end{array}$$

and see that it coincides with the vector field associated to the normal form of the focus-focus singularity.

References

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