List of exercises of the course "Geometry and Dynamics of Singular Symplectic manifolds"

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September 2021

The problems of this course mainly follow the articles by Victor Guillemin, Eva Miranda, Ana Rita Pires, Geoffrey Scott, Jonathan Weitsman and others. We would refer the reader to [GMP11], [GMP14], [GMPS15], [GMPS17], [GMW18b], [GMW18a], [GMW19], [GMW21] and [KMS16] to cite a few places to check the basics and not-so-basics of b-symplectic geometry.

1 Problem session 1. Introduction to symplectic geometry, b-symplectic geometry, Poisson manifolds, b-forms, the path method

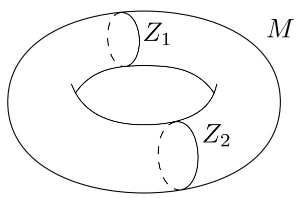
Exercise 1.1. *Check that the Circular Planar Restricted 3 Body Problem provides a b^3-symplectic structure. To do this, consider the symplectic form on $T^*\mathbb{R}^2$ in polar coordinates,*

$$\omega = dr \wedge d\alpha + dP_r \wedge dP_\alpha,$$

and apply to it the non-canonical McGehee change of coordinates, given by $r = \frac{2}{x^2}$, but without altering the momentum associated to r.

Exercise 1.2. *Let $Z = \{z_1, ..., z_k\} \subset S^1$ a finite collection of points within the circle. If we consider (S^1, Z) as a b-manifold, is it true that $TS^1 \cong {}^bTS^1$? This means, are the vector bundles isomorphic? Does this depend on the number of points k?*

Exercise 1.3. *Consider $M = \mathbb{T}^2$ the 2-torus. Let Z_1, Z_2 be two disjoint non-contractible circles embedded in M as in the picture:*



Does the b-manifold (M, Z_1) admit a b-symplectic structure? Does $(M, Z_1 \cup Z_2)$?
Hint: Use the results of the last exercise.

Exercise 1.4. *Let (M, ω) be a symplectic manifold and let Π be the corresponding bivector (i.e., $\omega(X_f, X_g) = \{f, g\} = \Pi(df, dg)$ for any smooth functions f, g). Suppose ω is locally given as*

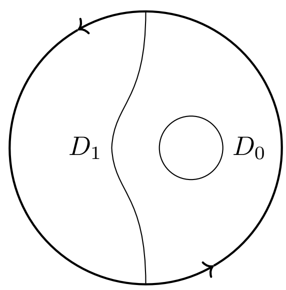
$$\omega = \sum_{i<j} \omega_{ij} dx_i \wedge dx_j.$$

Prove that the coefficients of Π satisfy $\pi_{i<j} = (\omega_{ij})^{-1}$.

Exercise 1.5. *Prove that the cubic polynomial $g(x) = x(x-1)(x-t)$, $0 < t < 1$, defines a Poisson structure on $\mathbb{R}^2$ given by*

$$\Pi = (g(x) - y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

Check that it extends smoothly to a b-symplectic structure on $\mathbb{R}P^2$ with critical set Z given by the real elliptic curve $y^2 = g(x)$. [GL14]



The critical set has two connected components: D_0, containing $\{(0,0),(t,0)\}$ and with trivial normal bundle, and D_1, containing $\{(1,0),(\infty,0)\}$ and with nontrivial normal bundle.

Exercise 1.6. *Prove that S^4 does not admit a b-symplectic structure.*

Exercise 1.7. *Prove the following result from [Cav17]. If a compact oriented manifold M^{2n}, with $n>1$, admits a b-symplectic structure, then there are classes $a,b \in H^2(M;\mathbb{R})$ such that $a^{n-1}b \neq 0$ and $b^2=0$.*

Exercise 1.8. *Prove the following corollaries*

1. *An orientable, compact, b-symplectic manifold M of dimension $2n$ has $b_{2i}(M) \geq 2$ for $0<i<n$.*
2. *For $n>1$, $\mathbb{C}P^n$ has no b-symplectic structure and, for $n>2$, the blow-up of $\mathbb{C}P^n$ along a symplectic submanifold of real codimension greater than 4 also does not carry b-symplectic structures.*

Exercise 1.9. *Compute the b-cohomology class of the b-torus of Radko with $2n$ connected components*

Exercise 1.10. *Let (R,π_R) be a Radko compact surface and (S,π_S) be a compact symplectic surface. Show that $(R\times S, \pi_R+\pi_S)$ is a b-Poisson manifold of dimension 4.*

Exercise 1.11. *Take S^2 with the b-Poisson structure $\Pi_1 = h\frac{\partial}{\partial h}\wedge\frac{\partial}{\partial \theta}$ and the symplectic torus $\mathbb{T}^2$ with dual Poisson structure $\Pi_2 = \frac{\partial}{\partial \theta_1}\wedge\frac{\partial}{\partial \theta_2}$. Prove that*

$$\hat{\Pi} = h\frac{\partial}{\partial h}\wedge\left(\frac{\partial}{\partial \theta}+\frac{\partial}{\partial \theta_1}\right)+\Pi_2$$

is a b-Poisson structure on $S^2\times\mathbb{T}^2$.

Exercise 1.12. *Let (N^{2n+1},π) be a regular corank-1 Poisson manifold, X be a Poisson vector field and $f: S^1\to\mathbb{R}$ a smooth function. Prove that the bivector field*

$$\Pi = f(\theta)\frac{\partial}{\partial \theta}\wedge X+\pi$$

is a b-Poisson structure on $S^1\times N$ if the function f vanishes linearly and the vector field X is transverse to the symplectic leaves of N.

Exercise 1.13. *Prove that the bracket $\{f,g\}=\omega(X_f,X_g)$ for $f,g\in C^\infty$ defines a Poisson structure on a symplectic manifold (M^{2n},ω). Hint: To check the Jacobi identity, expand $d\omega(X_f,X_g,X_h)$.*

References

[Cav17] Gil R. Cavalcanti. Examples and counter-examples of log-symplectic manifolds. *J. Topol.*, 10(1):1–21, 2017.

[GL14] Marco Gualtieri and Songhao Li. Symplectic groupoids of log symplectic manifolds. *Int. Math. Res. Not. IMRN*, pages 3022–3074, 2014.

[GMP11] Victor Guillemin, Eva Miranda, and Ana Rita Pires. Codimension one symplectic foliations and regular Poisson structures. *Bull. Braz. Math. Soc. (N.S.)*, 42(4):607–623, 2011.

[GMP14] Victor Guillemin, Eva Miranda, and Ana Rita Pires. Symplectic and Poisson geometry on *b*-manifolds. *Adv. Math.*, 264:864–896, 2014.

[GMPS15] Victor Guillemin, Eva Miranda, Ana Rita Pires, and Geoffrey Scott. Toric actions on *b*-symplectic manifolds. *Int. Math. Res. Not. IMRN*, pages 5818–5848, 2015.

[GMPS17] Victor Guillemin, Eva Miranda, Ana Rita Pires, and Geoffrey Scott. Convexity for Hamiltonian torus actions on *b*-symplectic manifolds. *Math. Res. Lett.*, 24(2):363–377, 2017.

[GMW18a] Victor W. Guillemin, Eva Miranda, and Jonathan Weitsman. Convexity of the moment map image for torus actions on b^m-symplectic manifolds. *Philos. Trans. Roy. Soc. A*, 376(2131):20170420, 6, 2018.

[GMW18b] Victor W. Guillemin, Eva Miranda, and Jonathan Weitsman. On geometric quantization of *b*-symplectic manifolds. *Adv. Math.*, 331:941–951, 2018.

[GMW19] Victor Guillemin, Eva Miranda, and Jonathan Weitsman. Desingularizing b^m-symplectic structures. *Int. Math. Res. Not. IMRN*, pages 2981–2998, 2019.

[GMW21] Victor W. Guillemin, Eva Miranda, and Jonathan Weitsman. On geometric quantization of b^m-symplectic manifolds. *Math. Z.*, 298(1-2):281–288, 2021.

[KMS16] Anna Kiesenhofer, Eva Miranda, and Geoffrey Scott. Action-angle variables and a KAM theorem for b-Poisson manifolds. *J. Math. Pures Appl. (9)*, 105(1):66–85, 2016.