Geometry and Dynamics of Singular Symplectic manifolds

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Space for notes

Symplectic

- Darboux theorem
- classification of surfaces $[\omega] \to$ De Rham cohomology
- action-angle coordinate — Liouville torus

- S op ca'pos
- toric actions, μ = moment map

$\mathbb{R}^n$

b^m-Symplectic $\subset$ **Poisson**

- b-Darboux theorem (Adv. Math 2014)
- classification of b-symplectic surface

$[\omega]$ ↑ b-form (restatement Radko's classification)

$[\omega] \in {}^bH^2(S) \cong H^2_{DR}(S) \oplus H^1(Z)$ (critical set)

$\cong H_n(S)$

Quantization

Poisson

splitting theorem (Weinstein)

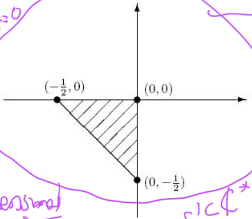
Delzant theorem as a classification scheme

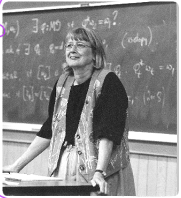
Some classification schemes are still possible when additional data is considered (toric manifolds). Toric manifolds are a particular example of integrable system.

Theorem (Delzant)

Toric manifolds are classified by Delzant's polytopes. More specifically, the bijective correspondence between these two sets is given by the image of the moment map:

$$\begin{array}{ccc} \{\textit{toric manifolds}\} & \longrightarrow & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, F) & \longrightarrow & F(M) \end{array}$$





Moment map for the $\mathbb{T}^2$-action on $\mathbb{C}P^2$ given by $(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] := [z_0 : e^{i\theta_1} z_1 : e^{i\theta_1} z_2]$

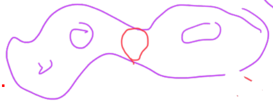
Special submanifolds

Definition (Lagrangian submanifold)

Given a symplectic manifold (M, ω), a submanifold $L \subset M$ is called Lagrangian if $i^*(\omega) = 0$ with $i : L \hookrightarrow M$ the inclusion. Lagrangian submanifolds satisfy: $T_pS^\omega = T_pS$

Examples:

- A curve on an orientable surface.
- A fiber of the moment map on a toric manifold.
- The zero section of the cotangent bundle T^*M.
- The fibers of an integrable system.



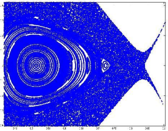
Other important submanifolds are:

- coisotropic when $T_pS^\omega \subset T_pS$.
- isotropic when $T_pS \subset T_pS^\omega$.

Integrability and dynamical systems

Importance of integrability of the associated system and dynamical properties of its solutions such as stability.

$$
\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p} \\
\dot{p} &= -\frac{\partial H}{\partial q}
\end{aligned}
$$



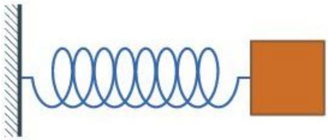
Integrable system

A set of functions $f_1, \ldots, f_n$ on (M^{2n}, ω) such that

- $f_1, \ldots, f_n$ Poisson commute, i.e., $\{f_i, f_j\} = 0, \forall i, j$.
- $df_1 \wedge \cdots \wedge df_n \neq 0$.

The mapping $F : M^{2n} \longrightarrow R^n$ given by $F = (f_1, \ldots, f_n)$ is called moment map.

Coupling two simple harmonic oscillators



- Phase space: $(T^*(\mathbb{R}^2), \omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$.
- Total energy:

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(x_1^2 + x_2^2)$$

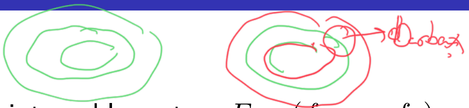
.

- $H = h$ is a sphere S^3.

We have rotational symmetry on this sphere $\rightsquigarrow$ the angular momentum is a constant of motion, $L = x_1 y_2 - x_2 y_1$, $X_L = (-x_2, x_1, -y_2, y_1)$ and

$$X_L(H) = \{L, H\} = 0.$$

Classical integrable systems



The compact regular level sets of an integrable system $F = (f_1, \ldots, f_n)$ on a symplectic manifold are tori (**Liouville tori**).

Theorem (Liouville-Mineur-Arnold)

Semilocally around a Liouville torus:

- *There exist coordinates (action-angle)* $(p_1, \ldots, p_n, \theta_1, \ldots, \theta_n)$ *with values in* $B^n \times \mathbf{T}^n$ *such that* $\omega = \sum_{i=1}^n dp_i \wedge d\theta_i$.
- *The level sets of the coordinates* $p_1, \ldots, p_n$ *correspond to the Liouville tori of* F.

The problem of global existence of action-angle variables is related to monodromy and the Chern class of the fibration given by the moment map.

The Liouville-Mineur-Arnold theorem for symplectic manifolds

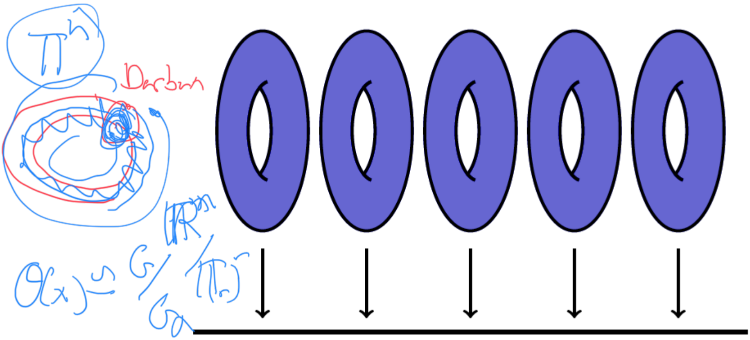


Figure: Liouville tori

In action-angle coordinates (p_i, θ_i) the fibers of F are the tori $\{p_i = c_i\}$ and the symplectic structure is *simple* (Darboux) $\omega = \sum_{i=1}^n dp_i \wedge d\theta_i$.

Definition (Hamiltonian action)

Let G be a compact Lie group acting symplectically on (M, ω).
The action is **Hamiltonian** if there exists an equivariant map $\mu : M \to \mathfrak{g}^*$ such that for each element $X \in \mathfrak{g}$,

$$-d\mu^X = \iota_{X^\#}\omega, \quad (1)$$

with $\mu^X = < \mu, X >$.
The map μ is called the **moment map**.

Moment maps and reduction provide an effective tool to study symmetries in geometrical models in mechanics.

Space for notes

$\mathbb{R}^n \times M(L) \to M(L)$ (compact)

$((t_1, \dots, t_n), p) \to \phi_{X_1}^{t_1} \circ \phi_{X_2}^{t_2} \circ \dots \circ \phi_{X_n}^{t_n}$

$\phi_{X_1}^{t_1}$: t_1-flow of f_1

$F = (f_1, \dots, f_n)$

1. Prove that this defines an $\mathbb{R}^n$-action
2. fiber $\Theta(x) = \mathbb{R}^n / \Lambda_x \simeq \mathbb{T}^k \times \mathbb{R}^{n-k}$ — compact — $n = k$ — $\mathbb{T}^n$ → Liouville theorem; Pablo's theorem
3. Get an induced $\mathbb{T}^n$-action → Hamiltonian

Torus actions in the proof

1. **Topology of the foliation. The fibration in a neighbourhood of a compact connected fiber is a trivial fibration by compact fibers**
2. **These compact fibers are tori: We recover a $\mathbb{T}^n$-action tangent to the leaves of the foliation This implies a process of uniformization of periods.**

$$\begin{array}{rcl} \Phi \ : \ \mathbf{R}^r \times (\mathbf{T}^r \times B^s) & \to & \mathbf{T}^r \times B^s \\ ((t_1, \dots, t_r), m) & \mapsto & \Phi^{(1)}_{t_1} \circ \dots \circ \Phi^{(r)}_{t_r}(m). \end{array} \qquad (2)$$

3. **We prove that this action is symplectic (we use the fact that if Y is a complete vector field of period 1 and ω is a symplectic form for which $\mathcal{L}_Y^2 \omega = 0$, then $\mathcal{L}_Y \omega = 0$).**
4. **As ω is exact in a neighbourhood of the Liouville torus the action is Hamiltonian.**
5. **To construct action-angle coordinates we use Darboux-Carathéodory theorem and the constructed Hamiltonian action of $\mathbb{T}^n$ to drag normal forms from a neighbourhood of a point to a neighbourhood of a fiber.**

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Space for notes

- Integrable systems are semilocally (in the neighbourhood of a Liouville torus) toric

Toric manifolds $\subset$ Integrable systems

$\mu = (f_1, \ldots, f_n)$ $\quad$ $\iota_{X_{f_i}} \omega = -df_i$

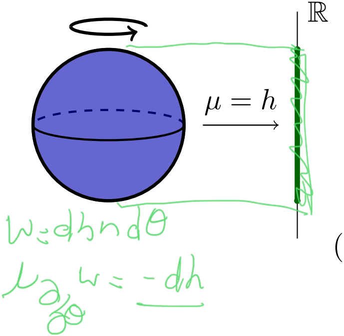
μ is "equivariant" invariant G abelian $G = T^n$

Toric symplectic manifolds

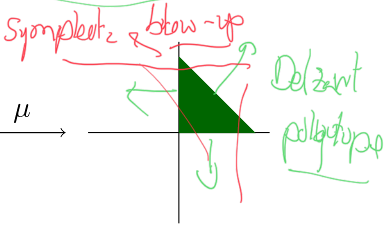
Theorem (Delzant)

Toric manifolds are classified by Delzant's polytopes. The bijective correspondence between these two sets is given by the image of the moment map:

$$\begin{array}{ccc} \{\text{toric manifolds}\} & \longrightarrow & \{\text{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, F) & \longrightarrow & F(M) \end{array}$$



$$\mathbb{CP}^2 \xrightarrow{\ \mu\ }$$



$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it_1} z_1 : e^{it_2} z_2]$$

Liouville-Mineur-Arnold via cotangent lifts

Cotangent lifts

Given a Lie group action $\rho : G \times M \longrightarrow M$, its cotangent lift to T^*M is Hamiltonian with moment map

$$\mu : T^*M \to \mathfrak{g}^*, \quad \langle \mu(\alpha), X \rangle = \langle \lambda|_\alpha, X^\#_{T^*M}|_\alpha \rangle$$

with λ the Liouville one-form.

Example

For $M = G = \mathbf{T}^n$ and the action is translations of the torus on itself the moment map is

$$\mu : \mathbf{T}^n \times \mathbf{R}^n \cong T^*\mathbf{T}^n \to \mathfrak{t}^* \cong \mathbf{R}^n : (\theta, p) \mapsto p.$$

In particular:

Cotangent model in the symplectic case

Semilocally around a Liouville torus, an integrable system is equivalent to the moment map of the cotangent lift of the action by translations of $\mathbf{T}^n$ on itself.

``We revolve around the Sun like any other planet.''

1514, Nicolaus Copernicus.

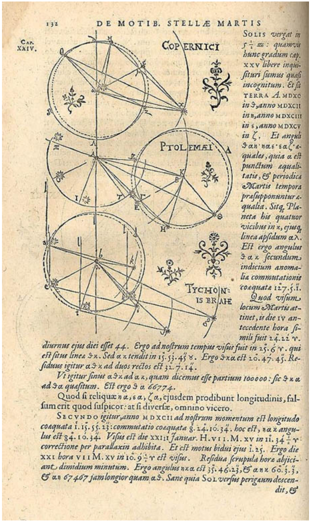

NICOLAI COPERNICI

net, in quo terram cum orbe lunari tanquam epicyclo contineri diximus. Quinto loco Venus nono menſe reducitur. Sextum denique locum Mercurius tenet, octuaginta dierum ſpacio circũ currens. In medio uero omnium reſidet Sol. Quis enim in hoc

pulcherrimo templo lampadem hanc in alio uel meliori loco poneret, quàm unde totum ſimul poſsit illuminare? Siquidem non inepte quidam lucernam mundi, alij mentem, alij rectorem uocant. Trimegiſtus uiſibilem Deum, Sophoclis Electra intuentẽ omnia. Ita profecto tanquam in ſolio regali Sol reſidens circum agentem gubernat Aſtrorum familiam. Tellus quoque minime fraudatur lunari miniſterio, ſed ut Ariſtoteles de animalibus ait,

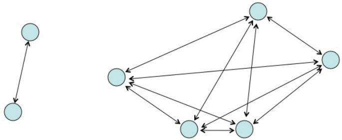
The solar system

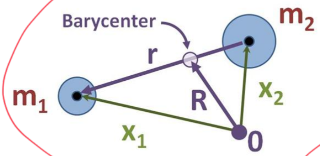
Kepler: Planets spin in elliptical orbits.



The n-body problem

The n-body problem describes the movement of n bodies under mutual attraction.





Integrable for $n = 2$: The Hamiltonian function is

$$H(x_1, x_2, p_1, p_2) := E_{kin} - U = \frac{\|p_1\|^2}{2m_1} + \frac{\|p_2\|^2}{2m_2} - U.$$

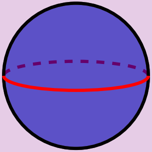
where $U := \mathcal{G} m_1 m_2 \frac{1}{\|x_2 - x_1\|}$ is the gravitational potential. Integrals:

1. Total linear momentum: The problem reduces to determining the relative position $r = x_2 - x_1$.
2. Total angular momentum: makes the problem *planar*.

Space for notes

Examples to keep in mind of b-Poisson manifolds

- A Radko surface.

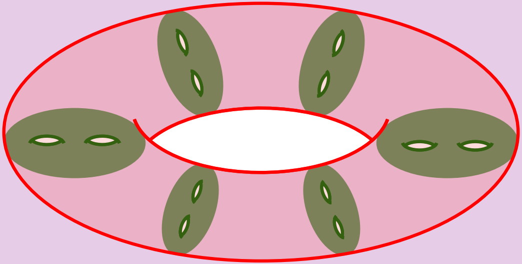
- The product of (R, π_R) a Radko compact surface with a compact symplectic manifold (S, ω) is a b-Poisson manifold.
- corank 1 Poisson manifold (N, π) and X Poisson vector field $\Rightarrow$ $(S^1 \times N, f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi)$ is a b-Poisson manifold if,
  1. f vanishes linearly.
  2. X is transverse to the symplectic leaves of N.

  We then have as many copies of N as zeroes of f.

Poisson Geometry of the critical hypersurface

This last example is semilocally the *canonical* picture of a b-Poisson structure .

1. The critical hypersurface Z has an induced regular Poisson structure of corank 1.
2. There exists a Poisson vector field v transverse to the symplectic foliation induced on Z (modular vector field).
3. (Guillemin-M. Pires) Z is a mapping torus with glueing diffeomorphism the flow of v.



Recall: Singular forms

- A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The **b-tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$

- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are **b-forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

- A **b-symplectic form** is a closed, nondegenerate, b-form of degree 2.
- This dual point of view, allows to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path method because we can play the same tricks as in the symplectic case.

Space for notes

Geometrical invariants

Theorem (Mazzeo-Melrose)

The b-cohomology groups of a compact M are computable by

$$^{b}H^*(M) \cong H^*(M) \oplus H^{*-1}(Z).$$

(Guillemin - M - Pires, 2014)

Corollary (Classification of b-symplectic surfaces à la Moser)

Two b-symplectic forms ω_0 and ω_1 on an orientable compact surface are b-symplectomorphic if and only if $[\omega_0] = [\omega_1]$.

Indeed,

$$^{b}H^*(M) \cong H^*_{\Pi}(M)$$

Space for notes

b-integrable systems

Definition

b-integrable system A set of b-functions[a] $f_1, \dots, f_n$ on (M^{2n}, ω) such that

- $f_1, \dots, f_n$ Poisson commute.
- $df_1 \wedge \dots \wedge df_n \neq 0$ as a section of $\Lambda^n({}^bT^*(M))$ on a dense subset of M and on a dense subset of Z

[a] $c \log|x| + g$

Example

The symplectic form $\frac{1}{h} dh \wedge d\theta$ defined on the interior of the upper hemisphere H_+ of S^2 extends to a b-symplectic form ω on the double of H_+ which is S^2. The triple $(S^2, \omega, log|h|)$ is a b-integrable system.

Example

If $(f_1, \dots, f_n)$ is an integrable system on M, then $(\log|h|, f_1, \dots, f_n)$ on $H_+ \times M$ extends to a b-integrable on $S^2 \times M$.

Action-angle coordinates for b-integrable systems

The compact regular level sets of a b-integrable system are (Liouville) tori.

Theorem (Kiesenhofer-M.-Scott)

Around a Liouville torus there exist coordinates $(p_1, \dots, p_n, \theta_1, \dots, \theta_n) : U \to B^n \times \mathbf{T}^n$ *such that*

$$\omega|_U = \frac{c}{p_1} dp_1 \wedge d\theta_1 + \sum_{i=2}^{n} dp_i \wedge d\theta_i, \tag{3}$$

invariant modular weight

and the level sets of the coordinates $p_1, \dots, p_n$ *correspond to the Liouville tori of the system.*

Reformulation

Integrable systems semilocally $\leftrightsquigarrow$ twisted cotangent lift[a] of a $\mathbb{T}^n$ action by translations on itself to $(T^*\mathbb{T}^n)$.

[a] We replace the Liouville form by $\log|p_1| d\theta_1 + \sum_{i=2}^{n} p_i d\theta_i$.

Proof

1. Topology of the foliation. In a neighbourhood of a compact connected fiber the b-integrable system F is diffeomorphic to the b-integrable system on $W := \mathbf{T}^n \times B^n$ given by the projections $p_1, \dots, p_{n-1}$ and $\log|p_n|$.

2. Uniformization of periods: We want to define integrals whose (b-)Hamiltonian vector fields induce a $\mathbf{T}^n$ action. Start with $\mathbf{R}^n$-action:

$$\begin{array}{rcl} \Phi : \mathbf{R}^n \times (\mathbf{T}^n \times B^n) & \to & \mathbf{T}^n \times B^n \\ ((t_1, \dots, t_n), m) & \mapsto & \Phi^{(1)}_{t_1} \circ \dots \circ \Phi^{(n)}_{t_n}(m). \end{array}$$

   Uniformize to get a $\mathbf{T}^n$ action with fundamental vector fields Y_i.

3. The vector fields Y_i are Poisson vector fields (check $\mathcal{L}_{Y_i}\mathcal{L}_{Y_i}\omega = 0$).

4. The vector fields Y_i are Hamiltonian with primitives $\sigma_1, \dots, \sigma_n \in {}^bC^\infty(W)$. In this step the properties of b-cohomology are essential.Use this action to drag a local normal form (Darboux-Carathéodory) in a whole neighbourhood.

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4. The vector fields Y_i are Hamiltonian with primitives $\sigma_1, \dots, \sigma_n \in {}^b C^\infty(W)$. In this step the properties of b-cohomology are essential.Use this action to drag a local normal form (Darboux-Carathéodory) in a whole neighbourhood.

Proof

1. Topology of the foliation. In a neighbourhood of a compact connected fiber the b-integrable system F is diffeomorphic to the b-integrable system on $W := \mathbf{T}^n \times B^n$ given by the projections $p_1, \dots, p_{n-1}$ and $\log |p_n|$.
2. Uniformization of periods: We want to define integrals whose (b-)Hamiltonian vector fields induce a $\mathbf{T}^n$ action. Start with $\mathbf{R}^n$-action:

$$\begin{array}{rcccl} \Phi & : & \mathbf{R}^n \times (\mathbf{T}^n \times B^n) & \to & \mathbf{T}^n \times B^n \\ & & ((t_1, \dots, t_n), m) & \mapsto & \Phi^{(1)}_{t_1} \circ \dots \circ \Phi^{(n)}_{t_n}(m). \end{array}$$

   Uniformize to get a $\mathbf{T}^n$ action with fundamental vector fields Y_i.
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4. The vector fields Y_i are Hamiltonian with primitives $\sigma_1, \dots, \sigma_n \in {}^b C^\infty(W)$. In this step the properties of b-cohomology are essential.Use this action to drag a local normal form (Darboux-Carathéodory) in a whole neighbourhood.

Surfaces and circle actions

The only orientable compact surfaces admitting an effective action by circles are $\mathbb{S}^2$ and $\mathbb{T}^2$ and the action is equivalent to the standard action by rotations.

In the symplectic case the rotation on the $\mathbb{T}^2$ cannot be Hamiltonian (only symplectic).

$$d\theta_1 \wedge d\theta_2(\frac{\partial}{\partial\theta_1}, \cdot) = d\theta_2.$$

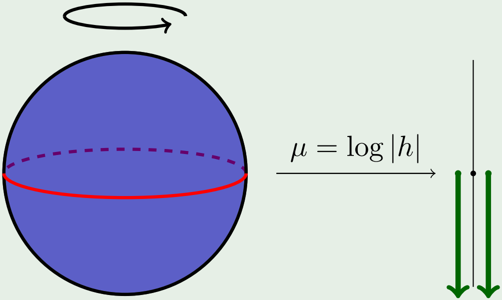
In the b-symplectic case, the toric surfaces are either the sphere or the torus.

Space for notes

The S^1-b-sphere

Example

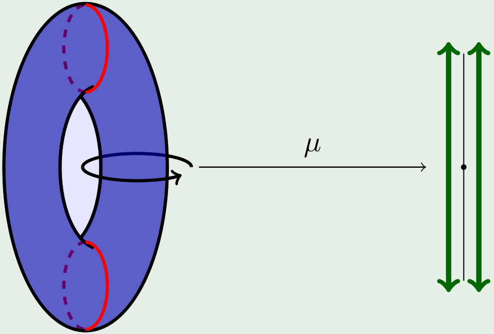
$(\mathbb{S}^2, \omega = \frac{dh}{h} \wedge d\theta)$, with coordinates $h \in [-1,1]$ and $\theta \in [0, 2\pi]$. The critical hypersurface Z is the equator, given by $h = 0$. For the $\mathbb{S}^1$-action by rotations, the moment map is $\mu(h,\theta) = \log|h|$.



The S^1-b-torus

Example

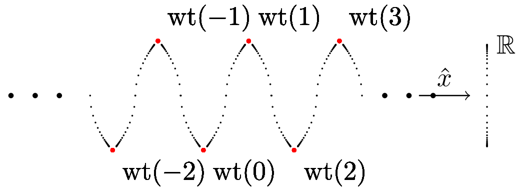
On $(\mathbb{T}^2, \omega = \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2)$, with coordinates: $\theta_1, \theta_2 \in [0, 2\pi]$. The critical hypersurface Z is the union of two disjoint circles, given by $\theta_1 = 0$ and $\theta_1 = \pi$. Consider rotations in θ_2 the moment map is $\mu : \mathbb{T}^2 \longrightarrow \mathbb{R}^2$ is given by $\mu(\theta_1, \theta_2) = \log\left|\frac{1+\cos(\theta_1)}{\sin(\theta_1)}\right|$.



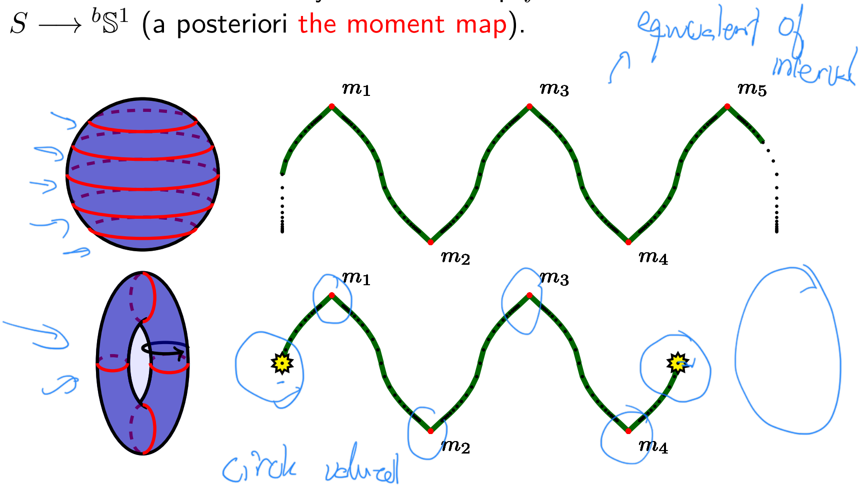
Consider the topological space
$^b\mathbb{R} \cong (\mathbb{Z} \times \overline{\mathbb{R}})/\{(a,(-1)^a\infty) \sim (a+1,(-1)^a\infty)\}$. and the local charts $\{\hat{x}\big|_{\{a\}\times\mathbb{R}}, \hat{y}_a\}_{a\in\mathbb{Z}}$ where $\hat{x}(a,x) = x$ and $\hat{y}_a : ((a-1,0),(a,0)) \to \mathbb{R}$,

$$\hat{y}_a = \begin{cases} -\exp\left((-1)^a \hat{x}/\mathrm{w}(a)\right) & \text{in } ((a-1,0),(a-1,(-1)^{a-1}\infty)) \\ 0 & \text{at } (a-1,(-1)^{a-1}\infty) \\ \exp\left((-1)^a \hat{x}/\mathrm{w}(a)\right) & \text{in } ((a,(-1)^{a-1}\infty),(a,0)) \end{cases}.$$

the function $\mathrm{w} : \mathbb{Z} \to \mathbb{R}_{>0}$ associates some weights to the connected components of the critical hypersurface and is determined by the modular periods of each component.



A toric b-surface is defined by a smooth map $f : S \longrightarrow {}^b\mathbb{R}$ or $f : S \longrightarrow {}^b\mathbb{S}^1$ (a posteriori the moment map).



Classification of toric b-surfaces

Theorem (Guillemin, M., Pires, Scott)

A toric b-symplectic surface is equivariantly b-symplectomorphic to either $(\mathbb{S}^2, Z)$ or $(\mathbb{T}^2, Z)$, where Z is a collection of latitude circles.

The action is the standard rotation, and the b-symplectic form is determined by the modular periods of the critical curves and the regularized Liouville volume.

The weights $w(a)$ of the codomain of the moment map are given by de modular periods of the connected components of the critical hypersurface.

b-toric actions

Definition

An action of $\mathbb{T}^n$ on a b-symplectic manifold (M, ω) is a **Hamiltonian action** if:

- for each $X \in \mathfrak{t}$, the b-one-form $\iota_{X^\#}\omega$ is exact (i.e., has a primitive $H_X \in {}^bC^\infty(M) = \{c_i \log|f| + g\}$).
- for any $X, Y \in \mathfrak{t}$, we have $\omega(X^\#, Y^\#) = 0$.

The action is **toric** if it is effective and the dimension of the torus is half the dimension of M.

b-moment map μ such that

$$< \mu(p), X >= H_X(p),$$

but we have to allow $\mu(p)$ to take values of $\pm\infty$, so we need to extend the pairing to accommodate that as we did in the case of circle actions.

Existence of b-moment maps

Theorem (Guillemin, M., Pires, Scott)

Let $(M, Z, \omega, \mathbb{T}^n)$ be a b-symplectic manifold with an effective Hamiltonian toric action. For an appropriately-chosen ${}^b\mathfrak{t}^*$ or ${}^b\mathfrak{t}^*/\langle a \rangle$, there is a moment map $\mu : M \to {}^b\mathfrak{t}^*$ or $\mu : M \to {}^b\mathfrak{t}^*/\langle a \rangle$.

Example

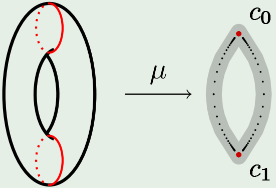
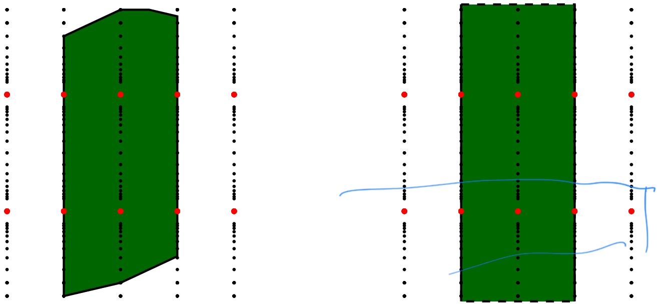


Figure: The moment map μ surjects onto ${}^b\mathfrak{t}^*/\langle 2 \rangle$.

Space for notes

From local to global....

We can reconstruct the b-Delzant polytope from the Delzant polytope on a mapping torus via *symplectic cutting* in a neighbourhood of the critical hypersurface.



This information can be recovered by doing reduction in stages: Hamiltonian reduction of an action of $\mathbb{T}_Z^{n-1}$ and the classification of toric b-surfaces.

The semilocal model

Fix ${}^b\mathfrak{t}^*$ with $wt(1) = c$.
For any Delzant polytope $\Delta \subseteq \mathfrak{t}_Z^*$ with corresponding symplectic toric manifold $(X_\Delta, \omega_\Delta, \mu_\Delta)$, the **semilocal model** of the b-symplectic manifold as

$$M_{\mathrm{lm}} = X_\Delta \times \mathbb{S}^1 \times \mathbb{R} \qquad \omega_{\mathrm{lm}} = \omega_\Delta + c\frac{dt}{t} \wedge d\theta$$

where θ and t are the coordinates on $\mathbb{S}^1$ and $\mathbb{R}$ respectively. The $\mathbb{S}^1 \times \mathbb{T}_Z$ action on M_{lm} given by $(\rho, g) \cdot (x, \theta, t) = (g \cdot x, \theta + \rho, t)$ has moment map $\mu_{\mathrm{lm}}(x, \theta, t) = (t, \mu_\Delta(x))$.

Space for notes

A b-Delzant theorem

Theorem (Guillemin, M., Pires, Scott)

The maps that send a b-symplectic toric manifold to the image of its moment map

$$\{(M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*)\} \to \{b\text{-}Delzant\ polytopes\ in\ {}^b\mathfrak{t}^*\} \tag{4}$$

and

$$\{(M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*/\langle N\rangle)\} \to \{b\text{-}Delzant\ polytopes\ in\ {}^b\mathfrak{t}^*/\langle N\rangle\} \tag{5}$$

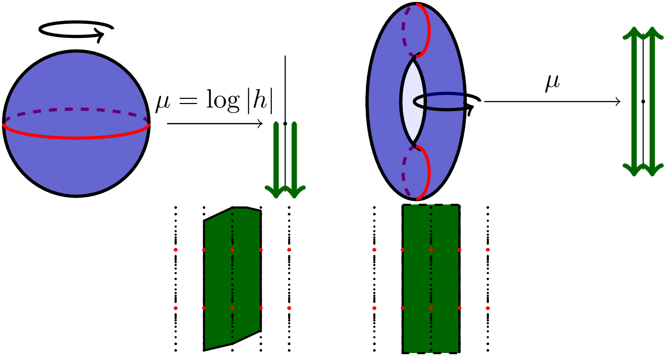
are bijections.

Toric b-manifolds can be of two types:

1. ${}^b\mathbb{T}^2 \times X$ (with X a toric symplectic manifold of dimension $(2n-2)$)
2. ${}^b\mathbb{S}^2 \times X$ and manifolds obtained via symplectic cutting (for instance, $m\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$, with $m, n \geq 1$).

Space for notes

A b-Delzant theorem



Guillemin-M.-Pires-Scott

There is a one-to-one correspondence between b-toric manifolds and b-Delzant polytopes. Toric b-manifolds are either:

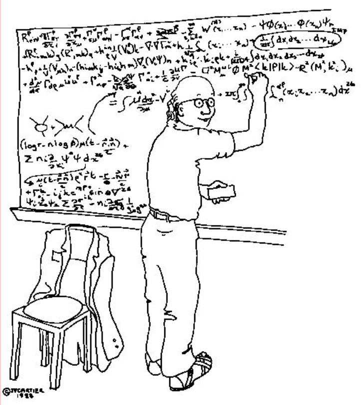
- ${}^b\mathbb{T}^2 \times X$ (X a toric symplectic manifold of dimension $(2n-2)$).
- obtained from ${}^b\mathbb{S}^2 \times X$ via symplectic cutting.

Space for notes

Classical vs. Quantum: Another love story.

1. Classical systems
2. Observables $C^\infty(M)$
3. Bracket $\{f, g\}$

1. Quantum System
2. Operators in $\mathcal{H}$ (Hilbert)
3. Commutator $[A, B]_h = \frac{2\pi i}{h}(AB - BA)$



"At this point we notice that this equation is beautifully simplified if we assume that space–time has 92 dimensions."

"I still don't understand quantum theory."

Geometric Quantization in a nutshell

- (M^{2n}, ω) symplectic manifold with integral $[\omega]$.
- $(\mathbb{L}, \nabla)$ a complex line bundle with a connection ∇ such that $curv(\nabla) = -i\omega$ (prequantum line bundle).
- A real polarization $\mathcal{P}$ is a Lagrangian foliation. Integrable systems provide natural examples of real polarizations.
- Flat sections equation: $\nabla_X s = 0$, $\forall X$ tangent to $\mathcal{P}$.

Definition

A Bohr-Sommerfeld leaf is a leaf of a polarization admitting global flat sections.

Example: Take $M = S^1 \times \mathbb{R}$ with $\omega = dt \wedge d\theta$, $\mathcal{P} =< \frac{\partial}{\partial\theta} >$, $\mathbb{L}$ the trivial bundle with connection 1-form $\Theta = td\theta \rightsquigarrow \nabla_X \sigma = X(\sigma) - i < \Theta, X > \sigma$ $\rightsquigarrow$Flat sections: $\sigma(t,\theta) = a(t).e^{it\theta}$ $\rightsquigarrow$ Bohr-Sommerfeld leaves are given by the condition $t = 2\pi k, k \in \mathbb{Z}$.
Liouville-Mineur-Arnold $\leftrightsquigarrow$ this example is the canonical one.

Quantization via action-angle coordinates

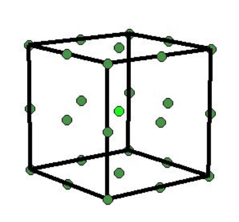
Theorem (Guillemin-Sternberg)

If the polarization is a regular fibration with compact leaves over a simply connected base B, then the Bohr-Sommerfeld set is given by, $BS = \{p \in M, (f_1(p), \dots, f_n(p)) \in \mathbb{Z}^n\}$ *where $f_1, \dots, f_n$ are global action coordinates on B.*

For toric manifolds the base B is the image of the moment map.

Quantization

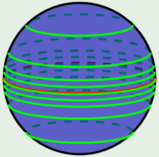
"Quantize" these systems counting Bohr-Sommerfeld leaves. For integrable systems Bohr-Sommerfeld leaves are just "integral" Liouville tori.



b-Bohr-Sommerfeld leaves

Example

Consider on the toric b-sphere: Bohr-Sommerfeld leaves near a connected component of Z in the local model $\omega_\Delta + c\frac{dt}{t} \wedge d\theta$ correspond to $c\log(|h|) = -n$ thus $h = e^{-n/c}$ or $h = -e^{-n/c}$.



Flat sections are given by $s(h, \theta) = f(h)e^{ic\log(|h|)\theta}$ with f analytically flat for $|h| = e^{-n/c}$ and c is the weight of the connected component of Z.

b-Bohr-Sommerfeld leaves

Theorem (Mir-M.-Weitsman, 2021)

Let (M, Z, ω) be a $2n$-dimensional b-toric symplectic manifold, and $\mu : M \to B$ its moment map with B simply connected. Then there exists a globally defined system of action coordinates $f_1, \dots, f_n$ on B; and, for any $p, q \in B$ in the Bohr-Sommerfeld set, we can assume that $f_1(p) = \cdots = f_n(p) = 0$ and $f_1(q), \dots, f_n(q)$ are integers.

Definition of formal quantization

Assume M is non-compact but ϕ proper:
Let $\mathbb{Z}_T \in \mathfrak{t}^*$ be the weight lattice of T and α a regular value of the moment map.
If T acts freely the reduced space $M_\alpha = \phi^{-1}(\alpha)/T$ is a prequantizable symplectic manifold and $[Q, R] = 0$ asserts that $Q(M)_\alpha = Q(M_\alpha)$ where $Q(M)_\alpha$ is the α-weight space of $Q(M)$. We define the formal quantization of M as $Q(M) = \bigoplus_\alpha Q(M_\alpha)$

Theorem (Braverman-Paradan)

$Q(M) = ind(\overline{\partial})$

Formal quantization of b-symplectic manifolds

A b-symplectic manifold is prequantizable if:

- $M \setminus Z$ is prequantizable
- The cohomology classes given under the Mazzeo-Melrose isomorphism applied to $[\omega]$ are integral.

Theorem (Guillemin-M.-Weitsman)

- $Q(M)$ *exists.*
- $Q(M)$ *is finite-dimensional.*

Idea of proof

$Q(M) = Q(M_+) \bigoplus Q(M_-)$

and an ϵ-neighborhood of Z does not contribute to quantization.

Formal quantization of b-symplectic manifolds

$$Q(M) = \bigoplus_{\alpha} Q(M//_{\alpha}T)\alpha,$$

where the sum is taken over all weights α of T.
also

Theorem (Braverman, Loizides, Song)

Formal geometric quantization of b-symplectic manifolds is the index of an operator.

A happy end

Theorem (Mir-M.-Weitsman, 2021)

For b-toric symplectic manifolds Formal geometric quantization= Geometric Quantization.