Geometry and Dynamics of Singular Symplectic manifolds

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Hénan University, Day 2
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The restricted 3-body problem

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem.
- After doing a change to Mc Gehee coordinates $(r = \frac{2}{x^2}, \quad x \in \mathbf{R}^+,)$ the symplectic structure becomes a singular object $\omega = -\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG.$ for $x > 0$

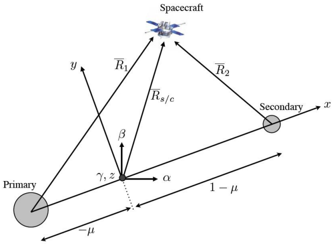


Figure: Circular 3-body problem

Model of these systems

b-Poisson / b-symplectic

$m = 1$

behavior change

parity m

$$\omega = \frac{1}{x_1^m} dx_1 \wedge dy_1 + \sum_{i \geq 2} dx_i \wedge dy_i$$

$\omega = x_1 \, dx_1 \wedge dy_1 + \sum dx_i \wedge dy_i$ ($\frac{1}{2} d(x_1^2)$)

$\omega^n \to x_1 \, dx_1 \wedge \dots \wedge dx_n$

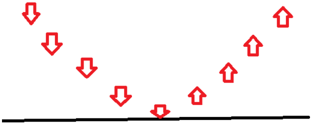
m even $\to$ symplectic structure

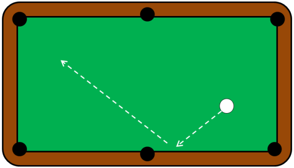
m odd $\to$ folded symplectic

Close to $x_1 = 0$, the systems behave like,

Critical set $\to$ induced dynamics on Z

and not like,





Symplectic surfaces with singularities (Radko's surfaces)

We want to modify the volume form on S by making it "explode" when we get close to a union of curves Z. We want this "blow up" process to be controlled.

b-symplectic → $\omega = \frac{c}{x} dx \wedge dy$

b-Poisson

$\Pi = \frac{x}{c} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$

Schouten bracket

smooth bivector field

$[\Pi, \Pi] = 0$ → Jacobi

$\{f, g\} := \Pi(df, dg)$

$C^\infty(M) \times C^\infty(M) \to C^\infty(M)$

$(f, g) \to \{f, g\}$

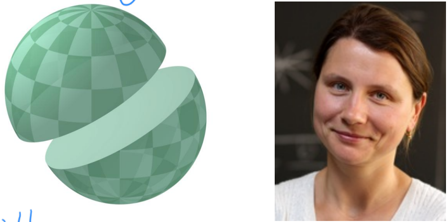
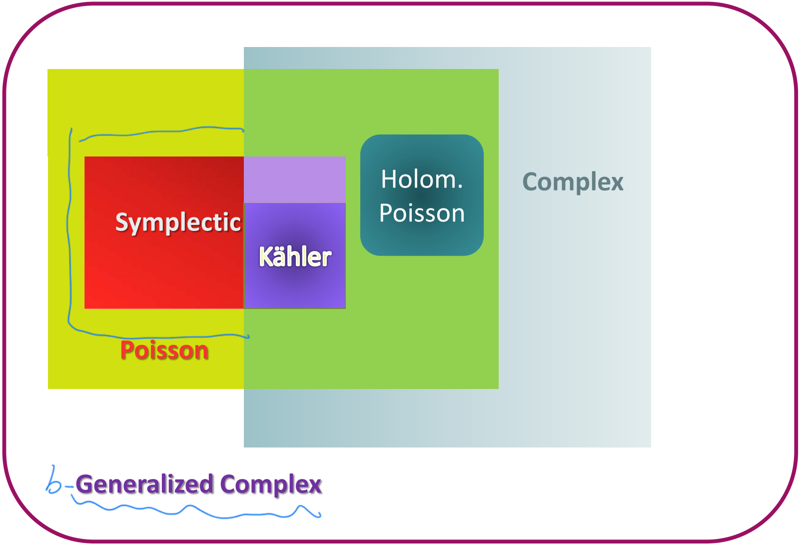


Figure: A Radko surface and Olga Radko

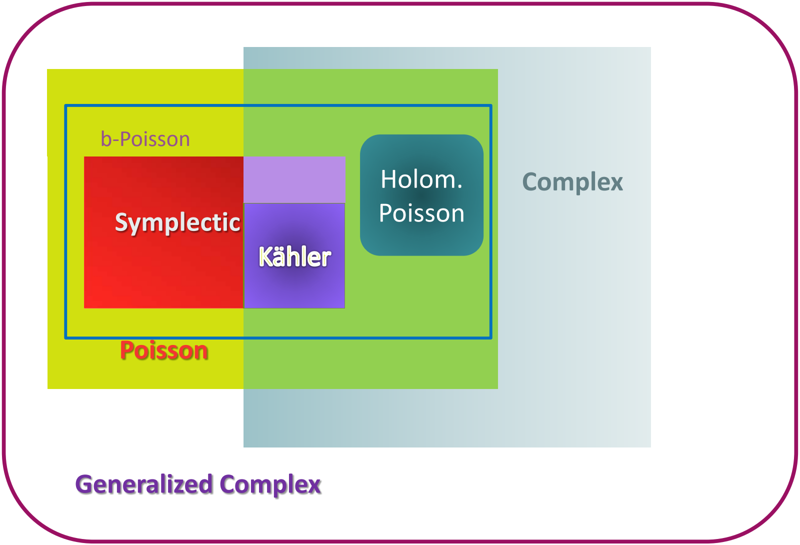
What does "controlled" mean here? We want that the 2-form looks locally $\omega = \frac{c}{x} dx \wedge dy$ (for points in Z).

Geometries involved



Zooming in...

b-Poisson close to symplectic



But sometimes it is good to zoom out...

Zooming out...to gain perspective

Poisson, 1809

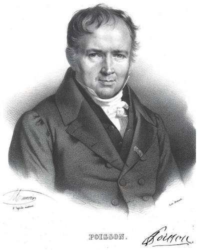
$\{f,g\} = \omega(X_f, X_g)$, $\iota_{X_f}\omega = -df$

MÉMOIRE

Sur la Variation des Constantes arbitraires dans les questions de Mécanique,

Lu à l'Institut le 16 Octobre 1809;

Par M. POISSON.



$(b,a) = \{b,a\}$
standard Poisson bracket

constante *a* ni la constante *b*; dans d'autres cas elle ne contiendra aucune constante arbitraire, et se réduira à une constante déterminée; mais, afin de rappeler l'origine de cette quantité, qui représente une certaine combinaison des différences partielles des valeurs de *a* et *b*, nous ferons usage de cette notation (b, a), pour la désigner; de manière que nous aurons généralement

$$\frac{db}{ds}\cdot\frac{da}{d\varphi} - \frac{da}{ds}\cdot\frac{db}{d\varphi} + \frac{db}{du}\cdot\frac{da}{d\psi} - \frac{da}{du}\cdot\frac{db}{d\psi} + \frac{db}{dv}\cdot\frac{da}{d\theta} - \frac{da}{dv}\cdot\frac{db}{d\theta} = (b, a).$$

Figure: Poisson bracket

Singular symplectic manifolds as Poisson manifolds

The local models ω

$$\omega = \frac{\mathbf{1}}{\mathbf{x_1^m}}\mathbf{dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

are formally not a smooth form but their dual defines a smooth Poisson structure! as their dual

$$\mathbf{\Pi} = \mathbf{x_1^m}\frac{\partial}{\partial \mathbf{x_1}} \wedge \frac{\partial}{\partial \mathbf{y_1}} + \sum_{\mathbf{i \geq 2}}^{\mathbf{n}} \frac{\partial}{\partial \mathbf{x_i}} \wedge \frac{\partial}{\partial \mathbf{y_i}}$$

is well-defined. The structure Π is a bivector field which satisfies the integrability equation $[\Pi, \Pi] = 0$. The Poisson bracket associated to Π is given by the equation

$$\{f, g\} := \Pi(df, dg)$$

Poisson Structure
Π 1) and 2)

Poisson structures as brackets

Symplectic case

$\{f,g\} = \omega(X_f, X_g) = \iota_{X_f}\omega(X_g) \overset{b}{=} X_g(-df) = X_f(dg)$

derivation

A Poisson bracket on a manifold is given by $\mathbb{R}$-bilinear operation

$$\begin{array}{rccc} \{\cdot,\cdot\}: & C^\infty(M) & \longrightarrow & C^\infty(M) \\ & (f,g) & \longmapsto & \{f,g\} \end{array}$$

$\{f,f\}=0 \Leftrightarrow$ Noether's theorem

which satisfies:

1. Anti-symmetry, $\{f,g\} = -\{g,f\}$ for any $f,g \in C^\infty(M)$
2. Leibnitz rule, $\{f, g\cdot h\} = g\cdot\{f,h\} + \{f,g\}\cdot h$
3. Jacobi identity, $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

Example $\mathbb{T}^4$ K Klein bottle → Any manifold admits a Poisson structure

$\{f,g\}=0$

$\Pi = \sum_{i,j} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$

Space for proofs

Question

$S^4 \rightarrow H^4(\ ) = 0$

S^2

$\omega = dh \wedge d\theta$

symplectic structures

S^4 cannot admit any symplectic structure

b-symplectic structure

Proposition (Stokes) A symplectic form on a compact manifold cannot be exact.

$\omega = d\alpha$ $\quad d\omega = 0 \rightarrow [\omega] \in H^2(M, \mathbb{R})$

$\neq 0$

Space for proofs

Poisson

Symplectic

Poisson manifold is a union of symplectic manifolds

(Poisson 1809)

Hamiltonian

Π Poisson

$X_f := \Pi(df, \ )$ → vector field

$f \in C^\infty(M)$

$D = \langle X_f, f \in C^\infty(M) \rangle$

distribution does not rank

Frobenius

$[X_f, X_g] = X_{\{f,g\}}$ (h)

Exercise

Stefan

→ D is integrable

$T_p(S) = D_x$ S is a symplectic manifold

$\{f,g\}$ → symplectic structure

$\mathfrak{g}^*$ $\mathfrak{so}(3,\mathbb{R})^*$

coadjoint orbits

ω

KKS

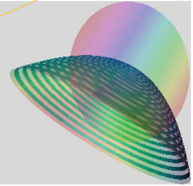
Poisson structures as bivector fields

Poisson structures

A Poisson structure is a bivector field Π with $[\Pi, \Pi] = 0$.

The Poisson manifold is locally a product of a symplectic manifold with a Poisson manifold with vanishing Poisson structure at the point (Weinstein's splitting theorem).

$$(P^n, \Pi, p) \approx (M^{2k}, \omega, p_1) \times (P_0^{n-2k}, \Pi_0, p_2)$$



This defines a symplectic foliation.

b-Poisson structures

Definition

Let (M^{2n}, Π) be an (oriented) Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface called *the critical hypersurface* and we say that Π is a b-**Poisson structure** on (M, Z).

Other singularities

It is possible to generalize this definition (M.-Planas-Scott) to consider more general Poisson structures.

Symplectic foliation of a b-Poisson manifold

The symplectic foliation has dense symplectic leaves and codimension 2 symplectic leaves whose union is Z.

b-Darboux theorem

Theorem

For all $p \in Z$, there exists a Darboux coordinate system $x_1, y_1, \ldots, x_n, y_n$ centered at p such that Z is defined by $x_1 = 0$ and

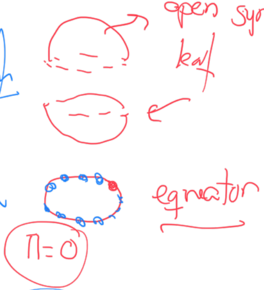
$$\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

$\Pi = h \frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}$

b-Poisson structure.

Symplectic foliation?

Symplectic foliation



higher dimension

$[\Pi, \Pi]$ trivector

$[\Pi, V]$ + & Poisson cohomology

Marcello's Theorem

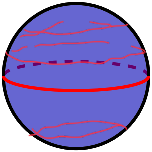
Π bivector $[\Pi, \Pi] = 0$ ← integrability condition. (Schouten)

on a 2-dimensional surface is a Poisson structure

(Ex 1.7)

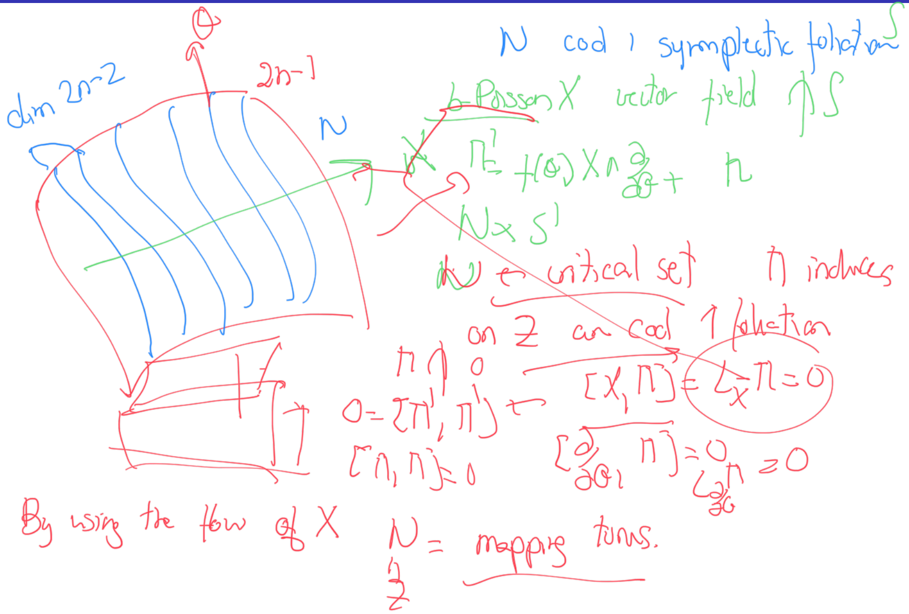
Examples

- A Radko surface.

- The product of (R, π_R) a Radko compact surface with a compact symplectic manifold (S, ω) is a b-Poisson manifold.
- corank 1 Poisson manifold (N, π) and X Poisson vector field $\Rightarrow$ $(S^1 \times N, f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi)$ is a b-Poisson manifold if,
  1. f vanishes linearly.
  2. X is transverse to the symplectic leaves of N.

  We then have as many copies of N as zeroes of f.

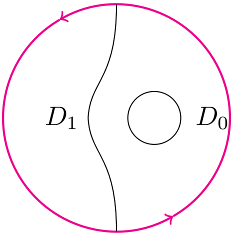

dim 2n-2
2n-1
N cod 1 symplectic foliation
b-Poisson X vector field
critical set
on Z a cod 1 foliation
By using the flow of X
mapping tori.

Another example (exercise in the list)

The cubic polynomial $g(x) = x(x-1)(x-t)$, $0 < t < 1$, defines a Poisson structure on $\mathbb{R}^2$ given by

$$\pi = (g(x) - y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},$$

which extends smoothly to a b-symplectic structure on $\mathbb{R}P^2$ with critical set Z given by the real elliptic curve $y^2 = g(x)$.



The critical set has two connected components: D_0, containing $\{(0,0),(t,0)\}$ and with trivial normal bundle, and D_1, containing $\{(1,0),(\infty,0)\}$ and with nontrivial normal bundle.

Modular vector field

Definition

Given a Poisson manifold (M, Π) and a volume form Ω, the modular vector field X_Π^Ω associated to the pair (Π, Ω) is the derivation given by the mapping

$$f \mapsto \frac{L_{X_f}\Omega}{\Omega}$$

1. $L_{X_\Pi^\Omega}(\Pi) = 0$ and $L_{X_\Pi^\Omega}(\Omega) = 0$.
2. $X^{H\Omega} = X^\Omega - X_{log(H)}$. $\rightsquigarrow$ its first cohomology class in Poisson cohomology does not depend on Ω.
3. Examples of unimodular (vanishing modular class) Poisson manifolds: symplectic manifolds.
4. In the case of b-Poisson manifolds in dimension 2, $\{x, y\} = y$ and the modular vector field is $\frac{\partial}{\partial x}$.

Modular vector fields

Modular vector field for Darboux form

The modular vector field of a local b-Poisson manifold with local normal form,

$$\Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

with respect to the volume form $\Omega = \sum_i dx_i \wedge dy_i$ is,

$$X^\Omega = \frac{\partial}{\partial x_1}.$$

The modular vector field of a b-Poisson manifold is tangent to the critical set Z and is transverse to the symplectic leaves of the induced symplectic foliation on Z.

Induced Poisson structures

Induced Poisson structures

a b-Poisson structure Π on M^{2n} induces a regular corank 1 Poisson structure on Z.

Given a Poisson manifold Z with codimension 1 symplectic foliation $\mathcal{L}$,

1. Does $(Z, \Pi_{\mathcal{L}})$ extend to a b-Poisson structure on a neighbourhood of Z in M?
2. If so to what extent is this structure unique?

The $\mathcal{L}$-De Rham complex

Choose $\alpha \in \Omega^1(Z)$ and $\omega \in \Omega^2(Z)$ such that for all $L \in \mathcal{L}$ (symplectic foliation) such that for all $L \in \mathcal{L}$, $i_L^*\alpha = 0$ and $i_L^*\omega = \omega_L$.

$$d\alpha = \alpha \wedge \beta, \beta \in \Omega^1(Z) \tag{1}$$

Therefore we can consider the complex

$$\Omega_{\mathcal{L}}^k = \Omega^K / \alpha\Omega^{k-1}$$

Consider $\Omega_0 = \alpha \wedge \Omega$ we get a short exact sequence of complexes

$$0 \longrightarrow \Omega_0 \xrightarrow{i} \Omega \xrightarrow{j} \Omega_{\mathcal{L}} \longrightarrow 0$$

By differentiation of 1 we get $0 = d(d\alpha) = d\beta \wedge \alpha - \beta \wedge \beta \wedge \alpha = d\beta \wedge \alpha$, so $d\beta$ is in Ω_0, i.e., $d(j\beta) = 0$.

First obstruction class

We define the **obstruction class** $c_1(\Pi_{\mathcal{L}}) \in H^1(\Omega_{\mathcal{L}})$ to be $c_1(\Pi_{\mathcal{L}}) = [j\beta]$

Notice that $c_1(\Pi_{\mathcal{L}}) = 0$ iff we can find a closed one form for the foliation.

The $\mathcal{L}$-De Rham complex

Assume now $c_1(\Pi_\mathcal{L}) = 0$ then, we obtain $d\omega = \alpha \wedge \beta_2$.

Second obstruction class

We define the **obstruction class** $c_2(\Pi_\mathcal{L}) \in H^2(\Omega_\mathcal{L})$ to be

$$c_2(\Pi_\mathcal{L}) = [j\beta_2]$$

Main property

$c_2(\Pi_\mathcal{L}) = 0 \Leftrightarrow$ there exists a **closed** 2-form, ω, such that $i_L^*(\omega) = \omega_L$.

The role of these invariants

The role of these invariants

$c_1(\Pi_{\mathcal{L}}) = c_2(\Pi_{\mathcal{L}}) = 0 \Leftrightarrow$ there exists a Poisson vector field v transversal to L.

Relation of v, ω and α:

1. $\iota_v \alpha = 1$.
2. $\iota_v \omega = 0$.

The fibration is a symplectic fibration and v defines an Ehresmann connection.

Dynamics of codimension-1 foliations on Poisson manifolds with vanishing invariants

Let β satisfy $d\alpha = \beta \wedge \alpha$. With respect to the volume form $\alpha \wedge \omega^n$ $\iota(v_{\mathrm{mod}})\omega_L = \beta_L$.

Theorem

A regular corank 1 Poisson manifold is unimodular iff we can choose closed defining one-form α for the symplectic foliation (i.e. if and only if $c_1(\Pi_{\mathcal{L}}) = 0$).

The b-Poisson case

The Poisson structure induced on the critical hypersurface of a b-Poisson structure manifold has vanishing invariants $c_1(\Pi_{\mathcal{L}})$ and $c_2(\Pi_{\mathcal{L}})$.

Summing up,

The foliation induced by a b-Poisson structure on its critical hypersurface satisfies,

- we can choose the defining one-form α to be closed.
- symplectic structure on leaves which extends to a closed 2-form ω on M

Given a symplectic foliation on a corank 1 regular Poisson manifold α and ω exist if and only if the invariants $c_1(\Pi_{\mathcal{L}})$ and $c_2(\Pi_{\mathcal{L}})$ vanish.

Question

Is every codimension one regular Poisson manifold with vanishing invariants the critical hypersurface of a b-Poisson manifold?
We will answer this question next week.

A theorem of Tischler: Foliations given by closed forms

Theorem

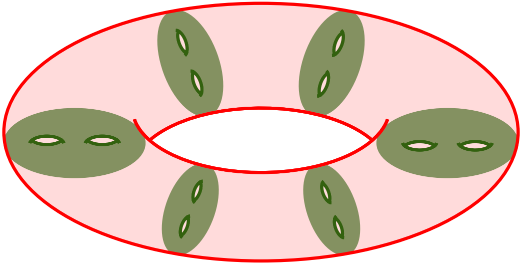
Let M be a compact manifold without boundary that admits a non-vanishing closed 1-form. Then M is a fibration over S^1.

The irrational flow

Observe that this is NOT telling us that the foliation given by α itself IS a fibration.

Theorem (Guillemin-M.-Pires)

If $\mathcal{L}$ contains a compact leaf L, then Z is the mapping torus of the symplectomorphism $\phi : L \to L$ determined by the flow of a Poisson vector field v transverse to the symplectic foliation.



This description also works for b^m-Poisson structures.

Invariants: Dimension 2

Radko classified b-Poisson structures on compact oriented surfaces giving a list of invariants:

- Geometrical: The topology of S and the curves γ_i where Π vanishes.
- Dynamical: The periods of the "modular vector field" along γ_i.
- Measure: The regularized Liouville volume of S, $V_h^\epsilon(\Pi) = \int_{|h|>\epsilon} \omega_\Pi$ for h a function vanishing linearly on the curves $\gamma_1, \ldots, \gamma_n$.

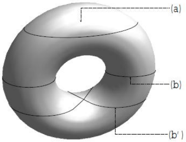


Figure: Two admissible vanishing curves (a) and (b) for Π; the ones in (b') is not admissible.

Singular forms

- A vector field v is a **b-vector field** if $v_p \in T_p Z$ for all $p \in Z$. The **b-tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$

- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are **b-forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

- A **b-symplectic form** is a closed, nondegenerate, b-form of degree 2.
- This dual point of view, allows to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path method because we can play the same tricks as in the symplectic case.

Space for notes

Example

$M = \mathbb{T}^4$ and $Z = \mathbb{T}^3 \times \{0\}$. Consider on Z the codimension 1 foliation given by $\theta_3 = a\theta_1 + b\theta_2 + k$, with rationally independent $a, b \in \mathbb{R}$. Then take

$$h = \log(\sin\theta_4),$$
$$\alpha = \frac{a}{a^2+b^2+1}\, d\theta_1 + \frac{b}{a^2+b^2+1}\, d\theta_2 - \frac{1}{a^2+b^2+1}\, d\theta_3,$$
$$\omega = d\theta_1 \wedge d\theta_2 + b\, d\theta_1 \wedge d\theta_3 - a\, d\theta_2 \wedge d\theta_3,$$

The 2-form $\omega_\Pi = dh \wedge \alpha + \omega$ defines a b-symplectic form in a neighbourhood of Z, which can be extended to M.

Geometrical invariants

Theorem (Mazzeo-Melrose)

The b-cohomology groups of a compact M are computable by

$$ {}^bH^*(M) \cong H^*(M) \oplus H^{*-1}(Z). $$

Corollary (Classification of b-symplectic surfaces à la Moser, Guillemin-M.-Pires)

Two b-symplectic forms ω_0 and ω_1 on an orientable compact surface are b-symplectomorphic if and only if $[\omega_0] = [\omega_1]$.

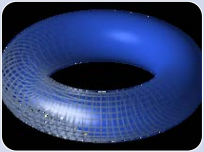
Indeed,

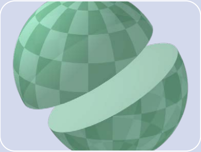
$$ {}^bH^*(M) \cong H^*_{\Pi}(M) $$

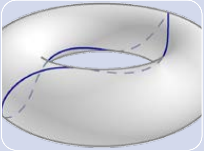
(Singular) symplectic manifolds

- b^m -Symplectic
- Symplectic
- Folded symplectic

Déjà-vu...







Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle coordinates

b-Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle theorem

Folded symplectic manifolds

- Darboux theorem (Martinet)
- Delzant-type theorems (Cannas da Silva-Guillemin-Pires)
- Action-agle theorem (M-Cardona)

Examples and counterexamples

Orientable Surface	CP^2	S^4
• Is symplectic • Is folded symplectic • (orientable or not) is b-symplectic	• Is symplectic • Is folded symplectic • Is not b-symplectic	• Is not symplectic • Is not b-symplectic • Is folded-symplectic

Desingularizing b^m-symplectic structures

Theorem (Guillemin-M.-Weitsman)

Given a b^m-symplectic structure ω on a compact manifold (M^{2n}, Z):

- *If $m = 2k$, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*
- *If $m = 2k+1$, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z.*

In particular:

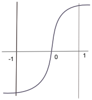
- Any b^{2k}-symplectic manifold admits a symplectic structure.
- Any b^{2k+1}-symplectic manifold admits a folded symplectic structure.
- The converse is not true: S^4 admits a folded symplectic structure but no b-symplectic structure.

Sketch of the proof: $m = 2k$

General principle: If you do not like something, just change it!

$$\omega = \frac{dx}{x^{2k}} \wedge \left(\sum_{i=0}^{2k-1} \alpha_i x^i \right) + \beta \qquad (2)$$

- $f \in \mathcal{C}^\infty(\mathbb{R})$ odd function s.t. $f'(x) > 0$ for $x \in [-1, 1]$,



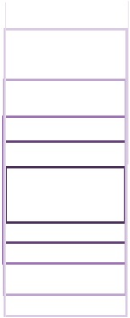
and such that outside $[-1, 1]$,

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \text{for} \quad x < -1 \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \text{for} \quad x > 1 \end{cases}$$

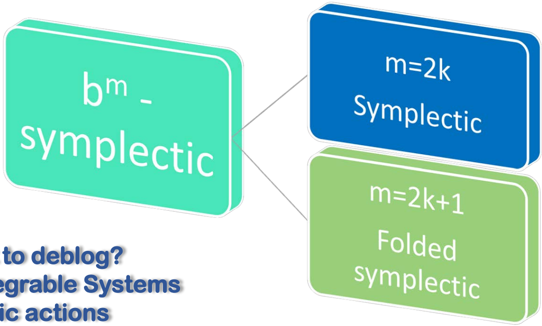
- Re-scale on ϵ.
- Replace $\frac{dx}{x^{2k}}$ by df_ϵ to obtain $\omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k-1} \alpha_i x^i) + \beta$ which is symplectic.

Applications of desingularization

- Convexity for $\mathbb{T}^k$-actions.



- Delzant theorem and Delzant-type theorem for semitoric systems (bolytopes).
- Applications to KAM.
- Periodic orbits of problems in celestial mechanics and applications to stability.



What to deblog?

- **Integrable Systems**
- **Toric actions**
- **Forms..**