# Geometry and Dynamics of Singular Symplectic manifolds

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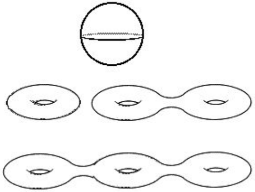
# Topological classification of compact surfaces

Any connected closed surface is homeomorphic to:

1. the sphere
2. the connected sum of $g$ tori, for $g \geq 1$
3. the connected sum of $k$ real projective planes, for $k \geq 1$.



Classification with additional structures may depend on new invariants.
Example: Riemannian structure $\rightsquigarrow$ curvature is an invariant.

# The antisymmetric case

oriented surface $\leftrightsquigarrow$ area form $\omega$ antisymmetric.

$\omega \in \Omega^2(M)$

$(S, \omega_0) \sim (S, \omega_1)$

surface

$d\omega = 0$

$[\omega] \in H^2_{DR}(M)$

$\boxed{\varphi^* \omega_1 = \omega_0}$ Equivalence

It is a closed 2-form and $\omega \neq 0$ (symplectic structure).

## Theorem (Moser)

*Two area forms on a surface $\omega_0$ and $\omega_1$ $[\omega_0] = [\omega_1]$ are equivalent.*

## Idea behind: Moser's path method

The linear path $\omega_t = (1-t)\omega_0 + t\omega_1$ is a path of symplectic structures $\rightsquigarrow$ (Moser's trick) integration of the flow of $X_t$ satisfying $\iota_{X_t}\omega_t = -\beta$ for $\omega_0 - \omega_1 = d\beta$, ($X_t(\phi_t) = \frac{d\phi_t}{dt}$) given by the path method yields the diffeomorphism.

# Space for proofs

$(\Leftarrow)$

$[\omega_0] = [\omega_1] \iff \omega_1 - \omega_0 = d\beta$ $\quad \boxed{\beta}$ 1-form

Idea of Moser: consider the following path $\boxed{\omega_t = t\omega_1 + (1-t)\omega_0}$

path of area forms $d\omega_t = 0$ $\quad d\omega_t = t d(\omega_1) + (1-t) d\omega_0$ $\quad 0 \quad 0$

((exercise) $\omega_1 \neq 0$ $\omega_0 > 0 \to \omega_t > 0$)

$\iota_{X_t}\omega_t = -\beta$ $\leftarrow$ Moser's equation $\quad \omega_t > 0$

For each $\beta$ and $\omega_t$ $\exists ! \, X_t$ solution

contraction $\omega_t$ with the vector field $X_t$

Take $t$-dependent flow $X_t(\varphi_t) = \frac{d\varphi_s}{ds}\Big|_{t=0}$

Claim $\boxed{\varphi_t^* \omega_t = \omega_0}$ $(t=1)$ $\boxed{\varphi_1^* \omega_1 = \omega_0}$

Proof of claim

Lie derivative

① $\frac{d}{dt}(\varphi_t^* \omega_t) = \varphi_t^*\left(\frac{d\omega_t}{dt} + \mathcal{L}_{X_t}\omega_t\right)$

② (Cartan's magic formula) $\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega$

$\frac{d}{dt}(\varphi_t^* \omega_t) \overset{?}{=} 0 \to ?$

$\varphi_t^* \omega_t$ constant

$t=0$ $\varphi_0 = Id$ $\varphi_0^* \omega_0 = \omega_0$

$? = \frac{d\omega_t}{dt} = \varphi_t^*\left(\frac{d\omega_t}{dt} + \mathcal{L}_{X_t}\omega_t\right) = 0$

$\frac{d\omega_t}{dt} = \frac{d}{dt}(t\omega_1 + (1-t)\omega_0) = \omega_1 - \omega_0 = d\beta$ $\Big\} = 0$

$\mathcal{L}_{X_t}\omega_t \overset{②}{=} d\iota_{X_t}\omega_t + \iota_{X_t} d\omega_t = -d\beta$ $\quad 0$

$\blacksquare$

# Symplectic structures

$\omega$ non-degenerate $\forall \beta\ \exists!\ X_\beta$ vector field $\boxed{\iota_{X_\beta}\omega = \beta}$

data

unknown $\iota_{X_\beta}\omega = \beta \leftarrow$ 1-form

Data

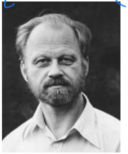
$d\omega = 0$ $\quad \det(\omega_{ij}) \neq 0$ $\quad \boxed{\omega = \sum \omega_{ij}\, dx_i \wedge dx_j}$

- A symplectic structure is a non-degenerate closed 2-form $\omega$.
- Non-degeneracy gives a natural isomorphism between $T^*(M)$ and $T(M)$.
- For every $f$, there is a unique vector field $X_f$ (Hamiltonian vector field),

$$\boxed{\iota_{X_f}\omega = -df}$$

Exercise: $\boxed{\omega_0 = \sum dp_i \wedge dq_i}$ $(\mathbb{R}^{2n}, \omega_0)$

The flow of the Hamiltonian vector field associated to $H: \mathbb{R}^{2n} \to \mathbb{R}$



$$\boxed{\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}}$$

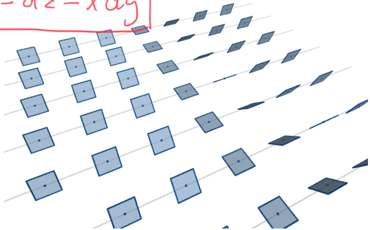
Figure: Sir William Rowan Hamilton, Jürgen Moser and Hamilton's equations.

Hamilton's equations are the equations of the flow of a Hamiltonian vector field in Darboux coordinates.

non-Squeezing theorem

Ker $\alpha$

$\boxed{\alpha = dz - x\,dy}$

| **Symplectic** | **Contact** |
|---|---|
| $\dim M = 2n$ | $\dim M = 2n+1$ |
| 2-form $\omega$, non-degenerate $d\omega = 0$ | 1-form $\alpha$, $\alpha \wedge (d\alpha)^n \neq 0$ |
| Hamiltonian $\iota_{X_H}\omega = -dH$ | Reeb $\alpha(R) = 1$, $\iota_R d\alpha = 0$<br>Ham. $\begin{cases} \iota_{X_H}\alpha = H \\ \iota_{X_H} d\alpha = -dH + R(H)\alpha. \end{cases}$ |

# Symplectic manifolds

even dimensional

- Locally, any symplectic form $\omega$ on a $2n$-dimensional manifold can be written as, $\omega = \sum_{i=1}^{2n} dx_i \wedge dy_i$, Darboux theorem.
- Any orientable surface is a symplectic manifold.
- Cotangent bundles $T^*(M)$ with symplectic form $\omega = -d\lambda$ ($\lambda$ is a Liouville one form).
- Classification problems for symplectic geometry in dimension $> 2$ are HARD.
- Moser's path method is still the most famous trick to construct symplectomorphisms.

# Space for proofs

Proof Darboux theorem

$\forall p \in (M^{2n}, \omega)$ $\exists$ $U \ni p$ local coordinates $(x_1, y_1, \dots, x_n, y_n)$ such that $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$

Proof Darboux local normal form $\boxed{\omega = \omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i}$ $\omega_0$ initial symplectic form

$\exp : T_pM \to M$ (w.r.t auxilliary metric)

constant form form on a vector space

$E = F \oplus F^*$ $\omega((x, \varphi), (x', \varphi')) = \varphi(x') - \varphi'(x)$

Because of linear algebra we can find a symplectic basis $\{e_i, f_i\}$ $e_i(f_j) = \delta_{ij}$ $\longrightarrow$ $\omega_0 = \sum dx_i \wedge dy_i$ $x_i\, y_i$ coordinate on $\{e_i, f_i\}$

Apply path method $\omega_0$ and $\omega_1$ $i_{X_t}\omega_t = -\beta$ $\beta = 0$ on $p$

$\longrightarrow$ $\boxed{\varphi_t^* \omega_t = \omega_0}$ $\varphi_1^* \omega_1 = \omega_0$ ∎

# Higher dimensions?

Some classification schemes are still possible when additional data is considered (toric manifolds). Toric manifolds are a particular example of integrable system.

**Theorem (Delzant)**

*Toric manifolds are classified by Delzant's polytopes. More specifically, the bijective correspondence between these two sets is given by the image of the moment map:*

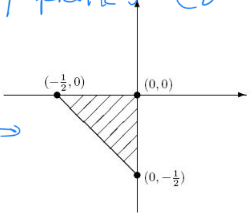
$$\begin{array}{ccc} \{\textit{toric manifolds}\} & \longrightarrow & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, F) & \longrightarrow & F(M) \end{array}$$

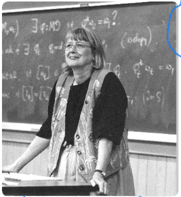
Del(T)

Symplectic geometry ⟶ linear algebra

$\mu : M \to \mathfrak{g}^* \cong \mathbb{R}$

$\mathbb{C}P^2$



$\langle \mu(x), X \rangle$, $X \in \mathfrak{g}$

Hamiltonian function of the fundamental field of the action

Moment map for the $\mathbb{T}^2$-action on $\mathbb{C}P^2$ given by $(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] := [z_0 : e^{i\theta_1} z_1 : e^{i\theta_1} z_2]$

# Some remarkable (open) problems in Symplectic/Contact

- Existence, obstructions and classification problems for Symplectic and contact structures.
- Periodic orbits of Reeb/Hamiltonian vector fields: Existence (Weinstein conjecture) and abundance (for star-shaped hypersurfaces of $\mathbb{R}^{2n}$/Arnold conjecture, Conley conjecture, the Hamiltonian-Seifert conjecture).
- Let algebra be your friend: Count periodic orbits through the eyes of algebraic topology (Floer and Contact Homology).

# Two guiding conjectures

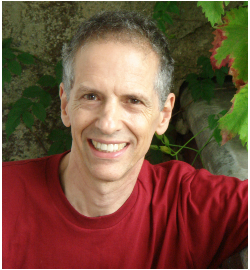
### Weinstein's conjecture

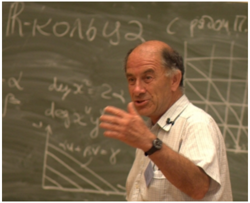
The Reeb vector field of a contact compact manifold admits at least one periodic orbit.

### Arnold's conjecture

Given a $t$-dependent Hamiltonian $H_t : \mathbb{R} \times M^{2n} \to \mathbb{R}$

$$\#\{\text{periodic orbits } X_{H_t}\} \geq \sum_{k=0}^{2n} \beta_k.$$





If $H$ is autonomous, Arnold conjecture is a consequence of Morse inequalities.

# Why periodic orbits?

*On peut alors avec avantage prendre [les] solutions périodiques comme première approximation, comme orbite intermédiaire [...].Ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénetrer dans une place jusqu'ici reputée inabordable.*

*H. Poincaré. Les méthodes nouvelles de la mécanique céleste Gauthier-Villars et fils, Paris, 1892.*

# Special submanifolds

## Definition (Lagrangian submanifold)

Given a symplectic manifold $(M, \omega)$, a submanifold $L \subset M$ is called Lagrangian if $i^*(\omega) = 0$ with $i : L \hookrightarrow M$ the inclusion. Lagrangian submanifolds satisfy: $T_pS^\omega = T_pS$

Examples:

- A curve on an orientable surface.
- A fiber of the moment map on a toric manifold.
- The zero section of the cotangent bundle $T^*M$.
- The fibers of an integrable system.

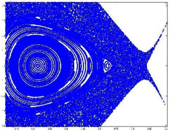
Other important submanifolds are:

- coisotropic when $T_pS^\omega \subset T_pS$.
- isotropic when $T_pS \subset T_pS^\omega$.

# Integrability and dynamical systems

Importance of integrability of a Hamiltonian system and dynamical properties of its solutions such as stability.

$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q}$$



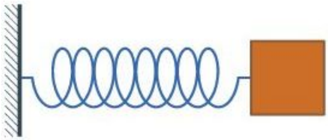
## Integrable system

A set of functions $f_1, \dots, f_n$ on $(M^{2n}, \omega)$ such that

- $f_1, \dots, f_n$ Poisson commute, i.e., $\{f_i, f_j\} = 0, \forall i, j$.
- $df_1 \wedge \cdots \wedge df_n \neq 0$ on an open dense set.

The mapping $F : M^{2n} \longrightarrow R^n$ given by $F = (f_1, \dots, f_n)$ is called moment map.

# Coupling two simple harmonic oscillators



- Phase space: $(T^*(\mathbb{R}^2), \omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$.
- Total energy:
$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(x_1^2 + x_2^2)$$
.
- $H = h$ is a sphere $S^3$.

We have rotational symmetry on this sphere $\rightsquigarrow$ the angular momentum is a constant of motion, $L = x_1y_2 - x_2y_1$, $X_L = (-x_2, x_1, -y_2, y_1)$ and

$$X_L(H) = \{L, H\} = 0.$$

# Classical integrable systems

The compact regular level sets of an integrable system $F = (f_1, \dots, f_n)$ on a symplectic manifold are tori (**Liouville tori**).

**Theorem (Liouville-Mineur-Arnold)**

*Semilocally around a Liouville torus:*

- *There exist coordinates (action-angle)* $(p_1, \dots, p_n, \theta_1, \dots, \theta_n)$ *with values in* $B^n \times \mathbf{T}^n$ *such that* $\omega = \sum_{i=1}^n dp_i \wedge d\theta_i$.
- *The level sets of the coordinates* $p_1, \dots, p_n$ *correspond to the Liouville tori of* $F$.
- *The flow of the Hamiltonian vector field on each Liouville torus is linear.*

The problem of global existence of action-angle variables is related to monodromy and the Chern class of the fibration given by the moment map.

# The Liouville-Mineur-Arnold theorem

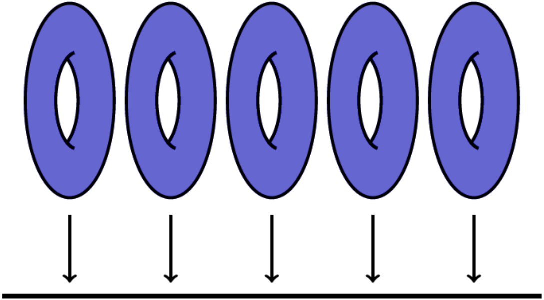
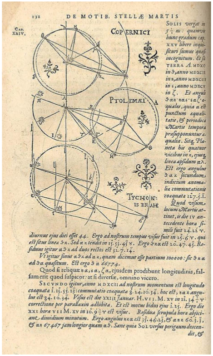


Figure: Liouville tori

- In action-angle coordinates $(p_i, \theta_i)$ the fibers of $F$ are the tori $\{p_i = c_i\}$ and the symplectic structure is *simple* (Darboux) $\omega = \sum_{i=1}^{n} dp_i \wedge d\theta_i$.
- This theorem can be reformulated as a cotangent lift (see next problem session).

# Action coordinates and Liouville 1-form


132 DE MOTIB. STELLÆ MARTIS

CAP. XXIV.

COPERNICI

PTOLEMAEI

TYCHONIS BRAHE

SOLIS vergat in 5 ½ ♒: quamvis hunc gradum cap. XXV libere inquisituri simus quasi incognitum. Et si TERRA A. MDXC in ϑ, anno MDXCII in η, anno MDXCIII in ε, anno MDXCV in ζ. Et anguli ϑαη, ηαε, εαζ æquales, quia α est punctum æqualitatis, & periodica Martis tempora præsupponuntur æqualia. Sitq; Planeta his quatuor vicibus in κ, ejusq; linea apsidum αλ. Est ergo angulus ϑακ secundum indicium anomaliæ commutationis coæquata 127.5.1. Quod visum locum Martis attinet, is die IV antecedente hora simili fuit 24.22 ♈. diurnus ejus diei esset 44. Ergo ad nostrum tempus visus fuit in 25.6 ♈. qui est situs lineæ ϑκ. Sed ακ tendit in 15.53.45 ♉. Ergo ϑκα est 20.47.45. Residuus igitur αϑκ ad duos rectos est 32.7.14.

Vt igitur sinus αϑκ ad ακ, quam dicemus esse partium 100000: sic ϑκα ad ϑα quæsitum. Est ergo ϑα 66774.

Quod si reliquæ ηα, εα, ζα, ejusdem prodibunt longitudinis, falsum erit quod suspicor: at si diversæ, omnino vicero.

SECVNDO igitur, anno MDXCII ad nostrum momentum est longitudo coæquata 1.15.55.23: commutatio coæquata 8.24.10.34. hoc est, ηακ angulus est 84.10.34. Visus est die XXIII Januar. H. VII. M. XV in 11.34 ½ ♈ correctione per parallaxin adhibita. Et est motus bidui ejus 1.25. Ergo die XXI hora VII M. XV in 10.9 ½ ♈ est visus. Residua scrupula horæ abjiciant dimidium minutum. Ergo angulus ηκα est 35.46.23, & αηκ 60.3.3, & αη 67467 jam longior quam αϑ. Sane quia SOL versus perigæum descendit, &

It was an astronomer and mathematician (Henri Mineur) who gave an explicit formula of action coordinates:

$$p_i = \int_{\gamma_i} \alpha$$

where $\gamma_i$ is a cycle of a Liouville torus and $(\omega = d\alpha)$.

# Semilocal and global toric actions

**Definition (Hamiltonian action)**

Let $G$ be a compact Lie group acting symplectically on $(M, \omega)$.
The action is **Hamiltonian** if there exists an equivariant map $\mu : M \to \mathfrak{g}^*$ such that for each element $X \in \mathfrak{g}$,

$$-d\mu^X = \iota_{X^\#}\omega, \tag{1}$$

with $\mu^X =< \mu, X >$.
The map $\mu$ is called the **moment map**.

Moment maps and reduction provide an effective tool to study symmetries in geometrical models in mechanics.

# Torus actions in the proof

1. **Topology of the foliation. The fibration in a neighbourhood of a compact connected fiber is a trivial fibration by compact fibers**
2. **These compact fibers are tori: We recover a $\mathbb{T}^n$-action tangent to the leaves of the foliation This implies a process of uniformization of periods.**

$$\begin{array}{rccl} \Phi & : & \mathbf{R}^r \times (\mathbf{T}^r \times B^s) & \rightarrow & \mathbf{T}^r \times B^s \\ & & ((t_1, \dots, t_r), m) & \mapsto & \Phi^{(1)}_{t_1} \circ \dots \circ \Phi^{(r)}_{t_r}(m). \end{array} \tag{2}$$

3. **We prove that this action is symplectic (we use the fact that if $Y$ is a complete vector field of period $1$ and $\omega$ is a symplectic form for which $\mathcal{L}^2_Y \omega = 0$, then $\mathcal{L}_Y \omega = 0$).**
4. **As $\omega$ is exact in a neighbourhood of the Liouville torus the action is Hamiltonian.**
5. **To construct action-angle coordinates we use Darboux-Carathéodory theorem and the constructed Hamiltonian action of $\mathbb{T}^n$ to drag normal forms from a neighbourhood of a point to a neighbourhood of a fiber.**

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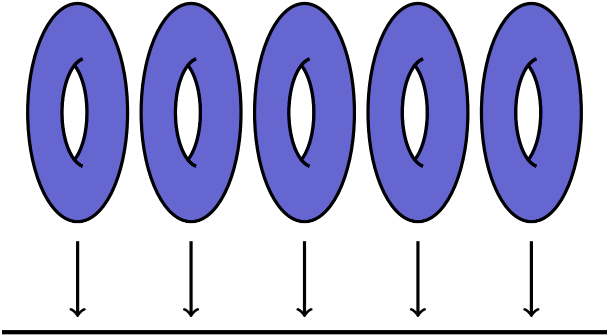
# Perturbations of integrable systems

KAM theory $\rightsquigarrow$ "some" of the Liouville torus survive under perturbations of the integrable system.



# Toric symplectic manifolds

## Theorem (Delzant)

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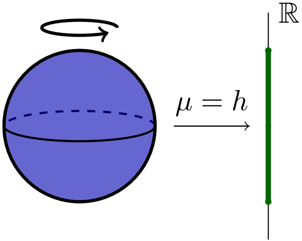
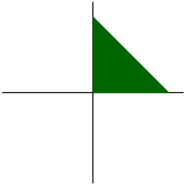



$\mathbb{CP}^2 \xrightarrow{\mu}$

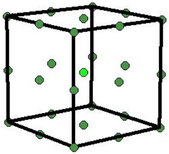


$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it_1} z_1 : e^{it_2} z_2]$$

# Applications to Quantization

**Theorem (Guillemin-Sternberg, Hamilton)**

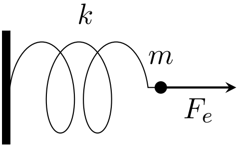
*The geometric quantization of a toric manifold is given by the Liouville tori over the integral points of the Delzant polytope (Bohr-Sommerfeld leaves).*



In the example of the sphere Bohr-Sommerfeld leaves are given by integer values of height (or, equivalently) leaves which divide out the manifold in integer areas.

# Singularities in physical examples

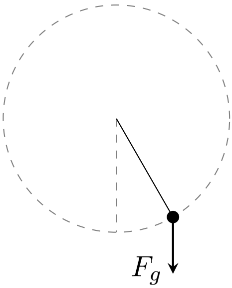
Harmonic oscillator




Elliptic singularity

$$f = x^2 + y^2$$

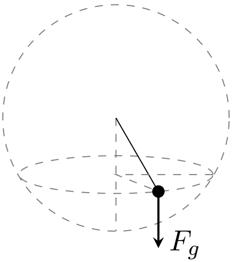
Simple pendulum




Hyperbolic singularity

$$f = xy$$
(or $f = x^2 - y^2$)

Spherical pendulum




Focus-focus singularity

$$f_1 = x_1 y_2 - x_2 y_1$$
$$f_2 = x_1 y_1 + x_2 y_2$$

# The solar system

''We revolve around the Sun like any other planet.''

1514, Nicolaus Copernicus.

NICOLAI COPERNICI

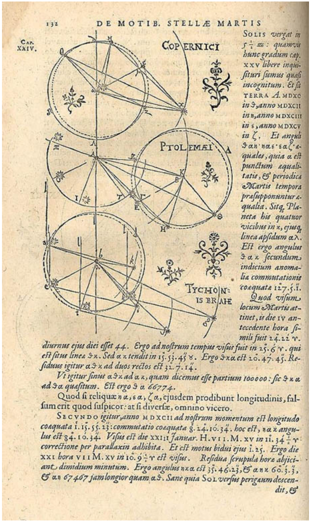
net, in quo terram cum orbe lunari tanquam epicyclo contineri diximus. Quinto loco Venus nono mense reducitur. Sextum deniq; locum Mercurius tenet, octuaginta dierum spacio circũ currens. In medio uero omnium residet Sol. Quis enim in hoc



pulcherrimo templo lampadem hanc in alio uel meliori loco poneret, quàm unde totum simul possit illuminare? Siquidem non inepte quidam lucernam mundi, alij mentem, alij rectorem uocant. Trimegistus uisibilem Deum, Sophoclis Electra intuentẽ omnia. Ita profecto tanquam in solio regali Sol residens circum agentem gubernat Astrorum familiam. Tellus quoque minime fraudatur lunari ministerio, sed ut Aristoteles de animalibus ait,

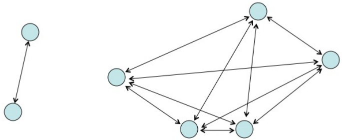
# The solar system

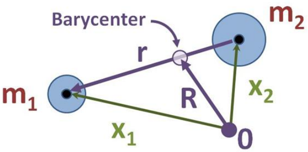
Kepler: Planets spin in elliptical orbits.



# The $n$-body problem

The $n$-body problem describes the movement of $n$ bodies under mutual attraction.






Integrable for $n = 2$: The Hamiltonian function is

$$H(x_1, x_2, p_1, p_2) := E_{kin} - U = \frac{\|p_1\|^2}{2m_1} + \frac{\|p_2\|^2}{2m_2} - U.$$

where $U := \mathcal{G} m_1 m_2 \frac{1}{\|x_2 - x_1\|}$ is the gravitational potential. Integrals:

1. Total linear momentum: The problem reduces to determining the relative position $r = x_2 - x_1$.
2. Total angular momentum: makes the problem *planar*.

# The restricted $3$-body problem

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem.

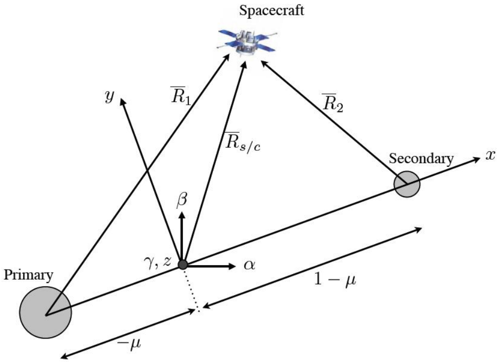



Figure: Circular 3-body problem

# Planar restricted 3-body problem

- The time-dependent self-potential of the small body is $U(q,t) = \frac{1-\mu}{|q-q_1|} + \frac{\mu}{|q-q_2|}$, with $q_1 = q_1(t)$ the position of the planet with mass $1-\mu$ at time $t$ and $q_2 = q_2(t)$ the position of the one with mass $\mu$.
- The Hamiltonian of the system is $H(q,p,t) = p^2/2 - U(q,t)$, $(q,p) \in \mathbf{R}^2 \times \mathbf{R}^2$, where $p = \dot{q}$ is the momentum of the planet.
- Consider the canonical change $(X, Y, P_X, P_Y) \mapsto (r, \alpha, P_r =: y, P_\alpha =: G)$.
- Introduce **McGehee coordinates** $(x, \alpha, y, G)$, where $\boxed{r = \frac{2}{x^2}}$, $x \in \mathbf{R}^+$, can be then extended to infinity ($x = 0$).
- The symplectic structure becomes a singular object $\boxed{\omega = -\frac{4}{x^3}dx \wedge dy + d\alpha \wedge dG.}$ for $x > 0$

not smooth $x=0$

$\Pi \Rightarrow$ Poisson structure

- The integrable 2-body problem for $\mu = 0$ is integrable with respect to the singular $\omega$.

$d\bar{x}$

# Model of these systems

$$\boxed{\omega = \frac{1}{x_1^m} dx_1 \wedge dy_1 + \sum_{i \geq 2} dx_i \wedge dy_i}$$

$b^m$- symplectic form

Close to $x_1 = 0$, the systems behave like,

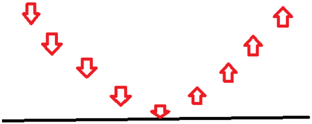
$\omega = x_1^m dx_1 \wedge dy_1 + \sum dx_i \wedge dy_i$ → folded forms

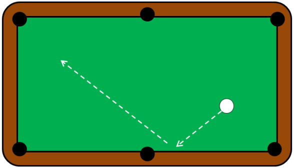
$\Pi = x_1^m \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$

$[\Pi, \Pi] = 0$

and not like,

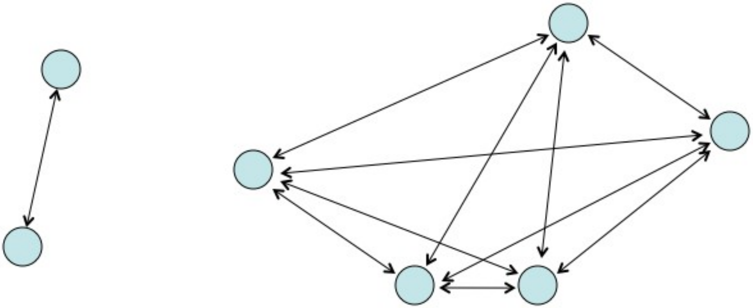
Poisson structure





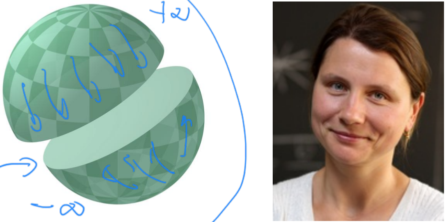
# Other examples

- Kustaanheimo-Stiefel regularization for $n$-body problem (useful for binary collisions) $\rightsquigarrow$ folded-type symplectic structures with hyperbolic singularities



- two fixed-center problem via Appell's transformation $\rightsquigarrow$ combination of folded-type and $b^m$-symplectic structures $\rightsquigarrow$ Dirac structures.

We want to modify the volume form on $S$ by making it "explode" when we get close to a union of curves $Z$. We want this "blow up" process to be controlled.



$+\infty$

$-\infty$

$+\infty - \infty$

finite volume

Figure: A Radko surface and Olga Radko

$\frac{dx \wedge dy}{xy}$    ${}^b T^*M$

What does "controlled" mean here? We want that the 2-form looks locally $\omega = \frac{c}{x} dx \wedge dy$ (for points in $Z$).

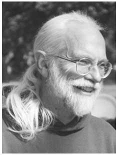
$\frac{dx \wedge dy}{x^2+y^2}$ ← E-symplectic manifold

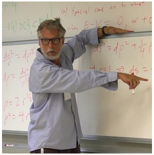
elliptic singularities

Bert Pym

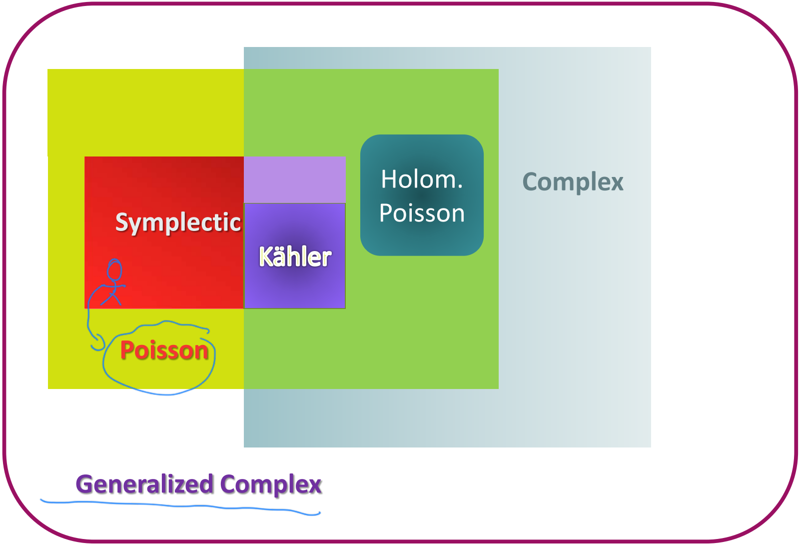



1. Because some non-compact symplectic manifolds can be compactified as compact singular symplectic manifolds.
2. Because these singularities often appear as regularization transformations in celestial mechanics.
3. Because they model certain manifolds with boundary.
4. Why not?

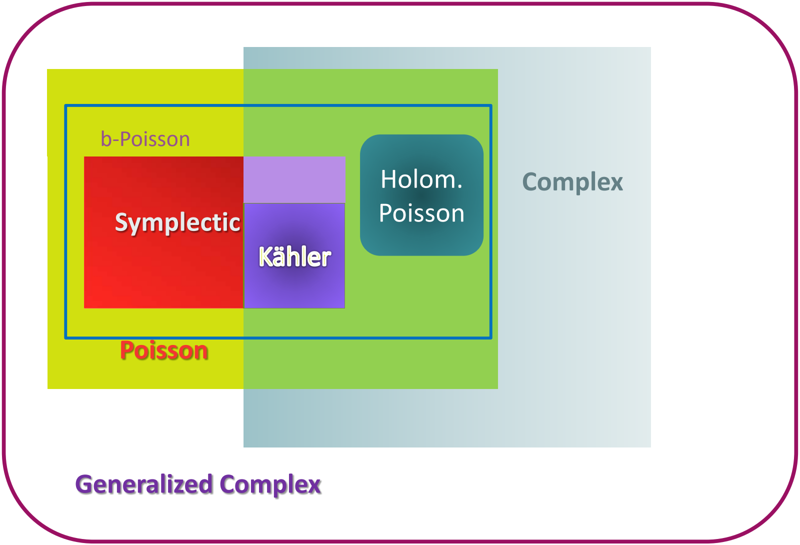


# Geometries involved



# Zooming in...

# $b$-Poisson close to symplectic



# But sometimes it is good to zoom out...

# Zooming out...to gain perspective

# Poisson, 1809

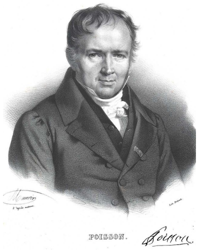
266 ANALYSE.

# MÉMOIRE

*Sur la Variation des Constantes arbitraires dans les questions de Mécanique,*

Lu à l'Institut le 16 Octobre 1809;

Par M. POISSON.




ANALYSE. 281

constante $a$ ni la constante $b$; dans d'autres cas elle ne contiendra aucune constante arbitraire, et se réduira à une constante déterminée; mais, afin de rappeler l'origine de cette quantité, qui représente une certaine combinaison des différences partielles des valeurs de $a$ et $b$, nous ferons usage de cette notation $(b, a)$, pour la désigner; de manière que nous aurons généralement

$$\frac{db}{ds}\cdot\frac{da}{d\varphi}-\frac{da}{ds}\cdot\frac{db}{d\varphi}+\frac{db}{du}\cdot\frac{da}{d\psi}-\frac{da}{du}\cdot\frac{db}{d\psi}+\frac{db}{dv}\cdot\frac{da}{d\theta}-\frac{da}{dv}\cdot\frac{db}{d\theta}=(b,a).$$

Figure: Poisson bracket

# Singular symplectic manifolds as Poisson manifolds

folded symplectic $\omega = x_1^m dx_1 \wedge dy_1 + \sum dx_i \wedge dy_i$ ← not Poisson

The local models

$b^m$-symplectic →

$$\boxed{\omega = \frac{\mathbf{1}}{\mathbf{x_1^m}} \mathbf{dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}}$$

are formally not a smooth form but their dual defines a smooth Poisson structure! as their dual

$$\boxed{\mathbf{\Pi} = \mathbf{x_1^m} \frac{\partial}{\partial \mathbf{x_1}} \wedge \frac{\partial}{\partial \mathbf{y_1}} + \sum_{\mathbf{i \geq 2}}^{\mathbf{n}} \frac{\partial}{\partial \mathbf{x_i}} \wedge \frac{\partial}{\partial \mathbf{y_i}}}$$

is well-defined. The structure $\Pi$ is a bivector field which satisfies the integrability equation $[\Pi, \Pi] = 0$. The Poisson bracket associated to $\Pi$ is given by the equation

$$\{f, g\} := \Pi(df, dg)$$

# Poisson structures as brackets

A Poisson bracket on a manifold is given by $\mathbb{R}$-bilinear operation

$$\{\cdot,\cdot\}: \begin{array}{ccc} C^\infty(M) & \longrightarrow & C^\infty(M) \\ (f,g) & \longmapsto & \{f,g\} \end{array}$$

which satisfies:

1. Anti-symmetry, $\{f,g\} = -\{g,f\}$ for any $f,g \in C^\infty(M)$
2. Leibnitz rule, $\{f, g\cdot h\} = g\cdot\{f,h\} + \{f,g\}\cdot h$
3. Jacobi identity, $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

Qun Wang    Guillemin - Sternberg    Atiyah

Convexity theorem

compact

$(M^{2n}, \omega)$ symplectic manifold
$G = \mathbb{T}^k$,  $k \leq n$
set-up

$G \circlearrowright (M^{2n}, \omega)$ Hamiltonian way  $\mu : M^{2n} \to \mathbb{R}^k$  $(\mathrm{Lie}(\mathbb{T}^k))^* \cong \mathbb{R}^k$

Theorem  $\mu(M^{2n})$ is a convex set of $\mathbb{R}^k$

not convex

not necessarily a polytope

$k=n$  Delzant not only is convex but it is a polytope

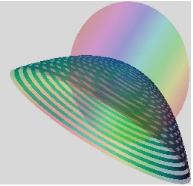
# Poisson structures as bivector fields

## Poisson structures

A Poisson structure is a bivector field $\Pi$ with $[\Pi, \Pi] = 0$.

The Poisson manifold is locally a product of a symplectic manifold with a Poisson manifold with vanishing Poisson structure at the point (Weinstein's splitting theorem).

$$(P^n, \Pi, p) \approx (M^{2k}, \omega, p_1) \times (P_0^{n-2k}, \Pi_0, p_2)$$



This defines a symplectic foliation.

# $b$-Poisson structures

## Definition

Let $(M^{2n}, \Pi)$ be an (oriented) Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface called *the critical hypersurface* and we say that $\Pi$ is a $b$-**Poisson structure** on $(M, Z)$.

## Other singularities

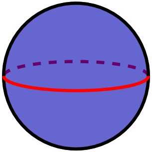
It is possible to generalize this definition (M.-Planas-Scott) to consider more general Poisson structures.

## Symplectic foliation of a $b$-Poisson manifold

The symplectic foliation has dense symplectic leaves and codimension $2$ symplectic leaves whose union is $Z$.

# Examples

- A Radko surface.

- The product of $(R, \pi_R)$ a Radko compact surface with a compact symplectic manifold $(S, \omega)$ is a $b$-Poisson manifold.
- corank $1$ Poisson manifold $(N, \pi)$ and $X$ Poisson vector field $\Rightarrow$ $(S^1 \times N, f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi)$ is a $b$-Poisson manifold if,
  1. $f$ vanishes linearly.
  2. $X$ is transverse to the symplectic leaves of $N$.

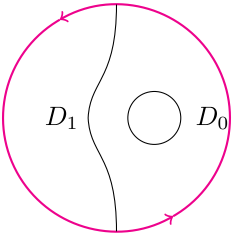
  We then have as many copies of $N$ as zeroes of $f$.

# Another example (exercise in the list)

The cubic polynomial $g(x) = x(x-1)(x-t)$, $0 < t < 1$, defines a Poisson structure on $\mathbb{R}^2$ given by

$$\pi = (g(x) - y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},$$

which extends smoothly to a $b$-symplectic structure on $\mathbb{R}P^2$ with critical set $Z$ given by the real elliptic curve $y^2 = g(x)$.




The critical set has two connected components: $D_0$, containing $\{(0,0),(t,0)\}$ and with trivial normal bundle, and $D_1$, containing $\{(1,0),(\infty,0)\}$ and with nontrivial normal bundle.

# Plan for the remaining lectures

- Melrose language of $b$-forms. $b$-symplectic forms on $b$-Poisson manifolds. The geometry of the critical set. More degenerate forms $b^m$-symplectic forms and $b^m$ contact forms. Desingularization of $b^m$-forms.
- The path method for $b^m$-symplectic structures. Local normal form ($b^m$-Darboux theorem) and extension theorems. $b^m$-Structures to the test: Examples in Fluid Dynamics and Celestial Mechanics. Application: Finding periodic orbits for trajectories of a satellite in the restricted three body problem.
- Exercise session.
- Some classical problems for $b^m$-symplectic and $b^m$-contact manifolds: The (singular) Weinstein conjecture. Connection to escape orbits in Celestial Mechanics.
- More symmetries: Toric actions, action-angle coordinates and Integrable systems on $b^m$-symplectic manifolds. Applications: KAM.
- Exercise session
- Open problems: Floer homology of Singular Symplectic Manifolds.