6. Tenint en compte la definició d'exponencial complexa: $e^{i\vartheta}=\cos\vartheta+i\sin\vartheta$, calculeu $\mathcal{L}\{e^{iat}\}_{(s)}$ per a $a\in\mathbb{R}$ i comproveu que s'obté el mateix que si s'apliqués, formalment, la transformada de Laplace a una exponencial real.

Solució.

$$\mathcal{L}\{e^{iat}\}_{(s)}=\mathcal{L}\{\cos(at)\}_{(s)}+i\,\mathcal{L}\{\sin(at)\}_{(s)}=\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2},\ s>0.$$

mentre que si transformem e^{iat} com si fos una exponencial real,

$$\mathcal{L}\{e^{iat}\}_{(s)}=\int_0^{+\infty}e^{iat}e^{-st}\,dt=\int_0^{+\infty}e^{-(s-ia)t}\,dt=\lim_{A\to+\infty}\int_0^{A}e^{-(s-ia)t}\,dt=$$

$$=\lim_{A\to+\infty}\left[-\frac{e^{-(s-ia)t}}{s-ia}\right]_{t=0}^{t=A}=\frac{1}{s-ia}-\lim_{A\to+\infty}\frac{e^{-(s-ia)A}}{s-ia}=\begin{cases}\frac{1}{s-ia}, & s>0\\ \text{indefinida}, & s\le 0\end{cases}$$

Llavors:

$$\mathcal{L}\{e^{iat}\}_{(s)}=\frac{1}{s-ia}\cdot\frac{s+ia}{s+ia}=\frac{s+ia}{s^2+a^2}=\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2},\ s>0$$

7. Calculeu la transformada de Laplace de les funcions següents:

a. $f(t)=e^{2-t}\,\mathcal{U}(t-2)$

b. $f(t)=e^{t}\cos^2 3t$

c. $f(t)=t\,e^{-3t}\cos(3t)$

d. $f(t)=t\int_0^t \tau e^{-\tau}\,d\tau$

e. $f(t)=e^{2t}*\sin t$

Solució.

a. $\mathcal{L}\{f(t)\}_{(s)}=\mathcal{L}\{e^{2-t}\,\mathcal{U}(t-2)\}_{(s)}=\mathcal{L}\{e^{-(t-2)}\,\mathcal{U}(t-2)\}_{(s)}=e^{-2s}\mathcal{L}\{e^{-t}\}_{(s)}$

$$=\boxed{\frac{e^{-2s}}{s+1},\ s>-1}.$$

Nota. Recordem que: $\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\}_{(s)}=e^{-as}\mathcal{L}\{f(t)\}_{(s)},\ a>0$

En efecte:

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\}_{(s)} = \int_0^{+\infty} e^{-st} f(t-a)\mathcal{U}(t-a)\,dt = \int_0^{+\infty} e^{-as} e^{-(t-a)s} f(t-a)\mathcal{U}(t-a)\,dt$$

$$= e^{-as}\int_0^{+\infty} e^{-(t-a)s} f(t-a)\mathcal{U}(t-a)\,dt = e^{-as}\int_a^{+\infty} e^{-(t-a)s} f(t-a)\,dt = \left\{\begin{array}{l} \text{c.v.} \\ u=t-a \\ du=dt \\ t=a \Rightarrow u=0 \\ t\to+\infty \Rightarrow u\to+\infty \end{array}\right.$$

$$= e^{-as}\int_0^{+\infty} e^{-st} f(t)\,dt = e^{-as}\mathcal{L}\{f(t)\}_{(s)}$$

b. $f(t) = e^t\cos^2(3t) = e^t\,\dfrac{1+\cos(6t)}{2}$

$$\mathcal{L}\{f(t)\}_{(s)} = \mathcal{L}\{e^t\cos^2(3t)\}_{(s)} = \mathcal{L}\left\{e^t\,\frac{1+\cos(6t)}{2}\right\}_{(s)} = \mathcal{L}\left\{\frac{1+\cos(6t)}{2}\right\}_{(s-1)}$$

$$= \frac{1/2}{s-1} + \frac{s-1}{2((s-1)^2+36)} = \frac{1}{2}\left(\frac{1}{s-1} + \frac{s-1}{(s-1)^2+36}\right),\ s>1$$

On hem fet servir que:

$$\mathcal{L}\{e^{at}f(t)\}_{(s)} = \int_0^{+\infty} e^{-st}e^{at}f(t)\,dt = \int_0^{} e^{-(s-a)t}f(t)\,dt = \mathcal{L}\{f(t)\}_{(s-a)},\ s>a+c,$$

on suposem que $f(t)$ és d'ordre exponencial, ie, que existeixen $M, t_0, c>0$ t.q.:

$|f(t)| \le Me^{ct}\ \forall t \ge t_0$.

c. $f(t) = te^{-3t}\cos(3t)$

$$\mathcal{L}\{f(t)\}_{(s)} = \mathcal{L}\{te^{-3t}\cos(3t)\}_{(s)} = \mathcal{L}\{t\cos(3t)\}_{(s+3)} = -\frac{d}{dr}\Big|_{r=s+3}\mathcal{L}\{\cos(3t)\}_{(r)}$$

$$= -\frac{d}{dr}\left(\frac{r}{r^2+9}\right)\Bigg|_{r=s+3} = -\frac{r^2+9-2r^2}{(r^2+9)^2}\Bigg|_{r=s+3} = \frac{(s+3)^2-9}{((s+3)^2+9)^2} = \frac{s^2+6s}{(s^2+6s+18)^2}$$

d. $f(t) = t\displaystyle\int_0^t \tau e^{-\tau}d\tau$

$$\mathcal{L}\{f(t)\}_{(s)} = \mathcal{L}\left\{t\int_0^t \tau e^{-\tau}d\tau\right\}_{(s)} = -\frac{d}{ds}\mathcal{L}\left\{\int_0^t \tau e^{-\tau}d\tau\right\}_{(s)} = -\frac{d}{ds}\,\frac{\mathcal{L}\{te^{-t}\}_{(s)}}{s}$$

$$= \frac{d}{ds}\left(\frac{1}{s}\frac{d}{ds}\mathcal{L}\{e^{-t}\}_{(s)}\right) = \frac{d}{ds}\left(\frac{1}{s}\frac{d}{ds}\left(\frac{1}{s+1}\right)\right) = -\frac{d}{ds}\left(\frac{1}{s(s+1)^2}\right)$$

$$= \frac{(s+1)^2+2s(s+1)}{s^2(s+1)^4} = \frac{1}{s^2(s+1)^2} + \frac{2}{s(s+1)^3} = \frac{1}{s(s+1)^2}\left(\frac{1}{s}+\frac{2}{s+1}\right) = \boxed{\frac{3s+1}{s^2(s+1)^3}}$$

e) $f(t)=e^{2t}*\sin t=\left(\exp(2\cdot)*\sin\right)(t)=\int_0^t e^{2u}\sin(t-u)\,du,$

$$\mathcal{L}\{f(t)\}_{(s)}=\mathcal{L}\left\{\left(\exp(2\cdot)*\sin\right)(t)\right\}(s)=\mathcal{L}\{e^{2t}\}_{(s)}\cdot\mathcal{L}\{\sin t\}_{(s)}=\boxed{\frac{1}{(s-2)(s^2+1)},\ s>2}$$

8. Calculeu l'antitransformada de Laplace de les funcions següents

a. $G(s)=\dfrac{2s+5}{s^2+6s+34}$

b. $G(s)=\dfrac{e^{-2s}}{s^2(s-1)}$

Solució

a. $$G(s)=\frac{2s+5}{s^2+6s+34}=\frac{2(s+3)-1}{(s+3)^2+5^2}=2\frac{s+3}{(s+3)^2+5^2}-\frac{1}{5}\cdot\frac{5}{(s+3)^2+5^2}$$

$$=2\mathcal{L}\{\cos(5t)\}_{(s+3)}-\frac{1}{5}\mathcal{L}\{\sin(5t)\}_{(s+3)}$$

$$=2\mathcal{L}\{e^{-3t}\cos(5t)\}_{(s)}-\frac{1}{5}\mathcal{L}\{e^{-3t}\sin(5t)\}_{(s)}$$

Aleshores:

$$g(t)=\mathcal{L}^{-1}\{G(s)\}_{(t)}=\left(2\cos(5t)-\frac{1}{5}\sin(5t)\right)e^{-3t}$$

b. $$G(s)=\frac{e^{-2s}}{s^2(s-1)}=e^{-2s}\left(\frac{A}{s^2}+\frac{B}{s}+\frac{C}{s-1}\right)\overset{(*)}{=}e^{-2s}\left(-\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s-1}\right)=(\&)$$

(*) $A(s-1)+Bs(s-1)+Cs^2=1$:

$s=1$: $C=1$

$s=0$: $-A=1\Leftrightarrow A=-1$

$s=2$: $A+2B+4C=-1+2B+4=1\Leftrightarrow 2B=-2\Leftrightarrow B=-1$

$$(\&)=-e^{-2s}\mathcal{L}\{t\}_{(s)}-e^{-2s}\mathcal{L}\{1\}_{(s)}+e^{-2s}\mathcal{L}\{e^t\}_{(s)}$$

$$=-\mathcal{L}\{(t-2)\mathcal{U}(t-2)\}_{(s)}-\mathcal{L}\{\mathcal{U}(t-2)\}_{(s)}+\mathcal{L}\{e^{t-2}\mathcal{U}(t-2)\}_{(s)}$$

Aleshores:

$$g(t)=\mathcal{L}^{-1}\{G(s)\}_{(t)}=-(t-2)\mathcal{U}(t-2)-\mathcal{U}(t-2)+e^{t-2}\mathcal{U}(t-2)$$

$$=\boxed{\left(1-t+e^{t-2}\right)\mathcal{U}(t-2)}\quad\square$$

11. Calculeu la antitransformada de Laplace de la funció $G(s) = \ln\left(\frac{s^2+1}{s^2+4}\right)$ fent servir que $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}(s)$.

Solució. Sigui $g(t)$ t.q.: $\mathcal{L}\{g(t)\}_{(s)} = G(s)$. Aleshores:

$$\mathcal{L}\{tg(t)\}_{(s)} = -\frac{d}{ds}\mathcal{L}\{g(t)\}_{(s)} = -\frac{dG}{ds}(s) = -\frac{s^2+4}{s^2+1}\cdot\frac{2s(s^2+4)-2s(s^2+1)}{(s^2+4)^2} = \frac{-6s}{(s^2+1)(s^2+4)}$$

$$= \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \overset{(\&)}{=} \frac{-2s}{s^2+1} + \frac{2s}{s^2+4} = -2\mathcal{L}\{\cos t\}_{(s)} + 2\mathcal{L}\{\cos(2t)\}_{(s)}$$

(&) Descomposició en fraccions simples,

$$(As+B)(s^2+4) + (Cs+D)(s^2+1) = -6, \ \forall s$$

$$\Leftrightarrow \left.\begin{array}{l} A+C=0 \\ 4A+C=-6 \end{array}\right\}: \begin{array}{l} 3A=-6 \Leftrightarrow A=-2 \\ C=-A=2 \end{array}$$

i

$$\left.\begin{array}{l} B+D=0 \\ 4B+D=0 \end{array}\right\}: B=0=D$$

Aleshores

$$g(t) = \mathcal{L}^{-1}\left\{\ln\left(\frac{s^2+1}{s^2+4}\right)\right\}_{(s)} = \frac{1}{t}\mathcal{L}^{-1}\left\{-\frac{d}{ds}\mathcal{L}\{g(t)\}_{(s)}\right\}_{(t)}$$

$$= \frac{1}{t}\mathcal{L}^{-1}\left\{-2\mathcal{L}\{\cos t\}_{(s)} + 2\mathcal{L}\{\cos(2t)\}_{(s)}\right\}$$

$$= \boxed{-\frac{2}{t}\left(\cos t - \cos(2t)\right)} \quad \square$$

12. Calculeu l'antitransformada de Laplace de la funció $F(s) = \frac{1}{(s+1)^2}$ de la forma següent

(i) Fent servir la transformada de Laplace de la convolució

(ii) Pels teoremes de translació

Solució.

(i) Recordem que $(f_1 * f_2)(t) = \int_0^t f_1(\tau) f_2(t-\tau)\,d\tau \xmapsto{\mathcal{L}} \mathcal{L}\{(f_1*f_2)(t)\}_{(s)} = \mathcal{L}\{f_1(t)\}_{(s)}\cdot\mathcal{L}\{f_2(t)\}_{(s)}$,

on suposem que existeixen $F_1(s) = \mathcal{L}\{f_1(t)\}_{(s)}$ i $F_2(s) = \mathcal{L}\{f_2(t)\}_{(s)}$.

Siguin:

$$F_1(s) = \mathcal{L}\{f_1(t)\}_{(s)} = \frac{1}{s+1} = \mathcal{L}\{f_2(t)\}_{(s)} = F_2(s),$$

per tant:

$$f(t) = \mathcal{L}^{-1}\{F(s)\}_{(t)} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}_{(t)} = \mathcal{L}^{-1}\left\{\mathcal{L}\{e^{-t}\}_{(s)}\cdot\mathcal{L}\{e^{-t}\}_{(s)}\right\}_{(t)} =$$

$$= \mathcal{L}^{-1}\left\{\mathcal{L}\{e^{-t} * e^{-t}\}_{(s)}\right\}_{(t)} = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\int_0^t e^{-\tau}e^{-(t-\tau)}\,d\tau\right\}_{(s)}\right\}_{(t)}$$

$$= \int_0^t e^{-\tau}e^{-t}e^{\tau}\,d\tau = e^{-t}\int_0^t dt = \boxed{te^{-t}}.$$

(ii) Recordem que $\mathcal{L}\left\{e^{at}f(t)\right\}(s) = \mathcal{L}\left\{f(t)\right\}(s-a)$, $s > a+c$;

on suposem que f és d'ordre exponencial, i.e, que $\exists\, M, t_0, c > 0$ t.q. $|f(t)| \leq M e^{ct}\ \forall t \geq t_0$.

Així:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t) = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{t\right\}(s+1)\right\}(t) = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{e^{-t}t\right\}(s)\right\}(t) = \boxed{te^{-t}}$$