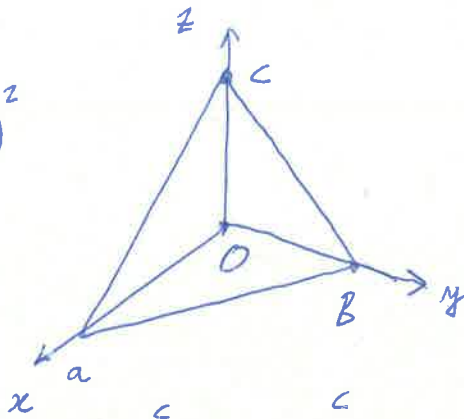
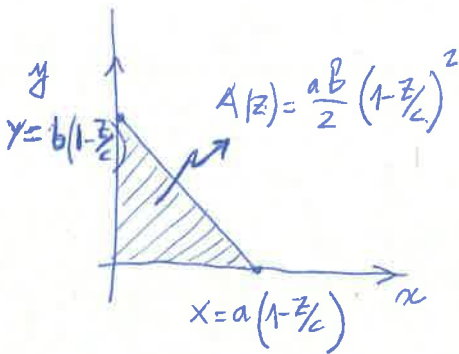


3c) Volum d'un tetraedre limitat pels plans $x=0, y=0, z=0$

P1

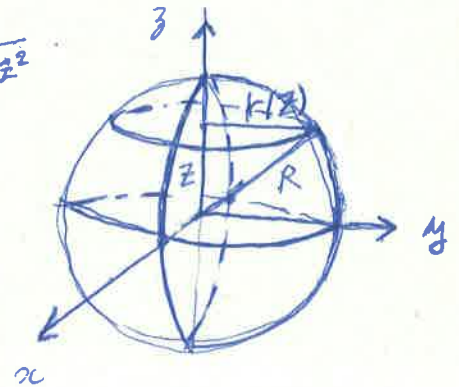
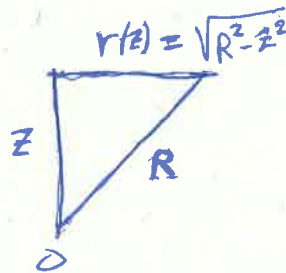
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, (a, b, c > 0)$$



$$\frac{x}{a} + \frac{y}{b} = 1 - \frac{z}{c}; \quad V(W) = \int_0^c A(z) dz = \int_0^c \frac{ab}{2} \left(1 - \frac{z}{c}\right)^2 dz = \frac{abc}{6} \left(1 - \frac{z}{c}\right)^3 \Big|_{z=0}^{z=c} = \frac{abc}{6}$$

3d) Volum envoltat per un casquet esfèric determinat per l'esfera $x^2 + y^2 + z^2 = R^2$ i la condició $R-h \leq z \leq R$

$$A(z) = \pi r(z)^2 = \pi \sqrt{R^2 - z^2}^2 = \pi(R^2 - z^2)$$



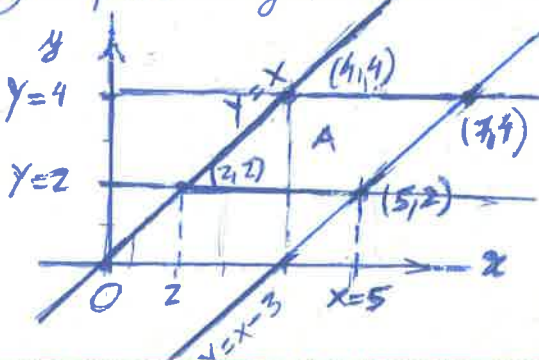
$$V(W) = \int_{R-h}^R \pi(R^2 - z^2) dz$$

$$= \pi \left(R^2 z - \frac{z^3}{3} \right) \Big|_{z=R-h}^{z=R} = \pi \left(R^3 - \frac{R^3}{3} - R^2(R-h) + \frac{(R-h)^3}{3} \right)$$

$$= \pi \left(R^3 - R^3 + hR^2 + \frac{R^3 - 3R^2h + 3Rh^2 - h^3 - R^3}{3} \right) = \frac{\pi h^2}{3} (3R - h)$$

Classe 15-11-2016 Grup 3D

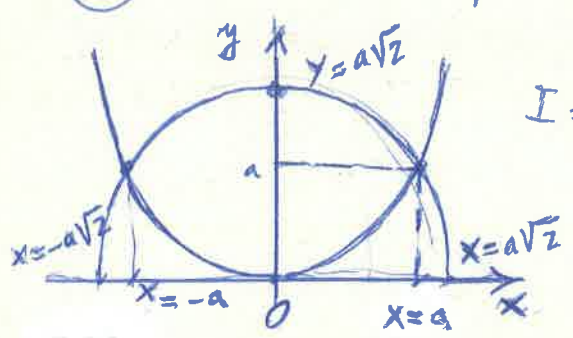
8b) A parallelogram limitat per les rectes $y=x, y=x-3, y=2, y=4$.



$$I_1 = \int_2^4 \int_2^x f(x,y) dy dx + \int_4^5 \int_2^4 f(x,y) dy dx + \int_5^7 \int_2^4 f(x,y) dy dx = \int_2^4 \int_{x-3}^{x+3} f(x,y) dx dy$$

problemes - tema 2 - fets - a - classe . pdf

8c) Regió limitada per les corbes $x^2+y^2=2a^2$, $x^2=ay$ ($y \geq 0, a > 0$)



$$I = \int_{-a\sqrt{2}}^{a\sqrt{2}} \left(\int_{\sqrt{ay}}^{\sqrt{2a^2-x^2}} f(x,y) dx \right) dy + \int_a^{a\sqrt{2}} \left(\int_{-\sqrt{2a^2-y^2}}^{\sqrt{ay}} f(x,y) dx \right) dy$$

$$\begin{cases} x^2+y^2=2a^2 \\ x^2=ay \end{cases} \Rightarrow ay+y^2=2a^2$$

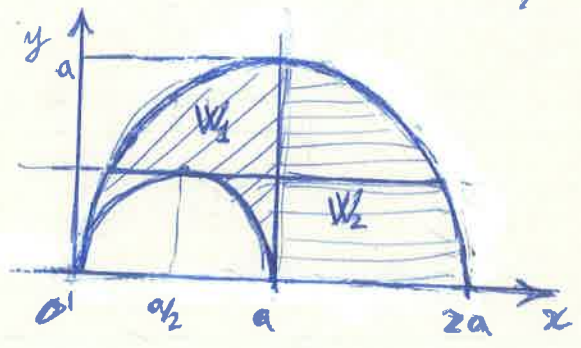
$$y = \frac{-a \pm \sqrt{a^2+8a^2}}{2} = \frac{-a \pm 3a}{2} = \begin{cases} -2a \text{ (No)} \\ a \end{cases}$$

$x = \pm a$

8e) A la regió limitada per les corbes: $x^2+y^2=ax$, $x^2+y^2=2ax$, $y=0$, ($y \geq 0, a > 0$)

$$I = \int_0^a \left(\int_{\sqrt{ax-x^2}}^{\sqrt{2ax-x^2}} f(x,y) dy \right) dx + \int_a^{2a} \left(\int_0^{\sqrt{2ax-x^2}} f(x,y) dy \right) dx$$

$$= \int_0^{a/2} \left(\int_{a-\sqrt{a^2-y^2}}^{a/2-\sqrt{a^2/4-y^2}} f(x,y) dx \right) dy + \int_0^{a/2} \left(\int_{a/2+\sqrt{a^2/4-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y) dx \right) dy + \int_{a/2}^a \left(\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y) dx \right) dy$$



W_1, W_2 - dominis x-elementals

$$x^2+y^2=ax \Leftrightarrow x^2-ax+y^2=0$$

$$x = \frac{a \pm \sqrt{a^2-4y^2}}{2} = \frac{a}{2} \pm \sqrt{\frac{a^2}{4}-y^2}$$

$$x^2+y^2=2ax$$

$$x = \frac{2a \pm \sqrt{4a^2-4y^2}}{2} = a \pm \sqrt{a^2-y^2}$$

(11) Investiu l'ordre d'integració de les integrals iterades següents

(11e)
$$I_1 = \int_0^{\pi} \left(\int_0^{\sin x} f(x,y) dy \right) dx$$

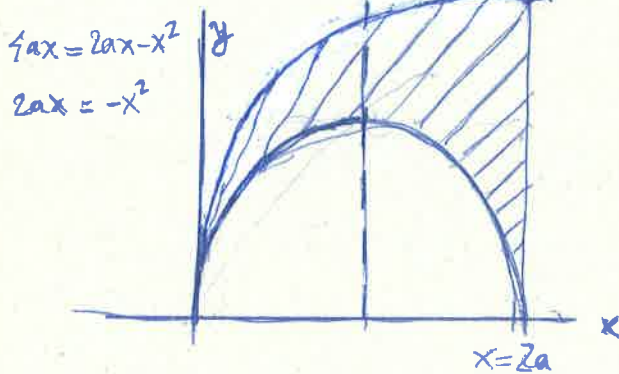
$$I_2 = \int_0^1 \left(\int_{\arcsin y}^{\pi - \arcsin y} f(x,y) dx \right) dy$$

Fubini: $I_1 = I_2$

(11g)
$$I_1 = \int_0^{2a} \left(\int_{\sqrt{2ax-x^2}}^{\sqrt{4ax}} f(x,y) dy \right) dx$$

$y = 2ax - x^2 \Leftrightarrow x^2 + y^2 = 2ax \Leftrightarrow x = a \pm \sqrt{a^2 - y^2}$

$y^2 = 4ax \Leftrightarrow x = \frac{y^2}{4a}$



$$I_2 = \int_0^a \left(\int_{y^2/4a}^{a - \sqrt{a^2 - y^2}} f(x,y) dx \right) dy + \int_0^{2\sqrt{2}a} \left(\int_{a + \sqrt{a^2 - y^2}}^{2a} f(x,y) dx \right) dy + \int_a^{2\sqrt{2}a} \left(\int_{y^2/4a}^{2a} f(x,y) dx \right) dy$$

Fubini: $I_1 = I_2$

(14) Per a les regions de \mathbb{R}^3 indicades escriu la integral triple $\iiint_A f(x,y,z) dx dy dz$ en termes d'integrals iterades preses en diferents ordres

(14a) Tetraedre limitat pels plans $x=0, y=0, z=0, 2x+3y+4z=12$

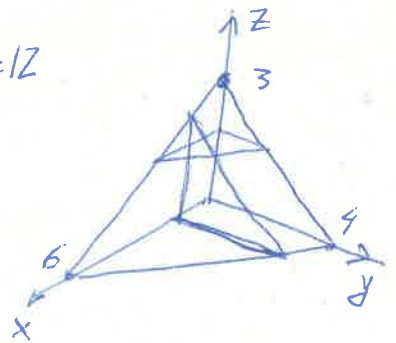
$0 \leq z \leq \frac{1}{4}(12 - 2x - 3y) = 3 - \frac{1}{2}x - \frac{3}{4}y, x \geq 0, y \geq 0$

$$\begin{cases} 0 \leq y \leq \frac{4}{3}(3 - \frac{1}{2}x) = 4 - \frac{2}{3}x \\ 0 \leq x \leq 6 \end{cases}$$

$0 \leq x \leq 2(3 - \frac{3}{4}y) = 6 - \frac{3}{2}y$

$0 \leq y \leq 4$

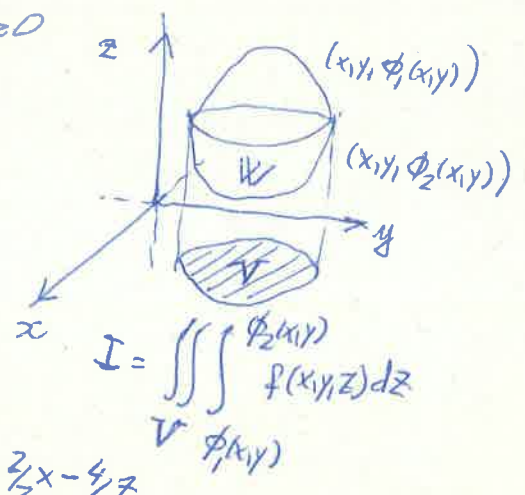
$$I_1 = \int_0^6 dx \int_0^{4 - \frac{2}{3}x} dy \int_0^{3 - \frac{x}{2} - \frac{3}{4}y} f(x,y,z) dz = \int_0^4 dy \int_0^{6 - \frac{3}{2}y} dx \int_0^{3 - \frac{x}{2} - \frac{3}{4}y} f(x,y,z) dz$$



$$0 \leq y \leq \frac{1}{3}(12 - 2x - 4z) = 4 - \frac{2}{3}x - \frac{4}{3}z, \quad x \geq 0, \quad z \geq 0$$

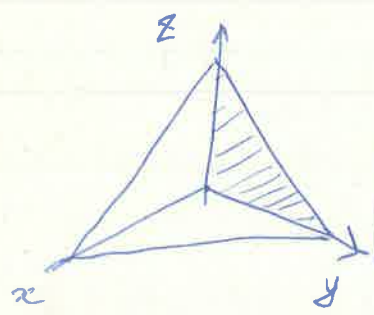
$$\begin{cases} 0 \leq z \leq \frac{3}{4}(4 - \frac{2}{3}x) = 3 - \frac{x}{2} \\ 0 \leq x \leq 6 \end{cases}$$

$$\begin{cases} 0 \leq x \leq \frac{3}{2}(4 - \frac{4}{3}z) = 6 - 2z \\ 0 \leq z \leq 3 \end{cases}$$



$$I_2 = \int_0^6 dx \int_0^{3-x/2} dz \int_0^{4-2/3x-4/3z} f(x,y,z) dy = \int_0^3 dz \int_0^{6-2z} dx \int_0^{4-2/3x-4/3z} f(x,y,z) dy$$

Tomelli-Fubini



$$0 \leq x \leq \frac{1}{2}(12 - 3y - 4z) = 6 - \frac{3}{2}y - 2z, \quad y \geq 0, \quad z \geq 0$$

$$\begin{cases} 0 \leq y \leq \frac{2}{3}(6 - 2z) = 4 - \frac{4}{3}z \\ 0 \leq z \leq 3 \end{cases}$$

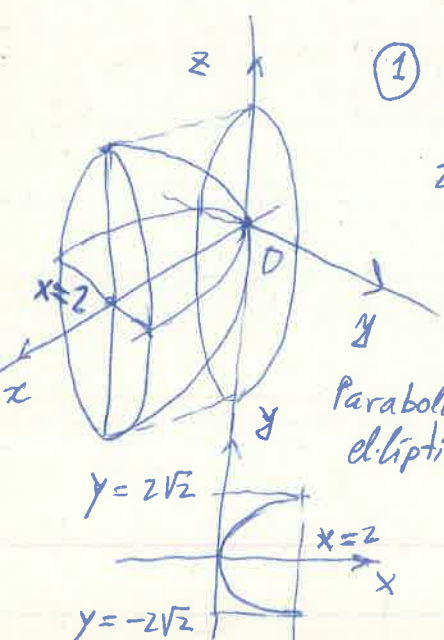
$$0 \leq z \leq \frac{1}{2}(6 - \frac{3}{2}y) = 3 - \frac{3}{4}y$$

$$I_3 = \int_0^3 dz \int_0^{4-4/3z} dy \int_0^{6-3/2y-2z} f(x,y,z) dx = \int_0^4 dy \int_0^{3-3/4y} dz \int_0^{6-3/2y-2z} f(x,y,z) dx$$

Tomelli-Fubini

Pel teorema de Fubini: $I_1 = I_2 = I_3$.

(14c) A cas limitat per les superfícies: $y^2 + 2z^2 = 4x, \quad x = z$ $y^2 + 2z^2 \leq 4x$
 $x \leq z$



$$(1) \quad z^2 = 2x - \frac{y^2}{2} : \quad -\sqrt{2x - \frac{y^2}{2}} \leq z \leq \sqrt{2x - \frac{y^2}{2}}$$

$$2x - \frac{y^2}{2} \geq 0, \text{ d'on: } \frac{y^2}{4} \leq x \leq z; \quad -2\sqrt{z} \leq y \leq 2\sqrt{z}$$

o bé:

$$-2\sqrt{x} \leq y \leq 2\sqrt{x}, \quad 0 \leq x \leq z$$

$$I_1 = \int_{-2\sqrt{z}}^{2\sqrt{z}} dy \int_{y^2/4}^z dx \int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}} f(x,y,z) dz =$$

Tonelli-Fubini

$$\int_0^2 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} \int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}} f(x,y,z) dz$$

② $-\sqrt{4x-2z^2} \leq y \leq \sqrt{4x-2z^2}, x \leq 2$

$4x-2z^2 \geq 0$, d'on: $z^2/2 \leq x \leq 2, -2 \leq z \leq 2$

o bé: $-\sqrt{2x} \leq z \leq \sqrt{2x}, 0 \leq x \leq 2$

$$I_2 = \int_{-2}^2 dz \int_{z^2/2}^2 dx \int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}} f(x,y,z) dy = \int_0^2 dx \int_{-\sqrt{2x}}^{\sqrt{2x}} dz \int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}} f(x,y,z) dy$$

Tonelli-Fubini

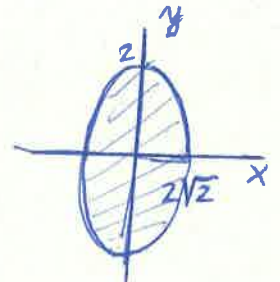
③ $y^2/4 + z^2/2 \leq x \leq 2;$

$y^2/8 + z^2/4 \leq 1 \Rightarrow -\sqrt{4-y^2/2} \leq z \leq \sqrt{4-y^2/2}, -2\sqrt{2} \leq y \leq 2\sqrt{2}$

$-\sqrt{8-2z^2} \leq y \leq \sqrt{8-2z^2}, -2 \leq z \leq 2$

$$I_3 = \int_{-2\sqrt{2}}^{2\sqrt{2}} dy \int_{-\sqrt{4-y^2/2}}^{2} dz \int_{y^2/4 + z^2/2}^2 f(x,y,z) dx = \int_{-2}^2 dz \int_{-\sqrt{8-2z^2}}^{\sqrt{8-2z^2}} dy \int_{y^2/4 + z^2/2}^2 f(x,y,z) dx$$

Tonelli-Fubini

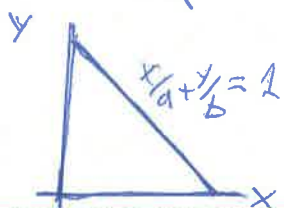


Tonelli-Fubini $I_4 = I_2 = I_3$

15) Calculen les integrals triples següents en les regions de \mathbb{R}^3 que s'indiquen

15e) $\iiint_A x dx dy dz$, A: tetraedre limitat pels plans $x=0, y=0, z=0, x/a + y/b + z/c = 1$

(a,b,c > 0),
 $0 \leq z \leq c(1 - x/a - y/b)$
 $0 \leq y \leq b(1 - x/a)$
 $0 \leq x \leq a$

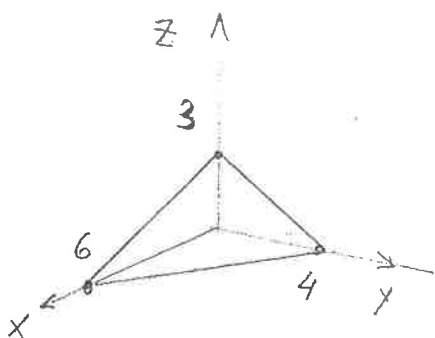


$$\begin{aligned} I &= \int_0^a dx \int_0^{b(1-x/a)} dy \int_0^{c(1-x/a-y/b)} x dz \\ &= c \int_0^a dx \int_0^{b(1-x/a)} x(1-x/a-y/b) dy \\ &= c \int_0^a \left(xy - \frac{x^2 y}{a} - \frac{xy^2}{2b} \right) \Big|_{y=0}^{y=b(1-x/a)} dx \\ &= bc \int_0^a x(1-x/a) \left(1 - x/a - \frac{1}{2} + \frac{x}{2a} \right) dx = \frac{bc}{2} \int_0^a x(1-x/a)^2 dx = \end{aligned}$$

$$\begin{aligned} &= \frac{bc}{z} \int_0^a \left(x - \frac{2x^2}{a} + \frac{x^3}{a^2} \right) dx = \frac{bc}{z} \left(\frac{x^2}{2} - \frac{2x^3}{3a} + \frac{x^4}{4a^2} \right) \Bigg|_{x=0}^{x=a} \\ &= \frac{bc}{z} \left(\frac{a^2}{2} - \frac{2}{3}a^2 + \frac{a^2}{4} \right) = \frac{bc}{z} \cdot \frac{6a^2 - 8a^2 - 3a^2}{12} = \boxed{\frac{bca^2}{24}} \end{aligned}$$

14) Per les regions de \mathbb{R}^3 indicades escriuiu la integral triple $\iiint_A f(x,y,z) dx dy dz =: I$ en termes d'integrals iterades preses en diferents ordres.

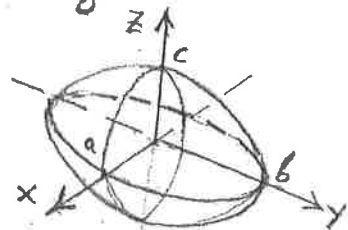
(a) A tetraedre limitat pels plans $x=0, y=0, z=0, 2x+3y+4z=12$.



$$\begin{aligned}
 I &= \int_0^6 dx \int_0^{4-\frac{2}{3}x} dy \int_0^{3-\frac{1}{2}x-\frac{3}{4}y} f(x,y,z) dz = \int_0^4 dy \int_0^{6-\frac{3}{2}y} dx \int_0^{3-\frac{1}{2}x-\frac{3}{4}y} f(x,y,z) dz \\
 &= \int_0^6 dx \int_0^{3-\frac{1}{2}x} dz \int_0^{4-\frac{2}{3}x-\frac{4}{3}z} f(x,y,z) dy = \int_0^3 dz \int_0^{6-2z} dx \int_0^{4-\frac{2}{3}x-\frac{4}{3}z} f(x,y,z) dy \\
 &= \int_0^4 dy \int_0^{3-\frac{3}{4}y} dz \int_0^{6-\frac{3}{2}y-2z} f(x,y,z) dx = \int_0^3 dz \int_0^{4-\frac{4}{3}z} dy \int_0^{6-\frac{3}{2}y-2z} f(x,y,z) dx.
 \end{aligned}$$

- $4z = 12 - 2x - 3y \rightarrow z = 3 - \frac{1}{2}x - \frac{3}{4}y$
- $z=0, 2x+3y=12 \rightarrow y = 4 - \frac{2}{3}x$
- $x = 6 - \frac{3}{2}y$

(b) A interior del el·lipsoide



- $3y = 12 - 2x - 4z \rightarrow y = 4 - \frac{2}{3}x - \frac{4}{3}z$
- $y=0, 2x+4z=12 \rightarrow z = 3 - \frac{1}{2}x$
- $x = 6 - 2z$

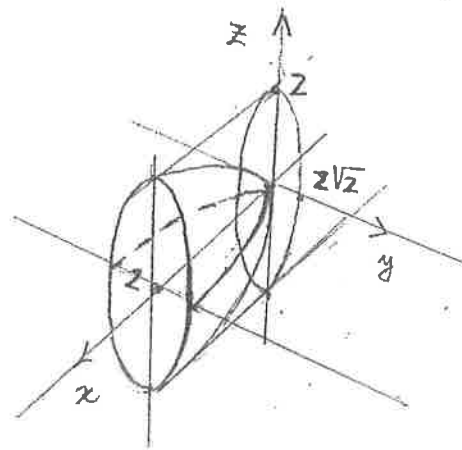
$$\begin{aligned}
 I &= \int_{-a}^a dx \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} dy \int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} f(x,y,z) dz \\
 &= \int_{-b}^b dy \int_{-a\sqrt{1-y^2/b^2}}^{a\sqrt{1-y^2/b^2}} dx \int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} f(x,y,z) dz \\
 &= \int_{-c}^c dz \int_{-a\sqrt{1-z^2/c^2}}^{a\sqrt{1-z^2/c^2}} dx \int_{-b\sqrt{1-x^2/a^2-z^2/c^2}}^{b\sqrt{1-x^2/a^2-z^2/c^2}} f(x,y,z) dy \\
 &= \int_{-c}^c dz \int_{-b\sqrt{1-z^2/c^2}}^{b\sqrt{1-z^2/c^2}} dy \int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z) dx. \square
 \end{aligned}$$

- $2x = 12 - 3y - 4z \rightarrow x = 6 - \frac{3}{2}y - 2z$
- $x=0: 3y+4z=12 \rightarrow z = 3 - \frac{3}{4}y$
- $y = 4 - \frac{4}{3}z$

$$\begin{aligned}
 &= \int_{-a}^a dx \int_{-c\sqrt{1-x^2/a^2}}^{c\sqrt{1-x^2/a^2}} dz \int_{-b\sqrt{1-x^2/a^2-z^2/c^2}}^{b\sqrt{1-x^2/a^2-z^2/c^2}} f(x,y,z) dy \\
 &= \int_{-b}^b dy \int_{-c\sqrt{1-y^2/b^2}}^{c\sqrt{1-y^2/b^2}} dz \int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z) dx
 \end{aligned}$$

$$\frac{y^2}{4} + \frac{z^2}{2} = x$$

(c) A cos limitat per les superfícies $y^2 + 2z^2 = 4x$, $x = 2$.



$$I = \int_{-2}^2 dz \int_{-\sqrt{8-2z^2}}^{\sqrt{8-2z^2}} dy \int_{\frac{y^2}{4} + \frac{z^2}{2}}^2 f(x,y,z) dx = \int_{-2\sqrt{2}}^{2\sqrt{2}} dy \int_{-\sqrt{4-y^2/2}}^{\sqrt{4-y^2/2}} dz \int_{\frac{y^2}{4} + \frac{z^2}{2}}^2 f(x,y,z) dx$$

$$= \int_0^2 dx \int_{-\sqrt{2x}}^{\sqrt{2x}} dz \int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}} f(x,y,z) dy = \int_{-2}^2 dz \int_{z^2/2}^2 dx \int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}} f(x,y,z) dy$$

$$= \int_{-2\sqrt{2}}^{2\sqrt{2}} dy \int_{\frac{1}{4}y^2}^2 dx \int_{-\sqrt{2x-\frac{1}{2}y^2}}^{\sqrt{2x-\frac{1}{2}y^2}} f(x,y,z) dz = \int_0^2 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} dy \int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}} f(x,y,z) dz. \quad \square$$

Problema 15 (Complet)

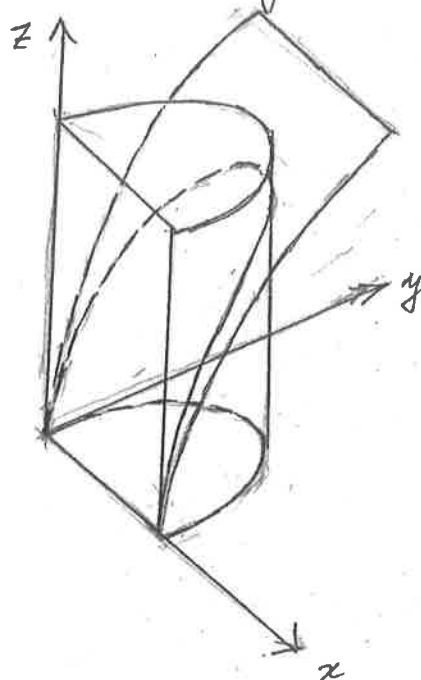
15) Calculeu les integrals triples següents en les regions de \mathbb{R}^3 que s'indiquen

a) $I = \iiint_A xz \, dx \, dy \, dz$, A limitat pel cilindre de base circular $x^2 + y^2 - 2x = 0$ i

la superfície: $z^2 = 2y$ ($y, z \geq 0$)

$$\begin{aligned}
 I &= \int_0^2 x \, dx \int_0^{\sqrt{2x-x^2}} dy \int_0^{\sqrt{2y}} z \, dz \\
 &= \int_0^2 x \, dx \int_0^{\sqrt{2x-x^2}} y \, dy \\
 &= \int_0^2 x \left(\frac{y^2}{2} \right)_0^{\sqrt{2x-x^2}} dx = \int_0^2 \left(x^2 - \frac{x^3}{3} \right) dx
 \end{aligned}$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{8} \right]_0^2 = \frac{8}{3} - \frac{16}{8} = \frac{8}{3} - 2 = \boxed{\frac{2}{3}} \quad \square$$



b) $I = \iiint_A zy \sqrt{x^2 + y^2} \, dz \, dy \, dx$, $A = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq x^2 + y^2, 0 \leq y \leq \sqrt{2x - x^2}\}$

$$\begin{aligned}
 I &= \int_0^2 dx \int_0^{\sqrt{2x-x^2}} y \sqrt{x^2 + y^2} \, dy \int_0^{x^2 + y^2} z \, dz = \int_0^2 dx \int_0^{\sqrt{2x-x^2}} y \sqrt{x^2 + y^2} \left[\frac{z^2}{2} \right]_0^{x^2 + y^2} dy \\
 &= \frac{1}{2} \int_0^2 dx \int_0^{\sqrt{2x-x^2}} y (x^2 + y^2)^{3/2} dy = \frac{1}{14} \int_0^2 dx \left[(x^2 + y^2)^{7/2} \right]_0^{\sqrt{2x-x^2}} = \frac{1}{14} \int_0^2 dx \left[(x^2 + 2x - x^2)^{7/2} - x^7 \right] \\
 &= \frac{1}{14} \left[2 \cdot 2^{7/2} \frac{x^{9/2}}{9} - \frac{x^8}{8} \right]_0^2 = \frac{1}{14} \cdot \frac{1}{72} \cdot (16 \cdot 2^8 - 9 \cdot 2^8) = \frac{1}{14} \cdot \frac{1}{72} \cdot 7 \cdot 2^8 = \boxed{\frac{16}{9}} \quad \square
 \end{aligned}$$

c) $I = \iiint_A dx \, dy \, dz$, $A = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x \leq 3, 1 \leq y \leq 3, 0 \leq z \leq xy\}$.

$$I = \int_1^3 dx \int_1^3 dy \int_0^{xy} dz = \int_1^3 x \, dx \int_1^3 y \, dy = \left(\int_1^3 x \, dx \right)^2 = \left(\left[\frac{x^2}{2} \right]_1^3 \right)^2 = \left(\frac{9}{2} - \frac{1}{2} \right)^2 = \boxed{16} \quad \square$$

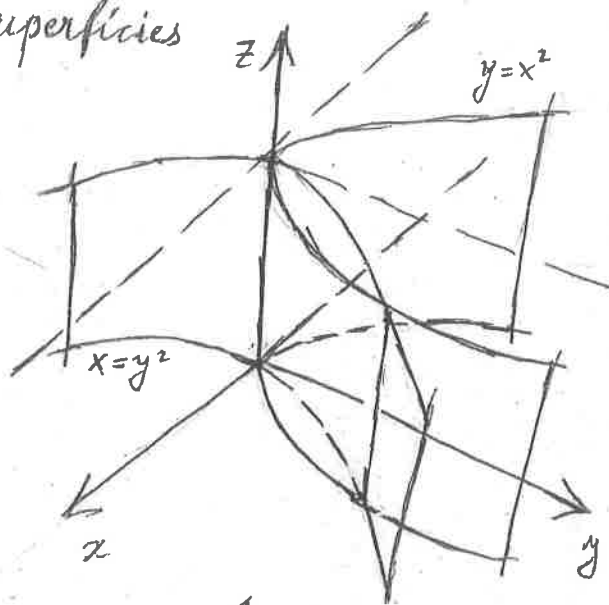
(d) $I = \iiint_A xyz \, dx \, dy \, dz$, A limitat per les superfícies

$$y = x^2, x = y^2, z = xy, z = 0$$

$$I = \int_0^1 x \, dx \int_{x^2}^{\sqrt{x}} y \, dy \int_0^{xy} z \, dz$$

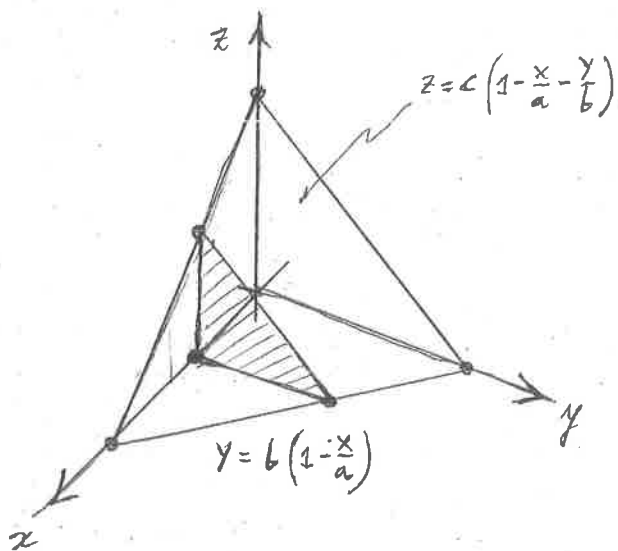
$$= \frac{1}{2} \int_0^1 x^3 \, dx \int_{x^2}^{\sqrt{x}} y^3 \, dy$$

$$= \frac{1}{8} \int_0^1 dx \, x^3 (x^2 - x^8) = \frac{1}{8} \int_0^1 (x^5 - x^{11}) \, dx = \frac{1}{8} \left(\frac{x^6}{6} - \frac{x^{12}}{12} \right) \Big|_0^1 = \frac{1}{8} \left(\frac{1}{6} - \frac{1}{12} \right) = \boxed{\frac{1}{96}} \quad \square$$



(e) $I = \iiint_A x \, dx \, dy \, dz$, A tetraedre format pels plans $x=0, y=0, z=0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$,

amb $a, b, c > 0$



$$I = \int_a^b x \, dx \int_0^{b(1-x/a)} dy \int_0^{c(1-x/a-y/b)} dz$$

$$= c \int_0^a x \, dx \int_0^{b(1-x/a)} dy (1 - \frac{x}{a} - \frac{y}{b})$$

$$= c \int_0^a dx \, x \left(y - \frac{xy}{a} - \frac{y^2}{2b} \right) \Big|_0^{b(1-x/a)}$$

$$= c \int_0^a x \left[b(1-x/a) - \frac{bx}{a}(1-x/a) - \frac{b}{2}(1-x/a)^2 \right] dx$$

$$(*) = \frac{cb}{2} \int_0^a x \left(1 - \frac{x}{a} \right)^2 dx = \frac{cb}{2} \int_0^a x \left(1 - \frac{2x}{a} + \frac{x^2}{a^2} \right) dx = \frac{cb}{2} \left(\frac{x^2}{2} - \frac{2x^3}{3a} + \frac{x^4}{4a^2} \right) \Big|_0^a$$

$$= \frac{cb}{2} \left(\frac{a^2}{2} - \frac{2a^2}{3} + \frac{a^2}{4} \right) = \boxed{\frac{cba^2}{24}}$$

$$(*) \left(1 - \frac{x}{a} \right) \left(1 - \frac{x}{a} - \frac{1}{2} + \frac{x}{2a} \right) = \frac{b}{2} \left(1 - \frac{x}{a} \right) \left(1 - \frac{x}{a} \right) = \frac{b}{2} \left(1 - \frac{x}{a} \right)^2$$

16. Usen coordenades polars per a calcular les següents integrals dobles

(16c) $\iint_A \frac{(x+y)^2}{x^2+y^2+2} dx dy, A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$

$$I = \iint_A \frac{(x+y)^2}{x^2+y^2+2} dx dy = \int_0^{2\pi} \int_0^1 r \frac{r^2+2r^2 \sin\theta \cos\theta}{r^2+2} dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \frac{r^2}{2+r^2} (1+\sin 2\theta) dr d\theta = \left(\int_0^{2\pi} (1+\sin 2\theta) d\theta \right) \cdot \int_0^1 \left(r - \frac{2r}{2+r^2} \right) dr$$

$$\stackrel{(a)}{=} 2\pi \left(r^2/2 - \ln(2+r^2) \right) \Big|_{r=0}^{r=1} = 2\pi \left(\frac{1}{2} - \ln 3 + \ln 2 \right) = \boxed{2\pi \left(\frac{1}{2} + \ln\left(\frac{2}{3}\right) \right)}$$

(a) $\int_0^{2\pi} \sin(2\theta) d\theta = 0.$

(16d) $\iint_A \frac{dx dy}{(1+x^2+y^2)^2 \sqrt{x^2+y^2}}, A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq R^2\}$

Indicació: Usen les propietats elementals del sin i cos per veure que $\sin(\arctan(R)) = \frac{R}{\sqrt{1+R^2}}$, $\cos(\arctan(R)) = \frac{1}{\sqrt{1+R^2}}$

$$\iint_A \frac{dx dy}{(1+x^2+y^2)^2 \sqrt{x^2+y^2}} = \int_0^{2\pi} \int_0^R \frac{r dr d\theta}{(1+r^2)^2 r} = 2\pi \int_0^R \frac{dr}{(1+r^2)^2} = \left. \begin{array}{l} \text{canvi de variables} \\ r = \tan u : dr = \frac{du}{\cos^2 u} \\ r = R \Rightarrow u = \arctan R \\ r = 0 \Rightarrow u = 0 \end{array} \right\}$$

$$= 2\pi \int_0^{\arctan R} \frac{1}{\left(\frac{1}{\cos^2 u}\right)^2} \frac{du}{\cos^2 u} = 2\pi \int_0^{\arctan R} \frac{\cos^4 u}{\cos^2 u} du = 2\pi \int_0^{\arctan R} \cos^2 u du$$

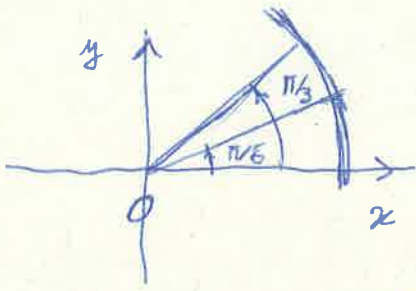
$$= 2\pi \int_0^{\arctan R} \frac{1+\cos 2u}{2} du = 2\pi \left(\frac{u}{2} + \frac{\sin 2u}{4} \right) \Big|_0^{\arctan R} =$$

$$= \pi \left(\arctan R + \sin(\arctan R) \cdot \cos(\arctan R) \right)$$

$$= \pi \left(\arctan R + \tan(\arctan R) \cdot \cos^2(\arctan R) \right)$$

$$= \pi \left(\arctan R + \frac{\tan(\arctan R)}{1+\tan^2(\arctan R)} \right) = \boxed{\pi \left(\arctan R + \frac{R}{1+R^2} \right)}$$

16g) $\iint_A x(x^2+y^2) dx dy$. A sector circular de centre $(0,0)$ i radi R formant angles entre $\pi/3$ i $\pi/6$ amb eix x-positiu.



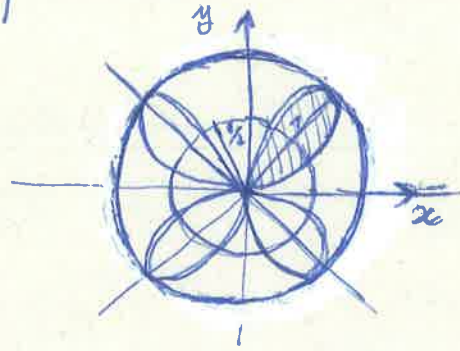
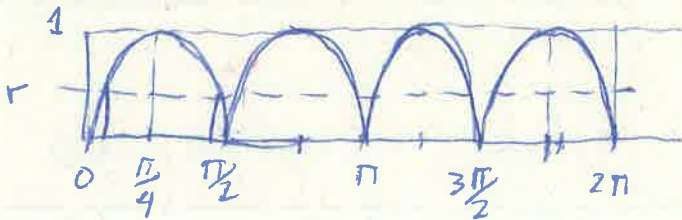
$$\iint_A x(x^2+y^2) dx dy = \int_{\pi/6}^{\pi/3} \int_0^R r^4 \cos \theta dr d\theta$$

$$= \left(\int_{\pi/6}^{\pi/3} \cos \theta d\theta \right) \cdot \left(\int_0^R r^4 dr \right) = \frac{R^5}{5} \cdot \left(\sin \theta \right)_{\theta=\pi/6}^{\theta=\pi/3}$$

$$= \frac{R^5}{5} \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right) = \boxed{\frac{\sqrt{3}-1}{10} R^5}$$

17. Calculen les àrees dels dominis $A \subset \mathbb{R}^2$ definits en coordenades polars $x=r \cos \theta$, $y=r \sin \theta$, que s'indiquen tot seguit.

17c) A regió definida per $\frac{1}{2} \leq r \leq |\sin(2\theta)|$ (Indicació: Cal $|\sin(2\theta)| \geq \frac{1}{2}$ perquè l'expressió tingui sentit).



$$A_{1/4} = \int_{\pi/2}^{5\pi/12} \int_{1/2}^{\sin 2\theta} r dr d\theta = \int_{\pi/2}^{5\pi/12} \left(\frac{\sin^2 2\theta}{2} - \frac{1}{8} \right) d\theta$$

$$= \int_{\pi/2}^{5\pi/12} \left(\frac{1}{8} - \frac{\cos(4\theta)}{4} \right) d\theta =$$

$$= \left(\frac{\theta}{8} - \frac{\sin(4\theta)}{16} \right) \Big|_{\theta=\pi/2}^{\theta=5\pi/12}$$

$$0 \leq \theta \leq \pi: \sin 2\theta \geq \frac{1}{2} \Leftrightarrow \frac{\pi}{6} \leq 2\theta \leq \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\Leftrightarrow \frac{\pi}{12} \leq \theta \leq \frac{5\pi}{12}$$

$$= \frac{1}{8} \left(\frac{5\pi}{12} - \frac{\pi}{2} \right) - \frac{1}{16} \left(\sin \left(2\pi - \frac{\pi}{3} \right) - \sin \frac{\pi}{3} \right)$$

$$= \frac{\pi}{24} + \frac{\sqrt{3}}{16} \cdot \text{Aleshores: } \boxed{A = \frac{\pi}{6} + \frac{\sqrt{3}}{4}}$$

19. Useu coordenades cilíndriques per calcular els següents integrals triples

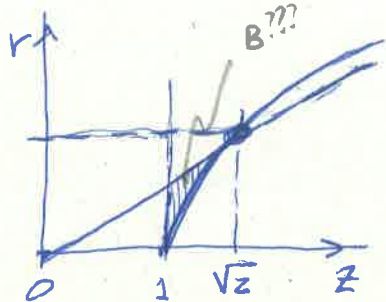
19b)
$$\iiint_B z e^{-(x^2+y^2)} dx dy dz, B = \{(x,y,z) \in \mathbb{R}^3, z^2 - 1 \leq x^2 + y^2 \leq z^2/2, z \geq 0\} \quad ???$$

$B: \sqrt{z^2 - 1} \leq r \leq z/\sqrt{2}$

$z \geq 0,$

$z/\sqrt{2} = \sqrt{z^2 - 1}, \Leftrightarrow z^2/2 = z^2 - 1 \Leftrightarrow z^2/2 - 1 = 0$

$(z = \sqrt{2}, r = 1)$



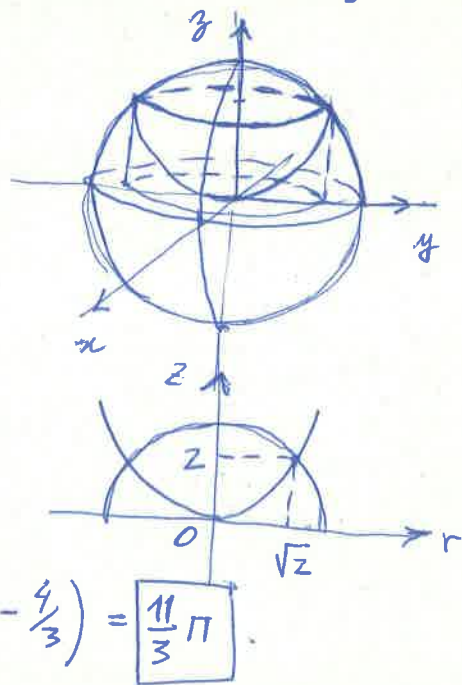
$$I = \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{\sqrt{z^2-1}}^{z/\sqrt{2}} z r e^{-r^2} dr dz d\theta = \int_0^{2\pi} d\theta \int_1^{\sqrt{2}} z dz \left[-\frac{1}{2} e^{-r^2} \right]_{r=\sqrt{z^2-1}}^{r=z/\sqrt{2}}$$

$$= \int_0^{2\pi} d\theta \int_1^{\sqrt{2}} z \left(-\frac{1}{2} e^{-z^2/2} + \frac{1}{2} e^{-(z^2-1)} \right) dz = 2\pi \cdot \left(\frac{e^{-z^2/2}}{2} - \frac{1}{4} e^{-(z^2-1)} \right) \Big|_{z=1}^{z=\sqrt{2}}$$

$= 2\pi \left(\frac{1}{2e} - \frac{1}{4e} - \frac{1}{2\sqrt{e}} + \frac{1}{4} \right) = \boxed{\frac{\pi}{2e} - \frac{\pi}{\sqrt{e}} + \frac{\pi}{2}}$

19e)
$$\iiint_B z dx dy dz, B = \{(x,y,z) \in \mathbb{R}^3, x^2 + y^2 + z^2 \leq 6, x^2 + y^2 \leq z, z \geq 0\}$$

$$I = \int_0^{2\pi} d\theta \int_0^{\sqrt{z}} r dr \int_{r^2}^{\sqrt{6-r^2}} z dz = \int_0^{2\pi} d\theta \int_0^{\sqrt{z}} r \left[\frac{z^2}{2} \right]_{z=r^2}^{z=\sqrt{6-r^2}} dr$$



$r^2 + z^2 = 6$

$z = r^2$

$z + z^2 = 6$

$\Leftrightarrow z^2 + z - 6 = 0$

$z = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} = \begin{cases} -3 \\ 2 \end{cases}$

$r = \sqrt{z} = \sqrt{2}$

$= \pi \int_0^{\sqrt{z}} r (6 - r^2 - r^4) dr$

$= \pi \left(6r^2/2 - r^4/4 - r^6/6 \right) \Big|_{r=0}^{r=\sqrt{2}}$

$= \pi \left(3 \cdot 2 - \frac{4}{4} - \frac{4}{6} \right)$

$= \pi \left(6 - 1 - \frac{4}{3} \right) = \pi \left(5 - \frac{4}{3} \right) = \boxed{\frac{11}{3} \pi}$

20. Useu coordenades esfèriques per a calcular les següents integrals triples.

P14

20c)
$$\iiint_B \frac{dx dy dz}{(x^2 + y^2 + z^2)^{3/2}}, \quad B = \{(x, y, z) \in \mathbb{R}^3 : a^2 \leq x^2 + y^2 + z^2 \leq b^2\}$$

$$I = \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi d\varphi \int_a^b r^2 / r^3 dr = \boxed{4\pi \ln(b/a)}$$

21. Calculeu els volums dels dominis $B \subset \mathbb{R}^3$ definits en coordenades esfèriques $x = r \cos\varphi$, $y = r \sin\varphi \sin\theta$, $z = r \sin\varphi \cos\theta$ ($0 < \theta < 2\pi$, $-\pi/2 < \varphi < \pi/2$) que s'indiquen tot seguit.

21b) B volums tancat per l'esfera deformada definida per $r = 1 + 0.2 \sin(8\theta) \sin\varphi$ (Sòlids d'aquesta mena s'utilitzen com a models de tumors).

$$\begin{aligned} I &= \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{1+0.2 \sin(8\theta) \sin\varphi} r^2 \cos\varphi dr = \frac{1}{3} \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi \cdot (1 + 0.2 \sin(8\theta) \sin\varphi)^3 d\varphi \\ &= \frac{1}{3} \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi (1 + 0.12 \sin^2(8\theta) \sin^2\varphi + 0.6 \sin(8\theta) \sin\varphi + 0.2^3 \sin^3(8\theta) \sin^3\varphi) d\varphi \\ &= \frac{1}{3} \left(\int_0^{2\pi} d\theta \right) \cdot \left(\int_{-\pi/2}^{\pi/2} \cos\varphi d\varphi \right) + \frac{1}{3} \cdot 0.12 \left(\int_0^{2\pi} \sin^2(8\theta) d\theta \right) \cdot \left(\int_{-\pi/2}^{\pi/2} \sin^2\varphi \cos\varphi d\varphi \right) \\ &= \frac{4\pi}{3} + \frac{0.12}{3} \cdot \pi \cdot \frac{2}{3} = \frac{\pi}{3} (4 + 0.08) = \frac{4.08}{3} \pi = \boxed{1.36\pi} \end{aligned}$$

21c)
$$\iiint_B \frac{dx dy dz}{\sqrt{x^2 + y^2 + z^2}}, \quad \text{on } B \text{ és la regió del 1er octant de } \mathbb{R}^3 \text{ acotada pels còns } \varphi = \pi/4 \text{ i } \varphi = \arctan z \text{ i l'esfera } r = \sqrt{6} \text{ (Recordem:}$$

$$\sin(\arctan(a)) = \frac{a}{\sqrt{1+a^2}},$$

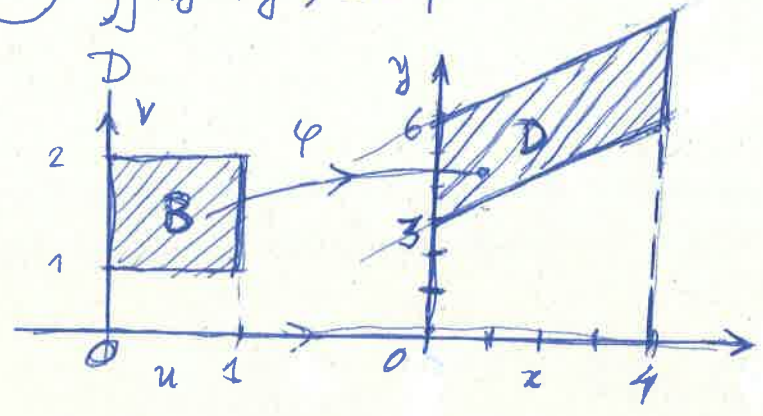
$$I = \int_0^{\pi/2} d\theta \int_{\pi/4}^{\arctan z} \cos\varphi d\varphi \int_0^{\sqrt{6}} \frac{r^2 dr}{r} = \left(\int_0^{\pi/2} d\theta \right) \left(\int_{\pi/4}^{\arctan z} \cos\varphi d\varphi \right) \cdot \left(\int_0^{\sqrt{6}} r dr \right) =$$

$$= \frac{\pi}{2} \left(\sin(\arctan 2) - \sin\left(\frac{\pi}{4}\right) \right) \cdot \left[\frac{r^2}{2} \right]_{r=0}^{r=\sqrt{6}} = \frac{\pi}{2} \left(\frac{\tan(\arctan 2)}{\sqrt{1+\tan^2(\arctan 2)}} - \frac{\sqrt{2}}{2} \right) \cdot 3$$

$$= \frac{3\pi}{2} \left(\frac{2}{\sqrt{5}} - \frac{\sqrt{2}}{2} \right)$$

18. Calenleu los integrals dobles següents mitjançant el canvi de variables que s'indiquen a cada cas.

18a) $\iint_D xy \, dx \, dy$; $D = \{(x,y) \in \mathbb{R}^2 : 6 \leq 2y-x \leq 12, 0 \leq x \leq 4\}$, $x=4u$; $y=2u+3v$.



$$B: 6 \leq 2y-x = 2u+6v-4u = 6v \leq 12$$

$$\Leftrightarrow 1 \leq v \leq 2,$$

$$0 \leq x = 4u \leq 4 \Leftrightarrow 0 \leq u \leq 1$$

$$D = \varphi(B), \text{ on: } \varphi(u,v) = (4u, 2u+3v)$$

$$B = \{(u,v) \in \mathbb{R}^2 : 0 \leq u \leq 1, 1 \leq v \leq 2\} = [0,1] \times [1,2]$$

$$I = \iint_{D=\varphi(B)} xy \, dx \, dy = \iint_B x(u,v) \cdot y(u,v) \cdot \left| \det \frac{\partial(x,y)}{\partial(u,v)}(u,v) \right| \, du \, dv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix}$$

$$\text{d'on: } \det \frac{\partial(x,y)}{\partial(u,v)} = 12$$

$$= \int_0^1 du \int_1^2 4u(2u+3v) \cdot 12 \, dv$$

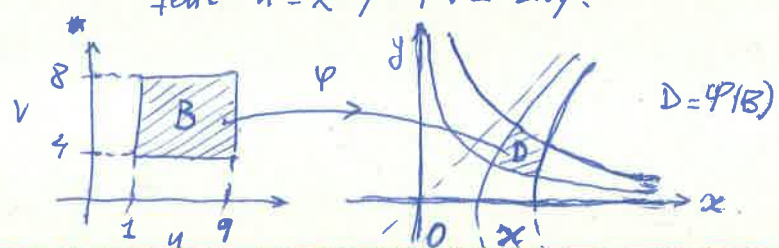
$$= 96 \left(\int_0^1 u^2 \, du \right) \cdot \left(\int_1^2 dv \right) + 144 \left(\int_0^1 u \, du \right) \left(\int_1^2 v \, dv \right)$$

$$= \frac{96}{3} + \frac{144}{4} \cdot 3 = 32 + 108 = \boxed{140}$$

18f) $\iint_D (x^2+y^2) \, dx \, dy$, $D = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2+y^2 \leq 9, 2 \leq xy \leq 4, x \geq 0, y \geq 0\}$.
 fent $u = x^2-y^2$; $v = 2xy$.

$$4 \leq v = 2xy \leq 8,$$

$$1 \leq u = x^2-y^2 \leq 9$$



$$\iint \dots = \iiint (x^2(u,v) + y^2(u,v)) \cdot \left| \det \frac{\partial(x,y)}{\partial(u,v)}(u,v) \right| du dv$$

$D = \varphi(B)$ $B = [1,9] \times [5,8]$

(2) $= \frac{1}{4} \int_1^9 du \int_5^8 \frac{x^2(u,v) + y^2(u,v)}{x^2(u,v) + y^2(u,v)} dv = \frac{1}{4} \left(\int_1^9 du \right) \cdot \left(\int_5^8 dv \right) = (9-1)(8-5) \frac{1}{4} = \boxed{8}$

(2) $\det \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\det \frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}} = \frac{1/4}{x^2(u,v) + y^2(u,v)}$

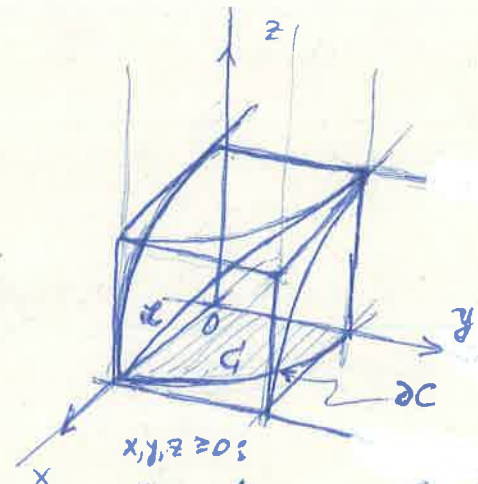
- (23a), (23e), (23f), (23j), (23l)

23) Usen coordenades cartesianes, cilíndriques o esfèriques (o bé el principi de Cavalieri) per a calcular el volum dels dominis de \mathbb{R}^3 limitats per les superfícies que s'indiquen.

23a) $x^2 + z^2 = 1, x^2 + y^2 = 1$:

$$V/8 = \iiint_G \left(\int_0^{\sqrt{1-x^2}} dz \right) dx dy = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2}} dz$$

$$= \int_0^1 dx \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2} dy = \int_0^1 (1-x^2) dx = \left(x - \frac{x^3}{3} \right)_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$



d'on, el volum buscat és: $V = \frac{16}{3}$

Nota: també surt (molt fàcil) per Cavalieri. En efecte, considerem talls segons $x = \text{const.}$, $0 \leq x \leq 1$. Aleshores $S(x) = 1 - x^2$ i llavors:

$$V/8 = \int_0^1 S(x) dx = \int_0^1 (1-x^2) dx = 1 - \frac{1}{3} = \frac{2}{3} \Rightarrow V = \frac{16}{3}$$

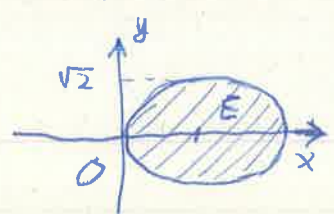
23e

$z = x^2 - 4x + 1, 1 - z = x^2 + y^2$

Projecció sobre el pla $z=0$ de la intersecció de les dues gràfiques

$$x^2 - 4x + 1 = 1 - x^2 - y^2 \Leftrightarrow 2(x^2 - 2x + 1) - 2 + y^2 = 0 \Leftrightarrow (x-1)^2 + \frac{y^2}{2} = 1 \quad \partial E, \text{ on:}$$

$E = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + \frac{y^2}{2} \leq 1\}$. Aleshores:



$$V = \iiint_E \left(\int_{x^2-4x+1}^{1-x^2-y^2} dz \right) dx dy = \iint_E (1-x^2-y^2-x^2+4x-1) dx dy =$$

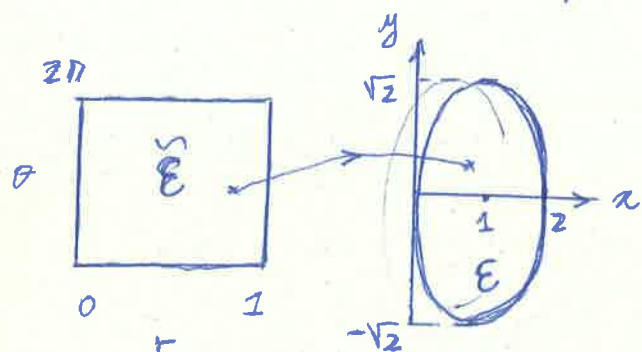
$$= 2 \cdot \iint_E \left(1 - (x-1)^2 - \frac{y^2}{2} \right) dx dy$$

Considerem coordenades polars "adaptades".

$$(x,y) = \varphi(r,\theta) = (1+r \cos \theta, \sqrt{2} r \sin \theta), \quad 0 < r < 1, \quad 0 < \theta < 2\pi.$$

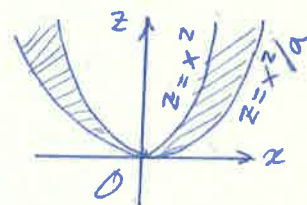
Lavors $E = \varphi(\tilde{E})$, amb $\tilde{E}: 0 < (x-1)^2 + \frac{y^2}{2} = r^2 < 1 \Leftrightarrow 0 < r < 1, 0 < \theta < 2\pi$

$$J_{\varphi}(r,\theta) = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sqrt{2} \sin \theta & r\sqrt{2} \cos \theta \end{pmatrix}, \text{ d'on: } \det J_{\varphi}(r,\theta) = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sqrt{2} \sin \theta & r\sqrt{2} \cos \theta \end{vmatrix} = \sqrt{2} r$$



$$\begin{aligned} V &= 2 \iint_E \dots = 2 \int_0^{2\pi} d\theta \int_0^1 (1-r^2) r \sqrt{2} dr \\ &= 2\sqrt{2} \left(\int_0^{2\pi} d\theta \right) \cdot \left(\int_0^1 (r-r^3) dr \right) \\ &= 4\sqrt{2} \pi \left(\frac{1}{2} - \frac{1}{4} \right) = \boxed{\sqrt{2} \pi} \end{aligned}$$

$$\tilde{E} = [0,1] \times [0,2\pi] \quad E: \frac{(x-1)^2}{1} + \frac{y^2}{\sqrt{2}^2} \leq 1$$



23f) $x^2 = z, y^2 = x, z^2 = y, x^2 = az, y^2 = ax, z^2 = ay \quad (a > 1).$

$$\frac{y^2}{a} \leq x \leq y^2 : \quad \frac{1}{a} \leq u = \frac{x}{y^2} \leq 1$$

$$\frac{z^2}{a} \leq y \leq z^2 : \quad \frac{1}{a} \leq v = \frac{y}{z^2} \leq 1$$

$$\frac{x^2}{a} \leq z \leq x^2 : \quad \frac{1}{a} \leq w = \frac{z}{x^2} \leq 1$$

$$\det \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\det \frac{\partial(u,v,w)}{\partial(x,y,z)}} = \frac{1}{\begin{vmatrix} 1/y^2 & -2x/y^3 & 0 \\ 0 & 1/z^2 & -2y/z^3 \\ -2z/x^3 & 0 & 1/x^2 \end{vmatrix}} = \frac{1}{\frac{-7}{x^2 y^2 z^2}}$$

$$= -\frac{1}{7} x(u,v,w)^2 y(u,v,w)^2 z(u,v,w)^2 = \frac{-1/7}{u^2 v^2 w^2}$$

$$V = \iiint_{\Omega} dx dy dz = \iiint_{\Gamma} 1 \cdot \left| \det \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du dv dw = \iiint_{\Gamma} \frac{1}{7} x \frac{du dv dw}{u^2 v^2 w^2} =$$

$\Gamma = [1/a, 1]^3$

$$= \frac{1}{7} \left(\int_{1/a}^1 \frac{du}{u^2} \right) \times \left(\int_{1/a}^1 \frac{dv}{v^2} \right) \times \left(\int_{1/a}^1 \frac{dw}{w^2} \right) = \frac{1}{7} \left[-\frac{1}{u} \right]_{1/a}^1 \times \left[-\frac{1}{v} \right]_{1/a}^1 \times \left[-\frac{1}{w} \right]_{1/a}^1$$

$$= \frac{(a-1)^3}{7}$$

23j) $x^2 + y^2 + z^2 = 2a^2, z = \frac{x^2 + y^2}{a} \quad (z \geq 0, a > 0)$

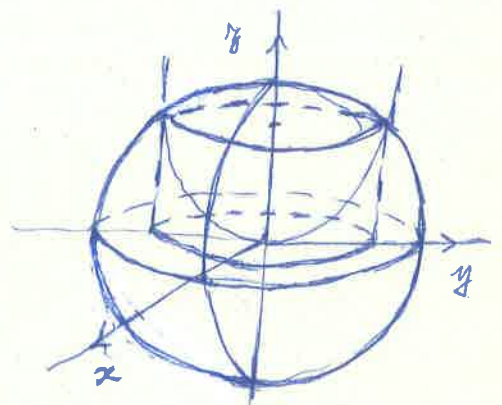
$$\frac{r^2}{a^2} \leq z \leq \sqrt{2a^2 - r^2}$$

$$V = \int_0^{2\pi} d\theta \int_0^a dr \int_{r^2/a^2}^{\sqrt{2a^2 - r^2}} r dz$$

$$= \int_0^{2\pi} d\theta \int_0^a \left(r \sqrt{2a^2 - r^2} - r^3/a \right) dr$$

$$= \int_0^{2\pi} \left[-\frac{1}{3} (2a^2 - r^2)^{3/2} - \frac{r^4}{4a} \right]_{r=0}^{r=a} d\theta = 2\pi \cdot \left(-\frac{1}{3} a^3 + \frac{1}{3} 2\sqrt{2} a^3 - \frac{a^3}{4} \right)$$

$$= 2\pi a^3 \left(\frac{2\sqrt{2}}{3} - \frac{7}{12} \right)$$



$$r^2/a^2 = 2a^2 - r^2 \Leftrightarrow r^2 + a^2 r^2 - 2a^2 = 0$$

$$r = \pm \frac{-a^2 \pm \sqrt{a^4 + 8a^4}}{2} = \pm \frac{-a^2 \pm 3a^2}{2}$$

$r > 0$, favors $r = a$.

23l)



Con de gelat definit per $x^2 + y^2 \leq z^2/5, 0 \leq z \leq 5 + \sqrt{5 - x^2 - y^2}$

$$(z-5)^2 + x^2 + y^2 \leq 5$$

$$x^2 + y^2 \leq z^2/5$$

$$V = \int_0^{2\pi} d\theta \int_0^{\sqrt{5}} dr \int_{\sqrt{5}r}^{5 + \sqrt{5 - r^2}} r dz = 2\pi \int_0^{\sqrt{5}} (5r - \sqrt{5}r^2 + r\sqrt{5 - r^2}) dr$$

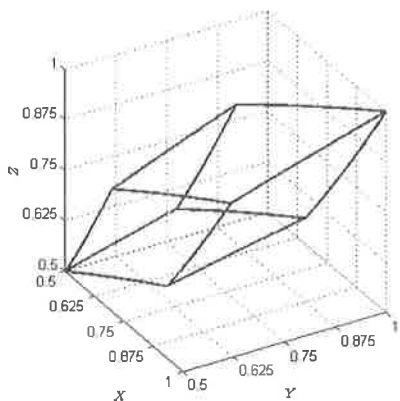
$$= 2\pi \left(\frac{5}{2} r^2 - \frac{\sqrt{5}}{3} r^3 - \frac{1}{3} (5 - r^2)^{3/2} \right) \Big|_0^{\sqrt{5}}$$

$$= 2\pi \left(\frac{25}{2} - \frac{25}{3} + \frac{5\sqrt{5}}{3} \right)$$

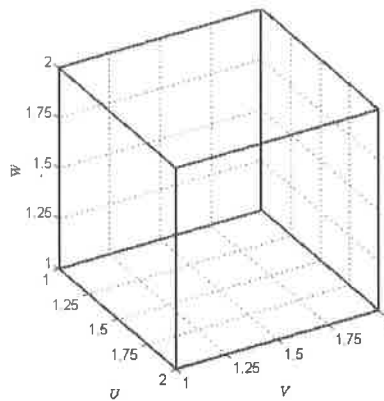
$$= \frac{25\pi}{3} + \frac{10\sqrt{5}\pi}{3}$$

$$V = \frac{2}{3} \pi \sqrt{5}^3 + \frac{\pi}{3} \sqrt{5}^2 \cdot 5 = \frac{25\pi}{3} + \frac{10\sqrt{5}\pi}{3}, \quad (= \frac{1}{2} \text{ Volum esfera} + \text{Volum con})$$

NOTA AL PROBLEMA 23f.



(a) \mathcal{D} : Domini definit per les superfícies,
 $x^2 = z, \quad y^2 = x, \quad z^2 = y,$
 $x^2 = az, \quad y^2 = ax, \quad z^2 = ay.$



(b) $\mathcal{D}' : 1 \leq u \leq a, 1 \leq v \leq a, 1 \leq w \leq a.$

FIGURA 1. Transformació del domini \mathcal{D} en \mathcal{D}' pel canvi (1).

NOTA (al problema 23f)

Alternativament, es comprova d'immediat que el canvi

$$u = \frac{y^2}{x}, \quad v = \frac{z^2}{y}, \quad w = \frac{x^2}{z}, \tag{1}$$

transforma el domini original \mathcal{D} , en $\mathcal{D}' = [1, a] \times [1, a] \times [1, a]$. És a dir, en un cub d'aresta $a - 1$ (veure Figura 1). Aleshores el Jacobià corresponent surt més senzill. En efecte:

$$\det \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} -\frac{y^2}{x^2} & 2\frac{y}{x} & 0 \\ 0 & -\frac{z^2}{y^2} & 2\frac{z}{y} \\ 2\frac{x}{z} & 0 & -\frac{x^2}{z^2} \end{vmatrix} = -7,$$

d'on:

$$\left| \det \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \frac{1}{\left| \det \frac{\partial(u, v, w)}{\partial(x, y, z)} \right|} = \frac{1}{7}.$$

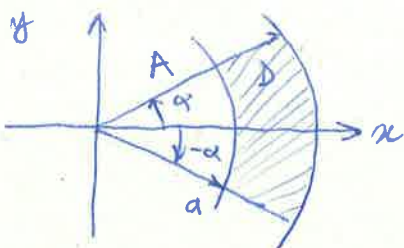
Lavors el càlcul del volum es simplifica encara més:

$$\begin{aligned} V &= \iiint_{\mathcal{D}} dx \, dy \, dz = \iiint_{\mathcal{D}'} \left| \det \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du \, dv \, dw \\ &= \frac{1}{7} \int_1^a du \int_1^a dv \int_1^a dw = \frac{1}{7} \left(\int_1^a du \right)^3 = \frac{(a-1)^3}{7}. \end{aligned}$$

Problemes Aplicacions: 25(a), 25(b), 26(a), 26(c) i 28

25. Troben el centre de masses de les regions planes següents amb les densitats que s'indiquen

(25a) Sector pla definit per una coromna de radi interior a i radi exterior A , un angle d'obertura 2α i que és simètrica respecte de l'eix x positiva suposant densitat constant $\rho(x,y)=1$.



Per simetria el CG (\bar{x}, \bar{y}) es troba sobre l'eix x , i.e. $\bar{y}=0$.

$$A(D)\bar{x} = \iint_D x \rho(x,y) dx dy = \int_{-\alpha}^{\alpha} d\theta \int_a^A r^2 \cos\theta dr$$

$$= \left(\int_{-\alpha}^{\alpha} \cos\theta d\theta \right) \cdot \left(\int_a^A r^2 dr \right) = \frac{2}{3} (A^3 - a^3) \sin\alpha.$$

$$A(D) = \alpha (A^2 - a^2)$$

Alleshores: $\bar{x} = \frac{2}{3} \frac{\sin\alpha}{\alpha} \frac{A^3 - a^3}{A^2 - a^2}$, $\bar{y} = 0$, i.e. $(\bar{x}, \bar{y}) = \left(\frac{2}{3} \frac{\sin\alpha}{\alpha} \frac{A^3 - a^3}{A^2 - a^2}, 0 \right)$.

(25b) Regió entre $y=x^2$, $y=x$ amb $\rho(x,y)=x+y$

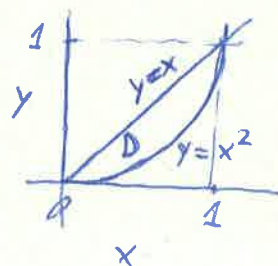
$$y=x^2$$

$$y=x$$

tall

$$x=0, x=1$$

$$y=0, y=1$$



$$m(D)\bar{x} = \iint_D x \rho(x,y) dx dy = \int_0^1 dx \int_{x^2}^x x(x+y) dy$$

$$= \int_0^1 \left[x^2 y + x y^2 / 2 \right]_{y=x^2}^{y=x} dx = \int_0^1 \left(x^3 + \frac{x^3}{2} - x^4 - \frac{x^5}{2} \right) dx$$

$$= \left(\frac{3x^4}{8} - \frac{x^5}{5} - \frac{x^6}{12} \right) \Big|_0^1 = \frac{3}{8} - \frac{1}{5} - \frac{1}{12} = \frac{45-24}{120} = \frac{11}{120}$$

$$m(D)\bar{y} = \iint_D y \rho(x,y) dx dy = \int_0^1 dx \int_{x^2}^x y(x+y) dy = \int_0^1 \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_{y=x^2}^{y=x} dx = \int_0^1 \left(\frac{5}{6} x^3 - \frac{x^5}{2} - \frac{x^6}{3} \right) dx$$

$$= \frac{5}{24} x^4 - \frac{1}{12} x^6 - \frac{1}{21} x^7 \Big|_{x=0}^{x=1} = \frac{5}{24} - \frac{1}{12} - \frac{1}{21} = \frac{35-14-8}{168} = \frac{35-22}{168} = \frac{13}{168}$$

$$m(D) = \iint_D \rho(x,y) dx dy = \int_0^1 dx \int_{x^2}^x (x+y) dy = \int_0^1 \left(xy + \frac{y^2}{2} \right) \Big|_{y=x^2}^{y=x} dx = \int_0^1 \left(x^2 + \frac{x^2}{2} - x^3 - \frac{x^4}{2} \right) dx$$

$$= \int_0^1 \left(\frac{3}{2} x^2 - x^3 - \frac{x^4}{2} \right) dx = \left(\frac{1}{2} x^3 - \frac{1}{4} x^4 - \frac{1}{10} x^5 \right) \Big|_0^1 = \frac{1}{4} - \frac{1}{10} = \frac{3}{20}$$

Així doncs:

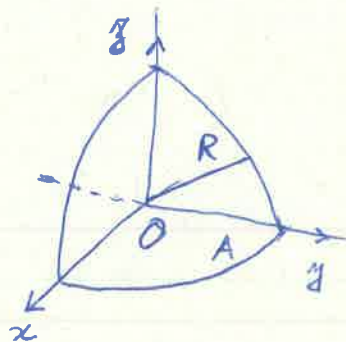
$$\bar{x} = \frac{11/120}{3/20} = \frac{11 \cdot 20}{3 \cdot 120} = \frac{11}{18}, \quad \bar{y} = \frac{13/168}{3/20} = \frac{13 \cdot 20}{3 \cdot 168} = \frac{13 \cdot 2^2 \cdot 5}{3^2 \cdot 2^3 \cdot 7} = \frac{65}{126}$$

i les coordenades de CM del cas són $(\bar{x}, \bar{y}) = \left(\frac{11}{18}, \frac{65}{126}\right)$

26. En cadascun dels casos següents, troben el centre de masses dels sòlids A, suposant distribució de masses homogènia.

26a) $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq R^2, x \geq 0, y \geq 0, z \geq 0\}$

Suposem $\rho(x, y, z) \equiv \rho_0 \in \mathbb{R}, \rho_0 \text{ cnt.}$



$$m(A) \bar{x} = \iiint_A x \rho(x, y, z) dx dy dz =$$

$$= \rho_0 \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\varphi \int_0^R r^3 \cos \varphi \cos \theta dr = \rho_0 \left(\int_0^{\pi/2} \cos \theta d\theta \right) \cdot \left(\int_0^{\pi/2} \cos^2 \varphi d\varphi \right) \cdot \left(\int_0^R r^3 dr \right)$$

$$= \rho_0 \frac{\pi}{4} \cdot \frac{R^4}{4} = \rho_0 \frac{\pi R^4}{16}$$

$$m(A) \bar{y} = \rho_0 \left(\int_0^{\pi/2} \sin \theta d\theta \right) \cdot \left(\int_0^{\pi/2} \cos^2 \varphi d\varphi \right) \cdot \left(\int_0^R r^3 dr \right) = \rho_0 \frac{\pi}{4} \cdot \frac{R^4}{4} = \rho_0 \frac{\pi R^4}{16}$$

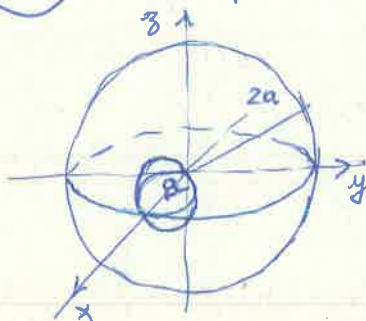
$$m(A) \bar{z} = \rho_0 \left(\int_0^{\pi/2} d\theta \right) \cdot \left(\int_0^{\pi/2} \cos \varphi \sin \varphi d\varphi \right) \cdot \left(\int_0^R r^3 dr \right) = \rho_0 \frac{\pi}{4} \cdot \frac{R^4}{4} = \rho_0 \frac{\pi R^4}{16}$$

com que $m(A) = \rho_0 \frac{1}{6} \pi R^3$. Aleshores: $\bar{x} = \frac{\rho_0 \frac{\pi R^4}{16}}{\rho_0 \frac{\pi R^3}{6}} = \frac{3R}{8} = \bar{y} = \bar{z}$,

Les coordenades del CG són doncs:

$$\left(\bar{x}, \bar{y}, \bar{z}\right) = \left(\frac{3R}{8}, \frac{3R}{8}, \frac{3R}{8}\right).$$

26c) $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4a^2, (x-a)^2 + y^2 + z^2 \geq a^2\}, \rho(x, y, z) \equiv \rho_0 \in \mathbb{R} \text{ cnt.}$



$$A_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4a^2\},$$

$$A_2 = \{(x, y, z) \in \mathbb{R}^3 : (x-a)^2 + y^2 + z^2 < a^2\}$$

$$A = A_1 \setminus A_2, \quad A_1 = A_1 \setminus A_2 \cup A_2$$

Signim $m(A_1), m(A_2); (\bar{x}_1, \bar{y}_1, \bar{z}_1), (\bar{x}_2, \bar{y}_2, \bar{z}_2)$ les masses i les posicions dels CG de A_1 i A_2 respectivament.

Anomenem $m(A) = m(A_1 \setminus A_2)$ i $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$, respectivament, a la massa i a les coordenades del CM del cos A . Aleshores:

$$\bar{x}_1 = \frac{m(A_1 \setminus A_2)\bar{x} + m(A_2)\bar{x}_2}{m(A_1 \setminus A_2) + m(A_2)} = \frac{m(A_1 \setminus A_2)\bar{x} + m(A_2)\bar{x}_2}{m(A_1)}$$

$$\bar{x} = \frac{1}{m(A_1 \setminus A_2)} (m(A_1)\bar{x}_1 - m(A_2)\bar{x}_2) \stackrel{(*)}{=} \frac{-\frac{4}{3}\pi a^3 \cdot a}{\frac{28}{3}\pi a^3} = -\frac{a}{7}$$

$$\begin{aligned} (*) \quad m(A_1 \setminus A_2) &= m(A_1) - m(A_2) \\ &= \frac{4}{3}\pi 8a^3 - \frac{4}{3}\pi a^3 \\ &= \frac{4}{3}\pi (8-1)a^3 = \frac{28}{3}\pi a^3 \end{aligned}$$

Com que l'eix x és un eix de simetria del cos A , el CG li pertany. Aleshores $\bar{y} = 0 = \bar{z}$ i les coordenades del CG són

$$\boxed{(\bar{x}, \bar{y}, \bar{z}) = \left(-\frac{a}{7}, 0, 0\right)} \quad \square$$

28. Troben el centre de masses de la semiesfera definida per $x^2 + y^2 + z^2 \leq R^2$ i $z \geq 0$ si la densitat a cada punt és proporcional a la distància d'aquest punt al centre.

$$\rho(x, y, z) = c(x^2 + y^2 + z^2)^{1/2} = cr$$

$$m(W)z = \iiint_W z \rho(x, y, z) dx dy dz$$

$$= c \int_0^{2\pi} d\theta \int_0^{\pi/2} d\varphi \int_0^R r^4 \cos\varphi \sin\varphi dr$$

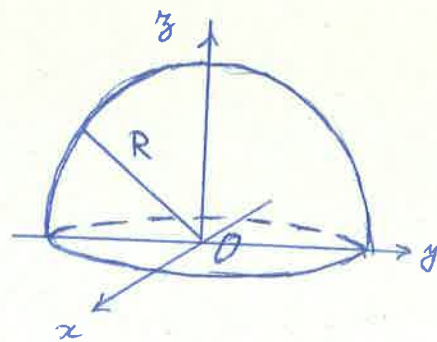
$$= c \left(\int_0^{2\pi} d\theta \right) \times \left(\int_0^{\pi/2} \sin\varphi \cos\varphi d\varphi \right) \times \left(\int_0^R r^4 dr \right) = c \frac{\pi}{5} R^5$$

$$m(W) = \iiint_W \rho(x, y, z) dx dy dz = c \int_0^{2\pi} d\theta \int_0^{\pi/2} d\varphi \int_0^R r^3 \cos\varphi dr$$

$$= c \left(\int_0^{2\pi} d\theta \right) \cdot \left(\int_0^{\pi/2} \cos\varphi d\varphi \right) \cdot \left(\int_0^R r^3 dr \right) = 2\pi c \frac{R^4}{4} = c \frac{\pi R^2}{2}$$

Aleshores:

$$\bar{z} = \frac{c \frac{\pi}{5} R^5}{c \frac{\pi}{2} R^2} = \frac{2}{5} R$$

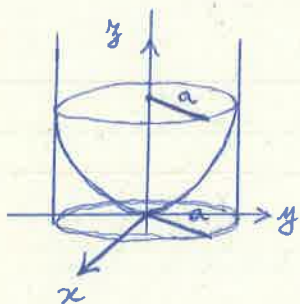


i, per simetria: $\bar{x} = 0 = \bar{y}$; d'on tenim que les coordenades del CM són $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 2R/5)$. \square

34(c), 34(d)

34. Pels sòlids següents, calculen els moments d'inèrcia que es demanen en cada cas tot suposant densitat homogènia igual a 1.

34c) Calculen I_z pel sòlid limitat pel paraboloid $z = x^2 + y^2$ i el cilindre $x^2 + y^2 = a^2$, $z \geq 0$.

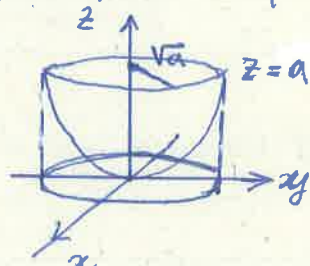


$\rho(x, y, z) \equiv 1$:

$$I_z = \iiint_W (x^2 + y^2) \rho(x, y, z) dx dy dz = \int_0^{2\pi} d\theta \int_0^a dr \int_0^{r^2} r^3 dz$$

$$= \int_0^{2\pi} d\theta \int_0^a r^5 dr = \left(\int_0^{2\pi} d\theta \right) \cdot \left(\int_0^a r^5 dr \right) = \frac{2\pi a^6}{6} = \boxed{\frac{\pi a^6}{3}}. \square$$

34d) Calculen I_x, I_y, I_z pel sòlid tancat pel paraboloid $z = x^2 + y^2$ i el pla $z = a$ ($a > 0$).



$I_x = \iiint_W (y^2 + z^2) \rho(x, y, z) dx dy dz$

$= \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} dr \int_{r^2}^a (r^2 \sin^2 \theta + z^2) r dz$

$= \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} \left(r^3 z \sin^2 \theta + r z^3 / 3 \right) \Big|_{z=r^2}^{z=a} dr = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} \left(ar^3 \sin^2 \theta + a^3 r / 3 - r^5 \sin^2 \theta - r^7 / 3 \right) dr$

$= \int_0^{2\pi} \left(\frac{a^4}{4} r^4 \sin^2 \theta + \frac{a^3 r^2}{6} - \frac{r^6}{6} \sin^2 \theta - \frac{r^8}{24} \right) \Big|_{r=0}^{r=\sqrt{a}} d\theta = \int_0^{2\pi} \left(\frac{a^3}{4} \sin^2 \theta + \frac{a^4}{6} - \frac{a^3}{6} \sin^2 \theta - \frac{a^4}{24} \right) d\theta$

$= \frac{a^4}{8} \int_0^{2\pi} d\theta + \frac{a^3}{12} \int_0^{2\pi} \sin^2 \theta d\theta = \frac{\pi a^4}{4} + \frac{\pi a^3}{12} = \frac{\pi a^3}{12} (1 + 3a) = I_y$ (per simetria).

$I_z = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} dr \int_{r^2}^a r^3 dz = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r^3 z \Big|_{z=r^2}^{z=a} dr = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} (ar^3 - r^5) dr$

$= \left(\int_0^{2\pi} d\theta \right) \cdot \left(\int_0^{\sqrt{a}} (ar^3 - r^5) dr \right) = 2\pi \cdot \left(ar^4 / 4 - r^6 / 6 \right) \Big|_{r=0}^{r=\sqrt{a}} = \pi a^3 \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{\pi a^3}{6}$

llavors: $I_x = I_y = \frac{\pi a^3}{12} (1 + 3a)$, $I_z = \frac{\pi a^3}{6}$. \square