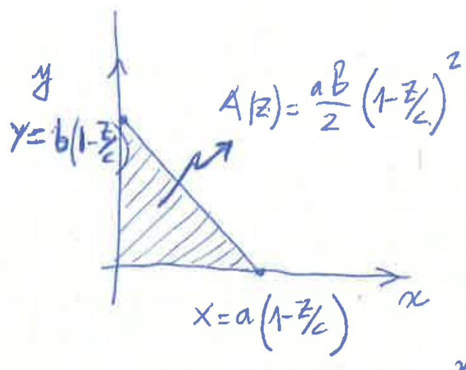
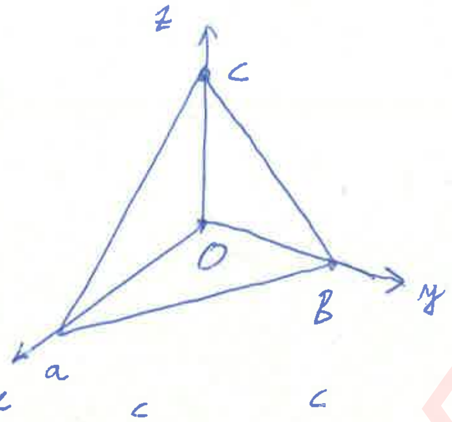
(3c) Volum d'un tetraedre limitat pels plans $x=0, y=0, z=0$

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1, \ (a,b,c>0)$

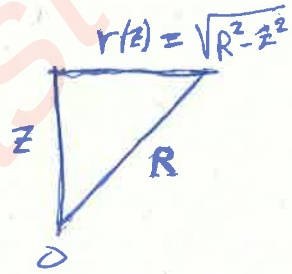


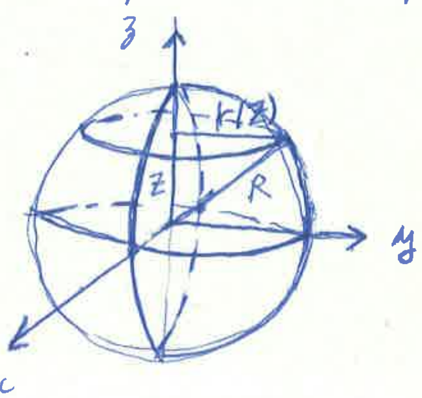


$$\frac{x}{a}+\frac{y}{b}=1-\frac{z}{c}\ ; \quad V(W)=\int_0^c A(z)\,dz=\int_0^c \frac{ab}{2}\left(1-\frac{z}{c}\right)^2 dz = \frac{abc}{6}\left(1-\frac{z}{c}\right)^3\Bigg|_{z=0}^{z=c}$$

$$=\boxed{\frac{abc}{6}}$$

(3d) Volum envoltat per un casquet esfèric determinat per l'esfera $x^2+y^2+z^2=$ $=R^2$ i la condició $R-h\leq z\leq R$





$$A(z)=\pi r(z)^2=\pi\sqrt{R^2-z^2}^{\,2}$$
$$=\pi(R^2-z^2),$$

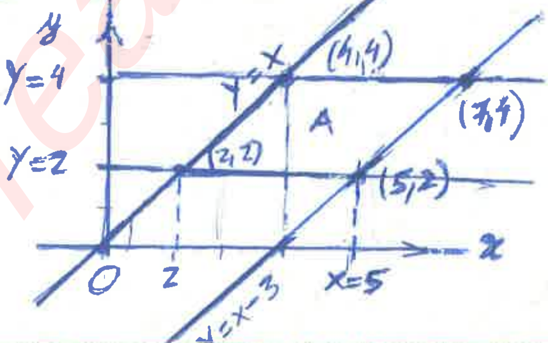
$$V(W)=\int_{R-h}^{R}\pi(R^2-z^2)\,dz$$

$$=\pi\left(R^2z-\frac{z^3}{3}\right)\Bigg|_{z=R-h}^{z=R} = \pi\left(R^3-\frac{R^3}{3}-R^2(R-h)+\frac{(R-h)^3}{3}\right)$$

$$=\pi\left(\cancel{R^3}-\cancel{R^3}+hR^2+\frac{\cancel{R^3}-3R^2h+3Rh^2-h^3-\cancel{R^3}}{3}\right)=\boxed{\frac{\pi h^2}{3}(3R-h)}.$$

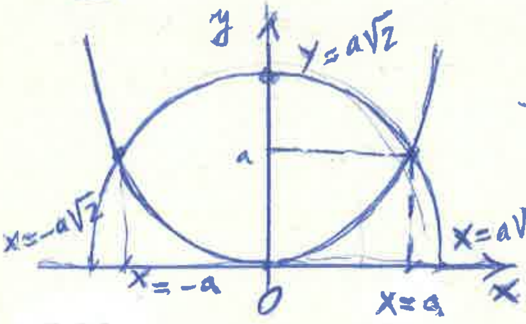
Classe 15-11-2016 Grup 30

(8b) A paral·lelogram limitat per les rectes $y=x,\ y=x-3,\ y=2,\ y=4$.



$$I_1=\int_2^4\left(\int_2^x f(x,y)\,dy\right)dx+\int_4^5\left(\int_2^4 f(x,y)\,dy\right)dx$$
$$+\int_5^7\left(\int_{x-3}^4 f(x,y)\,dy\right)dx=\int_2^4\left(\int_y^{y+3} f(x,y)\,dx\right)dy$$

(8c) Regió limitada per les corbes $x^2+y^2=2a^2$, $x^2=ay$ $(y\geq 0, a>0)$



$$I=\int_{-a}^{a}\left(\int_{x^2/a}^{\sqrt{2a^2-x^2}} f(x,y)\,dx\right)$$

$$=\int_{0}^{a}\left(\int_{-\sqrt{ay}}^{\sqrt{ay}} f(x,y)\,dx\right)dy+\int_{a}^{a\sqrt{2}}\left(\int_{-\sqrt{2a^2-y^2}}^{\sqrt{2a^2-y^2}} f(x,y)\,dx\right)dy$$

$$\left.\begin{array}{l} x^2+y^2=2a^2 \\ x^2=ay \end{array}\right\} ;\ ay+y^2=2a^2$$

$$y=\frac{-a\pm\sqrt{a^2+8a^2}}{2}=\frac{-a\pm 3a}{2}=\left\{\begin{array}{l} -2a \text{ (NO)} \\ a \end{array}\right.$$

$$x=\pm a$$

(8e) A la regió limitada per les corbes: $x^2+y^2=ax$, $x^2+y^2=2ax$, $y=0$, $(y\geq 0, a>0)$

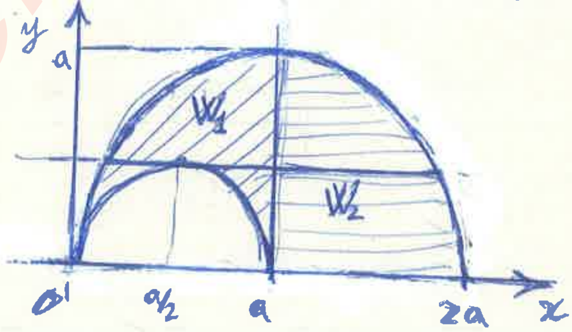
$$I=\int_{0}^{a}\left(\int_{\sqrt{ax-x^2}}^{\sqrt{2ax-x^2}} f(x,y)\,dy\right)dx$$

$$+\int_{a}^{2a}\left(\int_{0}^{\sqrt{2ax-x^2}} f(x,y)\,dy\right)dx$$

$$=\int_{0}^{a/2}\left(\int_{a-\sqrt{a^2-y^2}}^{a/2-\sqrt{a^2/4-y^2}} f(x,y)\,dx\right)dy$$

$$+\int_{0}^{a/2}\left(\int_{a/2+\sqrt{a^2/4-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx\right)dy$$

$$+\int_{a/2}^{a}\left(\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx\right)dy$$



W_1, W_2 · dominis x-elementals

$$x^2+y^2=ax \Leftrightarrow x^2-ax+y^2=0$$

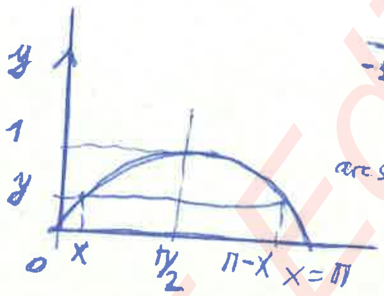
$$x=\frac{a\pm\sqrt{a^2-4y^2}}{2}=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-y^2}$$

$$x^2+y^2=2ax$$

$$x=\frac{2a\pm\sqrt{4a^2-4y^2}}{2}=a\pm\sqrt{a^2-y^2}$$

(11) Invertiu l'ordre d'integració de les integrals iterades següents

(11e) $$I_1 = \int_0^{\pi} \left(\int_0^{\sin x} f(x,y)\,dy \right) dx$$



$$I_2 = \int_0^1 \left(\int_{\arcsin y}^{\pi - \arcsin y} f(x,y)\,dx \right) dy$$

$\arcsin : (-1,1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$

$\sin x = \sin(\pi - x) = y$

$x = \arcsin y$

$\pi - x = \arcsin y \Leftrightarrow x = \pi - \arcsin y$

Fubini: $I_1 = I_2$

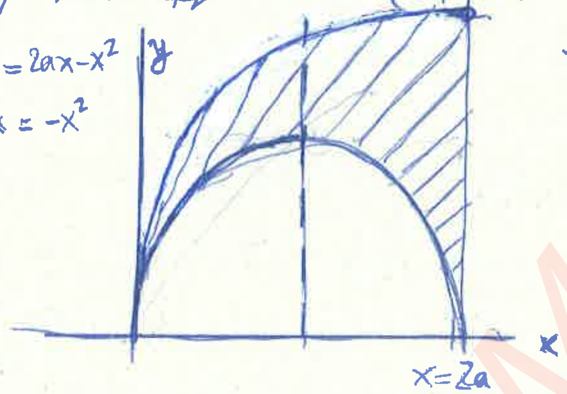
(11g) $$I_1 = \int_0^{2a} \left(\int_{\sqrt{2ax-x^2}}^{\sqrt{4ax}} f(x,y)\,dy \right) dx$$

$y^2 = 2ax - x^2 \Leftrightarrow x^2 + y^2 = 2ax \Leftrightarrow x = a \pm \sqrt{a^2 - y^2}$

$y^2 = 4ax \Leftrightarrow$

$4ax = 2ax - x^2$

$2ax = -x^2$

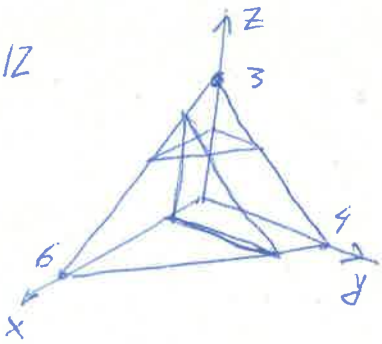


$$I_2 = \int_0^a \left(\int_{y^2/4a}^{a - \sqrt{a^2-y^2}} f(x,y)\,dx \right) dy + \int_0^a \left(\int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dx \right) dy + \int_a^{2\sqrt{2}a} \left(\int_{y^2/4a}^{2a} f(x,y)\,dx \right)$$

Fubini: $I_1 = I_2$

(14) Per a les regions de $\mathbb{R}^3$ indicades escriviu la integral triple $\iiint_A f(x,y,z)\,dx\,dy\,dz$ en termes d'integrals iterades preses en diferents ordres

(14a) Tetraedre limitat pels plans $x=0, y=0, z=0, 2x+3y+4z=12$



$0 \le z \le \frac{1}{4}(12 - 2x - 3y) = 3 - \frac{1}{2}x - \frac{3}{4}y, \; x \ge 0, y \ge 0$

$$\left[\begin{array}{l} 0 \le y \le \frac{4}{3}(3 - \frac{1}{2}x) = 4 - \frac{2}{3}x \\ 0 \le x \le 6 \end{array} \right.$$

$0 \le x \le 2(3 - \frac{3}{4}y) = 6 - \frac{3}{2}y$

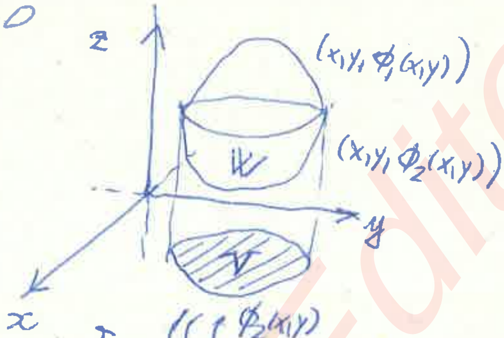
$0 \le y \le 4$

$$I_1 = \int_0^6 dx \int_0^{4 - \frac{2}{3}x} dy \int_0^{3 - \frac{x}{2} - \frac{3}{4}y} f(x,y,z)\,dz = \int_0^4 dy \int_0^{6 - \frac{3}{2}y} dx \int_0^{3 - \frac{x}{2} - \frac{3}{4}y} f(x,y,z)\,dz$$

$0\le y\le \frac{1}{3}(12-2x-4z)=4-\frac{2}{3}x-\frac{4}{3}z,\quad x\ge 0,\ z\ge 0$

$$\left[\begin{array}{l} 0\le z\le \frac{3}{4}\left(4-\frac{2}{3}x\right)=3-\frac{x}{2} \\ 0\le x\le 6 \end{array}\right.$$

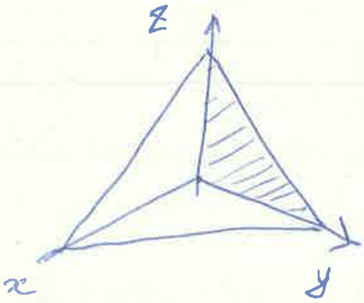
$$\left[\begin{array}{l} 0\le x\le \frac{3}{2}\left(4-\frac{4}{3}z\right)=6-2z \\ 0\le z\le 3 \end{array}\right.$$



$$I=\iint_V\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz$$

$$I_2=\int_0^6 dx\int_0^{3-x/2} dz\int_0^{4-\frac{2}{3}x-\frac{4}{3}z} f(x,y,z)\,dy=\int_0^3 dz\int_0^{6-2z} dx\int_0^{4-\frac{2}{3}x-\frac{4}{3}z} f(x,y,z)\,dy$$

Tonelli-Fubini



$0\le x\le \frac{1}{2}(12-3y-4z)=6-\frac{3}{2}y-2z,\ y\ge 0,\ z\ge 0$

$$\left[\begin{array}{l} 0\le y\le \frac{2}{3}(6-2z)=4-\frac{4}{3}z \\ 0\le z\le 3 \end{array}\right.$$

$0\le z\le \frac{1}{2}\left(6-\frac{3}{2}y\right)=3-\frac{3}{4}y$

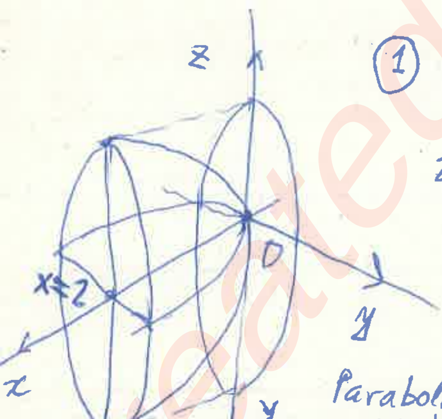
$0\le y\le 4$

$$I_3=\int_0^3 dz\int_0^{4-\frac{4}{3}z} dy\int_0^{6-\frac{3}{2}y-2z} f(x,y,z)\,dx=\int_0^4 dy\int_0^{3-\frac{3}{4}y} dz\int_0^{6-\frac{3}{2}y-2z} f(x,y,z)\,dx$$

Tonelli-Fubini

Pel teorema de Fubini: $I_1=I_2=I_3$.

(14c) A cos limitat per les superfícies: $y^2+2z^2=4x,\ x=2$ $\qquad y^2+2z^2\le 4x,\ x\le 2$



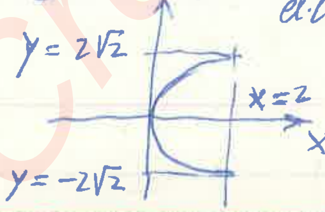
Paraboloide el·líptic

(1) $z^2=2x-y^2/2$; $-\sqrt{2x-y^2/2}\le z\le \sqrt{2x-y^2/2}$

$2x-y^2/2\ge 0$, d'on: $y^2/4\le x\le 2$; $-2\sqrt{2}\le y\le 2\sqrt{2}$

o bé:

$-2\sqrt{x}\le y\le 2\sqrt{x},\ 0\le x\le 2$



$$I_1=\int_{-2\sqrt{2}}^{2\sqrt{2}} dy\int_{y^2/4}^{2} dx\int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}} f(x,y,z)\,dz=$$

Tonelli-Fubini
$$\overset{\downarrow}{=} \int_0^2 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} dy \int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}} f(x,y,z)\,dz$$

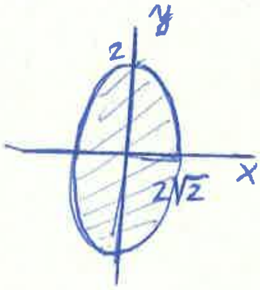
② $-\sqrt{4x-2z^2} \le y \le \sqrt{4x-2z^2}$, $x \le 2$

$4x-2z^2 \ge 0$, d'on: $z^2/2 \le x \le 2$, $-2 \le z \le 2$

o bé: $-\sqrt{2x} \le z \le \sqrt{2x}$, $0 \le x \le 2$

$$I_2 = \int_{-2}^{2} dz \int_{z^2/2}^{2} dx \int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}} f(x,y,z)\,dy \underset{\text{Tonelli-Fubini}}{=} \int_0^2 dx \int_{-\sqrt{2x}}^{\sqrt{2x}} dz \int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}} f(x,y,z)\,dy$$

③ $y^2/4 + z^2/2 \le x \le 2$;



$y^2/8 + z^2/4 \le 1$: $-\sqrt{4-y^2/2} \le z \le \sqrt{4-y^2/2}$, $-2\sqrt{2} \le y \le 2\sqrt{2}$

$-\sqrt{8-2z^2} \le y \le \sqrt{8-2z^2}$, $-2 \le z \le 2$

$$I_3 = \int_{-2\sqrt{2}}^{2\sqrt{2}} dy \int_{-\sqrt{4-y^2/2}}^{\sqrt{4-y^2/2}} dz \int_{y^2/4+z^2/2}^{2} f(x,y,z)\,dx \underset{\text{Tonelli-Fubini}}{=} \int_{-2}^{2} dz \int_{-\sqrt{8-2z^2}}^{\sqrt{8-2z^2}} dy \int_{y^2/4+z^2/2}^{2} f(x,y,z)\,dx$$

Tonelli-Fubini $I_1 = I_2 = I_3$

(15) Calculeu les integrals triples següents en les regions de $\mathbb{R}^3$ que s'indiquen

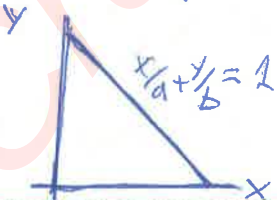
(15e) $\iiint_A x\,dx\,dy\,dz$, A: tetraedre limitat pels plans $x=0, y=0, z=0, x/a + y/b + z/c = 1$

$(a,b,c>0)$;

$0 \le z \le c(1 - x/a - y/b)$

$0 \le y \le b(1 - x/a)$

$0 \le x \le a$



$$I = \int_0^a dx \int_0^{b(1-x/a)} dy \int_0^{c(1-x/a-y/b)} x\,dz$$

$$= c\int_0^a dx \int_0^{b(1-x/a)} x\left(1 - x/a - y/b\right) dy$$

$$= c\int_0^a \left(xy - \frac{x^2y}{a} - \frac{xy^2}{2b}\right)\Bigg|_{y=0}^{y=b(1-x/a)} dx$$

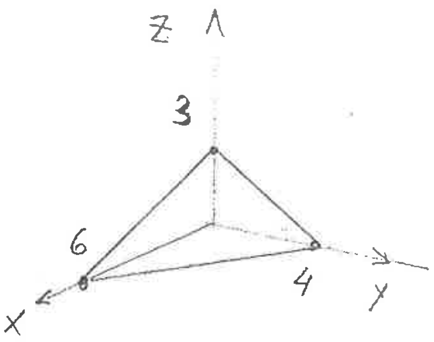
$$= bc\int_0^a x\left(1-\frac{x}{a}\right)\left(1 - \frac{x}{a} - \frac{1}{2} + \frac{x}{2a}\right) dx = \frac{bc}{2}\int_0^a x\left(1-\frac{x}{a}\right)^2 dx =$$

$$= \frac{bc}{2}\int_0^a \left(x - \frac{2x^2}{a} + \frac{x^3}{a^2}\right)dx = \frac{bc}{2}\left(x^2/2 - 2x^3/3a + x^4/4a^2\right)\Big|_{x=0}^{x=a}$$

$$= \frac{bc}{2}\left(\frac{a^2}{2} - \frac{2}{3}a^2 + \frac{a^2}{4}\right) = \frac{bc}{2}\cdot\frac{6a^2-8a^2-3a^2}{12} = \boxed{\frac{bca^2}{24}}$$

14) Per les regions de $\mathbb{R}^3$ indicades escriviu la integral triple $\iiint_A f(x,y,z)\,dx\,dy\,dz =: I$ en termes d'integrals iterades preses en diferents ordres.

(a) A tetraedre limitat pels plans $x=0,\ y=0,\ z=0,\ 2x+3y+4z=12$.



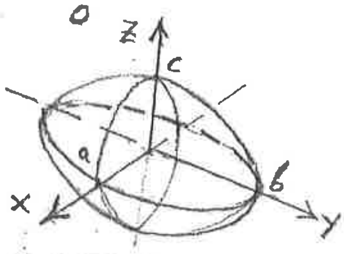
$$I=\int_0^6 dx\int_0^{4-\frac{2}{3}x} dy\int_0^{3-\frac{1}{2}x-\frac{3}{4}y} f(x,y,z)\,dz=\int_0^4 dy\int_0^{6-\frac{3}{2}y} dx\int_0^{3-\frac{1}{2}x-\frac{3}{4}y} f(x,y,z)\,dz$$

$$=\int_0^6 dx\int_0^{3-\frac{1}{2}x} dz\int_0^{4-\frac{2}{3}x-\frac{4}{3}z} f(x,y,z)\,dy=\int_0^3 dz\int_0^{6-2z} dx\int_0^{4-\frac{2}{3}x-\frac{4}{3}z} f(x,y,z)\,dy$$

$$=\int_0^4 dy\int_0^{3-\frac{3}{4}y} dz\int_0^{6-\frac{3}{2}y-2z} f(x,y,z)\,dx=\int_0^3 dz\int_0^{4-\frac{4}{3}z} dy\int_0^{6-\frac{3}{2}y-2z} f(x,y,z)\,dx.$$

- $4z=12-2x-3y \to z=3-\frac{1}{4}x-\frac{3}{4}y$
  $z=0,\ 2x+3y=12 \to y=4-\frac{2}{3}x$
  $x=6-\frac{3}{2}y$
- $3y=12-2x-4z \to y=4-\frac{2}{3}x-\frac{4}{3}z$
  $y=0,\ 2x+4z=12 \to z=3-\frac{1}{2}x$
  $x=6-2z$
- $2x=12-3y-4z \to x=6-\frac{3}{2}y-2z$
  $x=0:\ 3y+4z=12 \to z=3-\frac{3}{4}y$
  $y=4-\frac{4}{3}z$

(b) A interior del el·lipsoide



$$I=\int_{-a}^{a} dx\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} dy\int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} f(x,y,z)\,dz$$

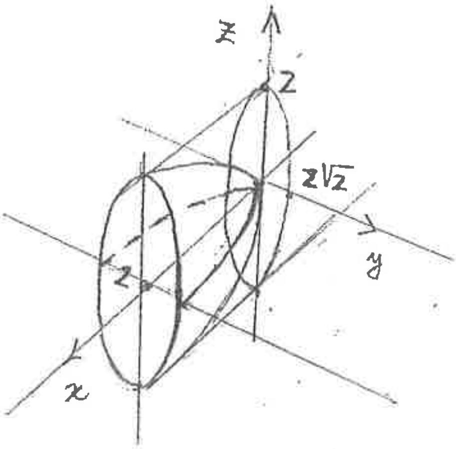
$$=\int_{-b}^{b} dy\int_{-a\sqrt{1-y^2/b^2}}^{a\sqrt{1-y^2/b^2}} dx\int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} f(x,y,z)\,dz$$

$$=\int_{-a}^{a} dx\int_{-c\sqrt{1-x^2/a^2}}^{c\sqrt{1-x^2/a^2}} dz\int_{-b\sqrt{1-x^2/a^2-z^2/c^2}}^{b\sqrt{1-x^2/a^2-z^2/c^2}} f(x,y,z)\,dy=\int_{-c}^{c} dz\int_{-a\sqrt{1-z^2/c^2}}^{a\sqrt{1-z^2/c^2}} dx\int_{-b\sqrt{1-x^2/a^2-z^2/c^2}}^{b\sqrt{1-x^2/a^2-z^2/c^2}} f(x,y,z)\,dy$$

$$=\int_{-b}^{b} dy\int_{-c\sqrt{1-y^2/b^2}}^{c\sqrt{1-y^2/b^2}} dz\int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z)\,dx=\int_{-c}^{c} dz\int_{-b\sqrt{1-z^2/c^2}}^{b\sqrt{1-z^2/c^2}} dy\int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z)\,dx.\ \square$$

$$\frac{y^2}{4}+\frac{z^2}{2}=x$$

(c) A cos limitat per les superficies $y^2+2z^2=4x$, $x=2$.



$$I=\int_{-2}^{2}dz\int_{-\sqrt{8-2z^2}}^{\sqrt{8-2z^2}}dy\int_{\frac{y^2}{4}+\frac{z^2}{2}}^{2}f(x,y,z)\,dx=\int_{-2\sqrt{2}}^{2\sqrt{2}}dy\int_{-\sqrt{4-y^2/2}}^{\sqrt{4-y^2/2}}dz\int_{\frac{y^2}{4}+\frac{z^2}{2}}^{2}f(x,y,z)\,dx$$

$$=\int_{0}^{2}dx\int_{-\sqrt{2x}}^{\sqrt{2x}}dz\int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}}f(x,y,z)\,dy=\int_{-2}^{2}dz\int_{z^2/2}^{2}dx\int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}}f(x,y,z)\,dy$$

$$=\int_{-2\sqrt{2}}^{2\sqrt{2}}dy\int_{\frac{1}{4}y^2}^{2}dx\int_{-\sqrt{2x-\frac{1}{2}y^2}}^{\sqrt{2x-\frac{1}{2}y^2}}f(x,y,z)\,dz=\int_{0}^{2}dx\int_{-2\sqrt{x}}^{2\sqrt{x}}dy\int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}}f(x,y,z)\,dz.\ \square$$

Problema 15 (Complet)

15) Calculeu les integrals triples següents en les regions de $\mathbb{R}^3$ que s'indiquen

a) $I=\iiint_A xz\,dx\,dy\,dz$, A limitat pel cilindre de base circular $x^2+y^2-2x=0$ i la superfície: $z^2=2y$ $(y,z\geq 0)$

$$I=\int_0^2 x\,dx\int_0^{\sqrt{2x-x^2}}dy\int_0^{\sqrt{2y}} z\,dz$$

$$=\int_0^2 x\,dx\int_0^{\sqrt{2x-x^2}} y\,dy$$

$$=\int_0^2 x\left(\frac{y^2}{2}\right]_0^{\sqrt{2x-x^2}}dx=\int_0^2\left(x^2-\frac{x^3}{2}\right)dx$$

$$=\left[\frac{x^3}{3}-\frac{x^4}{8}\right]_0^2=\frac{8}{3}-\frac{16}{8}=\frac{8}{3}-2=\boxed{\frac{2}{3}}\ \square$$

b) $I=\iiint_A zy\sqrt{x^2+y^2}\,dz\,dy\,dz$, $A=\left\{(x,y,z)\in\mathbb{R}^3: 0\leq z\leq x^2+y^2,\ 0\leq y\leq\sqrt{2x-x^2}\right\}$

$$I=\int_0^2 dx\int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\,dy\int_0^{x^2+y^2} z\,dz=\int_0^2 dx\int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\left[\frac{z^2}{2}\right]_0^{x^2+y^2}dy$$

$$=\frac{1}{2}\int_0^2 dx\int_0^{\sqrt{2x-x^2}} y\left(x^2+y^2\right)^{5/2}dy=\frac{1}{14}\int_0^2 dx\left[\left(x^2+y^2\right)^{7/2}\right]_0^{\sqrt{2x-x^2}}=\frac{1}{14}\int_0^2 dx\left[\left(x^2+2x-x^2\right)^{7/2}-x^7\right]$$

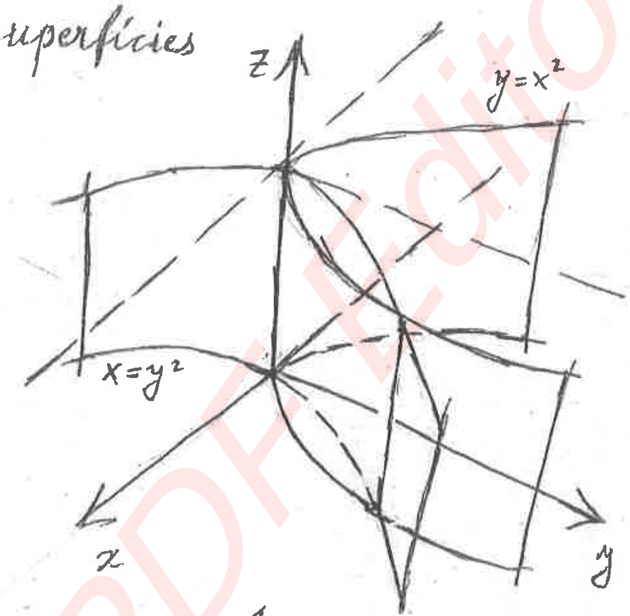
$$=\frac{1}{14}\left[2\cdot 2^{7/2}\frac{x^{9/2}}{9}-\frac{x^8}{8}\right]_0^2=\frac{1}{14}\cdot\frac{1}{72}\cdot\left(16\cdot 2^8-9\cdot 2^8\right)=\frac{1}{14}\cdot\frac{1}{2^3\cdot 9}\cdot 7\cdot 2^8=\boxed{\frac{16}{9}}\ \square$$

c) $I=\iiint_A dx\,dy\,dz$, $A=\left\{(x,y,z)\in\mathbb{R}^3: 1\leq x\leq 3,\ 1\leq y\leq 3,\ 0\leq z\leq xy\right\}$.

$$I=\int_1^3 dx\int_1^3 dy\int_0^{xy} dz=\int_1^3 x\,dx\int_1^3 y\,dy=\left(\int_1^3 x\,dx\right)^2=\left(\left[\frac{x^2}{2}\right]_1^3\right)^2=\left(\frac{9}{2}-\frac{1}{2}\right)^2=\boxed{16}\ \square$$

(d) $I = \iiint_A xyz\,dx\,dy\,dz$, A limitat per les superfícies

$y = x^2,\ x = y^2,\ z = xy,\ z = 0$

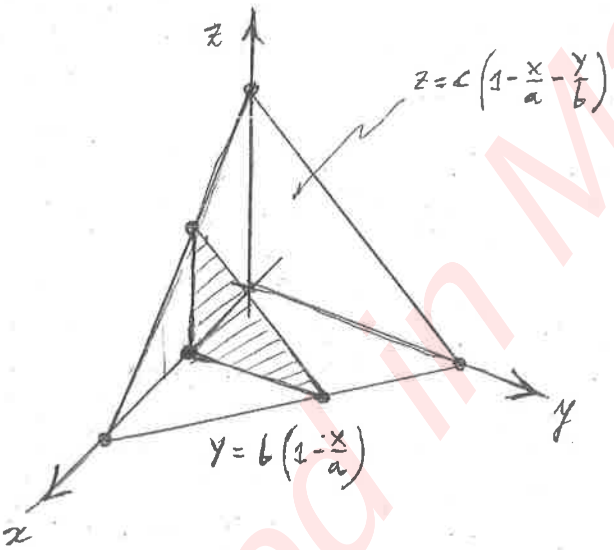


$$I = \int_0^1 x\,dx \int_{x^2}^{\sqrt{x}} y\,dy \int_0^{xy} z\,dz$$

$$= \frac{1}{2}\int_0^1 x^3 dx \int_{x^2}^{\sqrt{x}} y^3 dy$$

$$= \frac{1}{8}\int_0^1 dx\, x^3 (x^2 - x^8) = \frac{1}{8}\int_0^1 (x^5 - x^{11})\,dx = \frac{1}{8}\left(\frac{x^6}{6} - \frac{x^{12}}{12}\right)\Bigg|_0^1 = \frac{1}{8}\left(\frac{1}{6} - \frac{1}{12}\right) = \boxed{\frac{1}{96}} \quad \square$$

(e) $I = \iiint_A x\,dx\,dy\,dz$, A tetraedre format pels plans $x=0,\ y=0,\ z=0,\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, amb $a, b, c > 0$



$$I = \int_a^b x\,dx \int_0^{b(1-\frac{x}{a})} dy \int_0^{c(1-x/a-y/b)} dz$$

$$= c\int_0^a x\,dx \int_0^{b(1-x/a)} dy\left(1 - x/a - \frac{y}{b}\right)$$

$$= c\int_0^a dx\, x\left(y - \frac{xy}{a} - \frac{y^2}{2b}\right)\Bigg|_0^{b(1-\frac{x}{a})}$$

$$= c\int_0^a x\left[b\left(1 - \frac{x}{a}\right) - \frac{bx}{a}\left(1 - \frac{x}{a}\right) - \frac{b}{2}\left(1 - \frac{x}{a}\right)^2\right]dx$$

$$\overset{(*)}{=} \frac{cb}{2}\int_0^a x\left(1 - \frac{x}{a}\right)^2 dx = \frac{cb}{2}\int_0^a x\left(1 - \frac{2x}{a} - \frac{x^2}{a^2}\right)dx = \frac{cb}{2}\left(\frac{x^2}{2} - \frac{2x^3}{3a} + \frac{x^4}{4a^2}\right)\Bigg|_0^a$$

$$= \frac{cb}{2}\left(\frac{a^2}{2} - \frac{2a^2}{3} + \frac{a^2}{4}\right) = \boxed{\frac{cba^2}{24}}$$

(*) $\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{a} - \frac{1}{2} + \frac{x}{2a}\right) = \frac{b}{2}\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{a}\right) = \frac{b}{2}\left(1 - \frac{x}{a}\right)^2.$

16. Usen coordenades polars per a calcular les següents integrals dobles

16c $\iint_A \frac{(x+y)^2}{x^2+y^2+2}\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1\}$

$$I=\iint_A \frac{(x+y)^2}{x^2+y^2+2}\,dx\,dy=\int_0^{2\pi}\int_0^1 r\,\frac{r^2+2r^2\sin\theta\cos\theta}{r^2+2}\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 \frac{r^2}{2+r^2}(1+\sin 2\theta)\,dr\,d\theta=\left(\int_0^{2\pi}(1+\sin(2\theta))\,d\theta\right)\cdot\int_0^1\left(r-\frac{2r}{2+r^2}\right)dr$$

$$\overset{(\&)}{=}2\pi\left(r^2/2-\ln(2+r^2)\right)\Big|_{r=0}^{r=1}=2\pi\left(\tfrac{1}{2}-\ln 3+\ln 2\right)=\boxed{2\pi\left(\tfrac{1}{2}+\ln(2/3)\right)}\ \square$$

(&) $\int_0^{2\pi}\sin(2\theta)\,d\theta=0.$

16d $\iint_A \frac{dx\,dy}{(1+x^2+y^2)^2\sqrt{x^2+y^2}}$, $A=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le R^2\}$

Indicació: Usen les propietats elementals del sin i cos per veure que $\sin(\arctan(R))=\frac{R}{\sqrt{1+R^2}}$, $\cos(\arctan(R))=\frac{1}{1+R^2}$

$$\iint_A \frac{dx\,dy}{(1+x^2+y^2)^2\sqrt{x^2+y^2}}=\int_0^{2\pi}\int_0^R \frac{\cancel{r}\,dr\,d\theta}{(1+r^2)^2\cancel{r}}=2\pi\int_0^R\frac{dr}{(1+r^2)^2}=\left\{\begin{array}{l}\text{canvi de variables}\\ r=\tan u;\ dr=\frac{du}{\cos^2 u}\\ r=R\Rightarrow u=\arctan R\\ r=0\Rightarrow u=0\end{array}\right\}$$

$$=2\pi\int_0^{\arctan R}\frac{1}{\left(\frac{1}{\cos^2 u}\right)^2}\cdot\frac{du}{\cos^2 u}=2\pi\int_0^{\arctan R}\frac{\cos^4 u}{\cos^2 u}\,du=2\pi\int_0^{\arctan R}\cos^2 u\,du$$

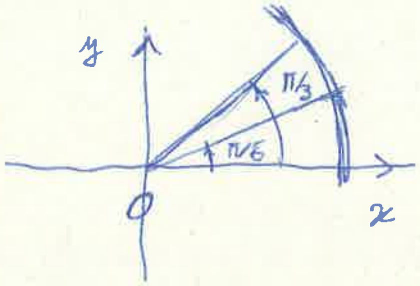
$$=2\pi\int_0^{\arctan R}\frac{1+\cos 2u}{2}\,du=2\pi\left(\frac{u}{2}+\frac{\sin 2u}{4}\right)\Big|_0^{\arctan R}=$$

$$=\pi\left(\arctan R+\sin(\arctan R)\cdot\cos(\arctan R)\right)$$

$$=\pi\left(\arctan R+\tan(\arctan R)\cdot\cos^2(\arctan R)\right)$$

$$=\pi\left(\arctan R+\frac{\tan(\arctan R)}{1+\tan^2(\arctan R)}\right)=\boxed{\pi\left(\arctan R+\frac{R}{1+R^2}\right)}$$

16g $\iint_A x(x^2+y^2)\,dx\,dy$. A: sector circular de centre $(0,0)$ i radi R formant angles entre $\pi/3$ i $\pi/6$ amb eix x-positiu.



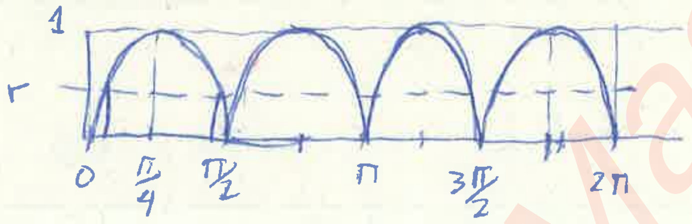
$$\iint_A x(x^2+y^2)\,dx\,dy = \int_{\pi/6}^{\pi/3}\int_0^R r^4\cos\theta\,dr\,d\theta$$

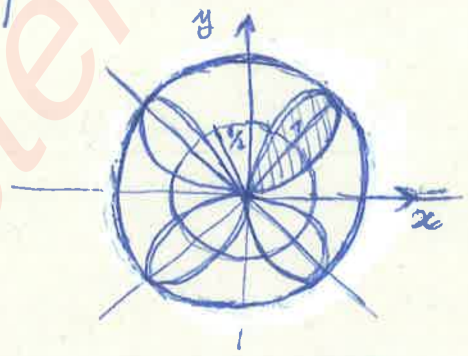
$$= \left(\int_{\pi/6}^{\pi/3}\cos\theta\,d\theta\right)\cdot\left(\int_0^R r^4\,dr\right) = \frac{R^5}{5}\cdot\left(\sin\theta\Big|_{\theta=\pi/6}^{\theta=\pi/3}\right)$$

$$= \frac{R^5}{5}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right) = \boxed{\frac{\sqrt{3}-1}{10}R^5}$$

17. Calculeu les àrees dels dominis $A\subset\mathbb{R}^2$ definits en coordenades polars $x=r\cos\theta$ $y=r\sin\theta$, que s'indiquen tot seguit.

17c A regió definida per $\frac{1}{2}\le r\le|\sin(2\theta)|$ (Indicació. Cal $|\sin(2\theta)|\ge\frac{1}{2}$ perquè l'expressió tingui sentit).





$0\le2\theta\le\pi \Leftrightarrow 0\le\theta\le\pi/2$,

$\pi\le2\theta\le2\pi \Leftrightarrow \pi/2\le\theta\le\pi$

$2\pi\le2\theta\le3\pi \Leftrightarrow \pi\le\theta\le\frac{3\pi}{2}$

$3\pi\le2\theta\le4\pi \Leftrightarrow \frac{3\pi}{2}\le\theta\le2\pi$

$0\le\theta\le\pi$: $\sin2\theta\ge\frac{1}{2} \Leftrightarrow \frac{\pi}{6}\le2\theta\le\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

$\Leftrightarrow \frac{\pi}{12}\le\theta\le\frac{5\pi}{12}$.

$$A/4 = \int_{\pi/12}^{5\pi/12}\int_{1/2}^{\sin2\theta} r\,dr\,d\theta = \int_{\pi/12}^{5\pi/12}\left(\frac{\sin^2 2\theta}{2}-\frac{1}{8}\right)d\theta$$

$$= \int_{\pi/12}^{5\pi/12}\left(\frac{1}{8}-\frac{\cos(4\theta)}{4}\right)d\theta =$$

$$= \left(\frac{\theta}{8}-\frac{\operatorname{sen}(4\theta)}{16}\right)\Bigg|_{\theta=\pi/12}^{\theta=5\pi/12}$$

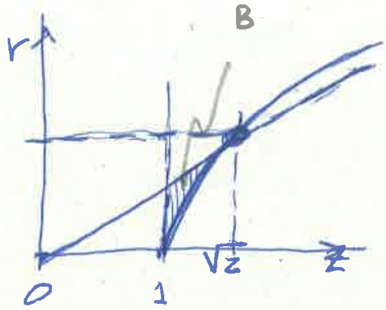
$$= \frac{1}{8}\left(\frac{5\pi}{12}-\frac{\pi}{12}\right)-\frac{1}{16}\operatorname{sen}\left(2\pi-\frac{\pi}{3}\right)+\frac{1}{16}\operatorname{sen}\frac{\pi}{3}$$

$= \frac{\pi}{24}+\frac{\sqrt{3}}{16}$. Aleshores: $\boxed{A=\frac{\pi}{6}+\frac{\sqrt{3}}{4}}$

19. Useu coordenades cilíndriques per calcular les següents integrals triples

(19b) $\iiint_B z e^{-(x^2+y^2)}\,dx\,dy\,dz$. $B=\{(x,y,z)\in\mathbb{R}^3;\ z^2-1\leq x^2+y^2\leq z^2/2,\ z\geq 1\}$

$B:\ \sqrt{z^2-1}\leq r\leq z/\sqrt{2}$



$z\geq 0$,

$z/\sqrt{2}=\sqrt{z^2-1}\iff z^2/2=z^2-1\iff z^2/2-1=0$

$(z=\sqrt{2},\ r=1)$

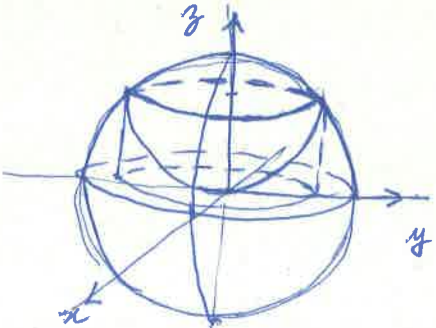
$$I=\int_0^{2\pi}\int_1^{\sqrt{2}}\int_{\sqrt{z^2-1}}^{z/\sqrt{2}} z\,r\,e^{-r^2}\,dr\,dz\,d\theta=\int_0^{2\pi}d\theta\int_1^{\sqrt{2}}z\,dz\left[-\tfrac{1}{2}e^{-r^2}\right]_{r=\sqrt{z^2-1}}^{r=z/\sqrt{2}}$$

$$=\int_0^{2\pi}d\theta\int_1^{\sqrt{2}}z\left(-\tfrac{1}{2}e^{-z^2/2}+\tfrac{1}{2}e^{-(z^2-1)}\right)dz=2\pi\cdot\left(\frac{e^{-z^2/2}}{2}-\frac{1}{4}e^{-(z^2-1)}\right]_{z=1}^{z=\sqrt{2}}=$$

$$=2\pi\left(\frac{1}{2e}-\frac{1}{4e}-\frac{1}{2\sqrt{e}}+\frac{1}{4}\right)=\boxed{\frac{\pi}{2e}-\frac{\pi}{\sqrt{e}}+\frac{\pi}{2}}$$

(19e) $\iiint_B z\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3;\ x^2+y^2+z^2\leq 6,\ x^2+y^2\leq z,\ z\geq 0\}$

$$I=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}r\,dr\int_{r^2}^{\sqrt{6-r^2}}z\,dz=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}r\left[\frac{z^2}{2}\right]_{z=r^2}^{z=\sqrt{6-r^2}}dr$$

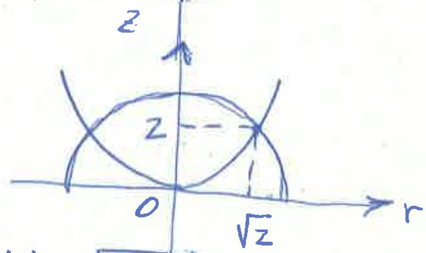


$$=\pi\int_0^{\sqrt{2}}r\left(6-r^2-r^4\right)dr$$

$$=\pi\left(6\frac{r^2}{2}-\frac{r^4}{4}-\frac{r^6}{6}\right|_{r=0}^{r=\sqrt{2}}$$

$$=\pi\left(3\cdot 2-\frac{4}{4}-\frac{4}{3}\right)$$

$$=\pi\left(6-1-\frac{4}{3}\right)=\pi\left(5-\frac{4}{3}\right)=\boxed{\frac{11}{3}\pi}$$



$r^2+z^2=6$

$z=r^2$

$z+z^2=6$

$\iff z^2+z-6=0$

$z=\frac{-1\pm\sqrt{1+24}}{2}=\frac{-1\pm 5}{2}=\begin{cases}-3\\ 2\end{cases}$

$r=\sqrt{z}=\sqrt{2}$.

20. Useu coordenades esfèriques per a calcular les següents integrals triples.

(20c) $$\iiint_B \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}, \quad B=\{(x,y,z)\in\mathbb{R}^3 : a^2\le x^2+y^2+z^2\le b^2\}$$

$$I=\int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2}\cos\varphi\, d\varphi \int_a^b \frac{r^2}{r^3}\,dr = \boxed{4\pi \ln\left(\frac{b}{a}\right)}$$

21. Calculeu els volums dels dominis $B\subset\mathbb{R}^3$ definits en coordenades esfèriques $x=r\cos\varphi\cos\theta$, $y=r\sin\varphi\sin\theta$, $z=r\sin\varphi$ $(0<\theta<2\pi,\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2})$ que s'indiquen tot seguit.

(21b) B volum tancat per l'esfera deformada definida per $r=1+0.2\sin(8\theta)\sin\varphi$ (Sòlids d'aquesta mena s'utilitzen com a models de tumors).

$$I=\int_0^{2\pi} d\theta\int_{-\pi/2}^{\pi/2} d\varphi\int_0^{1+0.2\sin(8\theta)\sin\varphi} r^2\cos\varphi\, dr = \frac{1}{3}\int_0^{2\pi} d\theta\int_{-\pi/2}^{\pi/2}\cos\varphi\cdot\left(1+0.2\sin(8\theta)\sin\varphi\right)^3 d\varphi$$

$$=\frac{1}{3}\int_0^{2\pi} d\theta\int_{-\pi/2}^{\pi/2}\cos\varphi\left(1+0'12\sin^2(8\theta)\sin^2\varphi+0.6\sin(8\theta)\sin\varphi+0.2^3\sin^3(8\theta)\sin^3\varphi\right)d\varphi$$

$$=\frac{1}{3}\left(\int_0^{2\pi}d\theta\right)\cdot\left(\int_{-\pi/2}^{\pi/2}\cos\varphi\, d\varphi\right)+\frac{1}{3}0'12\left(\int_0^{2\pi}\sin^2(8\theta)\,d\theta\right)\cdot\left(\int_{-\pi/2}^{\pi/2}\sin^2\varphi\cos\varphi\, d\varphi\right)$$

$$=\frac{4\pi}{3}+\frac{0'12}{3}\cdot\pi\cdot\frac{2}{3}=\frac{\pi}{3}\left(4+0'08\right)=\frac{4'08}{3}\pi=\boxed{1'36\pi}$$

(21c) $$\iiint_B \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}},$$ on B és la regió del 1er octant de $\mathbb{R}^3$ acotada pels cons $\varphi=\pi/4$ i $\varphi=\arctan 2$ i l'esfera $r=\sqrt{6}$ (Recordeu: $\sin(\arctan(a))=\frac{a}{\sqrt{1+a^2}}$).

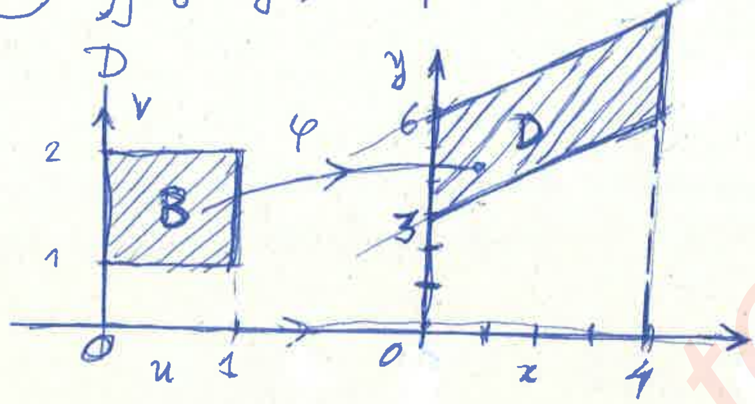
$$I=\int_0^{\pi/2}d\theta\int_{\pi/4}^{\arctan 2}\cos\varphi\, d\varphi\int_0^{\sqrt{6}}\frac{r^2\,dr}{r}=\left(\int_0^{\pi/2}d\theta\right)\left(\int_{\pi/4}^{\arctan 2}\cos\varphi\, d\varphi\right)\cdot\left(\int_0^{\sqrt{6}} r\,dr\right)=$$

$$= \frac{\pi}{2}\left(\operatorname{sen}(\arctan 2) - \operatorname{sen}\left(\frac{\pi}{4}\right)\right)\cdot\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{6}} = \frac{\pi}{2}\left(\frac{\tan(\arctan 2)}{\sqrt{1+\tan^2(\arctan 2)}} - \frac{\sqrt{2}}{2}\right)\cdot 3$$

$$= \boxed{\frac{3\pi}{2}\left(\frac{2}{\sqrt{5}} - \frac{\sqrt{2}}{2}\right)}.$$

18. Calculeu les integrals dobles següents mitjançant el canvi de variables que s'indiquen a cada cas.

(18a) $\iint xy\,dx\,dy$; $D=\{(x,y)\in\mathbb{R}^2 : 6\le 2y-x\le 12, 0\le x\le 4\}$, $x=4u$; $y=2u+3v$.



B: $6\le 2y-x = \cancel{4u}+6v-\cancel{4u} = 6v \le 12$

$\Leftrightarrow 1\le v\le 2$,

$0\le x = 4u \le 4 \Leftrightarrow 0\le u\le 1$

$D=\varphi(B)$, on: $\varphi(u,v) = (4u, 2u+3v)$

$B=\{(u,v)\in\mathbb{R}^2 : 0\le u\le 1, 1\le v\le 2\} = [0,1]\times[1,2]$

$$I = \iint_{D=\varphi(B)} \dots = \iint_B x(u,v)\cdot y(u,v)\cdot\left|\det\frac{\partial(x,y)}{\partial(u,v)}(u,v)\right| du\,dv =$$

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix}4 & 0\\ 2 & 3\end{pmatrix} \quad \text{d'on: } \det\frac{\partial(x,y)}{\partial(u,v)} = 12}$$

$$= \int_0^1 du \int_1^2 4u(2u+3v)\cdot 12\,dv$$

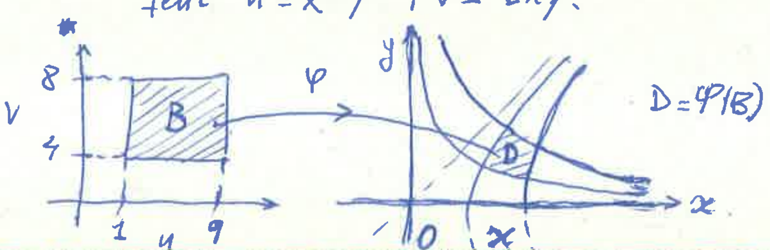
$$= 96\left(\int_0^1 u^2\,du\right)\cdot\left(\int_1^2 dv\right) + 144\left(\int_0^1 u\,du\right)\left(\int_1^2 v\,dv\right)$$

$$= \frac{96}{3} + \frac{144}{4}\cdot 3 = 32 + 108 = \boxed{140}$$

(18f) $\iint_D (x^2+y^2)\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2 : 1\le x^2-y^2\le 9, 2\le xy\le 4, x\ge 0, y\ge 0\}$, fent $u=x^2-y^2$; $v=2xy$.

$4\le v=2xy\le 8$,

$1\le u=x^2-y^2\le 9$



$$\iiint_{D=\varphi(B)} \cdots = \iint_{B=[1,9]\times[4,8]} \left(x^2(u,v)+y^2(u,v)\right)\cdot\left|\det \frac{\partial(x,y)}{\partial(u,v)}(u,v)\right| du\,dv$$

$$\overset{(\&)}{=} \frac{1}{4}\int_1^9 du \int_4^8 \frac{x^2(u,v)+y^2(u,v)}{x^2(u,v)+y^2(u,v)}\,dv = \frac{1}{4}\left(\int_1^9 du\right)\cdot\left(\int_4^8 dv\right) = (9-1)(8-4)\frac{1}{4} = \boxed{8}$$

(&)

$$\det\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\det\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 2x & -2y \\ 2y & 2x\end{vmatrix}} = \frac{1/4}{x^2(u,v)+y^2(u,v)}.$$

23a, 23e, 23f, 23j, 23l

23) Usen coordenades cartesianes, cilíndriques o esfèriques (o bé el principi de Cavalieri) per a calcular el volum dels dominis de $\mathbb{R}^3$ limitats per les superfícies que s'indiquen.

23a $x^2+z^2=1,\ x^2+y^2=1$:

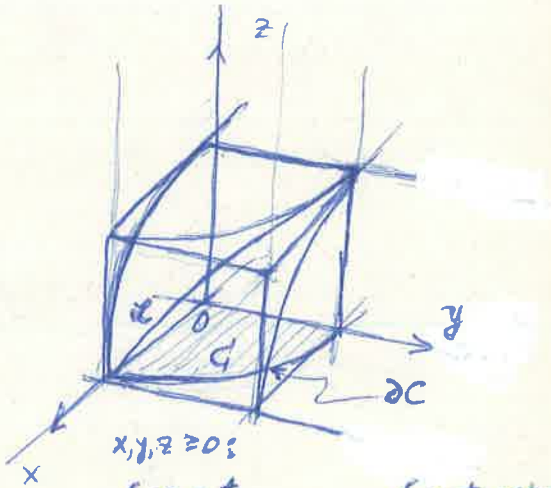
$$\frac{V}{8} = \iint_C \left(\int_0^{\sqrt{1-x^2}} dz\right) dx\,dy = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2}} dz$$

$C=\{x^2+y^2\le 1\}$

$$= \int_0^1 dx \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy = \int_0^1 (1-x^2)\,dx = \left(x-\frac{x^3}{3}\right]_0^1$$

$$= 1-\frac{1}{3} = \frac{2}{3}$$

d'on, el volum buscat és: $\boxed{V = \frac{16}{3}}$



$x,y,z\ge 0$:

$\gamma: \begin{cases} x=t \\ y=\sqrt{1-t^2} \\ z=\sqrt{1-t^2}\end{cases}$ $\partial C: \begin{cases} x=t,\ 0\le t\le 1 \\ y=\sqrt{1-t^2}\end{cases}$

Nota: també surt (molt fàcil) per Cavalieri. En efecte, considerem talls segons $x=\text{ctnt},\ 0\le x\le 1$. Aleshores $S(x)=1-x^2$ i llavors.

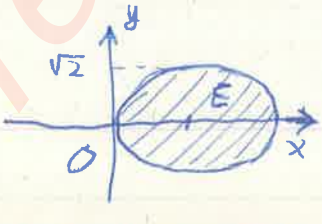
$$\frac{V}{8} = \int_0^1 S(x)\,dx = \int_0^1 (1-x^2)\,dx = 1-\frac{1}{3} = \frac{2}{3} \Rightarrow V = \frac{16}{3}$$

23e $z=x^2-4x+1,\ \ 1-z=x^2+y^2$

Projecció sobre el pla $z=0$ de la intersecció de les dues gràfiques

$$x^2-4x+1 = 1-x^2-y^2 \Leftrightarrow 2(x^2-2x+1)-2+y^2=0 \Leftrightarrow (x-1)^2+\frac{y}{\sqrt{2}^2}=1 \ :\partial E, \text{ on}:$$

$$E=\left\{(x,y)\in\mathbb{R}^2 :\ (x-1)^2+\frac{y}{\sqrt{2}^2}\le 1\right\}. \text{ Aleshores:}$$



$$V = \iint_E\left(\int_{x^2-4x+1}^{1-x^2-y^2} dz\right)dx\,dy = \iint_E \left(\cancel{1}-x^2-y^2-x^2+4x\cancel{-1}\right)dx\,dy =$$

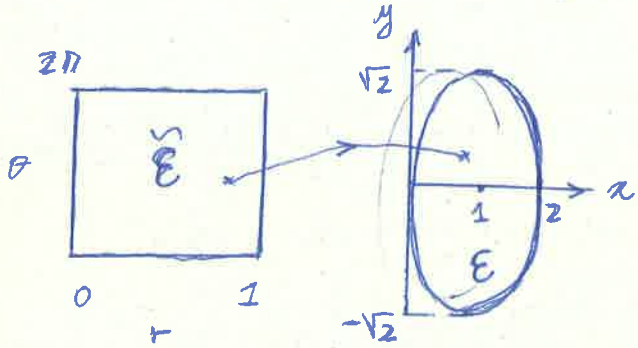
$$= 2 \cdot \iint_{\mathcal{E}} \left(1-(x-1)^2 - \frac{y^2}{\sqrt{2}^2}\right) dx\,dy$$

Considerem coordenades polars "adaptades":

$(x,y) = \varphi(r,\theta) = \left(1 + r\cos\theta, \sqrt{2}\, r \sin\theta\right)$, $0<r<1$, $0<\theta<2\pi$.

Llavors $\mathcal{E} = \varphi(\tilde{\mathcal{E}})$, amb $\tilde{\mathcal{E}} : 0 < (x-1)^2 + \dfrac{y^2}{\sqrt{2}^2} = r^2 < 1 \Leftrightarrow 0<r<1, 0<\theta<2\pi$

$$J_\varphi(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sqrt{2}\sin\theta & r\sqrt{2}\cos\theta \end{pmatrix}, \text{ d'on: } \det J_\varphi(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sqrt{2}\sin\theta & r\sqrt{2}\cos\theta \end{vmatrix} = \sqrt{2}\,r$$

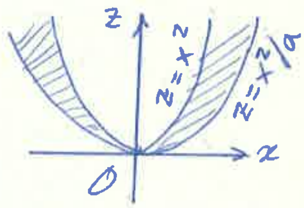


$$V = 2\iint_{\mathcal{E}} \cdots = 2\int_0^{2\pi} d\theta \int_0^1 (1-r^2)\, r\sqrt{2}\, dr$$

$$= 2\sqrt{2}\left(\int_0^{2\pi} d\theta\right)\cdot\left(\int_0^1 (r-r^3)\,dr\right)$$

$$= 4\sqrt{2}\,\pi\left(\tfrac{1}{2}-\tfrac{1}{4}\right) = \boxed{\sqrt{2}\,\pi}$$

$\tilde{\mathcal{E}} = [0,1]\times[0,2\pi]$ $\qquad \mathcal{E} : \dfrac{(x-1)^2}{1} + \dfrac{y^2}{\sqrt{2}^2} \leq 1$



23f) $x^2 = z$, $y^2 = x$, $z^2 = y$, $x^2 = az$, $y^2 = ax$, $z^2 = ay$ $(a>1)$.

$y^2/a \leq x \leq y^2 : \quad 1/a \leq u = x/y^2 \leq 1$

$z^2/a \leq y \leq z^2 ; \quad 1/a \leq v = y/z^2 \leq 1$

$x^2/a \leq z \leq x^2 : \quad 1/a \leq w = z/x^2 \leq 1$

$$\det \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\det \dfrac{\partial(u,v,w)}{\partial(x,y,z)}} = \frac{1}{\begin{vmatrix} 1/y^2 & -2x/y^3 & 0 \\ 0 & 1/z^2 & -2y/z^3 \\ -\dfrac{2z}{x^3} & 0 & 1/x^2 \end{vmatrix}} = \frac{1}{\dfrac{-7}{x^2y^2z^2}} =$$

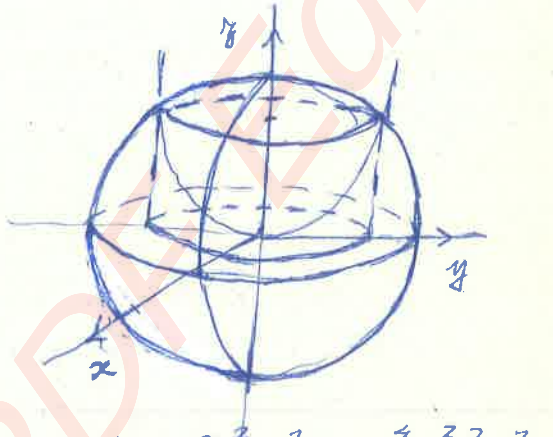
$$= -\frac{1}{7}\, x(u,v,w)^2\, y(u,v,w)^2\, z(u,v,w)^2 = \frac{-1/7}{u^2v^2w^2}.$$

$$V = \iiint_{\Omega} dx\,dy\,dz = \iiint_{\Gamma=[1/a,1]^3} 1 \cdot \left|\det \frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw = \iiint_{\Gamma=[1/a,1]^3} \frac{1}{7} \times \frac{du\,dv\,dw}{u^2v^2w^2} =$$

$$= \frac{1}{7}\left(\int_{1/a}^{1}\frac{du}{u^2}\right)\times\left(\int_{1/a}^{1}\frac{dv}{v^2}\right)\times\left(\int_{1/a}^{1}\frac{dw}{w^2}\right) = \frac{1}{7}\left[-\frac{1}{u}\right]_{1/a}^{1}\times\left[-\frac{1}{v}\right]_{1/a}^{1}\times\left[-\frac{1}{w}\right]_{1/a}^{1}$$

$$= \boxed{\frac{(a-1)^3}{7}}.$$

23j $x^2+y^2+z^2=2a^2,\ z=\dfrac{x^2+y^2}{a}\ (z\ge 0,\ a>0)$



$$r^2/a^2 \le z \le \sqrt{2a^2-r^2}$$

$$V=\int_0^{2\pi}d\theta\int_0^a dr\int_{r^2/a^2}^{\sqrt{2a^2-r^2}} r\,dz$$

$$= \int_0^{2\pi}d\theta\int_0^a\left(r\sqrt{2a^2-r^2}-r^3/a\right)dr$$

$$= \int_0^{2\pi}\left[-\tfrac{1}{3}(2a^2-r^2)^{3/2}-\frac{r^4}{4a}\right]_{r=0}^{r=a}d\theta = 2\pi\cdot\left(-\tfrac{1}{3}a^3+\tfrac{1}{3}2\sqrt{2}\,a^3-\frac{a^3}{4}\right)$$

$$= \boxed{2\pi a^3\left(\frac{2\sqrt{2}}{3}-\frac{7}{12}\right)}.$$

$r^4/a^2 = 2a^2-r^2 \Leftrightarrow r^4+a^2r^2-2a^2=0,$

$r=\pm\sqrt{\dfrac{-a^2\pm\sqrt{a^4+8a^4}}{2}}=\pm\sqrt{\dfrac{-a^2\pm3a^2}{2}}$

$r>0$, llavors $r=a$.

23l

Con de gelat definit per $x^2+y^2\le z^2/5,\ 0\le z\le 5+\sqrt{5-x^2-y^2}$

$(z-5)^2+x^2+y^2\le 5$

$x^2+y^2\le z^2/5$

$$V=\int_0^{2\pi}d\theta\int_0^{\sqrt5}dr\int_{\sqrt5 r}^{5+\sqrt{5-r^2}} r\,dz = 2\pi\cdot\int_0^{\sqrt5}\left(5r-\sqrt5 r^2+r\sqrt{5-r^2}\right)dr$$

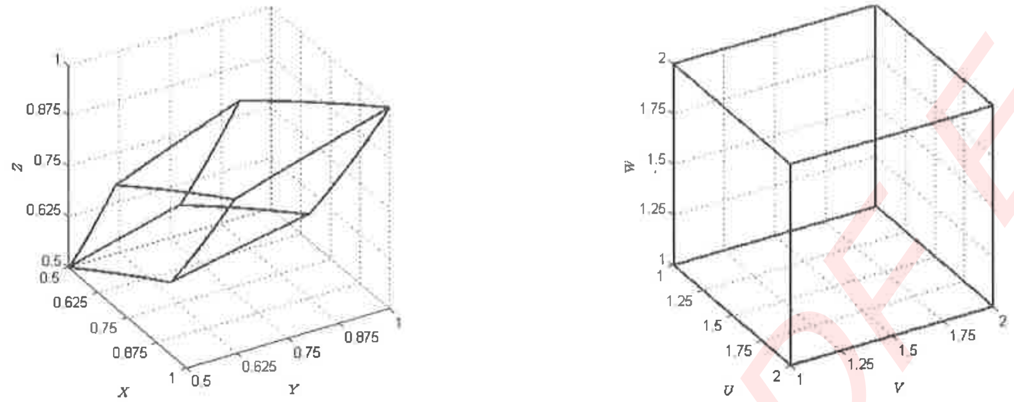
$$= 2\pi\left(\tfrac{5}{2}r^2-\tfrac{\sqrt5}{3}r^3-\tfrac{1}{3}(5-r^2)^{3/2}\right)\Big|_0^{\sqrt5}$$

$$= 2\pi\left(\frac{25}{2}-\frac{25}{3}+\frac{5\sqrt5}{3}\right)$$

$$= \boxed{\frac{25\pi}{3}+\frac{10\sqrt5}{3}\pi}$$

$V=\frac{2}{3}\pi\sqrt5^3+\frac{\pi}{3}\sqrt5^2\cdot5=\frac{25\pi}{3}+\frac{10\sqrt5}{3}\pi.$ $\left(=\tfrac12\text{ Volum esfera}+\text{Volum con}\right).$

NOTA AL PROBLEMA 23f.



(a) $\mathcal{D}$: Domini definit per les superfícies,
$x^2 = z, \quad y^2 = x, \quad z^2 = y,$
$x^2 = az, \quad y^2 = ax, \quad z^2 = ay.$

(b) $\mathcal{D}' : 1 \leq u \leq a,\ 1 \leq v \leq a,\ 1 \leq w \leq a.$

FIGURA 1. Transformació del domini $\mathcal{D}$ en $\mathcal{D}'$ pel canvi (1).

NOTA (al problema 23f)

Alternativament, es comprova d'immediat que el canvi

$$u = \frac{y^2}{x}, \qquad v = \frac{z^2}{y}, \qquad w = \frac{x^2}{z}, \tag{1}$$

transforma el domini original $\mathcal{D}$, en $\mathcal{D}' = [1,a] \times [1,a] \times [1,a]$. És a dir, en un cub d'aresta $a-1$ (veure Figura 1). Aleshores el Jacobià corresponent surt més senzill. En efecte:

$$\det \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -\dfrac{y^2}{x^2} & 2\dfrac{y}{x} & 0 \\ 0 & -\dfrac{z^2}{y^2} & 2\dfrac{z}{y} \\ 2\dfrac{x}{z} & 0 & -\dfrac{x^2}{z^2} \end{vmatrix} = -7,$$

d'on:

$$\left|\det \frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{\left|\det \dfrac{\partial(u,v,w)}{\partial(x,y,z)}\right|} = \frac{1}{7}.$$

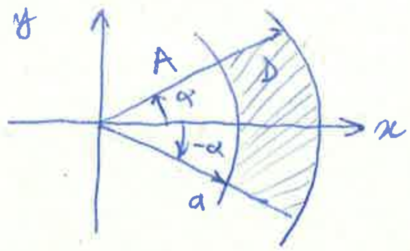
Llavors el càlcul del volum es simplifica encara més:

$$V = \iiint_{\mathcal{D}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{\mathcal{D}'} \left|\det \frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$
$$= \frac{1}{7}\int_1^a \mathrm{d}u \int_1^a \mathrm{d}v \int_1^a \mathrm{d}w = \frac{1}{7}\left(\int_1^a \mathrm{d}u\right)^3 = \frac{(a-1)^3}{7}.$$

Problemes Aplicacions: 25(a), 25(b), 26(a), 26(c) i 28

25. Trobeu el centre de masses de les regions planes següents amb les densitats que s'indiquen

(25a) Sector pla definit per una corona de radi interior a i radi exterior A, un angle d'obertura 2α i que és simètrica respecte de l'eix x positiu suposant densitat constant $\rho(x,y)=1$.



Per simetria el CG $(\bar{x},\bar{y})$ es troba sobre l'eix x, i.e. $\bar{y}=0$.

$$A(D)\bar{x} = \iint_D x\,\rho(x,y)\,dx\,dy = \int_{-\alpha}^{\alpha} d\theta \int_a^A r^2\cos\theta\,dr$$

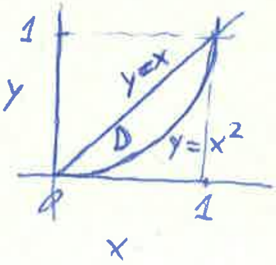
$$= \left(\int_{-\alpha}^{\alpha}\cos\theta\,d\theta\right)\cdot\left(\int_a^A r^2\,dr\right) = \tfrac{2}{3}(A^3-a^3)\operatorname{sen}\alpha.$$

$$A(D) = \alpha(A^2-a^2)$$

Aleshores: $\bar{x} = \frac{2}{3}\cdot\frac{\operatorname{sen}\alpha}{\alpha}\cdot\frac{A^3-a^3}{A^2-a^2}$, $\bar{y}=0$, i.e. $(\bar{x},\bar{y}) = \left(\frac{2}{3}\cdot\frac{\sin\alpha}{\alpha}\cdot\frac{A^3-a^3}{A^2-a^2},\ 0\right)$.

(25b) Regió entre $y=x^2$, $y=x$ amb $\rho(x,y)=x+y$

$y=x^2$
$y=x$
tall
$x=0, x=1$
$y=0, y=1$



$$m(D)\bar{x} = \iint_D x\,\rho(x,y)\,dx\,dy = \int_0^1 dx\int_{x^2}^{x} x(x+y)\,dy$$

$$= \int_0^1 \left[x^2y + x\tfrac{y^2}{2}\right]_{y=x^2}^{y=x} dx = \int_0^1 \left(x^3 + \tfrac{x^3}{2} - x^4 - \tfrac{x^5}{2}\right)dx$$

$$= \left(\frac{3x^4}{8} - \frac{x^5}{5} - \frac{x^6}{12}\right)\Big|_0^1 = \tfrac{3}{8} - \tfrac{1}{5} - \tfrac{1}{12} = \frac{45-24}{120} = \frac{11}{120}$$

$$m(D)\bar{y} = \iint_D y\,\rho(x,y)\,dx\,dy = \int_0^1 dx\int_{x^2}^{x} y(x+y)\,dy = \int_0^1\left[\frac{y^2x}{2} + \frac{y^3}{3}\right]_{y=x^2}^{y=x} dx = \int_0^1\left(\frac{5}{6}x^3 - \frac{x^5}{2} - \frac{x^6}{3}\right)dx$$

$$= \frac{5}{24}x^4 - \frac{1}{12}x^6 - \frac{1}{21}x^7\Big|_{x=0}^{x=1} = \frac{5}{24} - \frac{1}{12} - \frac{1}{21} = \frac{35-14-8}{168} = \frac{35-22}{168} = \frac{13}{168}$$

$$m(D) = \iint_D \rho(x,y)\,dx\,dy = \int_0^1 dx\int_{x^2}^{x}(x+y)\,dy = \int_0^1\left(xy + \tfrac{y^2}{2}\right)\Big|_{y=x^2}^{y=x} dx = \int_0^1\left(x^2 + \tfrac{x^2}{2} - x^3 - \tfrac{x^4}{2}\right)dx$$

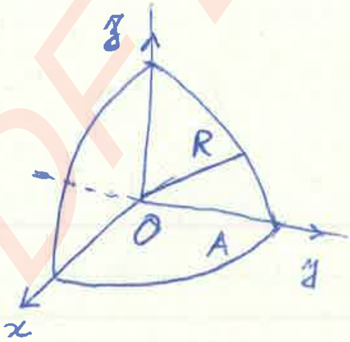
$$= \int_0^1\left(\tfrac{3}{2}x^2 - x^3 - \tfrac{x^4}{2}\right)dx = \left(\tfrac{1}{2}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{10}x^5\right)\Big|_0^1 = \frac{1}{4} - \frac{1}{10} = \frac{3}{20}$$

Així doncs:

$$\bar{x} = \frac{11/120}{3/20} = \frac{11\cdot 20}{3\cdot 120} = \frac{11}{18}\,,\quad \bar{y} = \frac{13/168}{3/20} = \frac{13\cdot 20}{3\cdot 168} = \frac{13\cdot 2^2\cdot 5}{3^2\cdot 2^3\cdot 7} = \frac{65}{126}$$

i les coordenades de CM del cos són $(\bar{x},\bar{y}) = \left(\frac{11}{18}, \frac{65}{126}\right)$

26. En cadascun dels casos següents, trobeu el centre de masses dels sòlids A, suposant distribució de masses homogènia.



26a $A = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 \le R^2,\ x\ge 0, y\ge 0, z\ge 0\}$

Suposem $\rho(x,y,z) \equiv \rho_0 \in\mathbb{R}$, ρ_0 ctnt.

$$m(A)\,\bar{x} = \iiint_A x\,\rho(x,y,z)\,dx\,dy\,dz =$$

$$= \rho_0 \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\varphi \int_0^R r^3\cos\varphi\cos\theta\,dr = \rho_0\left(\int_0^{\pi/2}\cos\theta\,d\theta\right)\cdot\left(\int_0^{\pi/2}\cos^2\varphi\,d\varphi\right)\cdot\left(\int_0^R r^3\,dr\right)$$

$$= \rho_0\,\frac{\pi}{4}\cdot\frac{R^4}{4} = \rho_0\,\frac{\pi R^4}{16}$$

$$m(A)\,\bar{y} = \rho_0\left(\int_0^{\pi/2}\sin\theta\,d\theta\right)\cdot\left(\int_0^{\pi/2}\cos^2\varphi\,d\varphi\right)\cdot\left(\int_0^R r^3\,dr\right) = \rho_0\,\frac{\pi}{4}\cdot\frac{R^4}{4} = \rho_0\,\frac{\pi R^4}{16}$$

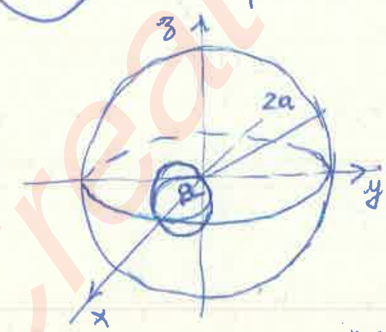
$$m(A)\,\bar{z} = \rho_0\left(\int_0^{\pi/2} d\theta\right)\cdot\left(\int_0^{\pi/2}\cos\varphi\sin\varphi\,d\varphi\right)\left(\int_0^R r^3\,dr\right) = \rho_0\,\frac{\pi}{4}\,\frac{R^4}{4} = \rho_0\,\frac{\pi R^4}{16}$$

com que $m(A) = \rho_0\,\frac{1}{6}\pi R^3$. Aleshores: $\bar{x} = \dfrac{\rho_0\frac{\pi R^4}{16}}{\rho_0\frac{\pi R^3}{6}} = \dfrac{3R}{8} = \bar{y} = \bar{z}$.

Les coordenades del CG són doncs:

$$(\bar{x},\bar{y},\bar{z}) = \left(\frac{3R}{8}, \frac{3R}{8}, \frac{3R}{8}\right).$$

26c $A = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 \le 4a^2,\ (x-a)^2+y^2+z^2 \ge a^2\}$, $\rho(x,y,z)\equiv\rho_0\in\mathbb{R}$ ctnt.



$A_1 = \{(x,y,z)\in\mathbb{R}^2 : x^2+y^2+z^2 \le 4a^2\}$,

$A_2 = \{(x,y,z)\in\mathbb{R}^2 : (x-a)^2+y^2+z^2 < a^2\}$

$A = A_1\setminus A_2\,;\quad A_1 = A_1\setminus A_2 \cup A_2$

Siguin $m(A_1), m(A_2)$; $(\bar{x}_1,\bar{y}_1,\bar{z}_1), (\bar{x}_2,\bar{y}_2,\bar{z}_2)$ les masses i les posicions dels CG de A_1 i A_2 respectivament.

Anomenem $m(A)=m(A_1\setminus A_2)$ i $(\bar{x},\bar{y},\bar{z})$, respectivament, a la massa i a les coordenades del CM del cos A. Aleshores:

$$\bar{x}_1=\frac{m(A_1\setminus A_2)\bar{x}+m(A_2)\bar{x}_2}{m(A_1\setminus A_2)+m(A_2)}=\frac{m(A_1\setminus A_2)\bar{x}+m(A_2)\bar{x}_2}{m(A_1)}$$

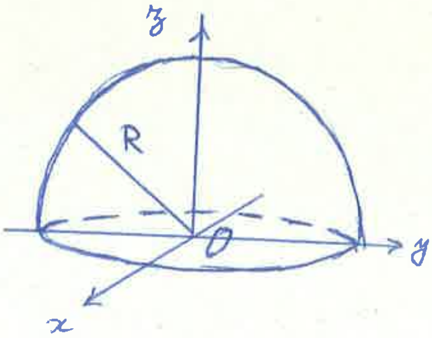
$$\bar{x}=\frac{1}{m(A_1\setminus A_2)}\Big(m(A_1)\bar{x}_1-m(A_2)\bar{x}_2\Big)\overset{(*)}{=}\frac{-4/3\,\pi a^3\cdot a}{28/3\,\pi a^3}=-a/7$$

(*) $$m(A_1\setminus A_2)=m(A_1)-m(A_2)$$
$$=4/3\,\pi\, 8a^3-4/3\,\pi a^3$$
$$=4/3\,\pi(8-1)a^3=\frac{28}{3}\pi a^3$$

Com que l'eix x és un eix de simetria del cos A, el CG li pertany. Aleshores $\bar{y}=0=\bar{z}$ i les coordenades del CG són

$$\boxed{(\bar{x},\bar{y},\bar{z})=(-a/7,0,0).}\quad\square$$

28. Troben el centre de masses de la semiesfera definida per $x^2+y^2+z^2\le R^2$ i $z\ge 0$ si la densitat a cada punt és proporcional a la distància d'aquest punt al centre.



$$\rho(x,y,z)=c(x^2+y^2+z^2)^{1/2}=cr$$

$$m(W)\,\bar{z}=\iiint_W z\,\rho(x,y,z)\,dx\,dy\,dz$$
$$=c\int_0^{2\pi}d\theta\int_0^{\pi/2}d\varphi\int_0^R r^4\cos\varphi\sin\varphi\,dr$$
$$=c\left(\int_0^{2\pi}d\theta\right)\cdot\left(\int_0^{\pi/2}\sin\varphi\cos\varphi\,d\varphi\right)\cdot\left(\int_0^R r^4dr\right)=c\,\pi/5\,R^5$$

$$m(W)=\iiint_W \rho(x,y,z)\,dx\,dy\,dz=c\int_0^{2\pi}d\theta\int_0^{\pi/2}d\varphi\int_0^R r^3\cos\varphi\,dr$$
$$=c\left(\int_0^{2\pi}d\theta\right)\cdot\left(\int_0^{\pi/2}\cos\varphi\,d\varphi\right)\cdot\left(\int_0^R r^3dr\right)=2\pi c\frac{R^4}{4}=c\frac{\pi R^2}{2}$$

Aleshores:

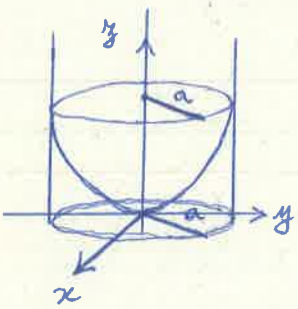
$$\bar{z}=\frac{c\,\pi/5\,R^5}{c\,\pi/2\,R^2}=\frac{2}{5}R,$$

i, per simetria: $\bar{x} = 0 = \bar{y}$; d'on tenim que les coordenades del CM són $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 2R/5)$. □

34(c), 34(d)

34. Pels sòlids següents, calculeu els moments d'inèrcia que es demanen en cada cas tot suposant densitat homogènia igual a 1.

34c Calculeu I_z pel sòlid limitat pel paraboloide $z = x^2+y^2$ i el cilindre $x^2+y^2 = a^2$, $z \geq 0$.

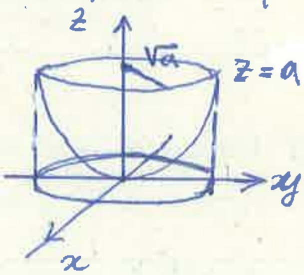


$\rho(x,y,z) \equiv 1$:

$$I_z = \iiint_W (x^2+y^2)\,\rho(x,y,z)\,dx\,dy\,dz = \int_0^{2\pi} d\theta \int_0^a dr \int_0^{r^2} r^3\,dz$$

$$= \int_0^{2\pi} d\theta \int_0^a r^5\,dr = \left(\int_0^{2\pi} d\theta\right)\cdot\left(\int_0^a r^5\,dr\right) = \frac{2\pi a^6}{6} = \boxed{\frac{\pi a^6}{3}}. \ \square$$

34d Calculeu I_x, I_y, I_z pel sòlid tancat pel paraboloide $z = x^2+y^2$ i el pla $z = a$ $(a>0)$.



$$I_x = \iiint_W (y^2+z^2)\,\rho(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} dr \int_{r^2}^{a} (r^2 \sin^2\theta + z^2)\,r\,dz$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} \left(r^3 z \sin^2\theta + r z^3/3 \right)\Big|_{z=r^2}^{z=a} dr = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} \left(a r^3 \sin^2\theta + \frac{a^3 r}{3} - r^5 \sin\theta - \frac{r^7}{3} \right) dr$$

$$= \int_0^{2\pi} \left(\frac{a}{4} r^4 \sin^2\theta + \frac{a^3 r^2}{6} - \frac{r^6}{6}\sin^2\theta - \frac{r^8}{24} \right)\Big|_{r=0}^{r=\sqrt{a}} d\theta = \int_0 \left(\frac{a^3}{4}\sin^2\theta + \frac{a^4}{6} - \frac{a^3}{6}\sin^2\theta - \frac{a^4}{24} \right) d\theta$$

$$= \frac{a^4}{8}\int_0^{2\pi} d\theta + \frac{a^3}{12}\int_0^{2\pi} \sin^2\theta\,d\theta = \frac{\pi a^4}{4} + \frac{\pi a^3}{12} = \frac{\pi a^3}{12}(1+3a) = I_y \text{ (per simetria)}.$$

$$I_z = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} dr \int_{r^2}^{a} r^3\,dz = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r^3 z \Big|_{z=r^2}^{z=a} dr = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} (a r^3 - r^5)\,dr$$

$$= \left(\int_0^{2\pi} d\theta\right)\cdot\left(\int_0^{\sqrt{a}} (a r^3 - r^5)\,dr\right) = 2\pi\cdot\left(a r^4/4 - \frac{r^6}{6} \right)\Big|_{r=0}^{r=\sqrt{a}} = \pi a^3\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi a^3}{6}$$

Llavors: $I_x = I_y = \dfrac{\pi a^3}{12}(1+3a)$, $I_z = \dfrac{\pi a^3}{6}$. □