

1. Usant el teorema del valor mig per integrals proveu que les següents desigualtats.

(a) $4e^5 \leq \iint_A e^{x^2+y^2} dx dy \leq 4e^{25}$, on $A = [1,3] \times [2,4]$

(b) $\frac{1}{e} \leq \frac{1}{4\pi^2} \iint_A e^{\sin(x+y)} dx dy \leq e$ on $A = [-\pi, \pi] \times [-\pi, \pi]$

(c) $\frac{1}{6} \leq \iint_A \frac{dx dy}{y-x+3} \leq \frac{1}{4}$, on A és el triangle de vèrtexs $(0,0), (3,1), (3,0)$.

Solució. Recordem el Teorema del valor mig (veure punta de teoria, Tema 2).

Si sigui $f: \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}$, Ω acotat i arc-convex, i $f \in C^0(\bar{\Omega})$, llavors $\exists c \in \bar{\Omega}$ tal que:

$$\int_{\Omega} f(x) = f(c) m(\Omega).$$

i on $m(\Omega)$ és la mesura del conjunt Ω (longitud en dimensió 1, àrea en dimensió 2, volumen en dimensió 3, ...). Llavors tenim les anàlisis següents:

$$m(\Omega) \inf_{x \in \Omega} f(x) \leq \int_{\Omega} f(x) \leq m(\Omega) \sup_{x \in \Omega} f(x). \quad (1)$$

(a) $f(x,y) = e^{x^2+y^2}$, d'on: $f_x(x,y) = 2xe^{x^2+y^2} = 0 \Leftrightarrow x=0$, $f_y(x,y) = 2ye^{x^2+y^2} = 0$.

Llavors l'únic punt crític de la funció és $(x,y) = (0,0) \notin A = [1,3] \times [2,4]$

• $x=1$: $f(1,y) = e^{1+y^2}$: $f'(1,y) = 2ye^{1+y^2} = 0 \Leftrightarrow y=0$: $(1,0) \notin A$.

• $x=3$: $f(3,y) = e^{9+y^2}$: $f'(3,y) = 2ye^{9+y^2} = 0 \Leftrightarrow y=0$: $(3,0) \notin A$.

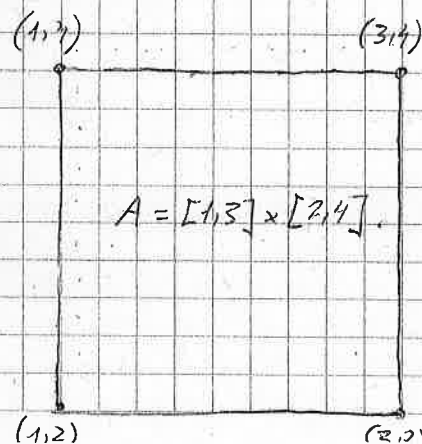
• $y=2$: $f(x,2) = e^{x^2+4}$: $f'(x,2) = 2xe^{x^2+4} = 0 \Leftrightarrow x=0$: $(0,2) \notin A$.

• $y=4$: $f(x,4) = e^{x^2+16}$: $f'(x,4) = 2xe^{x^2+16} = 0$

$\Leftrightarrow x=0$: $(0,4) \notin A$.

Verem que el punt crític de f no pertany a A i que les restriccions de f sobre els costats d' A no tenen punts crítics. Avaluem doncs f als vèrtexs d' A

$f(1,2) = e^5$, $f(3,2) = e^{13}$, $f(1,4) = e^{17}$, $f(3,4) = e^{25}$.



d'ora: $\max_{(x,y) \in A} e^{x^2+y^2} = e^{25}$ i $\min_{(x,y) \in A} e^{x^2+y^2} = e^5$ i s'atanyen als punts,

(3,4) i (1,2) respectivament. D'altra banda, $m(A) = \iint_A dx dy = (3-1) \cdot (4-2) = 4$.

Així doncs, aplicant el Teorema del Valor mitjà (fórmula (1)),

$$4e^5 \leq \iint_A e^{x^2+y^2} dx dy \leq 4 \cdot e^{25}$$

$$A = [1,3] \times [2,4]$$

(b) $\frac{1}{e} \leq \frac{1}{4\pi^2} \iint_A e^{\sin(x+y)} dx dy \leq e$, onent $A = [-\pi, \pi] \times [-\pi, \pi]$

Solució. Clarament $\max_{(x,y) \in A} e^{\sin(x+y)} = e$, i $\min_{(x,y) \in A} e^{\sin(x+y)} = \frac{1}{e}$, mentre que

$m(A) = 4\pi^2$. Aleshores, aplicant (1) resulta:

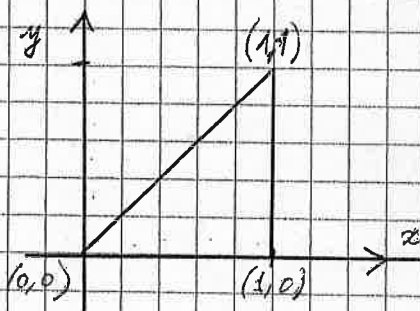
$$\frac{4\pi^2}{e} \leq \iint_A e^{\sin(x+y)} dx dy \leq 4\pi^2 e$$

$$A = [-\pi, \pi] \times [-\pi, \pi]$$

i dividint les dues desigualtats per $4\pi^2$ s'obtenen les acotacions buscades.

(c) $\frac{1}{6} \leq \iint_A \frac{dx dy}{y-x+3} \leq \frac{1}{9}$, on A és el triangle de vèrtexs (0,0), (1,1) i (1,0)

Solució.



El màxim de $y-x+3$ sobre el triangle A s'atany als vèrtexs (0,0) i (1,1) i val 3, mentre que el mínim té

lloc als vèrtexs (1,0) i val 2. Per tant:

$$\frac{1}{3} \leq \frac{1}{y-x+3} \leq \frac{1}{2}$$

per a tot punt del triangle donat, mentre que l'àrea corresponent és $m(A) = \frac{1}{2}$.

Finalment doncs, aplicant la fórmula (1) del Teorema del Valor Mitjà, obtenim:

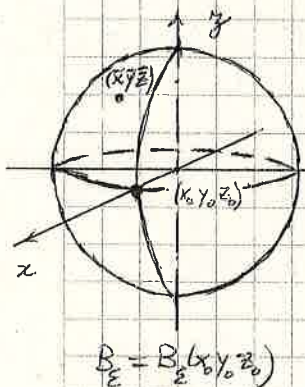
$$\frac{1}{6} \leq \iint_A \frac{dx dy}{y-x+3} \leq \frac{1}{4}$$

que són les desigualtats buscades

2) Sigui $f(x, y, z)$ una funció contínua i B_ε la bola de centre (x_0, y_0, z_0) i radi ε .

Prova que se satisfà que $f(x_0, y_0, z_0) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\text{volum}(B_\varepsilon)} \iiint_{B_\varepsilon} f(x, y, z) dx dy dz$.

Solució. Pel teorema del valor mitjà, existeix $(\bar{x}, \bar{y}, \bar{z}) \in B_\varepsilon = B_\varepsilon(x_0, y_0, z_0)$ t.q.:



$$\iiint_{B_\varepsilon} f(x, y, z) dx dy dz = f(\bar{x}, \bar{y}, \bar{z}) \text{Volum}(B_\varepsilon(x_0, y_0, z_0)) \quad (2)$$

D'altra banda, com que f és contínua i $(\bar{x}, \bar{y}, \bar{z}) \rightarrow (x_0, y_0, z_0)$ quan $\varepsilon \rightarrow 0^+$, prenent límits a (2) s'obté:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\text{Volum}(B_\varepsilon(x_0, y_0, z_0))} \iiint_{B_\varepsilon(x_0, y_0, z_0)} f(x, y, z) dx dy dz = \lim_{(\bar{x}, \bar{y}, \bar{z}) \rightarrow (x_0, y_0, z_0)} f(\bar{x}, \bar{y}, \bar{z}) = f(x_0, y_0, z_0)$$

que és el resultat que es buscava. □

3) Apliquen el principi de Cavalieri per a calcular els següents volums a partir de l'àrea de seccions amb plans paral·lels als plans coordenats (triades de forma adequada)

(a) Volum envoltant per l'el·lipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solució. — fixant $0 < x < a$: $\frac{y^2}{b^2(1-x^2/a^2)} + \frac{z^2}{c^2(1-x^2/a^2)} = 1$, d'on $S(x) = \pi bc \left(1 - \frac{x^2}{a^2}\right)$.

Nota: recordem que l'àrea d'una el·lipse ve donada per πAB , on A i B són les longituds dels seus semieixos. Llavors, aplicant el principi de Cavalieri:

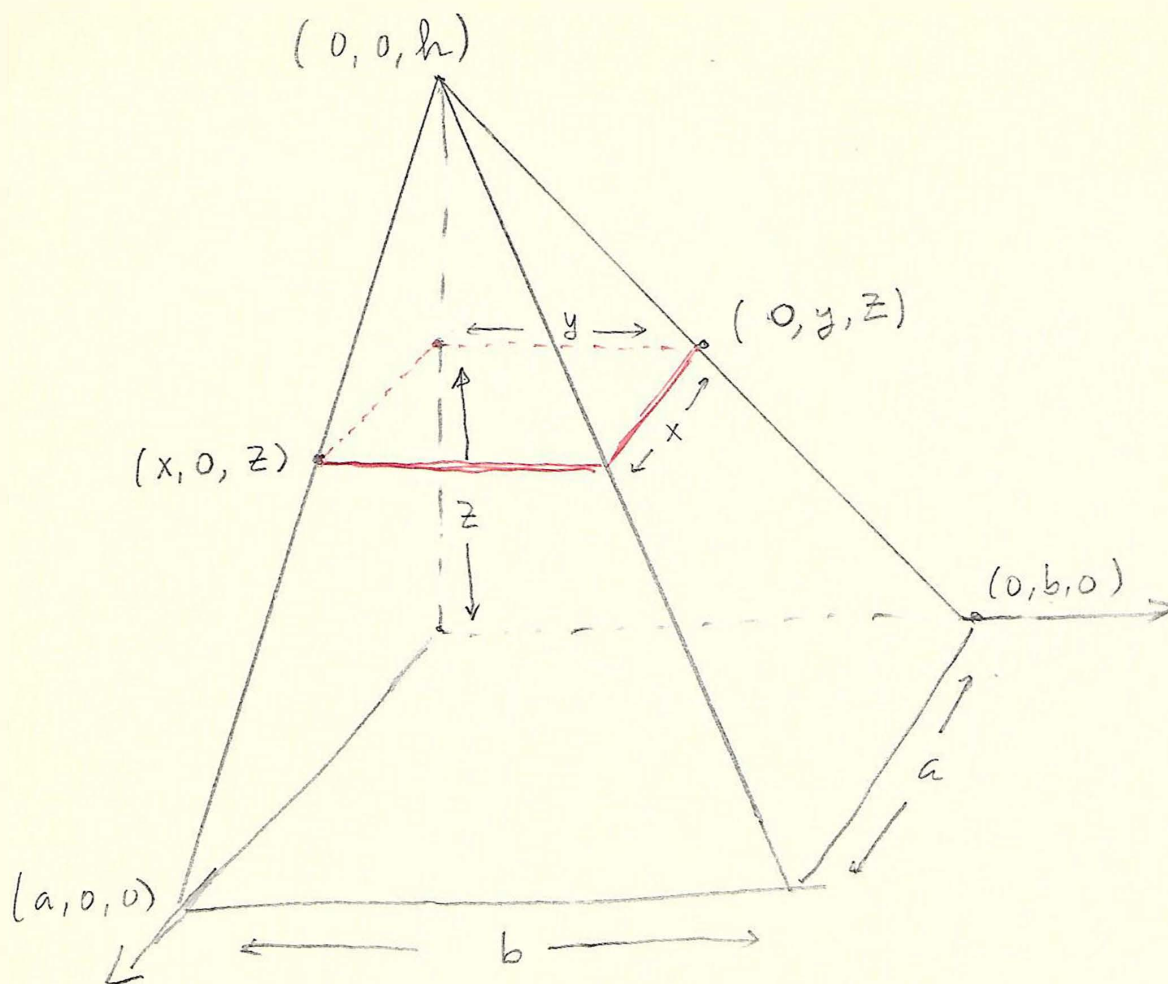
$$V = 2 \int_0^a S(x) dx = 2\pi bc \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi bc \left(x - \frac{x^3}{3a^2}\right) \Big|_{x=0}^{x=a} = 2\pi bc \left(a - \frac{a^3}{3a^2}\right) = \frac{4}{3}\pi abc$$

(b) Volum envoltat pel con de base el·líptica $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2$, amb $0 \leq z \leq h$.

Solució. — Fixant $0 \leq z \leq h$: $\frac{x^2}{(az)^2} + \frac{y^2}{(bz)^2} = 1$, d'on $S(z) = \pi ab z^2$

3(c)

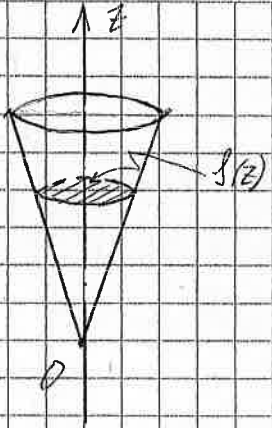
3 bis



Per Thales $\frac{x}{a} = \frac{y}{b} = \frac{h-z}{h}$

Área sección $A(z) = xy = \frac{ab}{h^2} (h-z)^2$

Volumen $\int_0^h A(z) dz = \frac{ab}{h^2} \int_0^h (h-z)^2 dz = \frac{ab}{h^2} \left[\frac{(h-z)^3}{3} \right]_h^0 = \frac{abh}{3}$

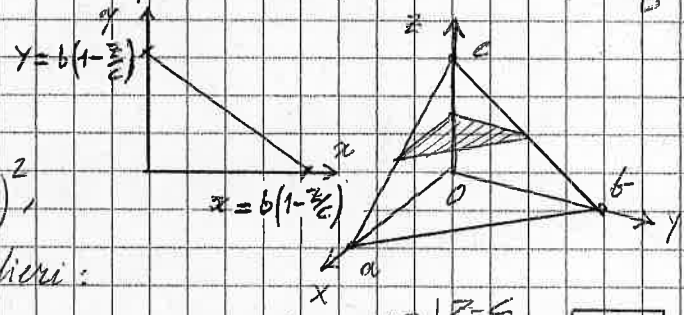


Alleshores, aplicant el principi de Cavalieri:

$$V = \int_0^h S(z) dz = \int_0^h \pi ab z^2 dz = \pi ab \left. \frac{z^3}{3} \right|_{z=0}^{z=h} = \boxed{\frac{\pi}{3} abh^3}$$

Nota: $V = \frac{1}{3} \pi (\underbrace{ab}_{\text{àrea de la base}}) \cdot (bh) \cdot h$

(c) Volum d'un tetraedre limitat pels plans $x=0, y=0, z=0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ($a, b, c > 0$).



Solució:-

Fixant z : $S(z) = \frac{1}{2} ab (1 - \frac{z}{c})^2$

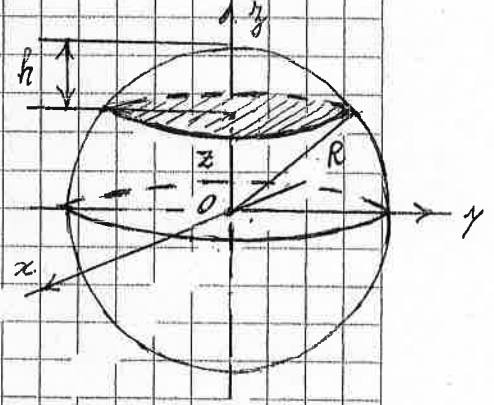
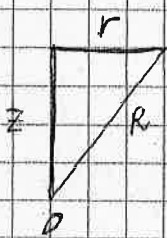
d'on, aplicant el principi de Cavalieri:

$$V = \int_0^c S(z) dz = \frac{1}{2} ab \int_0^c (1 - \frac{z}{c})^2 dz = -\frac{1}{6} abc \left(1 - \frac{z^3}{c^3}\right) \Big|_{z=0}^{z=c} = \boxed{\frac{abc}{6}}$$

(d) Volum envoltant pel corquet esfèric determinat per l'esfera $x^2 + y^2 + z^2 = R^2$, la condició $R-h \leq z \leq R$

Solució:

$$S(z) = \pi r^2(z) = \pi (R^2 - z^2)$$



d'on, aplicant el principi de Cavalieri:

$$V = \int_{R-h}^R S(z) dz = \pi \int_{R-h}^R (R^2 - z^2) dz$$

$$= \pi \left(R^3 z - \frac{z^3}{3} \right) \Big|_{R-h}^R = \pi \left(R^3 - \frac{R^3}{3} - R^3 + R^2 h + \frac{R^3}{3} - R^2 h + R h^2 - \frac{h^3}{3} \right) = \frac{\pi h^2}{3} (3R - h)$$

4) Generalitzem el principi de Cavalieri al càlcul de volums en \mathbb{R}^4 i calculem el volum de la bola 4-dimensional $B = \{(x,y,z,t) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + t^2 \leq R^2\}$

Solució Fixem t : $0 \leq x^2 + y^2 + z^2 \leq R^2 - t^2$, amb $-R \leq t \leq R$ Alleshores

$$V(t) = \frac{4}{3} \pi (R^2 - t^2)^{3/2}$$

D'on aplicant el principi de Cavalieri:

$$\begin{aligned}
 M &= \int_{-R}^R \sqrt{R^2 - t^2} dt = \frac{4}{3} \pi \int_{-R}^R (R^2 - t^2)^{3/2} dt = \frac{8}{3} \pi R^3 \int_0^R \left(1 - \frac{t^2}{R^2}\right)^{3/2} dt = \left\{ \begin{array}{l} \text{c.v.:} \\ t/R = \sin \theta \\ \dots \end{array} \right\} \\
 &= \frac{8}{3} \pi R^4 \int_0^{\pi/2} (1 - \sin^2 \theta)^{3/2} \cos \theta d\theta = \frac{8}{3} \pi R^4 \int_0^{\pi/2} \cos^4 \theta d\theta \stackrel{(*)}{=} \frac{8}{3} \pi R^4 \int_0^{\pi/2} \left(\frac{1 + \cos(2\theta)}{2} - \frac{1 - \cos(4\theta)}{8} \right) d\theta \\
 &= \frac{8}{3} \pi R^4 \left(\frac{\pi}{4} - \frac{\pi}{16} \right) = \frac{8}{3} \pi R^4 \cdot \frac{3\pi}{16} = \boxed{\frac{\pi^2 R^4}{2}}
 \end{aligned}$$

(*) $\cos^4 \theta = \cos^2 \theta (1 - \sin^2 \theta) = \cos^2 \theta - \frac{1}{4} \sin^2(2\theta) = \frac{1 + \cos(2\theta)}{2} - \frac{1 - \cos(4\theta)}{8}$, fent servir les fórmules de l'angle doble. \square

5) Troben les següents integrals dobles en els rectangles que s'indiquen.

(a) $I = \iint_R x^2 y dx dy, R = [0, 1] \times [0, 1]$

Solució: $I = \iint_R x^2 y dx dy = \left(\int_0^1 x^2 dx \right) \left(\int_0^1 y dy \right) = \left[\frac{x^3}{3} \right]_0^1 \cdot \left[\frac{y^2}{2} \right]_0^1 = \frac{1}{3} \cdot \frac{1}{2} = \boxed{\frac{1}{6}}$

(b) $I = \iint_R \frac{x^2}{1+y^2} dx dy, R = [0, 1] \times [0, 1]$

Solució: $I = \iint_{R=[0,1] \times [0,1]} \frac{x^2}{1+y^2} dx dy = \int_0^1 x^2 dx \int_0^1 \frac{dy}{1+y^2} = \left[\frac{x^3}{3} \right]_0^1 \cdot \left[\arctan y \right]_0^1 = \boxed{\frac{\pi}{12}}$

(c) $I = \iint_R y \ln x dx dy, R = [1, e] \times [1, e]$

Solució: $I = \iint_{R=[1,e] \times [1,e]} y \ln x dx dy = \left(\int_1^e \ln x dx \right) \cdot \left(\int_1^e y dy \right) \stackrel{(*)}{=} \boxed{\frac{e^2 - 1}{2}}$

(*) $\int_1^e \ln x dx = (x \ln x - x) \Big|_1^e = e - e + 1 = 1$, (primitivització per parts + regla de Barrow).

(d) $\iint_R (x^2 + y) dx dy, R = [0, 1] \times [0, 2]$

Solució: $I = \iint_{R=[0,1] \times [0,2]} (x^2 + y) dx dy = \int_0^1 dx \int_0^2 (x^2 + y) dy = \int_0^1 \left[x^2 y + \frac{y^2}{2} \right]_{y=0}^{y=2} dx = \int_0^1 (2x^2 + 2) dx$

$$= \left[\frac{2}{5}x^3 + 2x \right]_{x=0}^{x=1} = \frac{2}{5} + 2 = \boxed{\frac{8}{5}}$$

$$(e) I = \iint_R \frac{1}{(x+2y)^2} dx dy, \quad R = [2, 5] \times [1, 3]$$

$$\begin{aligned} \text{Solució } I &= \iint_R \frac{dx dy}{(x+2y)^2} = \int_2^5 dx \int_1^3 \frac{dy}{(x+2y)^2} = \int_2^5 \left[-\frac{1}{2} \frac{1}{x+2y} \right]_{y=1}^{y=3} dx \\ &= \frac{1}{2} \int_2^5 \left(\frac{1}{2+x} - \frac{1}{6+x} \right) dx = \frac{1}{2} \ln \left(\frac{2+x}{6+x} \right) \Big|_{x=2}^{x=5} \\ &= \frac{1}{2} \left(\ln \frac{7}{11} - \ln \frac{4}{8} \right) = \boxed{\frac{1}{2} \ln \left(\frac{14}{11} \right)} \end{aligned}$$

$$(f) I = \iint_R e^y \sin\left(\frac{x}{y}\right) dx dy, \quad R = [-\pi/2, \pi/2] \times [1, 2]$$

$$\begin{aligned} \text{Solució. } I &= \iint_R e^y \sin\left(\frac{x}{y}\right) dx dy = \int_1^2 e^y dy \int_{-\pi/2}^{\pi/2} \sin\left(\frac{x}{y}\right) dx \\ R &= [-\pi/2, \pi/2] \times [1, 2] \\ &= - \int_1^2 y e^y \left[\cos\left(\frac{x}{y}\right) \right]_{x=-\pi/2}^{x=\pi/2} dy = 0 \end{aligned}$$

$$(g) [\dots]_{x=-\pi/2}^{x=\pi/2} = \cos\left(\frac{\pi}{2y}\right) - \cos\left(-\frac{\pi}{2y}\right) = 0$$

$$(g) I = \iint_R (x+y)^{27} dx dy, \quad R = [-1, 1] \times [-1, 1]$$

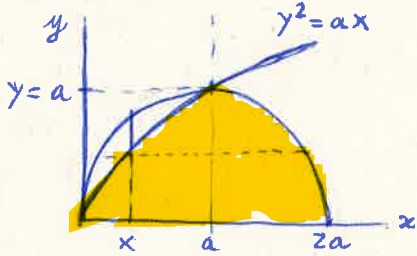
$$\begin{aligned} \text{Solució. } I &= \iint_R (x+y)^{27} dx dy = \int_{-1}^1 dx \int_{-1}^1 (x+y)^{27} dy = \frac{1}{28} \int_{-1}^1 (x+y)^{28} \Big|_{y=-1}^{y=1} dx \\ R &= [-1, 1] \times [-1, 1] \\ &= \frac{1}{28} \int_{-1}^1 \left[(x+1)^{28} - (x-1)^{28} \right] dx = \frac{1}{28} \cdot \frac{1}{29} \left[(x+1)^{29} - (x-1)^{29} \right] \Big|_{x=-1}^{x=1} \\ &= \frac{1}{28} \cdot \frac{1}{29} \left(2^{29} + (-2)^{29} \right) = \boxed{0} \end{aligned}$$

8) Per a les regions $A \subset \mathbb{R}^2$ indicades escrivim la integral doble $\iint_A f(x,y) dx dy$ en termes d'integrals iterades preses en diferents ordres, $\int \left(\int f(x,y) dx \right) dy$ i $\int \left(\int f(x,y) dy \right) dx$, donant quins són els extrems d'integració per a x i y en cada cas.

Apartats d) i e)

Solució:

d) A regió limitada per les corbes $y^2 = ax$, $x^2 + y^2 = 2ax$, $y = 0$ ($y \geq 0, a > 0$).



$$I_1 = \int_0^a dx \int_{\sqrt{ax-x^2}}^{a-x} f(x,y) dy + \int_a^{2a} dx \int_0^{\sqrt{2ax-x^2}} f(x,y) dy$$

$$I_2 = \int_0^a dy \int_{y^2/a}^{a+\sqrt{a^2-y^2}} f(x,y) dx$$

$$2ax - x^2 \leq ax, x > 0$$

$$\Leftrightarrow ax - x^2 = x(a-x) \leq 0$$

$$\Leftrightarrow x \geq a$$

$$(x-a)^2 + y^2 = a^2$$

$$x = a + \sqrt{a^2 - y^2}$$

Si suposem que f és contínua en D , llavors pel teorema de Fubini

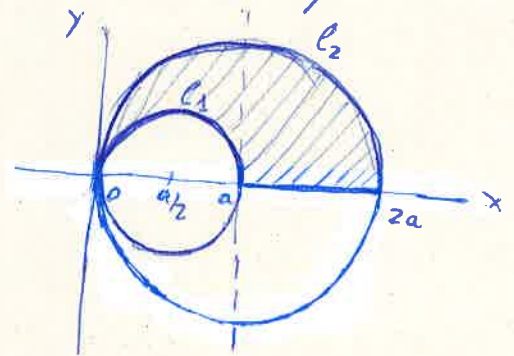
$$I_1 = I_2$$

Solució:

e) A regió limitada per les corbes $x^2 + y^2 = ax$, $x^2 + y^2 = 2ax$, $y = 0$ ($y \geq 0, a > 0$)

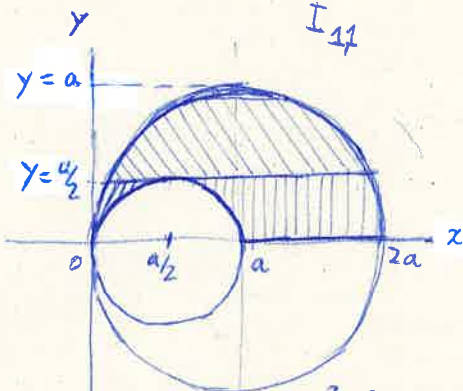
$$C_1: x^2 + y^2 = ax, y \geq 0$$

$$C_2: x^2 + y^2 = 2ax, y \geq 0$$



2-Domínis x-elementals

$$I_1 = \underbrace{\int_0^a dx \int_{\sqrt{ax-x^2}}^{\sqrt{2ax-x^2}} f(x,y) dy}_{I_{1,1}} + \underbrace{\int_a^{2a} dx \int_0^{\sqrt{2ax-x^2}} f(x,y) dy}_{I_{1,2}}$$



3-domínis y-elementals.

$$I_2 = \int_0^{a/2} dy \int_{a-\sqrt{a^2-y^2}}^{a/2-\sqrt{a^2/4-y^2}} f(x,y) dx + \int_0^{a/2} dy \int_{a/2+\sqrt{a^2/4-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y) dx$$

$$+ \int_{a/2}^a dy \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y) dx$$

$$C_1: x^2 + y^2 = ax \Leftrightarrow (x-a/2)^2 + y^2 = a^2/4$$

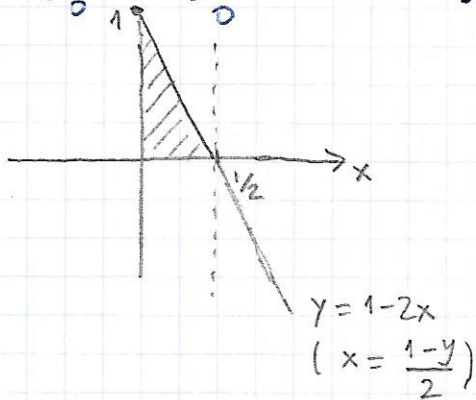
$$\Leftrightarrow x = a/2 \pm \sqrt{a^2/4 - y^2}$$

Si f és contínua en D : $I_1 = I_2$

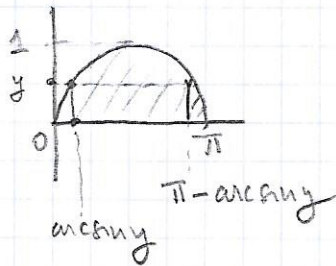
CANVI ORDRE INTEGRASI

(11)

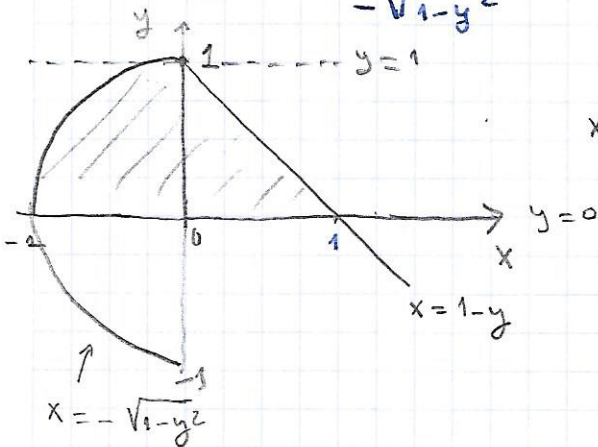
$$(a) \int_0^{1/2} dx \int_0^{1-2x} f dy = \int_0^1 dy \int_0^{\frac{1-y}{2}} f dx$$



$$(e) \int_0^{\pi} dx \int_0^{\sin x} f dy = \int_0^1 dy \int_{\arcsin y}^{\pi - \arcsin y} f dx$$



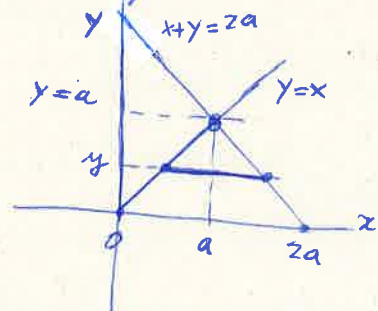
$$(f) \int_0^1 dy \int_{-\sqrt{1-y^2}}^{1-y} f dx = \int_{-1}^0 dx \int_0^{\sqrt{1-x^2}} f dy + \int_0^1 dx \int_0^{1-x} f dy$$



$$x = -\sqrt{1-y^2}; \quad x^2 = 1-y^2; \quad x^2 + y^2 = 1$$

12. Calculeu les següents integrals dobles en els dominis de \mathbb{R}^2 que s'indiquen

(a) $I = \iint_A (x^2 + y^2) dx dy$. A limitat per les rectes $y=x$, $x+y=2a$, $x=0$ ($a>0$)



Sol.
$$I = \int_0^a dy \int_y^{2a-y} (x^2 + y^2) dx = \int_0^a \left(\frac{x^3}{3} + y^2 x \right) \Big|_{x=y}^{x=2a-y} dy$$

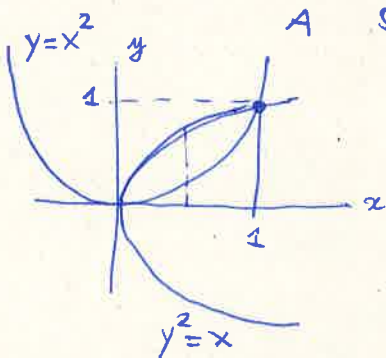
$$= \int_0^a \left[\frac{1}{3}(2a-y)^3 + y^2(2a-y) - \frac{y^3}{3} - y^3 \right] dy = \int_0^a \left(\frac{8}{3}a^3 - 4a^2y + 4ay^2 - \frac{8}{3}y^3 \right) dy$$

$$\frac{1}{3} (8a^3 - 12a^2y + 6ay^2 - y^3) + 2ay^2 - y^3 - \frac{y^3}{3} - y^3 = \left(\frac{8}{3}a^3y - 2a^2y^2 + \frac{4}{3}ay^3 - \frac{2}{3}y^4 \right) \Big|_{y=0}^{y=a}$$

$$= \frac{8}{3}a^3 - 4a^2y + 2ay^2 - \frac{1}{3}y^3 + 2ay^2 - \frac{y^3}{3} - 2y^3 = \left(\frac{8}{3} - 2 + \frac{4}{3} - \frac{2}{3} \right) a^4 = \boxed{\frac{4}{3}a^4}$$

$$= \frac{8}{3}a^3 - 4a^2y + 4ay^2 - \frac{8}{3}y^3$$

(b) $I = \iint_A (x+2y) dx dy$, A limitat per les corbes $y=x^2$, $y^2=x$.



Sol.
$$I = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (x+2y) dy = \int_0^1 (xy + y^2) \Big|_{y=x^2}^{y=\sqrt{x}} dx = \int_0^1 \left(x^{3/2} + x - x^3 - x^5 \right) dx$$

$$= \left(\frac{2}{5}x^{5/2} + \frac{x^2}{2} - \frac{x^4}{4} - \frac{x^5}{5} \right) \Big|_{x=0}^{x=1} = \frac{2}{5} + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} = \frac{8+10-5-4}{20} = \boxed{\frac{9}{20}}$$

$$x^5 = x \Leftrightarrow x(x^4 - 1) = 0$$

Sols: $x=0 \rightarrow y=0$
 $x=1 \rightarrow y=1$

(c) $I = \iint_A e^{x+y} dx dy$, A limitat per les corbes $y=e^x$, $x=0$, $y=2$

Sol.
$$I = \int_0^{\ln 2} dx \int_{e^x}^2 e^{x+y} dy = \int_0^{\ln 2} \left[e^{x+y} \right]_{y=e^x}^{y=2} dx$$

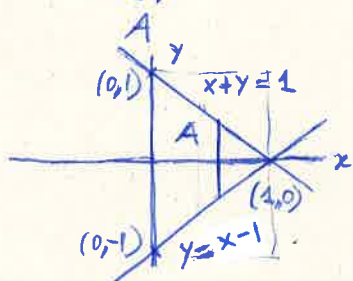
$$= \int_0^{\ln 2} (e^{x+2} - e^x e^{e^x}) dx = \left(e^{x+2} - e^x \right) \Big|_{x=0}^{x=\ln 2}$$

$$= e^{2+\ln 2} - e^{e^{\ln 2}} - e^2 + e = 2e^2 - e^2 - e^2 + e = \boxed{e}$$

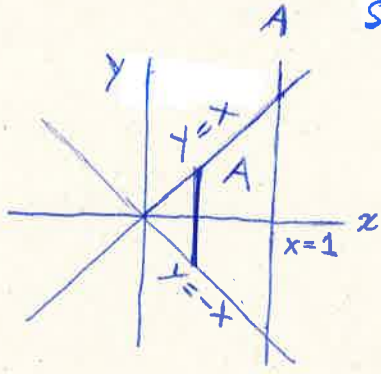
(d) $I = \iint_A e^y dx dy$, A triangle de vèrtexs $(1,0)$, $(0,1)$, $(0,-1)$.

Sol.
$$I = \int_0^1 dx \int_{x-1}^{1-x} e^y dy = \int_0^1 (e^{1-x} - e^{x-1}) dx = \left(-e^{1-x} - e^{x-1} \right) \Big|_{x=0}^{x=1}$$

$$= -1 - 1 + e + \frac{1}{e} = \boxed{e + \frac{1}{e} - 2}$$



(e) $I = \iint xy^2 dx dy$, A limitat per les rectes $x=1, x=y, x+y=0$.

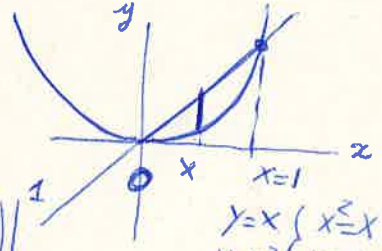


Sol.
$$I = \int_0^1 dx \int_{-x}^x xy^2 dy = \int_0^1 \left(x \cdot \frac{y^3}{3} \right) \Big|_{y=-x}^{y=x} dx = \frac{2}{3} \int_0^1 x^4 dx = \boxed{\frac{2}{15}}$$

(f) $I = \iint xy dx dy$, A limitat per les corbes $x=y, y=x^2$

$$I = \int_0^1 dx \int_{x^2}^x xy dy = \int_0^1 \left[\frac{xy^2}{2} \right]_{y=x^2}^{y=x} dx$$

$$= \frac{1}{2} \int_0^1 (x^3 - x^5) dx = \left(\frac{x^4}{8} - \frac{x^6}{12} \right) \Big|_0^1$$



$y=x \begin{cases} x \leq x \\ x=0 \end{cases}$
 $y=x^2 \begin{cases} x(x-1)=0 \\ x=1 \end{cases}$

$$= \frac{1}{8} - \frac{1}{12} = \frac{3-2}{24} = \boxed{\frac{1}{24}}$$

⑬ Calculen les intégrals itérées :

$$\begin{aligned}
 \text{(a)} \quad & \int_1^2 \int_0^1 \int_0^{\pi/2} x^2 y^3 \sin z \, dz \, dy \, dx = \\
 & = \int_1^2 x^2 \, dx \int_0^1 y^3 \, dy \int_0^{\pi/2} \sin z \, dz = \left[\frac{x^3}{3} \right]_1^2 \left[\frac{y^4}{4} \right]_0^1 \left[-\cos z \right]_{\pi/2}^0 = \\
 & = \left(\frac{8}{3} - \frac{1}{3} \right) \left(\frac{1}{4} - 0 \right) (1 - 0) = \frac{7}{12}
 \end{aligned}$$

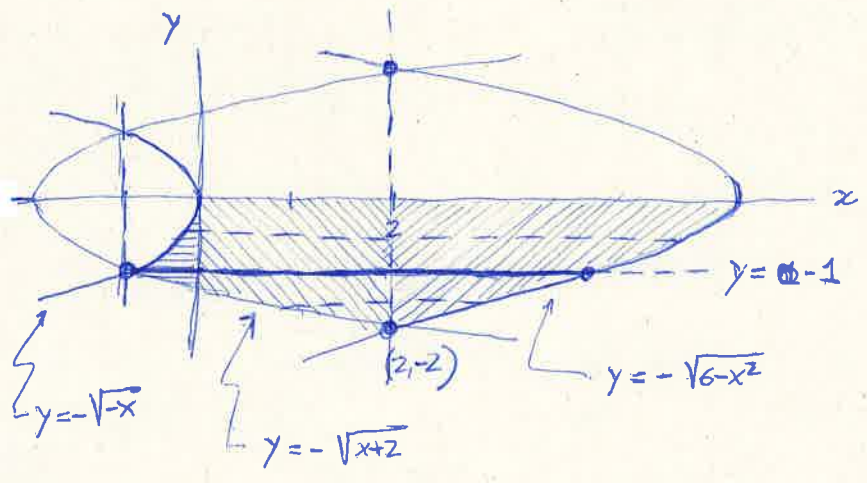
$$\begin{aligned}
 \text{(b)} \quad & \int_0^1 \int_0^x \int_0^{\sqrt{x^2+y^2}} z \, dz \, dy \, dx = \int_0^1 \int_0^x \left[\frac{z^2}{2} \right]_0^{\sqrt{x^2+y^2}} \, dy \, dx = \\
 & = \int_0^1 \int_0^x \frac{x^2+y^2}{2} \, dy \, dx = \frac{1}{2} \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^x \, dx = \\
 & = \frac{1}{2} \int_0^1 \left(x^3 + \frac{x^3}{3} \right) \, dx = \frac{2}{3} \int_0^1 x^3 \, dx = \frac{2}{3} \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \int_0^3 \int_0^{2x} \int_0^{\sqrt{xy}} z \, dz \, dy \, dx = \int_0^3 \int_0^{2x} \left[\frac{z^2}{2} \right]_0^{\sqrt{xy}} \, dy \, dx = \\
 & = \int_0^3 \int_0^{2x} \frac{xy}{2} \, dy \, dx = \frac{1}{2} \int_0^3 \left[\frac{xy^2}{2} \right]_0^{2x} \, dx = \int_0^3 x^3 \, dx = \left[\frac{x^4}{4} \right]_0^3 = \\
 & = \frac{81}{4}
 \end{aligned}$$

Genes 2005

Permuten l'ordre d'integració de la següent integral doble.

$$I = \int_{-1}^0 dx \int_{-\sqrt{x+2}}^{-\sqrt{-x}} f(x,y) dy + \int_0^2 dx \int_{-\sqrt{x+2}}^0 f(x,y) dy + \int_2^6 dx \int_{-\sqrt{6-x}}^0 f(x,y) dy$$



$$-\sqrt{-x} = -\sqrt{x+2}$$

$$-x = x+2 \Leftrightarrow 2x+2=0 \Leftrightarrow x=-1 \wedge y=-1$$

$$-\sqrt{x+2} = -\sqrt{6-x}$$

$$x+2 = 6-x \quad | \quad -y^2 \leq x \leq 6-y^2 \quad | \quad y^2-2 \leq x \leq 6-y^2$$

$$2x=4 \Leftrightarrow x=2 \quad | \quad -1 \leq y \leq 0 \quad | \quad -2 \leq y \leq -1$$

$$y=-2$$

$$I = \int_{-1}^0 dy \int_{-y^2}^{6-y^2} f(x,y) dx + \int_{-2}^{-1} dy \int_{y^2-2}^{6-y^2} f(x,y) dx$$

17. (Col·lecció de problemes) Apartats b, c. Per a les regions de \mathbb{R}^3 indicades escriure la integral triple $\iiint_A f(x,y,z) dx dy dz$ en termes d'integrals iterades preses en diferents ordres.

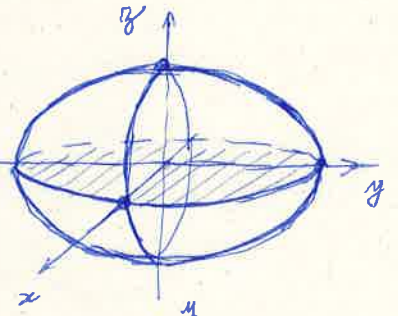
14b) $I = \iiint_E f(x,y,z) dx dy dz, E: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (a, b, c > 0)$

$I_{z,x} = \int_{-a}^a dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} f(x,y,z) dz$

$$-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \leq z \leq c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$$-b\sqrt{1-\frac{x^2}{a^2}} \leq y \leq b\sqrt{1-\frac{x^2}{a^2}}$$

$$-a \leq x \leq a$$

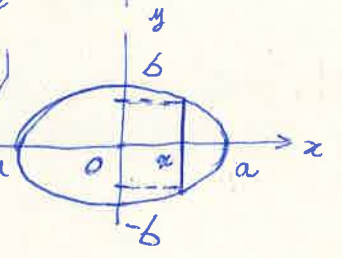


$I_{y,z} = \int_{-b}^b dy \int_{-a\sqrt{1-\frac{y^2}{b^2}}}^{a\sqrt{1-\frac{y^2}{b^2}}} dx \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} f(x,y,z) dz$

$$-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \leq z \leq c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

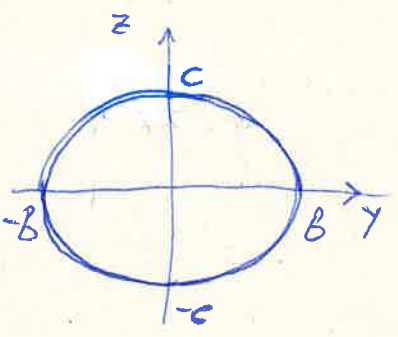
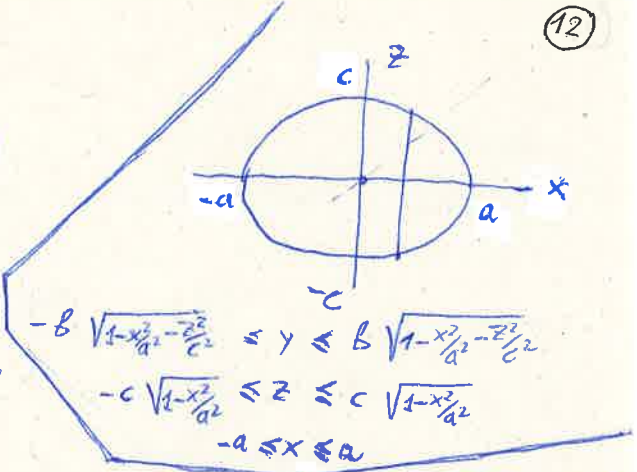
$$-a\sqrt{1-\frac{y^2}{b^2}} \leq x \leq a\sqrt{1-\frac{y^2}{b^2}}$$

$$-b \leq y \leq b$$



$$I_{2,1} = \int_{-a}^a dx \int_{-c\sqrt{1-x^2/a^2}}^{c\sqrt{1-x^2/a^2}} dz \int_{-b\sqrt{1-x^2/a^2-z^2/c^2}}^{b\sqrt{1-x^2/a^2-z^2/c^2}} f(x,y,z) dy$$

$$I_{2,2} = \int_{-c}^c dz \int_{-a\sqrt{1-z^2/c^2}}^{a\sqrt{1-z^2/c^2}} dx \int_{-b\sqrt{1-x^2/a^2-z^2/c^2}}^{b\sqrt{1-x^2/a^2-z^2/c^2}} f(x,y,z) dy$$



$$I_{3,1} = \int_{-b}^b dy \int_{-c\sqrt{1-y^2/b^2}}^{c\sqrt{1-y^2/b^2}} dz \int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z) dx$$

$$I_{3,2} = \int_{-c}^c dz \int_{-b\sqrt{1-z^2/c^2}}^{b\sqrt{1-z^2/c^2}} dy \int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z) dx$$

Fubini:

$$I_{1,1} = I_{1,2} = I_{2,1} = I_{2,2} = I_{3,1} = I_{3,2}$$

14c) $I = \iiint_D f(x,y,z) dx dy dz$; $D: y^2 + 2z^2 = 4x, x=2$



$$I_{1,1} = \int_0^2 dx \int_{-\sqrt{2x}}^{\sqrt{2x}} dz \int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}} f(x,y,z) dy$$

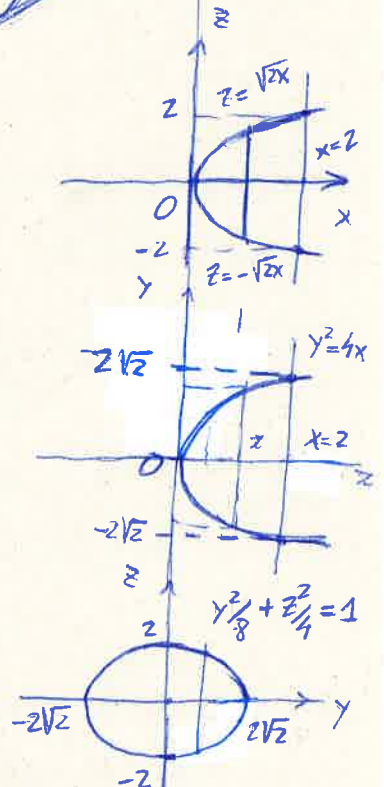
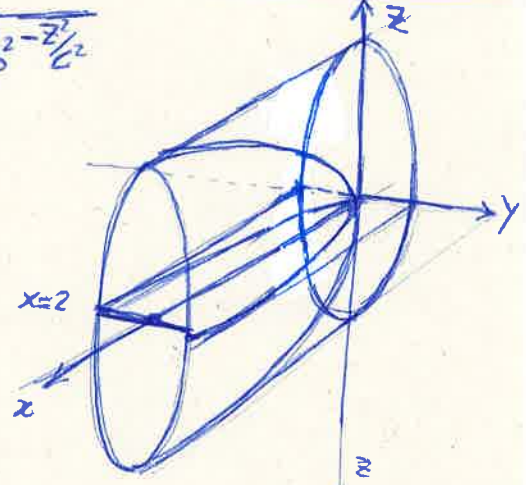
$$I_{1,2} = \int_{-2}^2 dz \int_{z^2/2}^2 dx \int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}} f(x,y,z) dy$$

$$I_{2,1} = \int_0^2 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} dy \int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}} f(x,y,z) dz$$

$$I_{2,2} = \int_{-2\sqrt{2}}^{2\sqrt{2}} dy \int_{y^2/4}^2 dx \int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}} f(x,y,z) dz$$

$$I_{3,1} = \int_{-2\sqrt{2}}^{2\sqrt{2}} dy \int_{-\sqrt{4-y^2/2}}^{\sqrt{4-y^2/2}} dz \int_{y^2/4 + z^2/2}^2 f(x,y,z) dx$$

$$I_{3,2} = \int_{-2}^2 dz \int_{-\sqrt{8-2z^2}}^{\sqrt{8-2z^2}} dy \int_{y^2/4 + z^2/2}^2 f(x,y,z) dx$$



Fubini:
 $I_{1,1} = I_{1,2} = I_{2,1} = I_{2,2} = I_{3,1} = I_{3,2}$

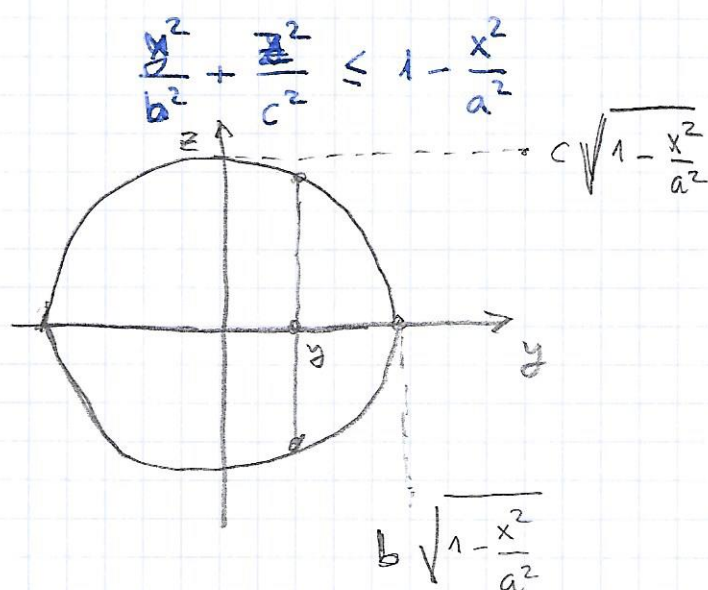
14b) Explicació d'un orbe, p. exemple x, y, z

$$A = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

El valor màxim de la variable x : $\frac{x^2}{a^2} = 1$ ($y=0, z=0$)

llavors $\boxed{-a \leq x \leq a}$

Les seccions $x = \text{constant}$ són interiors d'el·lipsoïdes:



Els valors límits de y : $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$ ($z=0$)

llavors $\boxed{-b \sqrt{1 - \frac{x^2}{a^2}} \leq y \leq b \sqrt{1 - \frac{x^2}{a^2}}}$

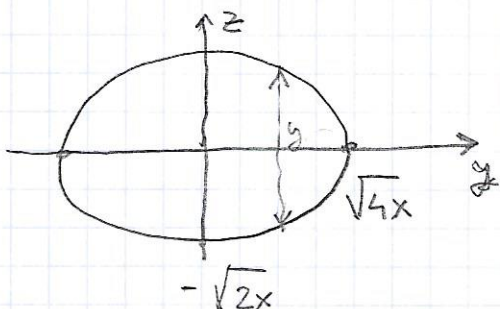
Les seccions $y = \text{constant}$ ($x = \text{constant}$) són:

$$\frac{z^2}{c^2} \leq 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

llavors $\boxed{-c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \leq z \leq c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}$

14c) Cos limitat per $y^2 + 2z^2 = 4x$, $x = 2$

① Seccions $x = \text{constant}$: són el·lipsoïdes sempre que $x \geq 0$



Perquè sigui limitat:

$$0 \leq x \leq 2$$

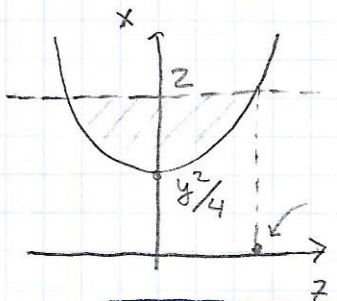
repetint el que ja sabem per una el·lipse:

$$\begin{aligned} -\sqrt{4x} &\leq y \leq \sqrt{4x} \\ -\sqrt{\frac{4x-y^2}{2}} &\leq z \leq \sqrt{\frac{4x-y^2}{2}} \end{aligned}$$

or

$$\begin{aligned} -\sqrt{2x} &\leq z \leq \sqrt{2x} \\ -\sqrt{4x-2z^2} &\leq y \leq \sqrt{4x-2z^2} \end{aligned}$$

② Seccions $y = \text{constant}$: $x = \frac{z^2}{2} + \frac{y^2}{4}$ són parabòles



Com que hi ha la limitació $x = 2$ perquè sigui acotat i hi hagi secció

$$\frac{y^2}{4} \leq 2 \rightarrow -\sqrt{8} \leq y \leq \sqrt{8}$$

lavors $\frac{y^2}{4} \leq x \leq 2$

$$-\sqrt{2x - \frac{y^2}{2}} \leq z \leq \sqrt{2x - \frac{y^2}{2}}$$

$$-\sqrt{4 - \frac{y^2}{2}} \leq z \leq \sqrt{4 - \frac{y^2}{2}}$$

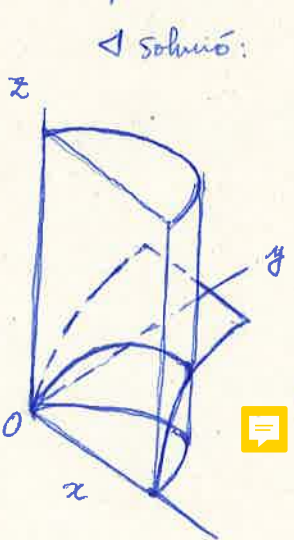
or

$$\frac{z^2}{2} + \frac{y^2}{4} \leq x \leq 2$$

15. Calcular los integrales triples siguientes en las regiones de \mathbb{R}^3 que s'indiquen,

(a) $I = \iiint_A xz \, dx \, dy \, dz$. A limitat pel cilindre de base circular $x^2 + y^2 - z = 0$ i

la superficie $z = 2y$ ($y, z \geq 0$)



◁ Solució:

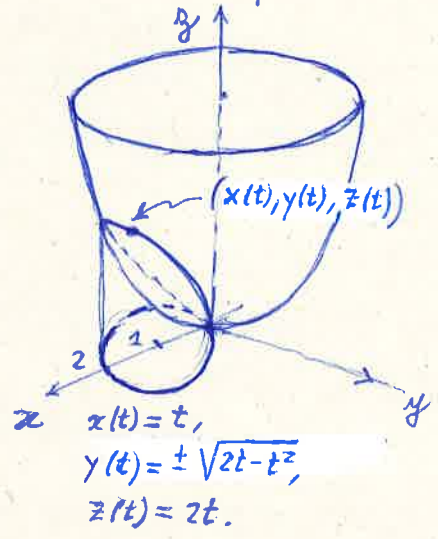
$$I = \int_0^2 x \, dx \int_0^{\sqrt{2x-x^2}} dy \int_0^{2y} z \, dz = \frac{1}{2} \int_0^2 x \, dx \int_0^{\sqrt{2x-x^2}} 2y \, dy = \frac{1}{2} \int_0^2 x(2x-x^2) \, dx$$

$$= \frac{1}{2} \left(\frac{2}{3}x^3 - \frac{x^4}{4} \right) \Big|_{x=0}^{x=2} = \frac{1}{2} \left(\frac{16}{3} - 4 \right) = \frac{2}{3} \quad \Delta$$

(b) $I = \iiint_A zy \sqrt{x^2+y^2} \, dx \, dy \, dz$ amb $A = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq z \leq x^2+y^2, 0 \leq y \leq \sqrt{2x-x^2}\}$

◁ Solució:

$$\begin{cases} z = x^2 + y^2 \\ z = x^2 + y^2 \end{cases} \rightarrow \begin{cases} z = 2x \\ y = \pm \sqrt{2x-x^2} \end{cases} \rightarrow \begin{cases} x = t, \\ y = \pm \sqrt{2t-t^2}, \\ z = 2t, \end{cases} \quad 0 \leq t \leq 2.$$



Projecció sobre el pla x-y ($z=0$):

$$\begin{cases} x = t \\ y = \pm \sqrt{2t-t^2} \end{cases} \Leftrightarrow (x-1)^2 + y^2 = 1,$$

amb $y \geq 0$ en el nostre cas. Aleshores:

$$I = \int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^{x^2+y^2} zy \sqrt{x^2+y^2} \, dz$$

$$= \int_0^2 dx \int_0^{\sqrt{2x-x^2}} \frac{z^2}{2} y \sqrt{x^2+y^2} \Big|_{z=0}^{z=x^2+y^2} dy = \frac{1}{2} \int_0^2 dx \int_0^{\sqrt{2x-x^2}} y (x^2+y^2)^{5/2} dy$$

$$= \frac{1}{2} \cdot \frac{1}{7} \int_0^2 (x^2+y^2)^{7/2} \Big|_{y=0}^{y=\sqrt{2x-x^2}} dx = \frac{1}{14} \int_0^2 \left[(x^2+2x-x^2)^{7/2} - x^7 \right] dx = \frac{1}{14} \left(2^{7/2} \cdot \frac{2}{9} x^{9/2} - \frac{x^8}{8} \right) \Big|_{x=0}^{x=2}$$

$$= \frac{1}{14} \left(2^8 \cdot \frac{2}{9} - \frac{2^8}{8} \right) = \frac{2^8}{2 \cdot 7} \left(\frac{16-9}{3^2 \cdot 2^3} \right) = \frac{2^8 \cdot 7}{2 \cdot 3^2 \cdot 7} = \frac{2^4}{3^2} = \frac{16}{9} \quad \Delta$$

$$(15)_b \quad I = \int_A z y \sqrt{x^2 + y^2} \, dx dy dz$$

$$A = \left\{ 0 \leq z \leq x^2 + y^2, 0 \leq y \leq \sqrt{2x - x^2} \right\}$$

z, y ja les tenim com volum, l'única que ens falta és trobar l'interval de variació de x .

Però, perquè $\sqrt{2x - x^2}$ tingui sentit

$$2x - x^2 \geq 0 \quad \Leftrightarrow \quad 0 \leq x \leq 2$$

Per tant,

$$\begin{aligned} I &= \int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^{x^2+y^2} z y \sqrt{x^2+y^2} \, dz = \\ &= \int_0^2 dx \int_0^{\sqrt{2x-x^2}} \frac{y (x^2+y^2)^{5/2}}{2} dy = \frac{1}{14} \int_0^2 \left[(x^2+y^2)^{7/2} \right]_0^{\sqrt{2x-x^2}} dx = \\ &= \frac{1}{14} \int_0^2 \left[(2x)^{7/2} - x^7 \right] dx = \dots = \frac{16}{9} \end{aligned}$$

(c) $I = \iiint_A dx dy dz$, $A = \{(x,y,z) \in \mathbb{R}^3 : 1 \leq x \leq 3, 1 \leq y \leq 3, 0 \leq z \leq xy\}$

◁ Solució:
$$I = \int_1^3 dx \int_1^3 dy \int_0^{xy} dz = \int_1^3 dx \int_1^3 xy dy = \frac{1}{2} \int_1^3 [xy^2]_{y=1}^{y=3} dx$$

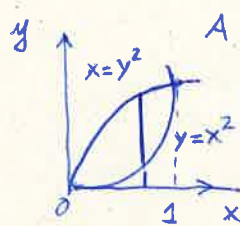
$$= \frac{1}{2} \int_1^3 8x dx = 4 \int_1^3 x dx = 4 \left[\frac{x^2}{2} \right]_{x=1}^{x=3} = 4 \left(\frac{9}{2} - \frac{1}{2} \right) = \boxed{16} \cdot \Delta$$

(d) $I = \iiint_A xyz dx dy dz$, A limitat per les superfícies $y=x^2, x=y^2, z=xy, z=0$

◁ Solució:
$$I = \int_0^1 x dx \int_{x^2}^{\sqrt{x}} dy \int_0^{xy} dz = \int_0^1 x dx \int_{x^2}^{\sqrt{x}} y \left[\frac{z^2}{2} \right]_0^{z=xy} dy = \frac{1}{2} \int_0^1 x dx \int_{x^2}^{\sqrt{x}} x^2 y^3 dy$$

$$= \frac{1}{8} \int_0^1 x^3 [y^4]_{y=x^2}^{y=\sqrt{x}} dx = \frac{1}{8} \int_0^1 x^3 (x^2 - x^8) dx$$

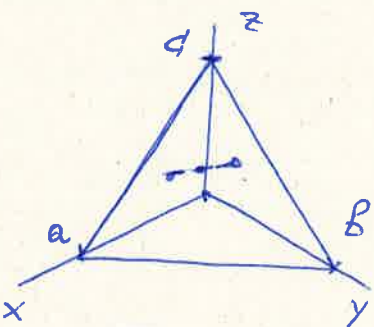
$$= \frac{1}{8} \left(-\frac{x^2}{12} + \frac{x^6}{6} \right) \Big|_{x=0}^{x=1} = \frac{1}{8} \cdot \frac{1}{12} = \boxed{\frac{1}{96}} \cdot \Delta$$



A: $0 \leq z \leq xy$
 $x^2 \leq y \leq \sqrt{x}$
 $0 \leq x \leq 1$

e) $I = \iiint_A x dx dy dz$, A: tetraedre limitat pels plans $x=0, y=0, z=0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, $(a,b,c > 0)$.

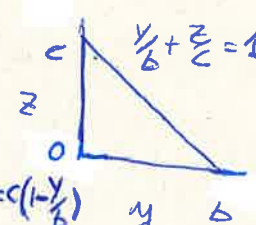
◁ Solució:
$$A: \begin{cases} 0 \leq x \leq a(1 - \frac{y}{b} - \frac{z}{c}) \\ 0 \leq z \leq c(1 - \frac{y}{b}) \\ 0 \leq y \leq b \end{cases}$$



$$I = \iiint_A x dx dy dz = \int_0^b dy \int_0^{c(1-y/b)} dz \int_0^{a(1-y/b-z/c)} x dx$$

$$= \frac{a^2}{2} \int_0^b dy \int_0^{c(1-y/b)} (1 - \frac{y}{b} - \frac{z}{c})^2 dz = -\frac{a^2}{2} \cdot \frac{c}{3} \cdot \left[(1 - \frac{y}{b} - \frac{z}{c})^3 \right]_{z=0}^{z=c(1-y/b)}$$

$$= \frac{a^2}{2} \cdot \frac{c}{3} \int_0^b (1 - \frac{y}{b})^3 dy = -\frac{a^2}{2} \cdot \frac{c}{3} \cdot \frac{b}{4} (1 - \frac{y}{b}) \Big|_{y=0}^{y=b} = \boxed{\frac{a^2 b c}{24}} \cdot \Delta$$



16) Useu coordenades polars per calcular les següents integrals dobles.

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r.$$

$$\iint_{T(D)} f(x,y) dx dy = \iint_D f(r \cos \theta, r \sin \theta) r dr d\theta.$$

a) $I = \iint_A (x^2 + y^2) dx dy, A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$

∇ Solució

$$I = \int_0^{2\pi} d\theta \int_0^2 r^3 dr = 2\pi \left[\frac{r^4}{4} \right]_0^2 = \boxed{8\pi}.$$

b) $I = \iint_A \cos(x^2 + y^2) dx dy, A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq \frac{\pi}{2}\}$

∇ Solució

$$I = \int_0^{2\pi} d\theta \int_0^{\sqrt{\pi/2}} r \cos(r^2) dr = 2\pi \frac{1}{2} \left[\sin(r^2) \right]_0^{\sqrt{\pi/2}} = \boxed{\pi}.$$

c) $I = \iint_A \frac{(x+y)^2}{x^2 + y^2 + 2} dx dy, A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

∇ Solució

$$\begin{aligned} I &= \int_0^{2\pi} d\theta \int_0^1 \frac{r^3}{r^2 + 2} dr = 2\pi \int_0^1 \left(1 - \frac{r}{r^2 + 2} \right) dr = 2\pi \left[\frac{r^2}{2} - \ln(r^2 + 2) \right]_0^1 \\ &= 2\pi \left[\frac{1}{2} - \ln 3 + \ln 2 \right] = \boxed{2\pi \left[\frac{1}{2} + \ln \left(\frac{2}{3} \right) \right]} \end{aligned}$$

d) $I = \iint_A \frac{dx dy}{(1+x^2+y^2)^2 \sqrt{x^2+y^2}}, A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$. (Indicació: useu

proprietats elementals del sin i cos per veure que $\sin(\arctan(R)) = \frac{R}{\sqrt{1+R^2}}$ i $\cos(\arctan(R)) = \frac{1}{\sqrt{1+R^2}}$).

16c

$$I = \int_A \frac{(x+y)^2}{x^2+y^2+2} dx dy, \quad A = \{(x,y) : x^2+y^2 \leq 1\}$$

Fem el canvi i queda:

$$I = \int_0^{2\pi} d\theta \int_0^1 \frac{\overbrace{r^2 \cos^2 \theta + r^2 \sin^2 \theta + 2r^2 \cos \theta \sin \theta}^{r^2}}{r^2 + 2} \cdot r dr =$$

funció

↓
|Jacobià|

$$= \int_0^1 dr \int_0^{2\pi} \left(\frac{r^3}{r^2+2} + \frac{2r^3 \cos \theta \sin \theta}{r^2+2} \right) d\theta =$$

com és un rectangle en (r, θ) puc canviar l'ordre d'integració

$$= 2\pi \int_0^1 \frac{r^3}{r^2+2} dr + \int_0^1 \frac{2r^3}{r^2+2} dr \int_0^{2\pi} \underbrace{\cos \theta \sin \theta}_{\text{"}} d\theta =$$

$$\left[\frac{\sin^2 \theta}{2} \right]_0^{2\pi} = 0$$

$$= 2\pi \int_0^1 \frac{r^3}{r^2+2} dr = \dots \text{ tal com està en el full } (15)$$

Solució.

$$I = \int_0^{2\pi} d\theta \int_0^R \frac{r dr}{(1+r^2)^2 \sqrt{r^2}} = 2\pi \int_0^R \frac{dr}{(1+r^2)^2} = \begin{cases} r = \tan t \Rightarrow dr = \frac{dt}{\cos^2 t} \\ r=0 \Rightarrow t=0 \\ r=R \Rightarrow t = \arctan(R) \end{cases}$$

$$= 2\pi \int_0^{\arctan(R)} \frac{\cos^4 t}{\cos^2 t} dt = 2\pi \int_0^{\arctan(R)} \frac{1+\cos(2t)}{2} dt = 2\pi \left(\frac{t}{2} + \frac{\sin(2t)}{4} \right) \Big|_0^{\arctan(R)}$$

$$\stackrel{(*)}{=} \boxed{\pi \left(\arctan(R) + \frac{R}{1+R^2} \right)}$$

(*) Notem que:
 $\sin(2t) = 2 \sin t \cos t = \frac{2 \sin t \cos^2 t}{\cos t}$
 $= 2 \tan t \cos^2 t = \frac{2 \tan t}{1+\tan^2 t}$

e) $I = \iint_A \sqrt{x^2+y^2-9} dx dy, A = \{(x,y) \in \mathbb{R}^2 : 9 \leq x^2+y^2 \leq 25\}$

Solució.

$$I = \int_0^{2\pi} d\theta \int_3^5 \sqrt{r^2-9} r dr = \frac{2\pi}{3} (r^2-9)^{3/2} \Big|_3^5 = \frac{2\pi}{3} 4^3 = \boxed{\frac{128\pi}{3}}$$

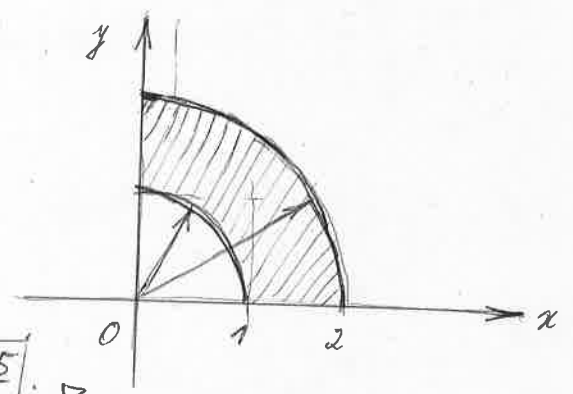
f) $I = \iint_A xy dx dy, A$ intersecció amb el 1er quadrant de la corona circular de centre (0,0), radi interior 1 i radi exterior 2

Solució.

$$I = \int_0^{\pi/2} d\theta \int_1^2 r^3 \cos\theta \sin\theta d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \sin(2\theta) d\theta \int_1^2 r^3 dr$$

$$= \left[-\frac{1}{2} \cos(2\theta) \right]_0^{\pi/2} \cdot \left[\frac{r^4}{4} \right]_1^2 = \frac{1}{2} \left(4 - \frac{1}{4} \right) = \boxed{\frac{15}{8}}$$

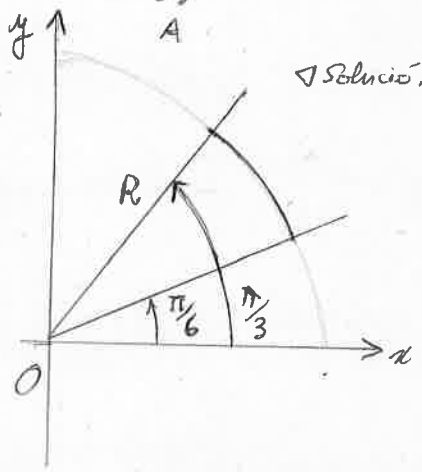


g) $I = \iint_A x(x^2+y^2) dx dy, A$ sector circular de centre (0,0) i radi R format per angles entre $\frac{\pi}{3}$ i $\frac{\pi}{6}$ amb l'eix x positiu.

Solució.

$$I = \int_{\pi/6}^{\pi/3} d\theta \int_0^R r^4 \cos\theta dr = \int_{\pi/6}^{\pi/3} \cos\theta d\theta \int_0^R r^4 dr$$

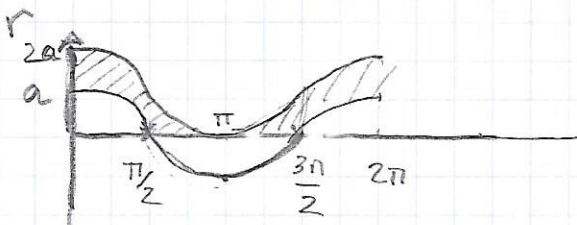
$$= \left[\sin\theta \right]_{\pi/6}^{\pi/3} \cdot \left[\frac{r^5}{5} \right]_0^R = \boxed{\frac{R^5}{10} (\sqrt{3}-1)}$$



(17) Àrees dels dominis $A \subset \mathbb{R}^2$ definits en polars per :

$$(a) \quad A = \{ (r, \theta) : a \cos \theta \leq r \leq a(1 + \cos \theta) \} \quad (a > 0)$$

Dibuxem les gràfiques de les dues funcions límit $r = a \cos \theta$ i $r = a(1 + \cos \theta)$ tenint en compte que $r \geq 0$.



Veiem que hi ha dues zones amb la mateixa àrea

$0 \leq \theta \leq \pi$ i $\pi \leq \theta \leq 2\pi$. Farem la 1^a;

la multiplicarem per 2:

$$A = 2 \int_0^{\pi} a(1 + \cos \theta)$$

$$A = 2 \left[\int_0^{\pi} \int_0^{a(1 + \cos \theta)} r \, dr - \int_0^{\pi/2} \int_0^{a \cos \theta} r \, dr \right] =$$

Hem fet la diferència entre les zones limitades per les corbes.

$$= \int_0^{\pi} a^2 (1 + \cos \theta)^2 \, d\theta - \int_0^{\pi/2} a^2 \cos^2 \theta \, d\theta =$$

$$= a^2 \left[\int_0^{\pi} (1 + 2\cos \theta + \cos^2 \theta) \, d\theta - \int_0^{\pi/2} \cos^2 \theta \, d\theta \right] =$$

$$= a^2 \left(\pi + \int_{\pi/2}^{\pi} \frac{1 + \cos 2\theta}{2} \, d\theta \right) = a^2 \left(\pi + \frac{\pi}{4} \right) = \frac{5\pi}{4} a^2$$

* Hem usat que $\int_0^{\pi} \cos \theta \, d\theta = 0$, $\int_{\pi/2}^{\pi} \cos 2\theta \, d\theta = 0$ i

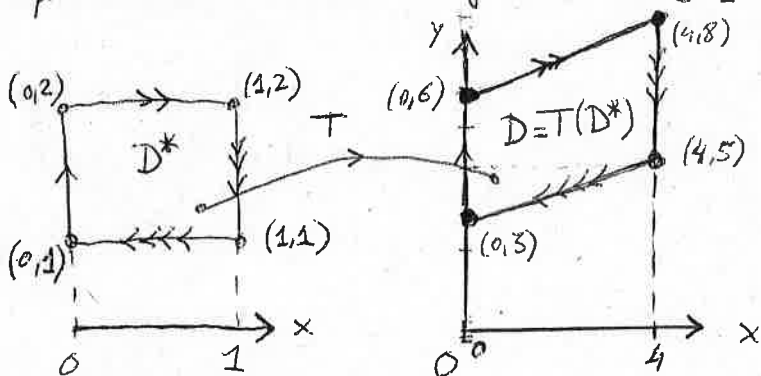
$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

Es poden veure en el dibuix o calcular-les

18. Calculeu les integrals següents mitjançant el canvi de variable que s'indica a cada cas

(a) $I = \iint_D xy \, dx \, dy$, $D = \{(x,y) \in \mathbb{R}^2; 6 \leq 2y-x, 0 \leq x \leq 4\}$, fent $x=4u$ i $y=2u+3v$.

◁ Solució. $(x(u,v), y(u,v)) = T(u,v) = (4u, 2u+3v)$ és una aplicació afí al pla. Es comprova que la seva inversa $(u(x,y), v(x,y)) = T^{-1}(x,y) = (x/4, \frac{1}{3}(y-x/2))$ transforma el quadrilàter D en el rectangle $D^* = [0,1] \times [1,2]$ (veure figura).

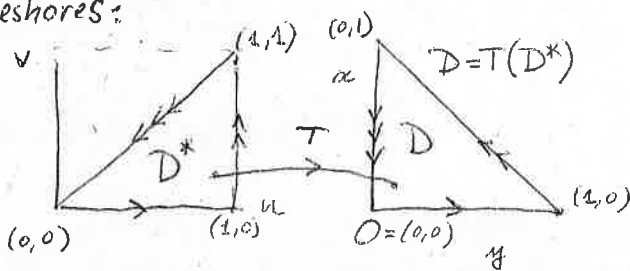


$$\begin{aligned} I &= \iint_{D=T(D^*)} xy \, dx \, dy = \iint_{D^*} x(u,v) \cdot y(u,v) \left| \det J_T(u,v) \right| \, du \, dv = \iint_{D^*} 4u \left| \det \begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix} \right| \, du \, dv \\ &= \int_0^1 du \int_1^2 4u(2u+3v) \cdot 12 \, dv = 48 \int_0^1 \left(2u^2v + \frac{3}{2}uv^2 \right) \Big|_{v=1}^{v=2} du \\ &= 48 \int_0^1 \left(2u^2 + \frac{9}{2}u \right) du = 48 \left(\frac{2}{3}u^3 + \frac{9}{4}u^2 \right) \Big|_{u=0}^{u=1} = 48 \left(\frac{2}{3} + \frac{9}{4} \right) \\ &= \frac{48}{12} (8+27) = 4 \cdot 35 = \boxed{140} \cdot \triangleright \end{aligned}$$

(b) $I = \iint_D \frac{dx \, dy}{(1+x+y)^5}$, $D = \{(x,y) \in \mathbb{R}^2; x \geq 0, y \geq 0, x+y \leq 1\}$, fent $u=x+y$, $v=y$.

◁ Solució. La transformació $(u,v) = f(x,y) = (x+y, y)$ és una transformació afí del pla, per tant porta el triangle D de vèrtexs $(0,0)$, $(1,0)$ i $(0,1)$ en el triangle D^* de vèrtexs $(0,0)$, $(1,0)$ i $(1,1)$. Sigui $T(u,v) = f^{-1}(u,v) = (u-v, v)$. Aleshores:

$$\begin{aligned} I &= \iint_{D=T(D^*)} \frac{dx \, dy}{(1+x+y)^5} = \iint_{D^*} \frac{1}{(1+u)^5} \left| \det J_T(u,v) \right| \, du \, dv \\ &= \int_0^1 \frac{du}{(1+u)^5} \int_0^u dv = \int_0^1 \frac{1+u}{(1+u)^5} du - \int_0^1 \frac{du}{(1+u)^5} \\ &= \int_0^1 \frac{du}{(1+u)^4} - \int_0^1 \frac{du}{(1+u)^5} = -\frac{1}{3}(1+u)^{-3} + \frac{1}{4}(1+u)^{-4} \Big|_{u=0}^{u=1} = -\frac{1}{3} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{1}{16} + \frac{1}{3} - \frac{1}{4} = \boxed{\frac{11}{192}} \cdot \triangleright \end{aligned}$$



$$18a) \quad I = \int_D xy \, dx \, dy, \quad D = \{(x,y): 6 \leq 2y-x \leq 12, 0 \leq x \leq 4\}$$

amb el canvi $x=4u, y=2u+3v$

$$T(u,v) = (x,y) = (4u, 2u+3v)$$

$$JT = \begin{vmatrix} 4 & 0 \\ 2 & 3 \end{vmatrix} = 12$$

D^* (D en les varzables (u,v)) quedarà

$$6 \leq 2(2u+3v) - 4u \leq 12, 0 \leq u \leq 1$$

és a dir: $2 \geq v \geq 1, 0 \leq u \leq 1$ que és un rectangle

$$18b) \quad J = \int_D \frac{dx \, dy}{(1+x+y)^5}; \quad D = \{(x,y): x \geq 0, y \geq 0, x+y \leq 1\}$$

amb el canvi $u=x+y, v=y$

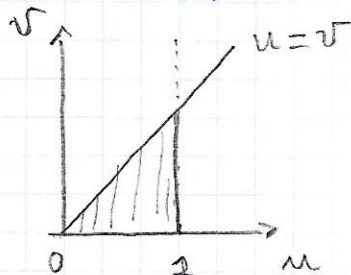
Fixeu-vos que ens donen la inversa de $T(u,v)=(x,y)$, és a dir, $T^{-1}(x,y) = (u,v) = (x+y, y)$.

En aquest cas podem fer fàcilment $T(u,v) = (x,y) = (u-v, v)$

i trobau el jacobí $JT = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$ però si no fos

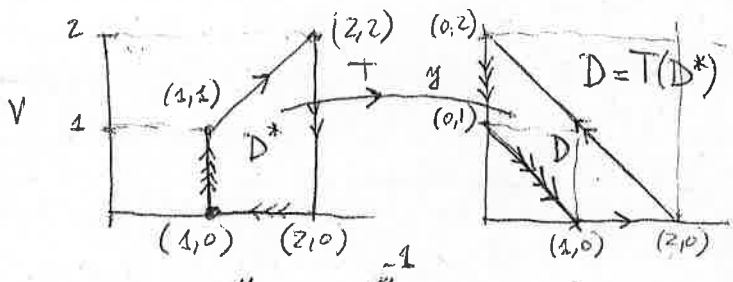
fàcil trobar T podem usar que $J(T^{-1}) = \frac{1}{JT}$.

D^* quedarà $u \geq v, v \geq 0, u \leq 1$



(c) $\iint_D \frac{dx dy}{(x+y)^{m+1}}$, $D = \{(x,y) \in \mathbb{R}^2; 1 \leq x+y \leq 2, x \geq 0, y \geq 0\}$, fent $u=x+y, v=x$.

◁ Solució. La transformació afí del pla $(u,v) = f(x,y) = (x+y, x)$ transforma el quadrilàter D en el quadrilàter D^* de vèrtexs $(u,v) = (1,0), (1,1), (2,2), (2,0)$ (veure figura).



Signifi: $T(u,v) = f^{-1}(u,v) = (v, u-v)$. Aleshores $D = T(D^*), i^*$

$$I = \iint_{D^*} \frac{dx dy}{(x+y)^{m+1}} = \iint_{D^*} \frac{1}{u^{m+1}} \left| \det J_{T^{-1}} \right| du dv$$

$$= \iint_{D^*} \frac{1}{u^{m+1}} \left| \det \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \right| du dv = \int_1^2 \frac{du}{u^{m+1}} \int_0^u dv = \int_1^2 \frac{du}{u^m} = \frac{1}{1-m} u^{1-m} \Big|_1^2 = \frac{1}{1-m} (2^{1-m} - 1)$$

$$= \frac{1}{m-1} \left(1 - \frac{1}{2^{m+1}} \right) \cdot \Delta$$

(e) $I = \iint_D \arctan\left(\frac{x^2+y^2}{2}\right) dx dy$, $D = \{(x,y) \in \mathbb{R}^2; x^2+y^2 \leq 1, x \geq 0, y \geq 0\}$, fent $x=r \cos \theta$, $y=r \sqrt{2} \sin \theta$.

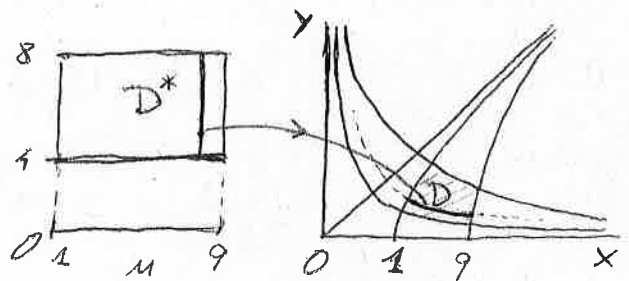
◁ Solució. $0 \leq x^2+y^2 = r^2 \leq 1$. La transformació $x=r \cos \theta, y=r \sqrt{2} \sin \theta$ porta la regió $x^2+y^2 \leq 1$ a disc $\bar{B}_1(0,0)$, amb centre l'origen i radi $r=1$. Com que d'altra banda $x \geq 0, y \geq 0$ en D , llavors $0 \leq \theta \leq \pi/2$, amb la qual cosa, en coordenades polars, el domini D s'expressa com $D^*: 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2$. Aleshores:

$$I = \iint_{D^*} \arctan\left(\frac{x^2+y^2}{2}\right) dx dy = \iint_{D^*} \arctan(r^2) \left| \det J_{T^{-1}} \right| dr d\theta = \iint_{D^*} \arctan(r^2) \left| \begin{matrix} \cos \theta & -\sin \theta \\ \sqrt{2} \sin \theta & -\sqrt{2} \cos \theta \end{matrix} \right| dr d\theta$$

$$= \sqrt{2} \int_0^{\pi/2} d\theta \int_0^1 \arctan(r^2) dr = \frac{\pi \sqrt{2}}{2} \left(\frac{r^2}{2} \arctan(r^2) \Big|_{r=0}^{r=1} - \int_0^1 \frac{r^3}{1+r^4} dr \right) = \frac{\pi \sqrt{2}}{2} \left(\frac{\pi}{8} - \frac{1}{4} \ln 2 \right) = \frac{\sqrt{2} \pi}{8} \left(\frac{\pi}{2} - \ln 2 \right) \cdot \Delta$$

(f) $I = \iint_D (x^2+y^2) dx dy$, $D = \{(x,y) \in \mathbb{R}^2; 1 \leq x^2-y^2 \leq 9, 2 \leq xy \leq 4, x \geq 0, y \geq 0\}$, fent $u=x^2-y^2, v=2xy$.

◁ Solució. Signifi $(u,v) = f(x,y) = (x^2-y^2, 2xy)$. Aquesta transformació estableix un difeomorfisme entre la regió $D \subseteq \mathbb{R}^2$ i el rectangle

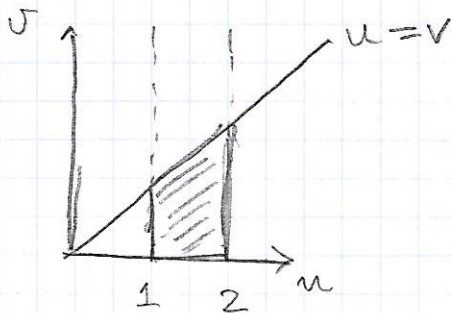


$$18c) I = \int \frac{dx dy}{(x+y)^{n+1}}, \quad D = \{(x,y) : 1 \leq x+y \leq 2, x \geq 0, y \geq 0\}$$

fent el canvi $T(x,y) = (u,v) = (x+y, x)$

$$JT = \frac{1}{JT^{-1}} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}^{-1} = -1 \rightarrow |JT| = 1$$

D^* quedaria: $1 \leq u \leq 2, v \geq 0, u-v \geq 0$



$D^* = [1,9] \times [4,8]$. Signi $T = f^{-1}$ l'aplicació inversa. Aleshores escrivim $D = T(D^*)$.

Nota: signi $(x,y) \in D$. Llavors: $1 \leq u = x^2 - y^2 \leq 9$, $4 \leq v = xy \leq 8$, $x > 0$, $y > 0$. Per tant: $D \cong T^{-1}(D^*)$

D'altra banda, per a cada $(u,v) \in D^* = [1,9] \times [4,8] \exists! (x,y) \in D$ t.q. $T^{-1}(x,y) = (u,v)$; la qual cosa implica també que $T^{-1}(D) \cong D^*$ i, per tant, $D^* = T^{-1}(D)$. $D^* = T^{-1}(D)$. Es comprova, fent els càlculs, que

$$x(u,v) = \sqrt{\frac{u + \sqrt{u^2 + 4v^2}}{2}}, \quad y(u,v) = \frac{\sqrt{2}v}{\sqrt{u + \sqrt{u^2 + 4v^2}}}, \quad (u,v) \in [1,9] \times [4,8]$$

Així:

$$I = \iint_D (x^2 + y^2) dx dy = \iint_{D^*} (x^2(u,v) + y^2(u,v)) |\det_{T^{-1}}(u,v)| du dv$$

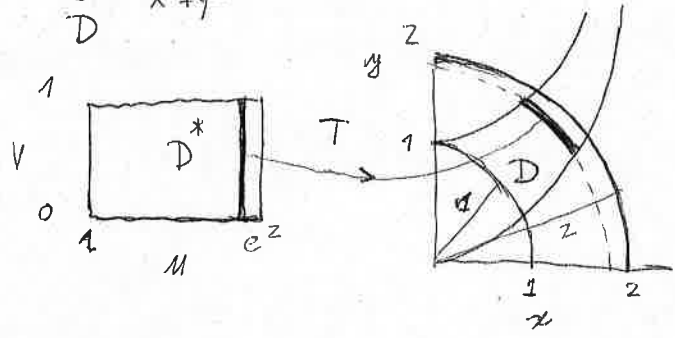
$$D = T(D^*) \quad D^* = [1,9] \times [4,8]$$

$$= \iint_{D^*} (x^2(u,v) + y^2(u,v)) \frac{du dv}{|\det J_{T^{-1}}(x(u,v), y(u,v))|} = \iint_{D^*} (x^2(u,v) + y^2(u,v)) \frac{du dv}{|\det \begin{pmatrix} 2x(u,v) & -2y(u,v) \\ 2y(u,v) & 2x(u,v) \end{pmatrix}|}$$

$$= \iint_{D^*} \frac{x^2(u,v) + y^2(u,v)}{4(x^2(u,v) + y^2(u,v))} du dv = \frac{1}{4} \iint_{D^*} du dv = \frac{1}{4} (9-1) \cdot (8-4) = \boxed{8} \cdot \Delta$$

$$D^* = [1,9] \times [4,8] \quad D^* = [1,9] \times [4,8]$$

(g) $I = \iint_D \frac{x+2xy}{x^2+y^2} dx dy$, $D = \{(x,y) \in \mathbb{R}^2 : x^2 \leq y \leq x^2+1, 1 \leq x^2+y^2 \leq e^2, x \geq 0\}$, fent $u = x^2+y^2$, $v = y-x^2$.



◁ Solució. L'aplicació $(u,v) = f(x,y) = (x^2+y^2, y-x^2)$ estableix un difeomorfisme de la regió D en el rectangle $D^* = [1, e^2] \times [0, 1]$. Signi $f^{-1} = T$. Escrivim $D = T(D^*)$. Aleshores:

$$I = \iint_{D^*} \frac{x(u,v) + 2x(u,v)y(u,v)}{x^2(u,v) + y^2(u,v)} |\det J_T(u,v)| du dv$$

$$= \iint_{D^*} \frac{x(u,v) + 2x(u,v)y(u,v)}{x^2(u,v) + y^2(u,v)} \frac{du dv}{|\det J_{T^{-1}}(x(u,v), y(u,v))|} = \iint_{D^*} \frac{x(u,v) + 2x(u,v)y(u,v)}{x^2(u,v) + y^2(u,v)} \frac{du dv}{|\det \begin{pmatrix} 2x(u,v) & 2y(u,v) \\ -2x(u,v) & 1 \end{pmatrix}|}$$

$$= \iint_{D^*} \frac{x(u,v) + 2x(u,v)y(u,v)}{x^2(u,v) + y^2(u,v)} \frac{du dv}{2(x(u,v) + 2x(u,v)y(u,v))} = \frac{1}{2} \iint_{D^*} \frac{du dv}{u} = \frac{1}{2} \int_1^{e^2} \frac{du}{u} \int_0^1 dv$$

$$= \frac{1}{2} \ln u \Big|_{u=1}^{u=e^2} = \boxed{1} \cdot \Delta$$

Examen final. Gener 2019

1 [2 punts] Troben el valor de les integrals dobles següents als dominis que s'indiquen

(a) $\iint_C \frac{|y|}{x^2+y^2} dx dy$. $C = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 2, x^2+y^2 \leq 2x\}$. [1 punt]

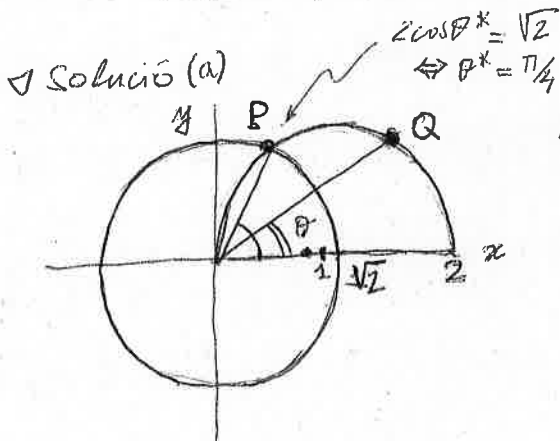
(b) $\iint_C \frac{xy(x^2+y^2)}{xy+1} (x^2-y^2)^{xy-1} dx dy$. $C = \{(x,y) \in \mathbb{R}^2 : y \geq 0, x \geq y, xy \leq 1, x^2-y^2 \leq 1\}$.

En aquest cas, considereu el canvi $u=x^2-y^2, v=xy$. Llavors:

(i) Calculeu el determinant de la seva matriu Jacobiana (0.25 punts)

(ii) Expressen el domini C en les noves variables (u,v) (0.25 punts)

(iii) Finalment, doneu el valor de la integral (0.5 punts)



En coordenades polars, el domini C ve determinat per

$C' : \sqrt{2} \leq r \leq 2 \cos \theta, 0 \leq \theta \leq \frac{\pi}{4}$. Amb això:

$$I = \iint_C \frac{|y|}{x^2+y^2} dx dy = \int_{-\pi/4}^{\pi/4} d\theta \int_{\sqrt{2}}^{2 \cos \theta} \frac{r |\sin \theta|}{r^2} r dr$$

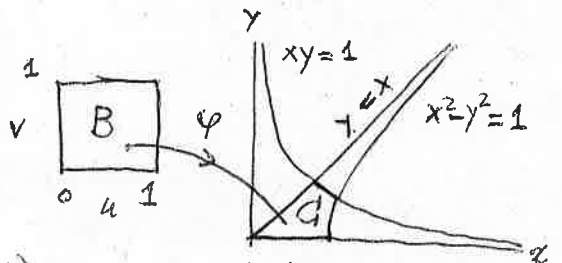
$$= \int_{-\pi/4}^{\pi/4} |\sin \theta| d\theta \int_{\sqrt{2}}^{2 \cos \theta} dr = \int_{-\pi/4}^{\pi/4} |\sin \theta| (2 \cos \theta - \sqrt{2}) d\theta$$

$$= 2 \int_0^{\pi/4} (2 \cos \theta \sin \theta - \sqrt{2} \sin \theta) d\theta = 2 \int_0^{\pi/4} \sin(2\theta) d\theta - 2\sqrt{2} \int_0^{\pi/4} \sin \theta d\theta = (-\cos(2\theta) + 2\sqrt{2} \cos \theta) \Big|_0^{\pi/4}$$

$$= 1 - \cos\left(\frac{\pi}{2}\right) + 2\sqrt{2} (\cos\left(\frac{\pi}{4}\right) - 1) = 1 - 0 + 2\sqrt{2} \left(\frac{1}{\sqrt{2}} - 1\right) = 1 + 2 - 2\sqrt{2} = \boxed{3 - 2\sqrt{2}} \cdot \Delta$$

◁ Solució (b).

Es comprova que $(u,v) = \varphi^{-1}(x,y) = (xy, x^2-y^2)$ estableix un difeomorfisme entre la regió $C \subseteq \mathbb{R}^2$ i el rectangle $B = [0,1]^2 = [0,1] \times [0,1]$. Aplicant aquest canvi tenim:



$$I = \iint_{C = \varphi(B)} \frac{xy(x^2+y^2)}{xy+1} (x^2-y^2)^{xy-1} dx dy = \iint_{B = [0,1]^2} \frac{v \cdot (x^2(u,v) + y^2(u,v))}{v+1} u^{v-1} \frac{du dv}{|\det J_{\varphi^{-1}}(x(u,v), y(u,v))|} =$$

$$= \iint_{B = [0,1]^2} \frac{v(x^2(u,v) + y^2(u,v))}{v+1} \cdot u^{v-1} \frac{du dv}{\begin{vmatrix} 2x(u,v) & -2y(u,v) \\ y(u,v) & x(u,v) \end{vmatrix}} = \frac{1}{2} \iint_{B = [0,1]^2} \frac{v(x^2(u,v) + y^2(u,v))}{v+1} \frac{u^{v-1} du dv}{x^2(u,v) + y^2(u,v)}$$

$$= \frac{1}{2} \int_0^1 \frac{dv}{v+1} \int_0^1 u^{v-1} du = \frac{1}{2} \int_0^1 \frac{v}{v+1} \left[\frac{u^v}{v} \right]_{u=0}^{u=1} dv = \frac{1}{2} \int_0^1 \frac{dv}{1+v} = \frac{1}{2} \ln(1+v) \Big|_{v=0}^{v=1} = \frac{\ln 2}{2} \cdot \Delta$$

20. Useu coordenades esfèriques per calcular les següents integrals triples

(a) $I = \iiint_B x^2 y^2 z^3 dx dy dz, B = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2\} = B_a(0,0,0) (a > 0)$

◁ Solució. $I = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \cos^4 \theta \sin^2 \theta d\theta \int_{-\pi/2}^{\pi/2} \cos^2 \varphi \sin^3 \varphi d\varphi \int_0^a r^{11} dr = 0. \triangleright$

(b) $I = \iiint_B z(x^2 + y^2) dx dy dz, B = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2, z \geq 0\}$

◁ Solució. $I = \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos^3 \varphi \sin \varphi \int_0^a r^5 dr = 2\pi \left[-\frac{\cos^4 \varphi}{4} \right]_0^{\pi/2} \cdot \left[\frac{r^6}{6} \right]_0^a = \frac{\pi a^6}{12}. \triangleright$

(c) $I = \iiint_B \frac{dx dy dz}{(x^2 + y^2 + z^2)^{3/2}}, B = \{(x,y,z) \in \mathbb{R}^3 : a^2 \leq x^2 + y^2 + z^2 \leq b^2\}$

◁ Solució. $I = \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos \varphi d\varphi \int_a^b \frac{r^2 dr}{r^3} = 4\pi \ln\left(\frac{b}{a}\right). \triangleright$

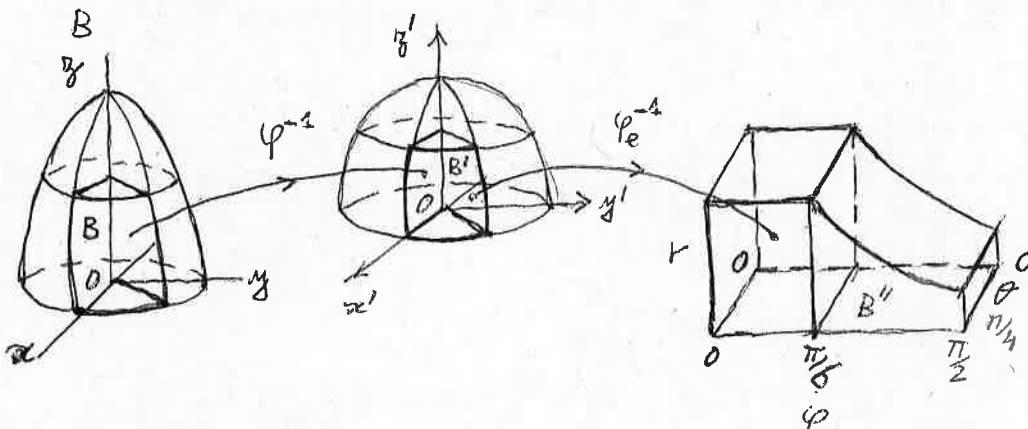
(d) $I = \iiint_B \sqrt{x^2 + y^2 + z^2} e^{-(x^2 + y^2 + z^2)} dx dy dz, B$ el domini de l'apartat anterior.

◁ Solució. $I = \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos \varphi d\varphi \int_a^b r^3 e^{-r^2} dr = 4\pi \int_a^b r^3 e^{-r^2} dr = 2\pi (e^{-a^2}(a^2 + 1) - e^{-b^2}(b^2 + 1)). \triangleright$

(*) Parts: $\int_a^b r^3 e^{-r^2} dr = -\frac{1}{2} \int_a^b r^2 d(e^{-r^2}) = \left[-\frac{r^2 e^{-r^2}}{2} \right]_a^b + \int_a^b r e^{-r^2} dr = \left[-\frac{1}{2} e^{-r^2} \right]_a^b$
 $= \frac{a^2 e^{-a^2}}{2} - \frac{b^2 e^{-b^2}}{2} + \frac{e^{-a^2}}{2} - \frac{e^{-b^2}}{2} = \frac{1}{2} (a^2 + 1) e^{-a^2} - \frac{1}{2} (b^2 + 1) e^{-b^2}$

22. Adapten les coordenades esfèriques per calcular les següents integrals triples

(a) $\iiint_B 16z dx dy dz, B = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, 0 \leq z \leq 1, 0 \leq y \leq x\}$



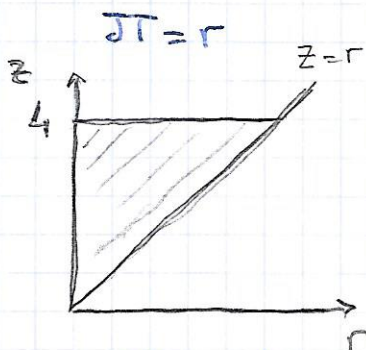
- $B = \varphi(B')$ amb $x = \varphi_1(x',y',z') = x', y = \varphi_2(x',y',z') = y', z = \varphi_3(x',y',z') = zz'$. Llavors $B' = \{(x',y',z') \in \mathbb{R}^3 : (x')^2 + (y')^2 + (z')^2 \leq 1, 0 \leq z' \leq 1/2, 0 \leq y' \leq x'\}$.

$I = \iiint_{B=\varphi(B')} 16z dx dy dz = \iiint_{B'} 32z' |\det J_\varphi(x',y',z')| dx' dy' dz' = 64 \iiint_{B'} z' dx' dy' dz'$

(19) Useu coordenades cilíndriques per calcular:

$$(a) I = \int_B \sqrt{x^2 + y^2 + z^2} dx dy dz, \quad B = \{ \sqrt{x^2 + y^2} \leq z \leq 4 \}$$

$$T \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad B \begin{cases} r \leq z \leq 4 \\ (0 \leq \theta \leq 2\pi) \end{cases}$$



Secció: $\theta = \text{constant}$ $\begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq 4 \end{cases}$

Per fer la iteració: $\begin{cases} 0 \leq r \leq z \end{cases}$

$$\begin{aligned} I &= \int_0^{2\pi} d\theta \int_0^4 dz \int_0^z \underbrace{\sqrt{r^2 + z^2}}_{\text{funció}} \cdot \underbrace{r}_{|\text{Jacobin}|} dr = \\ &= 2\pi \int_0^4 \left[(r^2 + z^2)^{3/2} \right]_0^z \frac{2}{3} \frac{1}{2} dz = \frac{2\pi}{3} \int_0^4 (2 \cdot z^{3/2} - z^3) dz = \\ &= \frac{2\pi}{3} (2\sqrt{2} - 1) \cdot \left[\frac{z^4}{4} \right]_0^4 = \frac{128\pi}{3} (2\sqrt{2} - 1) \end{aligned}$$

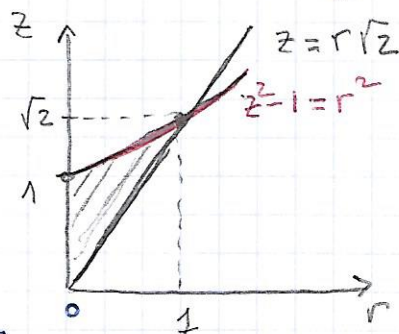
$$(b) I = \int_B z e^{-(x^2 + y^2)} dx dy dz, \quad B = \left\{ \frac{z^2}{2} - 1 \leq x^2 + y^2 \leq \frac{z^2}{2}, z \geq 0 \right\}$$

$$B \begin{cases} \frac{z^2}{2} - 1 \leq r^2 \leq \frac{z^2}{2} \\ z \geq 0 \quad (0 \leq \theta \leq 2\pi) \end{cases}$$

$$r^2 = \frac{z^2}{2} \rightarrow r = \pm \frac{\sqrt{z}}{\sqrt{2}} \rightarrow r = \frac{\sqrt{z}}{\sqrt{2}}$$

$$\text{Tall } \left\{ \begin{array}{l} r^2 = \frac{z^2}{2} - 1 \\ r^2 = \frac{z^2}{2} \end{array} \right\} \rightarrow z = \pm \sqrt{2} \rightarrow z = \sqrt{2} \quad (z \geq 0) \quad r = 1 \quad (\text{mirem verticalment per iterar})$$

$$I = \int_0^{2\pi} d\theta \int_0^1 r dr \int_{\frac{r^2}{\sqrt{2}}}^{\sqrt{r^2+1}} \underbrace{z e^{-r^2}}_{\text{funció}} dz = 2\pi \int_0^1 r e^{-r^2} \frac{1}{2} (r^2 + 1 - 2r^2) dr$$



$$= \cancel{2\pi} \int_0^1 r e^{-r^2} (1-r^2) dr = \frac{\pi}{2} \int_0^1 e^{-t} (1-t) dt =$$

$$\begin{array}{l} \downarrow \\ t=r^2 \quad | \quad t=1 \text{ si } r=1 \\ dt=2rdr \quad | \quad t=0 \text{ si } r=0 \end{array}$$

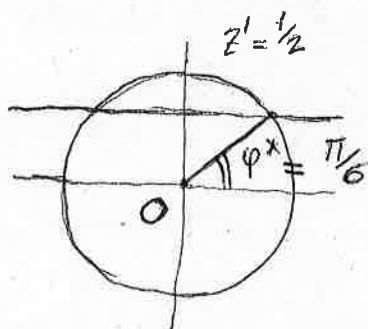
for parts $\left\{ \begin{array}{l} u=1-t \\ v'=e^{-t} \end{array} \right\} \rightarrow \begin{array}{l} u'=-1 \\ v=-e^{-t} \end{array}$

$$= \frac{\pi}{2} \left(\left[(1-t)e^{-t} \right]_1^0 - \int_0^1 e^{-t} dt \right) = \frac{\pi}{2} \left(e^{-0} + \left[e^{-t} \right]_0^1 \right) =$$

$$= \frac{\pi}{2} (2e^{-1} - 1)$$

A continuació canviem a coord. esfèriques

$$\begin{cases} x' = r \cos \theta \cos \varphi \\ y' = r \sin \theta \cos \varphi \\ z' = r \sin \varphi \end{cases} \left\{ \begin{array}{l} r=1, z=1/2 \\ \sin \varphi^* = 1/2 \text{ amb } 0 \leq \varphi^* \leq \pi/2 \\ \Leftrightarrow \varphi^* = \pi/6 \end{array} \right\}$$



$$B' = \varphi_e(B''), \quad B'' = \left\{ (r, \theta, \varphi) \in \mathbb{R}^+ \times [0, 2\pi) \times (-\pi/2, \pi/2) : 0 \leq \theta \leq \pi/4, 0 \leq \varphi \leq \pi/6, 0 < r \leq 1 \right\} \\ \cup \left\{ (r, \theta, \varphi) \in \mathbb{R}^+ \times [0, 2\pi) \times (-\pi/2, \pi/2) : 0 \leq \theta \leq \pi/4, \pi/2 \leq \varphi \leq \pi/6, 0 < r \leq \frac{1/2}{\sin \varphi} \right\}$$

$$I = 64 \int_0^{\pi/4} d\theta \int_0^{\pi/6} \cos \varphi \sin \varphi d\varphi \int_0^1 r^3 dr + 64 \int_0^{\pi/4} d\theta \int_{\pi/6}^{\pi/2} \sin \varphi \cos \varphi d\varphi \int_{\frac{1/2}{\sin \varphi}}^{1/2} r^3 dr \\ = 64 \cdot \frac{\pi}{4} \cdot \frac{1}{8} \sin^2\left(\frac{\pi}{6}\right) + \frac{\pi}{4} \cdot \frac{64}{64} \int_{\pi/6}^{\pi/2} \frac{\cos \varphi d\varphi}{\sin^3 \varphi} = \frac{\pi}{2} + \frac{\pi}{4} \left(\frac{-1}{\sin^2 \varphi} \right) \Big|_{\pi/6}^{\pi/2} = \frac{\pi}{2} + \frac{\pi}{4} (4-1) = \boxed{\frac{7\pi}{8}} \quad \triangleleft$$

$$(6) \iiint_B \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dx dy dz, \quad B = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

B

△ Solució. Apliquem coordenades esfèriques adaptades $\begin{cases} x = ar \cos \theta \cos \varphi \\ y = br \sin \theta \cos \varphi \\ z = cr \sin \varphi \end{cases}$. En aquestes coord. denades el domini B es transforma en:

$$B': 0 < r \leq 1, -\pi/2 \leq \varphi \leq \pi/2, 0 \leq \theta \leq 2\pi.$$

$$I = \iiint_{B'} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dx dy dz = \iiint_{B'} r^2 \left| \det J_{\varphi_e}(r, \theta, \varphi) \right| dr d\theta d\varphi \\ B = \varphi_e(B') \\ = abc \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos \varphi d\varphi \int_0^1 r^4 dr = abc (2\pi) 2 \left(\frac{r^5}{5} \right) \Big|_{r=0}^{r=1} = \boxed{\frac{4}{5} \pi abc} \quad \triangleleft$$

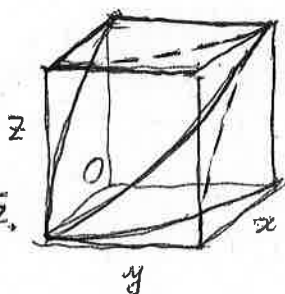
(8) En coordenades cilíndriques:

$$I = \int_0^{\pi/4} d\theta \int_0^1 16z dz \int_0^{\sqrt{1-z^2/4}} r dr = 16 \frac{\pi}{8} \int_0^1 z \left(1 - \frac{z^2}{4} \right) dz = 16 \cdot \frac{\pi}{8} \left(\frac{z^2}{2} - \frac{z^4}{16} \right) \Big|_{z=0}^{z=1} \\ = 2\pi \cdot \frac{7}{16} = \boxed{\frac{7\pi}{8}}.$$

... Veiem que el càlcul és més "directe".

23. Usen coordenades cartesianes, cilíndriques o esfèriques (o bé el principi de Cavalieri) per calcular el volum dels dominis de \mathbb{R}^3 limitats per les superfícies que s'indiquen.

a) $x^2 + z^2 = 1, x^2 + y^2 = 1,$



◁ Solució:

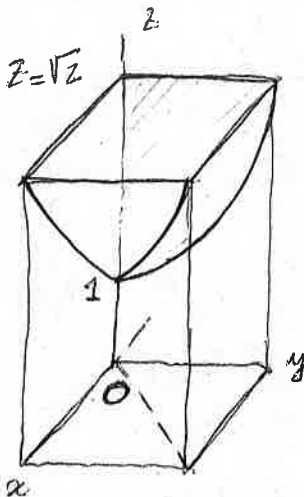
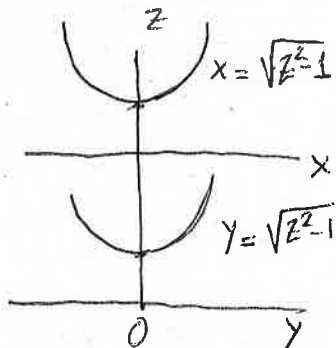
$$V_{\frac{1}{8}} = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2}} dz = \int_0^1 dx \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2} dy$$

$$= \int_0^1 (1-x^2) dx = \left(x - \frac{x^3}{3} \right) \Big|_{x=0}^{x=1}$$

$$= 1 - \frac{1}{3} = \frac{2}{3} \Rightarrow \boxed{V = \frac{16}{3}} \cdot \Delta$$

b) $z^2 - x^2 = 1, z^2 - y^2 = 1, z = \sqrt{2}.$

◁ Solució:



$$V_{\frac{1}{4}} = \int_1^{\sqrt{2}} dz \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dy \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dx = \int_1^{\sqrt{2}} dz \int_0^{\sqrt{1-z^2}} \sqrt{1-z^2} dx = \int_1^{\sqrt{2}} (1-z^2) dz = \left(z - \frac{z^3}{3} \right) \Big|_{z=1}^{z=\sqrt{2}}$$

$$= \frac{2\sqrt{2}}{3} - \sqrt{2} - \frac{1}{3} + 1 = \frac{2}{3} - \frac{\sqrt{2}}{3} = \frac{1}{3}(2 - \sqrt{2}) \Leftrightarrow \boxed{V = \frac{4}{3}(2 - \sqrt{2})} \cdot \Delta$$

c) $z^2 = x^2 + y^2, z = x^2 + y^2 (z \geq 0).$

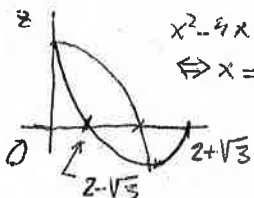
◁ Solució: $I = \int_0^{2\pi} d\theta \int_0^1 r dr \int_{r^2}^r dz = 2\pi \int_0^1 r(r-r^2) dr = 2\pi \left(\frac{r^3}{3} - \frac{r^4}{4} \right) \Big|_0^1 = 2\pi \frac{4-3}{12} = \frac{\pi}{6} \cdot \Delta$

d) Part de l'esfera $x^2 + y^2 + z^2 = a^2$ que és exterior al cilindre $x^2 + y^2 = b^2$ ($a > b > 0$).

◁ Solució: $V = \int_0^{2\pi} d\theta \int_b^a r dr \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} dz = 4\pi \int_b^a r \sqrt{a^2-r^2} dr = -\frac{4\pi}{3} (a^2-r^2)^{3/2} \Big|_b^a = \frac{4\pi}{3} (a^2-b^2)^{3/2} \cdot \Delta$

e) $z = x^2 - 4x + 1, 1 - z = x^2 + y^2$

◁ Solució:



$$x^2 - 4x + 1 = 0$$

$$\Leftrightarrow x = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

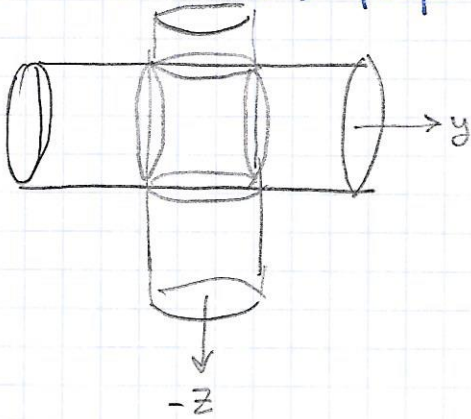
$$z = 1 - x^2 - y^2 = x^2 - 4x + 1 \Leftrightarrow 2x^2 - 4x + y^2 = 2(x-1)^2 + y^2 - z = 0$$

$$\Leftrightarrow (x-1)^2 + \frac{y^2}{2} = \frac{z}{2}$$

23) Calculeu el volum del cos limitat per

(a) $x^2 + z^2 = 1$, $x^2 + y^2 = 1$

Fixeu-vos que és el volum tancat per la intersecció de dos cilindres perpendiculars.



Com que prendre seccions $x = \text{constant}$ és el que simplifica millor les dues superfícies, és el que fem:

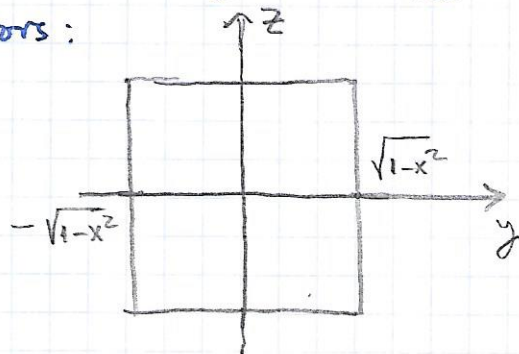
Secció $x = \text{constant}$:

$$\begin{cases} z = \pm \sqrt{1-x^2} & \text{parell de rectes (1)} \\ y = \pm \sqrt{1-x^2} & \text{" (2)} \end{cases}$$

existiran sempre que (1) i (2) tinguin sentit, és a dir:

$$\boxed{-1 \leq x \leq 1}$$

llavors:



la secció serà un quadrat de costats $2\sqrt{1-x^2}$

Per tant el volum :

$$Vol = \int_{-1}^1 (2\sqrt{1-x^2})^2 dx =$$

$$= 2 \cdot 4 \int_0^1 (1-x^2) dx = 8 \left(1 - \frac{1}{3}\right) = \frac{16}{3}$$

$$V = \iiint_D 1 \, dx \, dy \, dz = \iint_E dx \, dy \int_{x^2-4x+1}^{1-x^2-y^2} dz = 2 \iint_E [1 - (x-1)^2 - \frac{y^2}{2}] \, dx \, dy = \textcircled{8181}$$

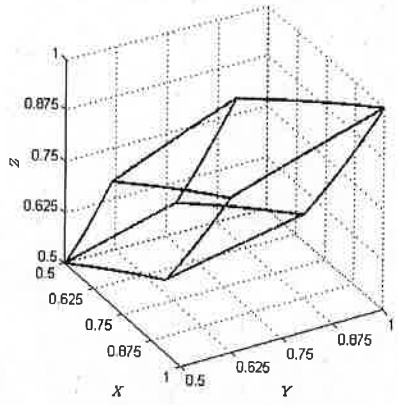
$E: (x,y) \in \mathbb{R}^2$
 $(x-1)^2 + \frac{y^2}{2} \leq 1$

Coordenades polars adaptades: $\varphi: \begin{cases} x-1 = r \cos \theta \\ y = \sqrt{2} r \sin \theta, \quad 0 \leq \theta \leq 2\pi, \quad 0 < r < 1 \end{cases}$

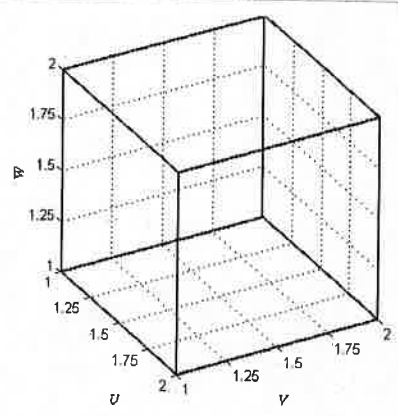
$$\det J_\varphi(r, \theta) = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sqrt{2} \sin \theta & \sqrt{2} r \cos \theta \end{vmatrix} = \sqrt{2} r.$$

$$\textcircled{8181} = 2\sqrt{2} \int_0^{2\pi} d\theta \int_0^1 (1-r^2) r \, dr = 2\sqrt{2} (2\pi) \int_0^1 (r-r^3) \, dr = 4\pi\sqrt{2} \left(\frac{r^2}{2} - \frac{r^4}{4} \right) \Big|_{r=0}^{r=1} = 4\sqrt{2} \frac{\pi}{4} = \boxed{\pi\sqrt{2}} \cdot \blacktriangleright$$

f) $x^2 = z, y^2 = x, z^2 = y, x^2 = az, y^2 = ax, z^2 = ay \ (a > 1)$.



(a) D : Domini definit per les superfícies,
 $x^2 = z, \quad y^2 = x, \quad z^2 = y,$
 $x^2 = az, \quad y^2 = ax, \quad z^2 = ay.$



(b) $D' : 1 \leq u \leq a, 1 \leq v \leq a, 1 \leq w \leq a.$

FIGURA 1. Transformació del domini D en D' pel canvi (1).

Solució: es comprova d'immediat que el canvi

$$u = \frac{y^2}{x}, \quad v = \frac{z^2}{y}, \quad w = \frac{x^2}{z}, \tag{1}$$

transforma el domini original D , en $D' = [1, a] \times [1, a] \times [1, a]$. És a dir, en un cub d'aresta $a - 1$ (veure Figura 1). Aleshores el Jacobià corresponent surt més senzill. En efecte:

$$\det \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} -\frac{y^2}{x^2} & 2\frac{y}{x} & 0 \\ 0 & -\frac{z^2}{y^2} & 2\frac{z}{y} \\ 2\frac{x}{z} & 0 & -\frac{x^2}{z^2} \end{vmatrix} = -7,$$

d'on:

$$\left| \det \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \frac{1}{\left| \det \frac{\partial(u, v, w)}{\partial(x, y, z)} \right|} = \frac{1}{7}.$$

Llavors el càlcul del volum es simplifica encara més:

$$V = \iiint_D dx \, dy \, dz = \iiint_{D'} \left| \det \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du \, dv \, dw = \frac{1}{7} \int_1^a du \int_1^a dv \int_1^a dw = \frac{1}{7} \left(\int_1^a du \right)^3 = \frac{(a-1)^3}{7} \cdot \blacktriangleright$$

g) $z^2 = y, x^2 = 1-y$

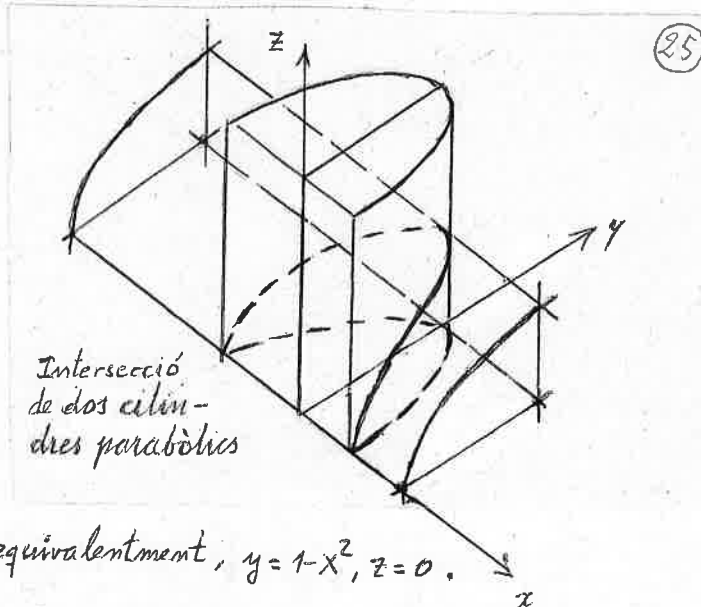
Solució.

parametrizació de la intersecció per a $x \geq 0, z \geq 0$

$$\left. \begin{aligned} x &= \sqrt{1-t} \\ y &= t \\ z &= \sqrt{t} \end{aligned} \right\} 0 \leq t \leq 1$$

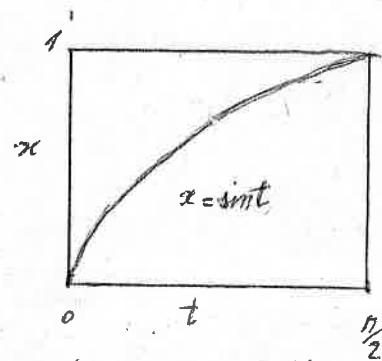
Projecció sobre el pla $z=0$:

$x = \sqrt{1-t}, y = t, z = 0$; o, equivalentment, $y = 1-x^2, z = 0$.



Aleshores:

$$\begin{aligned} V/4 &= \int_0^1 dx \int_0^{1-x^2} dy \int_0^{\sqrt{y}} dz = \int_0^1 dx \int_0^{1-x^2} dy \sqrt{y} = \frac{2}{3} \int_0^1 dx \left[y^{3/2} \right]_0^{1-x^2} \\ &= \frac{2}{3} \int_0^1 (1-x^2)^{3/2} dx = \left\{ \begin{array}{l} \text{c.v. (*)} \\ x = \sin t \\ dx = \cos t dt \end{array} \right\} = \frac{2}{3} \int_0^{\pi/2} \cos^4 t dt = \frac{\pi}{8} \Rightarrow \boxed{V = \frac{\pi}{2}} \end{aligned}$$

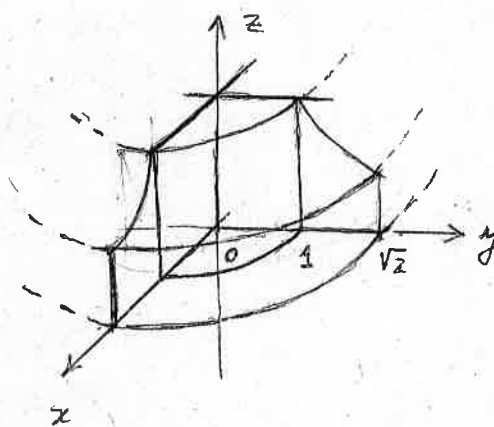


(*) Canvi de variables a la integral

h) $x^2+y^2=1, x^2+y^2=z, z(x^2+y^2)=1, z=0$.

Solució. En coordenades cilíndriques:

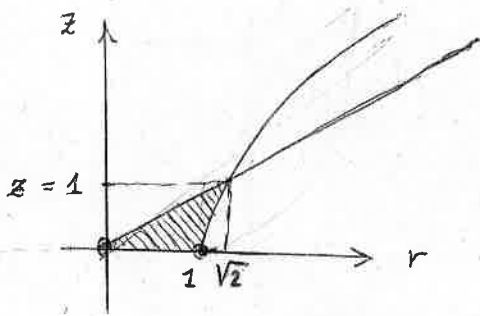
$1 \leq r \leq \sqrt{z}, 0 \leq z \leq \frac{1}{r^2}$



$$\begin{aligned} V &= \int_0^{2\pi} d\theta \int_1^{\sqrt{z}} r dr \int_0^{\frac{1}{r^2}} dz = 2\pi \int_1^{\sqrt{z}} \frac{dr}{r} \\ &= 2\pi \left[\ln r \right]_1^{\sqrt{z}} = \boxed{\pi \ln z} \end{aligned}$$

i) $x^2+y^2=z z^2, x^2+y^2=z^2+1 (x \geq 0, y \geq 0, z \geq 0)$

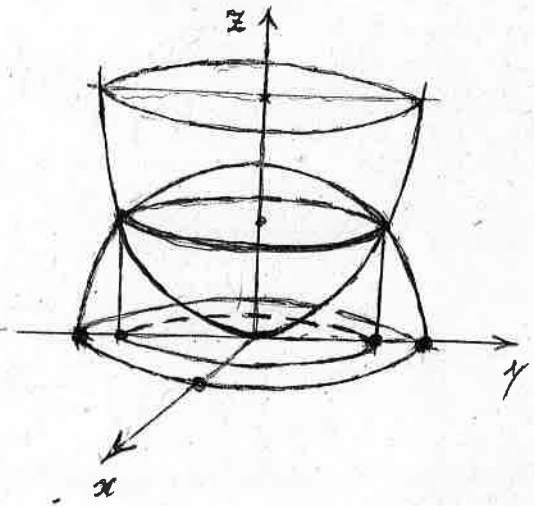
Solució Apliquem coordenades cilíndriques (pàg. següent)



$$\begin{aligned}
 V &= \int_0^{\pi/2} d\theta \int_0^1 r dr \int_0^{\sqrt{2}} dz \\
 &\quad + \int_0^{\pi/2} d\theta \int_1^{\sqrt{2}} r dr \int_{\sqrt{r^2-1}}^{\sqrt{2}} dz \\
 &= \frac{\pi}{2} \int_0^1 \frac{r^2}{\sqrt{2}} dr + \frac{\pi}{2} \int_1^{\sqrt{2}} \left(\frac{r^2}{\sqrt{2}} - r\sqrt{r^2-1} \right) dr \\
 &= \frac{\pi}{4} \sqrt{2} \left[\frac{r^3}{3} \right]_0^1 + \frac{\pi}{2} \left[\frac{\sqrt{2}}{2} \frac{r^3}{3} - \frac{1}{3} (r^2-1)^{3/2} \right]_1^{\sqrt{2}} = \frac{\sqrt{2}\pi}{12} + \frac{\pi}{2} \left[\frac{2}{3} - \frac{1}{3} - \frac{\sqrt{2}}{6} \right] = \boxed{\frac{\pi}{6}}
 \end{aligned}$$

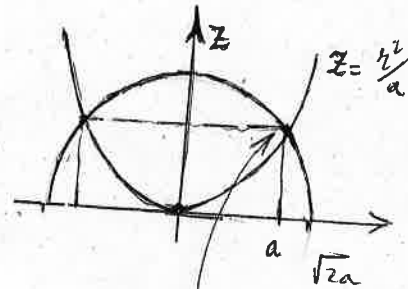
Alternativement (més 'facile'):

$$\begin{aligned}
 V &= \int_0^{\pi/2} d\theta \int_0^1 dz \int_{\sqrt{z^2+1}}^1 r dr = \frac{\pi}{4} \int_0^1 (z^2+1 - z z^2) dz = \frac{\pi}{4} \int_0^1 (1-z^2) dz \\
 &= \frac{\pi}{4} \left[z - \frac{z^3}{3} \right]_0^1 = \frac{\pi}{4} \left(1 - \frac{1}{3} \right) = \boxed{\frac{\pi}{6}}
 \end{aligned}$$



(f) $x^2 + y^2 + z^2 \leq 2a^2$, $z \geq \frac{x^2 + y^2}{a}$ ($a > 0$). En cylindriques:

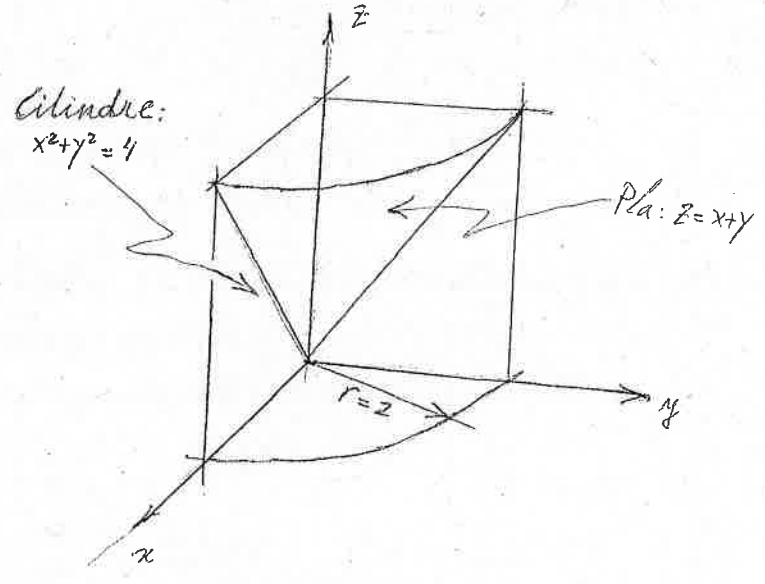
$$\begin{aligned}
 V &= \int_0^{2\pi} d\theta \int_0^a r dr \int_{\frac{r^2}{a}}^{\sqrt{2a^2-r^2}} dz = 2\pi \int_0^a \left(r\sqrt{2a^2-r^2} - \frac{r^3}{a} \right) dr \\
 &= 2\pi \left[-\frac{1}{3} (2a^2-r^2)^{3/2} - \frac{r^4}{4a} \right]_0^a \\
 &= 2\pi \left[\frac{1}{3} (2\sqrt{2}-1) - \frac{1}{4} \right] a^3 = \boxed{2\pi a^3 \left(\frac{2^{3/2}}{3} - \frac{7}{12} \right)}
 \end{aligned}$$



$$\begin{aligned}
 &az + z^2 = 2a^2 \\
 \Leftrightarrow &z^2 + az - 2a^2 = 0 \\
 \text{d'où:} & \\
 z = &\frac{-a \pm \sqrt{a^2 + 8a^2}}{2} = \\
 = &\frac{-a \pm 3a}{2} = \begin{cases} a \rightarrow r = a \\ -2a \text{ (No)} \end{cases}
 \end{aligned}$$

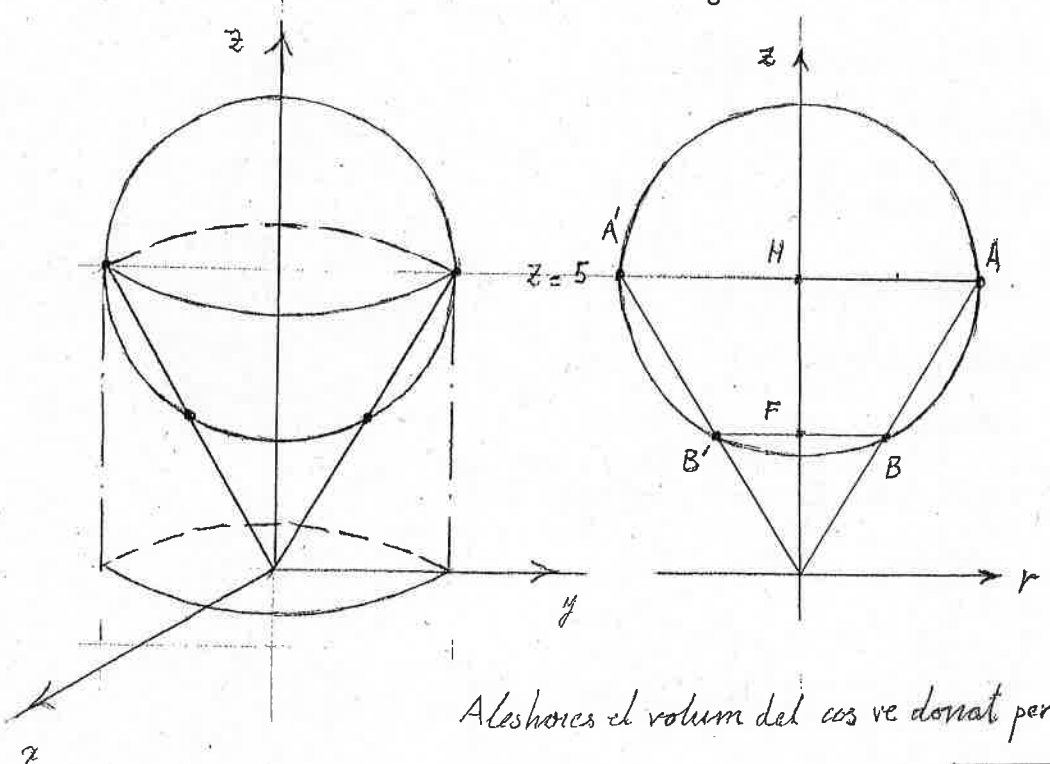
(k) $x^2 + y^2 = 4, z = x + y$ ($x \geq 0, y \geq 0, z \geq 0$)

$$\begin{aligned}
 V &= \int_0^{\pi/2} d\theta \int_0^z r dr \int_0^z r \cos\theta + r \sin\theta dz \\
 &= \int_0^{\pi/2} d\theta \int_0^z r^2 (\cos\theta + \sin\theta) dr \\
 &= \int_0^{\pi/2} (\cos\theta + \sin\theta) d\theta \int_0^z r^2 dr \\
 &= \left[\sin\theta - \cos\theta \right]_0^{\pi/2} \cdot \left[\frac{r^3}{3} \right]_0^z = \boxed{\frac{16}{3}} \quad \square
 \end{aligned}$$



Exercici: substituir $z = x + y \rightarrow z = x + y - 1$ i repetir el càlcul...

(l) Con de gelat definit per: $x^2 + y^2 \leq \frac{z^2}{5}, 0 \leq z \leq 5 + \sqrt{5 - x^2 - y^2}$



Alçada dels punts d'intersecció A, B
 A', B'

$$(z-5)^2 = 5 - \frac{z^2}{5}$$

$$\Leftrightarrow 6z^2 - 50z + 100 = 0$$

d'on, $z = 5$: alçada de A, A'
 $z = \frac{10}{3}$: " " B, B'

i els 'radis' respectivament són:

$$r = \sqrt{5} \text{ ; distància } \overline{HA} = \overline{A'H}$$

$$r = \frac{2\sqrt{5}}{3} \text{ ; " } \overline{FB} = \overline{B'F}$$

Aleshores el volum del cos ve donat per:

$$V = \frac{2}{3} \pi \sqrt{5}^3 + \frac{1}{3} \pi \sqrt{5}^2 \cdot 5 = \boxed{\left(\frac{10\sqrt{5}}{3} + \frac{25}{3} \right) \pi}$$

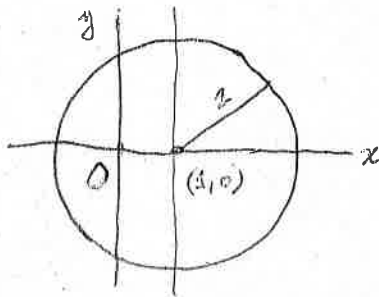
Volum de la semiesfera Volum del con.

Gemer 2017

(2a) Calculeu la integral doble $\iint_D (x^2+y^2) dx dy$ essent D -en coordenades cartesianes- la regió de la circumferència de radi 2 centrada en el punt $(x,y)=(1,0)$ i delimitada per les desigualtats $x>1, y>0$ [1.5 punts].

(2b) Calculeu el volum del cos determinat per la regió que està sobre el con $z=\sqrt{\frac{x^2+y^2}{3}}$ i dintre de l'esfera de radi $\frac{1}{2}$ centrada en $(x,y,z)=(0,0,\frac{1}{2})$. Ajut: considereu l'ús de coordenades cilíndriques o, alternativament, el principi de Cavalieri

△ Solució (2a)



$$(x-1)^2 + y^2 = 4$$

$$x-1 = 2r \cos \theta \quad 0 \leq r \leq 1$$

$$y = 2r \sin \theta \quad 0 \leq \theta \leq \pi$$

$$I = 4 \int_0^{\pi/2} d\theta \int_0^1 r [(1+2r \cos \theta)^2 + 4r^2 \sin^2 \theta] dr$$

$$= 4 \int_0^{\pi/2} d\theta \int_0^1 (r + 4r^2 \cos \theta + 4r^3) dr = 4 \int_0^{\pi/2} \left(\frac{r^2}{2} + \frac{4}{3} r^3 \cos \theta + r^4 \right) \Big|_{r=0}^{r=1} d\theta$$

$$= 4 \int_0^{\pi/2} \left(\frac{1}{2} + \frac{4}{3} \cos \theta + 1 \right) d\theta = 4 \cdot \frac{3\pi}{4} + 4 \cdot \frac{4}{3} = \boxed{3\pi + \frac{16}{3}} \cdot \Delta$$

△ Solució (2b). Fent servir coordenades esfèriques

$$x^2 + y^2 + (z - \frac{1}{2})^2 \leq \frac{1}{4}$$

$$z \geq \sqrt{\frac{x^2+y^2}{3}} \Leftrightarrow z^2 \geq \frac{x^2+y^2}{3}, \text{ amb } z \geq 0$$

$$r^2 \cos^2 \theta \cos^2 \varphi + r^2 \sin^2 \theta \cos^2 \varphi + (r \sin \varphi - \frac{1}{2})^2 \leq \frac{1}{4}$$

$$= r^2 \cos^2 \varphi + r^2 \sin^2 \varphi - r \sin \varphi + \frac{1}{4} \leq \frac{1}{4}$$

$$\Leftrightarrow r^2 \leq r \sin \varphi \Leftrightarrow r \leq \sin \varphi$$

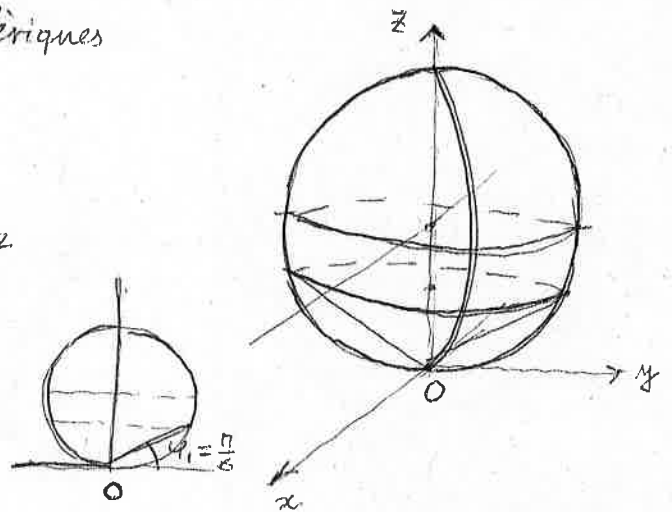
$$r^2 \sin^2 \varphi \geq \frac{r^2 \cos^2 \varphi}{3} \Leftrightarrow \tan \varphi \geq \frac{1}{\sqrt{3}}$$

$$0 \leq \varphi \leq \frac{\pi}{2}$$

$$\text{donc: } \frac{\pi}{6} \leq \varphi \leq \frac{\pi}{2}$$

$$V = \int_0^{2\pi} d\theta \int_{\pi/6}^{\pi/2} \cos \varphi d\varphi \int_0^{\sin \varphi} r^2 dr = \frac{1}{3} \int_0^{2\pi} d\theta \int_{\pi/6}^{\pi/2} \sin^3 \varphi \cos \varphi d\varphi = \frac{2\pi}{3} \left[\frac{\sin^4 \varphi}{4} \right]_{\varphi=\pi/6}^{\varphi=\pi/2}$$

$$= \frac{\pi}{6} \left(1 - \frac{1}{16} \right) = \frac{15\pi}{3 \cdot 2^4} = \boxed{\frac{5\pi}{32}}$$



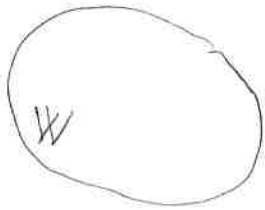
En coordenades cilíndriques:

$$V = \int_0^{2\pi} d\varphi \int_{1/4}^{1/4} dz \int_0^{\sqrt{z-z^2}} r dr + \int_0^{2\pi} d\varphi \int_{1/4}^1 dz \int_0^{\sqrt{z-z^2}} r dr = 2\pi \int_0^{1/4} \frac{3}{2} z^2 dz + 2\pi \int_{1/4}^1 \frac{1}{2} (z-z^2) dz$$

$$= \pi z^3 \Big|_0^{1/4} + \left(\frac{z^2}{2} - \frac{z^3}{3} \right) \Big|_{1/4}^1 = \pi \frac{3+96-64-6+4}{102} = \boxed{\frac{5\pi}{32}}$$

CM/CG

$$X = \frac{1}{m(W)} \iiint_W x \rho(x,y,z) dx dy dz, \text{ on } m(W) = \iiint_W \rho(x,y,z) dx dy dz$$

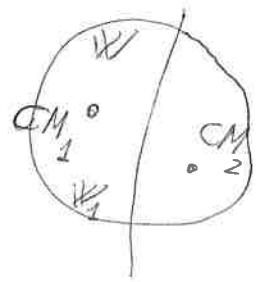


id. per Y i per Z, id en el cas 2D. / CM amb $\rho(x,y,z) \equiv 1 \Rightarrow$ CG.

Propietats del CM i del CG.:

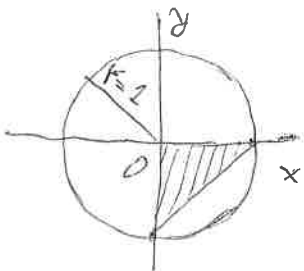
1. Si dividim un cos en dues o més parts, el seu CM és el mateix que s'obtingria si les masses fossin puntuals i estiguessin situades en els CM corresponents.

$$\begin{aligned} X_{CM} &= \frac{1}{m(W)} \iiint_W x \rho(x,y,z) dx dy dz \\ &= \frac{m_1(W)}{m(W)} \left(\frac{1}{m_1(W)} \iiint_{W_1} x \rho(x,y,z) dx dy dz \right) \\ &\quad + \frac{m_2(W)}{m(W)} \left(\frac{1}{m_2(W)} \iiint_{W_2} x \rho(x,y,z) dx dy dz \right) = \frac{m(W_1) X_{CM}(W_1) + m(W_2) X_{CM}(W_2)}{m(W)} \end{aligned}$$



Id. per Y_{CM}, Z_{CM} id en el cas 2D.

Juny 2014: Troben el centre geomètric del conjunt que resulta de treure el triangle de vèrtexs (0,0), (1,0) i (0,-1) del disc $B_1^2(0,0)$.



$$X_D = \frac{m(C) X_C + m(T) X_T}{m(D)} = 0 \Rightarrow X_C = -\frac{m(T)}{m(C)} X_T = -\frac{\frac{1}{2}}{\pi - \frac{1}{2}} \cdot \frac{1}{3}(0+1)$$

$$m(T) = A(T) = \frac{1}{2}$$

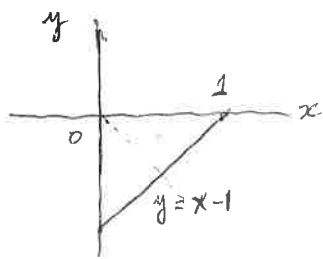
$$m(C) = A(C) = \pi - \frac{1}{2}$$

$$Y_C = -\frac{m(T)}{m(C)} Y_T = -\frac{\frac{1}{2}}{\pi - \frac{1}{2}} \cdot \frac{1}{3}(0-1)$$

$$(X_T, Y_T) = \frac{1}{3} (X_1 + X_2 + X_3, Y_1 + Y_2 + Y_3)$$

on $(X_i, Y_i), i=1,2,3$ són les coordenades dels vèrtexs del triangle.

Aleshores: $(X_C, Y_C) = \left(-\frac{1}{6\pi-3}, \frac{1}{6\pi-3} \right)$.



$$A(T) = \frac{1}{2}$$

$$A(T) X_T = \int_0^1 x dx \int_{x-1}^0 dy = \int_0^1 x(1-x) dx = \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_{x=0}^{x=1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

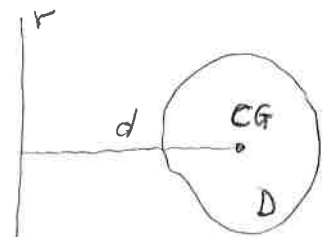
$$X_T = \frac{1}{A(T)} \cdot \frac{1}{6} = \frac{1}{3}, \quad Y_T = -X_T \text{ (per simetria)}$$

2on Teorema de Pappus-Guldin

El volum de revolució d'un cos generat per una placa uniforme al girar al voltant d'un eix contingut en el mateix pla que conté la placa i que no talla la placa, és igual al producte de l'àrea de la placa per la longitud de la circumferència descrita pel seu CG al girar al voltant de l'eix de revolució

$$d = d(r, CG)$$

$$V(W) = 2\pi d(r, CG) \cdot A(D)$$



Exercici. Calculen el CG d'un quart de disc de radi R centrat a l'origen.



$$2\pi \bar{x} \cdot \frac{1}{4} \pi R^2 = \frac{2}{3} \pi R^3 \Leftrightarrow \bar{x} = \frac{4R}{3\pi} \quad (= \bar{y}, \text{ per simetria}).$$

$$\text{Aleshores: } (\bar{x}, \bar{y}) = \left(\frac{4R}{3\pi}, \frac{4R}{3\pi} \right).$$

Gener 2016

1 (2 punt) Signi Ω el triangle de vèrtexs $(0,0), (2,0), (0,2)$, on se li ha extret el sector circular

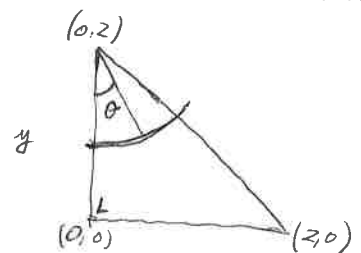
$$C = \{ x^2 + (y-2)^2 \leq 1, x \geq 0, x+y \leq 2 \}.$$

Solució:

$$x = r \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{4}, \quad 0 \leq r \leq 1$$

$$y = 2 - r \cos \theta$$

$$A(S) X_S = \int_0^{\pi/4} \sin \theta d\theta \int_0^1 r^2 dr = \left(\cos \theta \right) \Big|_{\theta=0}^{\theta=\pi/4} \times \left(\frac{r^3}{3} \right) \Big|_{r=0}^{r=1} = \left(1 - \cos\left(\frac{\pi}{4}\right) \right) \cdot \frac{1}{3} = \frac{2-\sqrt{2}}{6}$$

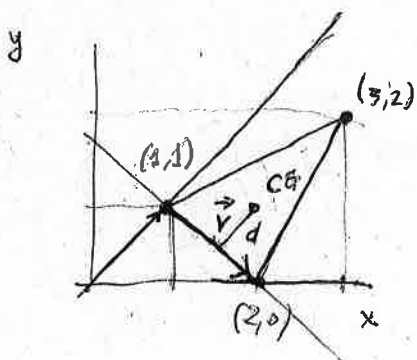


$$A(T) \cdot X_T = \frac{2}{3} (0+2+0) = \frac{4}{3} = A(\Omega) \cdot X_{\Omega} + A(S) X_S = A(\Omega) X_{\Omega} + \frac{2-\sqrt{2}}{6}$$

$$A(\Omega) = 2 - \frac{\pi}{8} : \quad X_{\Omega} = \frac{1}{A(\Omega)} \left(\frac{4}{3} - \frac{2-\sqrt{2}}{6} \right) = \frac{1}{16-\pi} \cdot \frac{8-2+\sqrt{2}}{6} = \frac{4(6+\sqrt{2})}{3(16-\pi)}$$

Genes 2015. (1P) Quin és el volum del cos generat pel triangle de vèrtexs $(1,1)$, $(2,0)$ i $(3,2)$ i al girar al voltant de l'eix que passa pel punt $(2,0)$ i és perpendicular a la bisectriu del 1er quadrant?

Solució:



$$(\bar{x}, \bar{y}) = \frac{1}{3}((1,1) + (2,0) + (3,2)) = \frac{1}{3}(6,3) = (2,1)$$

$$\vec{V} = (1,0) \quad \vec{m} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$a = \langle \vec{V}, \vec{m} \rangle = \frac{1}{\sqrt{2}}$$

$$d = \sqrt{\|\vec{V}\|^2 - a^2} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\left\{ \begin{array}{l} \vec{V} = (2,0) - (2,1) \\ = (0,-1) \perp \text{bisectriu del} \\ \text{1er quadrant.} \end{array} \right.$$

$$A(T) = \left| \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix} \right| = \left| -\frac{3}{2} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - \frac{1}{2} \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \right| = \left| -1 - \frac{1}{2} \right| = \boxed{\frac{3}{2}}$$

Aplicant el 2on teorema de Pappus-Guldinus

$$V(T) = 2\pi d \cdot A(T) = 2\pi \cdot \frac{1}{\sqrt{2}} \cdot \frac{3}{2} = \boxed{\frac{3\pi}{\sqrt{2}}}$$

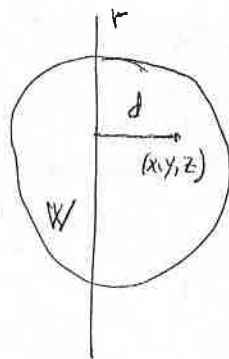
Fer el problema 28.

" " problema 34 (e)

" " " 26 (e)

Moments d'inèrcia respecte d'un eix.

$$I_r = \iiint_W d^2(r, (x,y,z)) \rho(x,y,z) dx dy dz$$

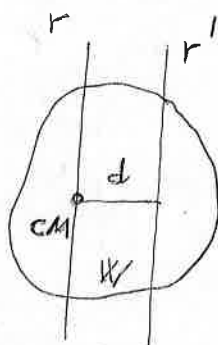


Moments d'inèrcia respecte dels eixos principals.

$$I_x = \iiint_W (y^2+z^2) \rho(x,y,z) dx dy dz,$$

$$I_y = \iiint_W (x^2+z^2) \rho(x,y,z) dx dy dz,$$

$$I_z = \iiint_W (x^2+y^2) \rho(x,y,z) dx dy dz.$$



Teorema de Steiner.

$$I_{r'} = I_r + M(W) d^2$$

Gener 2017

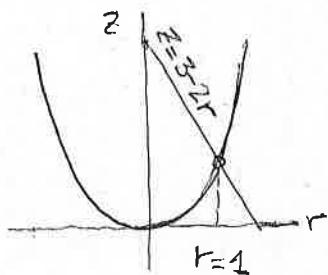
(a) [1 punt] Calculen la massa del sòlid limitat pel paraboloide $Z = X^2 + Y^2$ i el con $Z + 2\sqrt{X^2 + Y^2} = 3$ si es suposa que la seva densitat és constant igual a 1

(b) [1 punt] Calculen el moment d'inèrcia del cos de l'apartat anterior respecte de l'eix Z.

(c) [1/2 punt] Troben el moment d'inèrcia del mateix cos respecte de l'eix perpendicular al pla $Z=0$ que passa pel punt $(x,y,z) = (\sqrt{2}/2, \sqrt{2}/2, 0)$

Solució

$Z = r^2, Z = 3 - 2r$



$$3 - 2r = r^2$$

$$\Leftrightarrow r^2 + 2r - 3 = 0$$

$$r = \frac{-2 \pm \sqrt{4 + 12}}{2} = \frac{-2 \pm 4}{2}$$

$$= \begin{cases} -3 & \text{No} \\ 1 & \end{cases}$$

$$\begin{aligned} (a) M &= \int_0^{2\pi} d\theta \int_0^1 r dr \int_{r^2}^{3-2r} dz = 2\pi \int_0^1 r(3-2r-r^2) dr = 2\pi \int_0^1 (3r - 2r^2 - r^3) dr \\ &= 2\pi \left(\frac{3}{2} r^2 - \frac{2}{3} r^3 - \frac{r^4}{4} \right) \Big|_{r=0}^{r=1} = 2\pi \left(\frac{3}{2} - \frac{2}{3} - \frac{1}{4} \right) \\ &= \frac{\pi}{6} (18 - 8 - 3) = \boxed{\frac{7\pi}{6}} \end{aligned}$$

$$\begin{aligned} (b) I_z &= \int_0^{2\pi} d\theta \int_0^1 r^3 dr \int_{r^2}^{3-2r} dz = 2\pi \int_0^1 (3r^3 - 2r^4 - r^5) dr \\ &= 2\pi \left(\frac{3}{4} r^4 - \frac{2}{5} r^5 - \frac{1}{6} r^6 \right) \Big|_{r=0}^{r=1} = \frac{\pi}{30} (45 - 24 - 10) \\ &= \boxed{\frac{11\pi}{30}} \end{aligned}$$

$$(c) I_{z'} = I_z + d^2 M = \frac{11\pi}{30} + 1 \cdot \frac{7\pi}{6} = \frac{11\pi + 35\pi}{30} = \frac{46\pi}{30} = \boxed{\frac{23\pi}{15}}$$

$$d^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

2 [1p] Troben el moment d'inèrcia del cos homogeni $W = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, x^2 + z^2 \leq 1\}$ amb densitat ρ , respecte de l'eix $y = \frac{z}{5\sqrt{3}}, z = \frac{1}{5\sqrt{3}}$

Solució:

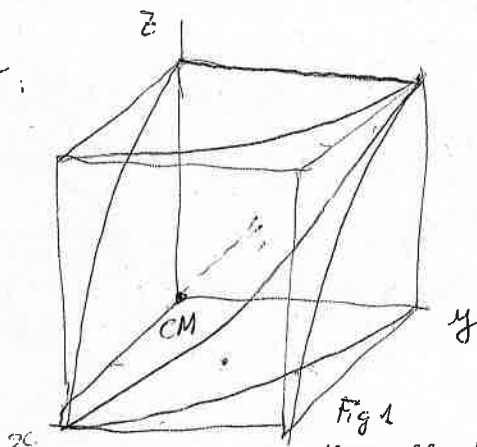


Fig 1
Només per octant!

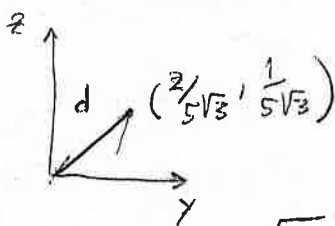
$$I_x = k \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (y^2 + z^2) dz$$

$$= k \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(y^2 z + \frac{z^3}{3} \right) \Big|_{z=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy$$

$$= 2k \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[y^2 \sqrt{1-x^2} + \frac{1}{3} (1-x^2)^{3/2} \right] dy = 4k \int_{-1}^1 \left[\frac{1}{3} (1-x^2)^2 + \frac{1}{3} (1-x^2)^2 \right] dx$$

$$= \frac{16}{3} k \int_0^1 (1-2x^2+x^4) dx = \frac{16}{3} k \left(x - \frac{2}{3}x^3 + \frac{x^5}{5} \right) \Big|_0^1 = \frac{16}{3} k \frac{15-10+3}{15} =$$

$$= \frac{16}{3} k \cdot \frac{8}{15} = \frac{2^7}{3^2 \cdot 5} k = \boxed{\frac{128k}{45}}$$



$$I_r = I_x + M d^2 = \frac{128k}{45} + \frac{16k}{3} \left(\left(\frac{2}{5\sqrt{3}} \right)^2 + \left(\frac{1}{5\sqrt{3}} \right)^2 \right)$$

$$= \frac{128k}{45} + \frac{16k}{45} = \frac{144k}{45} = \frac{3^2 \cdot 2^4 k}{3^2 \cdot 5} = \boxed{\frac{16k}{5}}$$

$$M_y = k \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2}} dz$$

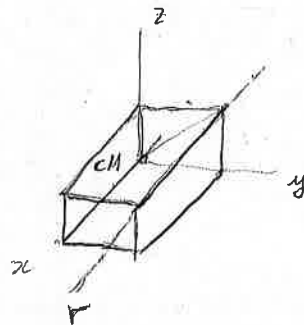
$$= k \int_0^1 (1-x^2) dx$$

$$= k \left(x - \frac{x^3}{3} \right) \Big|_{x=0}^{x=1}$$

$$= k \left(1 - \frac{1}{3} \right) = \frac{2k}{3} \Rightarrow M = \frac{16k}{3}$$

$$d^2 = \left(\frac{2}{5\sqrt{3}} \right)^2 + \left(\frac{1}{5\sqrt{3}} \right)^2 = \frac{4}{75} + \frac{1}{75}$$

$$= \frac{5}{75 \cdot 3} = \boxed{\frac{1}{15}}$$



Genes 2018

34

13

4) Signi $B \subseteq \mathbb{R}^3$ el cos homogèni de densitat 1 limitat per les superfícies $y^2 = z^2 - 1$, $x^2 = z^2 - 1$ i $z = 2$

Calculen

(a) [1/2 punt] La massa del cos Sol. $M = 16/3$

(b) [1/2 punt] les coordenades del centre geomètric Sol. $(x_c, y_c, z_c) = (0, 0, 27/16)$

(c) [1 punt] Els seus moments d'inèrcia respecte dels eixos x i y

Solució:

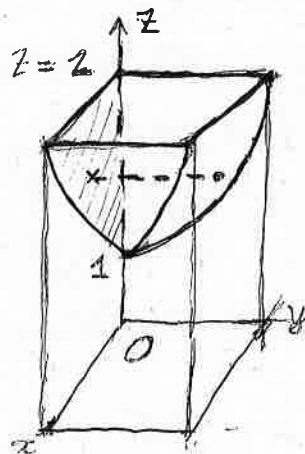
$$(a) y^2 = z^2 - 1 \Leftrightarrow y = \pm \sqrt{z^2 - 1}$$

$$x^2 = z^2 - 1 \Leftrightarrow x = \pm \sqrt{z^2 - 1}$$

$$B: 1 \leq z \leq 2, -\sqrt{z^2 - 1} \leq x \leq \sqrt{z^2 - 1}, -\sqrt{z^2 - 1} \leq y \leq \sqrt{z^2 - 1}.$$

$$M = \int_1^2 dz \int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} dx \int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} dy = \int_1^2 dz \int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} 2\sqrt{z^2-1} dx = 4 \int_1^2 (z^2-1) dz = 4 \left(\frac{z^3}{3} - z \right) \Big|_{z=1}^{z=2}$$

$$= 4 \left(\frac{8}{3} - 2 - \frac{1}{3} + 1 \right) = 4 \left(\frac{7}{3} - 1 \right) = \frac{16}{3}$$



(b) Per simetria: $x_c = 0 = y_c$

$$M \cdot z_c = \int_1^2 z dz \int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} dx \int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} dy = 2 \int_1^2 z dz \int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} \sqrt{z^2-1} dx = 4 \int_1^2 z(z^2-1) dz$$

$$= 4 \left(\frac{z^4}{4} - \frac{z^2}{2} \right) \Big|_{z=1}^{z=2} = 4 \left(4 - 2 - \frac{1}{4} + \frac{1}{2} \right) = 4 \left(2 + \frac{1}{4} \right) = \frac{36}{4} = 9$$

$$z_c = \frac{36/4}{16/3} = \frac{9 \cdot 3}{16} = \frac{27}{16} \quad \text{Aleshores: } (x_c, y_c, z_c) = \left(0, 0, \frac{27}{16} \right)$$

(c) Per simetria $I_x = I_y = \int_1^2 dz \int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} dx \int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} (y^2 + z^2) dy = \int_1^2 dz \int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} \left[\frac{2}{3} (z^2-1)^{3/2} + 2z^2 (z^2-1)^{1/2} \right] dx$

$$= \int_1^2 \left[\frac{4}{3} (z^2-1)^2 + 4z^2 (z^2-1) \right] dz = 4 \int_1^2 \left(\frac{4}{3} z^4 - \frac{5}{3} z^2 + \frac{4}{3} \right) dz = 4 \left(\frac{4}{15} z^5 - \frac{5}{9} z^3 + \frac{4}{3} z \right) \Big|_{z=1}^{z=2}$$

$$\frac{4}{3} (z^4 - z^2 + 1) + 4z^4 - 4z^2 = \frac{4}{15} (12z^5 - 25z^3 + 15z) \Big|_{z=1}^{z=2} = \frac{4}{15} (12 \times 32 - 25 \times 8 + 30 - 12 + 25 - 15)$$

$$= \frac{16}{3} z^4 - \frac{20}{3} z^2 + \frac{4}{3} = \frac{848}{15}$$