TEMA 2

1. Usant el teorema del valor mig per integrals proveu que les següents desigualtats.

(a) $4e^5 \leq \iint_A e^{x^2+y^2}\,dx\,dy \leq 4e^{25}$, on $A=[1,3]\times[2,4]$

(b) $\frac{1}{e} \leq \frac{1}{4\pi^2}\iint_A e^{\sin(x+y)}\,dx\,dy \leq e$ on $A=[-\pi,\pi]\times[-\pi,\pi]$

(c) $\frac{1}{6} \leq \iint_A \frac{dx\,dy}{y-x+3} \leq \frac{1}{4}$, on A és el triangle de vèrtexs $(0,0),(1,1),(1,0)$.

Solució. Recordem el Teorema del valor mig (veure apunts de teoria, Tema 2).

▷ Sigui $f:\Omega\subset\mathbb{R}^n\to\mathbb{R}$, Ω acotat i arc-connex, i $f\in\mathcal{C}^0(\overline{\Omega})$, llavors $\exists\, c\in\overline{\Omega}$ tal que:

$$\int_\Omega f(x) = f(c)\, m(\Omega).$$

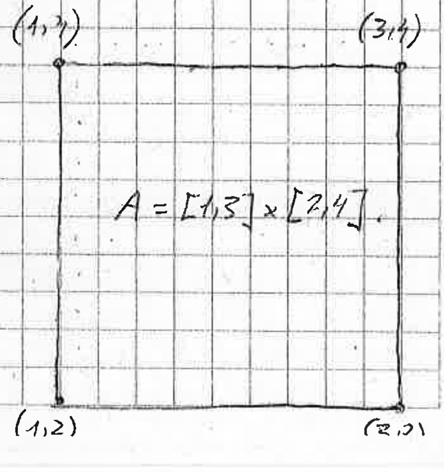
i on $m(\Omega)$ és la mesura del conjunt Ω (longitud en dimensió 1, àrea en dimensió 2, volumen en dimensió 3, ...) Llavors tenim les acotacions següents

$$m(\Omega)\inf_{x\in\Omega} f(x) \leq \int_\Omega f(x) \leq m(\Omega)\sup_{x\in\Omega} f(x). \qquad (1)$$

(a) $f(x,y)=e^{x^2+y^2}$, d'on: $f_x(x,y)=2xe^{x^2+y^2}=0 \Leftrightarrow x=0$, $f_y(x,y)=2ye^{x^2+y^2}=0$,

Llavors l'únic punt crític de la funció és $(x,y)=(0,0)\notin A=[1,3]\times[2,4]$

- $x=1$: $f(1,y)=e^{1+y^2}$: $f'(1,y)=2ye^{1+y^2}=0 \Leftrightarrow y=0$: $(1,0)\notin A$.
- $x=3$: $f(3,y)=e^{9+y^2}$: $f'(3,y)=2ye^{9+y^2}=0 \Leftrightarrow y=0$: $(3,0)\notin A$.
- $y=2$: $f(x,2)=e^{x^2+4}$: $f'(x,2)=2xe^{x^2+4}=0 \Leftrightarrow x=0$: $(0,2)\notin A$.
- $y=4$: $f(x,4)=e^{x^2+16}$: $f'(x,4)=2xe^{x^2+16}=0$ $\Leftrightarrow x=0$: $(0,4)\notin A$.



Veiem que el punt crític de f no pertany a A i que les restriccions de f sobre els costats d'A no tenen punts crítics. Avaluem doncs f als vèrtexs d'A

$f(1,2)=e^5$, $f(3,2)=e^{13}$, $f(1,4)=e^{17}$, $f(3,4)=e^{25}$.

②

d'on: $\max_{(x,y)\in A} e^{x^2+y^2} = e^{25}$ i $\min_{(x,y)\in A} e^{x^2+y^2} = e^{5}$ i s'atanyen als punts (3,4) i (1,2) respectivament. D'altra banda: $m(A) = \iint_A dx\,dy = (3-1)\cdot(4-2) = 4$.

Així doncs, aplicant el Teorema del Valor mig (fórmula (1)),

$$4e^5 \leq \iint_{A=[1,3]\times[2,4]} e^{x^2+y^2}\,dx\,dy \leq 4\cdot e^{25}. \qquad \square$$

(b) $\dfrac{1}{e} \leq \dfrac{1}{4\pi^2}\iint_A e^{\sin(x+y)}\,dx\,dy \leq e$, essent $A = [-\pi,\pi]\times[-\pi,\pi]$

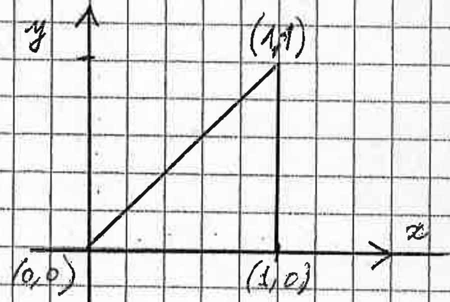
Solució. Clarament $\max_{(x,y)\in A} e^{\sin(x+y)} = e$, i $\min_{(x,y)\in A} e^{\sin(x+y)} = \frac{1}{e}$, mentre que $m(A) = 4\pi^2$. Aleshores, aplicant (1) resulta:

$$\frac{4\pi^2}{e} \leq \iint_{A=[-\pi,\pi]\times[-\pi,\pi]} e^{\sin(x+y)}\,dx\,dy \leq 4\pi^2 e,$$

i dividint les dues desigualtats per $4\pi^2$ s'obtenen les acotacions buscades.

(c) $\dfrac{1}{6} \leq \iint_A \dfrac{dx\,dy}{y-x+3} \leq \dfrac{1}{4}$, on A és el triangle de vèrtexs (0,0), (1,1) i (1,0)

Solució.



El màxim de $y-x+3$ sobre el triangle A s'ateny als vèrtexs (0,0) i (1,1) i val 3, mentre que el mínim té lloc als vèrtexs (1,0) i val 2. Per tant:

$$\frac{1}{3} \leq \frac{1}{y-x+3} \leq \frac{1}{2}$$

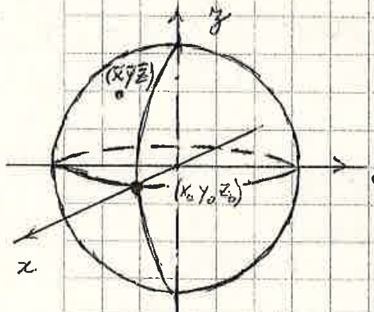
per a tot punt del triangle donat, mentre que l'àrea corresponent és $m(A) = \frac{1}{2}$. Finalment doncs, aplicant la fórmula (1) del Teorema del Valor Mitjà, obtenim:

$$\frac{1}{6} \leq \iint_A \frac{dx\,dy}{y-x+3} \leq \frac{1}{4},$$

que són les desigualtats buscades. $\square$

2) Sigui $f(x,y,z)$ una funció contínua i B_ε la bola de centre (x_0,y_0,z_0) i radi ε. Proveu que se satisfà que $f(x_0,y_0,z_0) = \lim_{\varepsilon\to 0^+} \frac{1}{\text{Volum}(B_\varepsilon)} \iiint_{B_\varepsilon} f(x,y,z)\, dx\,dy\,dz$.

Solució. Pel teorema del valor mitjà, existeix $(\bar{x},\bar{y},\bar{z}) \in B_\varepsilon = B_\varepsilon(x_0,y_0,z_0)$ t.q.:



$B_\varepsilon = B_\varepsilon(x_0,y_0,z_0)$

$$\iiint_{B_\varepsilon(x_0,y_0,z_0)} f(x,y,z)\,dx\,dy\,dz = f(\bar{x},\bar{y},\bar{z})\ \text{Volum}\ B_\varepsilon(x_0,y_0,z_0) \qquad (z)$$

D'altra banda, com que f és contínua i $(\bar{x},\bar{y},\bar{z}) \to (x_0,y_0,z_0)$ quan $\varepsilon \to 0^+$, prenent límits a (z) s'obté:

$$\lim_{\varepsilon\to 0} \frac{1}{\text{Volum}(B_\varepsilon(x_0,y_0,z_0))} \iiint_{B_\varepsilon(x_0,y_0,z_0)} f(x,y,z)\,dx\,dy\,dz = \lim_{(\bar{x},\bar{y},\bar{z})\to(x_0,y_0,z_0)} f(\bar{x},\bar{y},\bar{z}) = f(x_0,y_0,z_0)$$

que és el resultat que es buscava. $\square$

3) Apliqueu el principi de Cavalieri per a calcular els següents volums a partir de l'àrea de seccions amb plans paral·lels als plans coordenats (triades de forma adequada).

(a) Volum envoltant per l'el·lipsoide: $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

Solució. Fixant $0<x<a$: $\dfrac{y^2}{b^2(1-x^2/a^2)} + \dfrac{z^2}{c^2(1-x^2/a^2)} = 1$, d'on $S(x) = \pi bc\left(1 - x^2/a^2\right)$.

Nota: recordem que l'àrea d'una el·lipse ve donada per πAB, on A i B són les longituds dels seus semieixos. Llavors, aplicant el principi de Cavalieri:

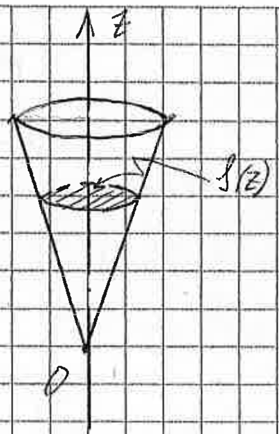
$$V = 2\int_0^a S(x)\,dx = 2\pi bc \int_0^a \left(1 - x^2/a^2\right) dx = 2\pi bc \left(x - \frac{x^3}{3a^2}\right)\Bigg|_{x=0}^{x=a} = 2\pi bc\left(a - \frac{a^3}{3a^2}\right)$$

$$= \boxed{\tfrac{4}{3}\pi abc}$$

$\square$

(b) Volum envoltat pel con de base el·líptica $x^2/a^2 + y^2/b^2 = z^2$, amb $0 \le z \le h$.

Solució. Fixant $0 \le z \le h$: $\dfrac{x^2}{(az)^2} + \dfrac{y^2}{(bz)^2} = 1$, d'on $S(z) = \pi ab z^2$.

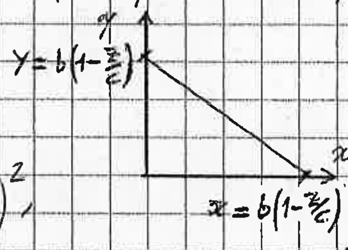


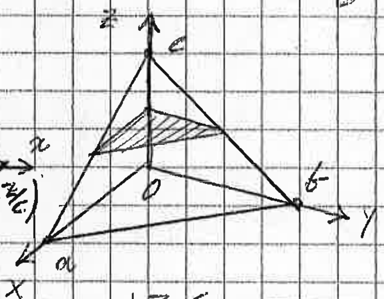
Alesbores, aplicant el principi de Cavalieri:

$$V=\int_0^h S(z)\,dz = \int_0^h \pi ab z^2\,dz = \pi ab \frac{z^3}{3}\Big|_{z=0}^{z=h} = \boxed{\frac{\pi}{3}abh^3}$$

Nota: $V=\frac{1}{3}\underbrace{\pi(ah)\cdot(bh)}_{\substack{\shortparallel\\ \text{àrea de la base}}}\cdot h$

(c) Volum d'un tetraedre limitat pels plans $x=0$, $y=0$, $z=0$, $x/a+y/b+z/c=1$ $(a,b,c>0)$.





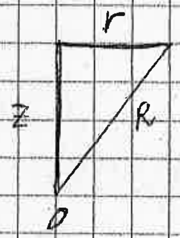
Solució.-

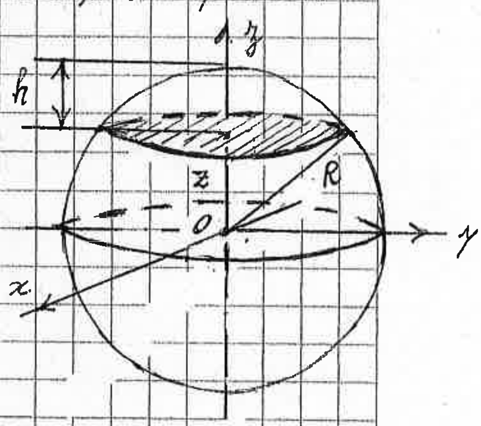
Fixant z: $S(z)=\frac{1}{2}ab\left(1-\frac{z}{c}\right)^2$,

d'on, aplicant el principi de Cavalieri:

$$V=\int_0^c S(z)\,dz = \frac{1}{2}ab\int_0^c\left(1-\frac{z}{c}\right)^2 dz = -\frac{1}{6}abc\left(1-\frac{z}{c}\right)^3\Big|_{z=0}^{z=c} = \boxed{\frac{abc}{6}}$$

(d) Volum envoltant pel casquet esfèric determinat per l'esfera $x^2+y^2+z^2=R^2$ i la condició $R-h\le z\le R$





Solució:

$$S(z)=\pi r^2(z)=\pi(R^2-z^2)$$

D'on, aplicant el principi de Cavalieri:

$$V=\int_{R-h}^{R} S(z)\,dz = \pi\int_{R-h}^{R}(R^2-z^2)\,dz$$

$$=\pi\left(R^2z-z^3/3\right)\Big|_{R-h}^{R} = \pi\left(R^3-\frac{R^3}{3}-R^3+R^2h+\frac{R^3}{3}-R^2h+Rh^2-\frac{h^3}{3}\right)=$$

$$=\frac{\pi h^2}{3}(3R-h).$$

4) Generalitzeu el principi de Cavalieri al càlcul de volums en $\mathbb{R}^4$ i calculeu el volum de la bola 4-dimensional $B=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2+t^2\le R^2\}$

Solució. Fixem t: $0\le x^2+y^2+z^2\le R^2$, amb $-R\le t\le R$. Alshores

$$V(t)=\frac{4}{3}\pi(R^2-t^2)^{3/2},$$

D'on aplicant el principi de Cavalieri:

$$M = \int_{-R}^{R} V(t)\,dt = \frac{4}{3}\pi \int_{-R}^{R} (R^2-t^2)^{3/2}\,dt = \frac{8}{3}\pi R^3 \int_0^R \left(1-t^2/R^2\right)^{3/2} dt = \left\{ \begin{array}{l} C.V.: \\ t/R = \sin\theta \\ \dots \end{array}\right\}$$

$$= \frac{8}{3}\pi R^4 \int_0^{\pi/2} (1-\sin^2\theta)^{3/2}\cos^2\theta\,d\theta = \frac{8}{3}\pi R^4 \int_0^{\pi/2} \cos^4\theta\,d\theta \overset{(\&)}{=} \frac{8}{3}\pi R^4 \int_0^{\pi/2} \left(\frac{1+\cos(2\theta)}{2} - \frac{1-\cos(4\theta)}{8}\right) d\theta$$

$$= \frac{8}{3}\pi R^4 \left(\frac{\pi}{4} - \frac{\pi}{16}\right) = \frac{8}{\cancel{3}}\pi R^4 \cdot \frac{\cancel{3}\pi}{16} = \boxed{\frac{\pi^2 R^4}{2}}.$$

(&) $\cos^4\theta = \cos^2\theta(1-\sin^2\theta) = \cos^2\theta - \frac{1}{4}\sin^2(2\theta) = \frac{1+\cos(2\theta)}{2} - \frac{1-\cos(4\theta)}{8}$, fent servir les fórmules de l'angle doble. □

5) Trobeu les següents integrals dobles en els rectangles que s'indiquen.

(a) $I = \iint_R x^2 y\,dx\,dy$, $R = [0,1]\times[0,1]$

Solució: $I = \iint_R x^2 y\,dx\,dy = \left(\int_0^1 x^2 dx\right)\left(\int_0^1 y\,dy\right) = \left[\frac{x^3}{3}\right]_0^1 \cdot \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{3}\cdot\frac{1}{2} = \boxed{\frac{1}{6}}$

(b) $I = \iint_R \frac{x^2}{1+y^2}\,dx\,dy$, $R = [0,1]\times[0,1]$

Solució: $I = \iint_{R=[0,1]\times[0,1]} \frac{x^2}{1+y^2}\,dx\,dy = \int_0^1 x^2 dx \int_0^1 \frac{dy}{1+y^2} = \left[\frac{x^3}{3}\right]_0^1 \cdot \left[\arctan y\right]_0^1 = \boxed{\frac{\pi}{12}}$

(c) $I = \iint_R y \ln x\,dx\,dy$, $R = [1,e]\times[1,e]$.

Solució: $I = \iint_{R=[1,e]\times[1,e]} y\ln x\,dx\,dy = \left(\int_1^e \ln x\,dx\right)\cdot\left(\int_1^e y\,dy\right) \overset{(\&)}{=} \boxed{\frac{e^2-1}{2}}$

(&) $\int_1^e \ln x\,dx = (x\ln x - x)\Big|_1^e = \cancel{e} - \cancel{e} + 1 = 1$, (primitivització per parts + regla de Barrow).

(d) $\iint_R (x^2+y)\,dx\,dy$, $R = [0,1]\times[0,2]$

Solució. $I = \iint_{R=[0,1]\times[0,2]} (x^2+y)\,dx\,dy = \int_0^1 dx \int_0^2 (x^2+y)\,dy = \int_0^1 \left[x^2 y + \frac{y^2}{2}\right]_{y=0}^{y=2} dx = \int_0^1 (2x^2+2)\,dx$

$$= \left[\frac{2}{3}x^3 + 2x \right]_{x=0}^{x=1} = \frac{2}{3} + 2 = \boxed{\frac{8}{3}}$$

(e) $I = \iint_R \frac{1}{(x+2y)^2}\, dx\,dy, \quad R = [2,5] \times [1,3]$

Solució $I = \iint_R \frac{dx\,dy}{(x+2y)^2} = \int_2^5 dx \int_1^3 \frac{dy}{(x+2y)^2} = \int_2^5 \left[\frac{-1/2}{x+2y} \right]_{y=1}^{y=3} dx$

$$= \frac{1}{2} \int_2^5 \left(\frac{1}{2+x} - \frac{1}{6+x} \right) dx = \frac{1}{2} \ln\left(\frac{2+x}{6+x} \right) \Bigg|_{x=2}^{x=5}$$

$$= \frac{1}{2} \left(\ln \frac{7}{11} - \ln \frac{4}{8} \right) = \boxed{\frac{1}{2} \ln\left(\frac{14}{11} \right)}$$

(f) $I = \iint_R e^y \sin\left(\frac{x}{y} \right) dx\,dy, \quad R = [-\pi/2, \pi/2] \times [1,2]$

Solució. $I = \iint_{R = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [1,2]} e^y \sin\left(\frac{x}{y} \right) dx\,dy = \int_1^2 e^y\, dy \int_{-\pi/2}^{\pi/2} \sin\left(\frac{x}{y} \right) dx$

$$= -\int_1^2 y e^y \left[\cos\left(\frac{x}{y} \right) \right]_{x=-\pi/2}^{x=\pi/2} d\cancel{y} \overset{(\&)}{=} 0$$

(&) $[\ldots]_{x=-\pi/2}^{x=\pi/2} = \cos\left(\frac{\pi}{2y} \right) - \cos\left(-\frac{\pi}{2y} \right) = 0.$

(g) $I = \iint_R (x+y)^{27}\, dx\,dy, \quad R = [-1,1] \times [-1,1]$

Solució. $I = \iint_{R = [-1,1] \times [-1,1]} (x+y)^{27}\, dx\,dy = \int_{-1}^1 dx \int_{-1}^1 (x+y)^{27}\, dy = \frac{1}{28} \int_{-1}^1 (x+y)^{28} \Big|_{y=-1}^{y=1} dx$

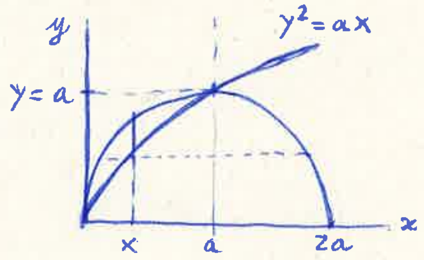
$$= \frac{1}{28} \int_{-1}^1 \left[(x+1)^{28} - (x-1)^{28} \right] dx = \frac{1}{28} \cdot \frac{1}{29} \left[(x+1)^{29} - (x-1)^{29} \right]_{x=-1}^{x=1}$$

$$= \frac{1}{28} \cdot \frac{1}{29} \left(2^{29} + (-2)^{29} \right) = \boxed{0}.$$

8 Per a les regions $A \subset \mathbb{R}^2$ indicades escriviu la integral doble $\iint_A f(x,y)\,dxdy$ en termes d'integrals iterades preses en diferents ordres, $\int\left(\int f(x,y)dx\right)dy$ i $\int\left(\int f(x,y)dy\right)dx$, donant quins són els extrems d'integració per a x i y en cada cas.

Solució. Apartats d) i e)

d) A regió limitada per les corbes $y^2 = ax$, $x^2+y^2 = 2ax$, $y=0$ $(y \geq 0, a>0)$.



$$I_1 = \int_0^a dx \int_{\sqrt{ax}}^{\sqrt{2ax-x^2}} f(x,y)\,dy + \int_a^{2a} dx \int_0^{\sqrt{2ax-x^2}} f(x,y)\,dy.$$

$$I_2 = \int_0^a dy \int_{y^2/a}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx.$$

$2ax - x^2 \leq ax$, $x>0$

$\Leftrightarrow$ $ax - x^2 = x(a-x) \leq 0$

$\Leftrightarrow$ $x \geq a$

$(x-a)^2 + y^2 = a^2$

$x = a + \sqrt{a^2-y^2}$

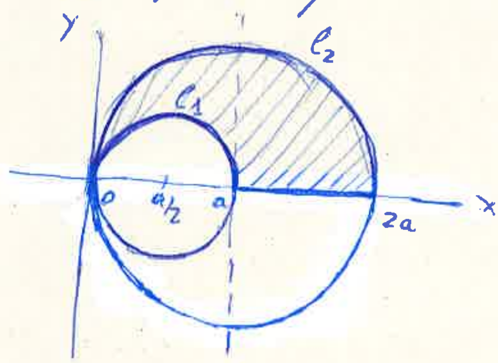
Si suposem que f és contínua en D, llavors pel teorema de Fubini $I_1 = I_2$

Solució.

e) A regió limitada per les corbes $x^2+y^2 = ax$, $x^2+y^2 = 2ax$, $y=0$ $(y\geq 0, a>0)$

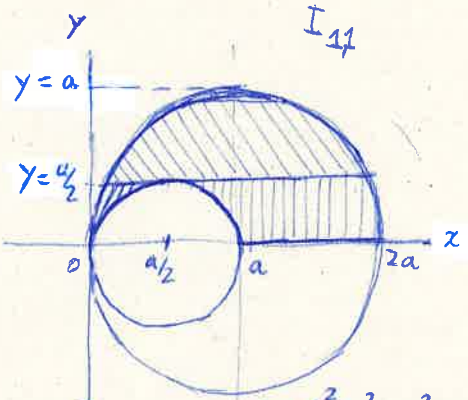
$\mathcal{C}_1: x^2+y^2 = ax$, $y \geq 0$

$\mathcal{C}_2: x^2+y^2 = 2ax$, $y\geq 0$.



2-Dominis x-elementals

$$I_1 = \underbrace{\int_0^a dx \int_{\sqrt{ax-x^2}}^{\sqrt{2ax-x^2}} f(x,y)\,dy}_{= I_{11}} + \underbrace{\int_a^{2a} dx \int_0^{\sqrt{2ax-x^2}} f(x,y)\,dy}_{= I_{1,2}}$$



3-dominis y-elementals.

$$I_2 = \int_0^{a/2} dy \int_{a-\sqrt{a^2-y^2}}^{a/2-\sqrt{a^2/4-y^2}} f(x,y)\,dx + \int_0^{a/2} dy \int_{a/2+\sqrt{a^2/4-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx$$

$$+ \int_{a/2}^{a} dy \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx$$

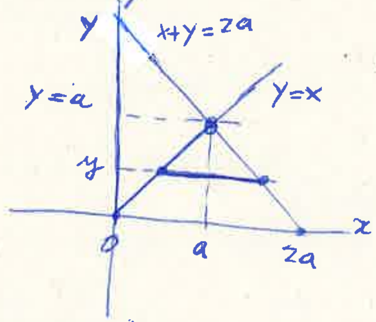
$\mathcal{C}_1: x^2+y^2 = ax \Leftrightarrow (x-a/2)^2+y^2 = a^2/4$

$\Leftrightarrow x = a/2 \pm \sqrt{a^2/4-y^2}$

Si f és contínua en D: $I_1 = I_2$

12. Calculeu les següents integrals dobles en els dominis de $\mathbb{R}^2$ que s'indiquen

(a) $I = \iint_A (x^2+y^2)\,dx\,dy$ A limitat per les rectes $y=x$, $x+y=2a$, $x=0$ $(a>0)$



Sol.

$$I = \int_0^a dy \int_y^{2a-y} (x^2+y^2)\,dx = \int_0^a \left(x^3/3 + y^2x\right)\Big|_{x=y}^{x=2a-y} dy$$

$$= \int_0^a \left[\frac{1}{3}(2a-y)^3 + y^2(2a-y) - y^3/3 - y^3\right] dy = \int_0^a \left(8/3\,a^3 - 4a^2y + 4ay^2 - 8/3\,y^3\right) dy$$

$$\frac{1}{3}\left(8a^3 - 12a^2y + 6ay^2 - y^3\right) + 2ay^2 - y^3 - y^3/3 - y^3$$

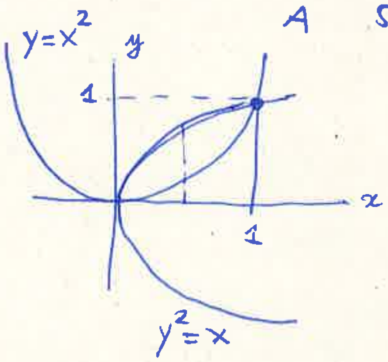
$$= 8/3\,a^3 - 4a^2y + 2ay^2 - \frac{1}{3}y^3 + 2ay^2 - y^3/3 - 2y^3$$

$$= 8/3\,a^3 - 4a^2y + 4ay^2 - \frac{8}{3}y^3$$

$$= \left(8/3\,a^3y - 2a^2y^2 + 4/3\,ay^3 - 2/3\,y^4\right)\Big|_{y=0}^{y=a}$$

$$= \left(8/3 - 2 + 4/3 - 2/3\right)a^4 = \boxed{\frac{4}{3}a^4}$$

(b) $I = \iint_A (x+2y)\,dx\,dy$, A limitat per les corbes $y=x^2$, $y^2=x$.



$x^4 = x \Leftrightarrow x(x^3-1)=0$

Sols: $x=0 \rightarrow y=0$

$x=1 \rightarrow y=1$

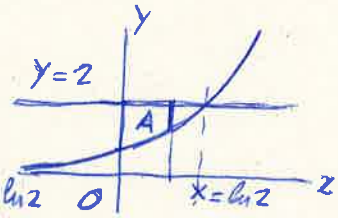
Sol.

$$I = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (x+2y)\,dy = \int_0^1 \left(xy + y^2\right)\Big|_{y=x^2}^{y=\sqrt{x}} dx = \int_0^1 \left(x^{3/2} + x - x^3 - x^4\right) dx$$

$$= \left(2/5\,x^{5/2} + x^2/2 - x^4/4 - x^5/5\right)\Big|_{x=0}^{x=1} = \frac{2}{5} + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} = \frac{8+10-5-4}{20} = \boxed{\frac{9}{20}}$$

Sol.

(c) $I = \iint_A e^{x+y}\,dx\,dy$, A limitat per les corbes $y=e^x$, $x=0$, $y=2$



$e^x = 2$

$x = \ln 2$

Sol.

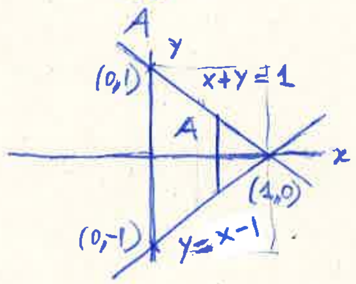
$$I = \int_0^{\ln 2} dx \int_{e^x}^{2} e^{x+y}\,dy = \int_0^{\ln 2} \left[e^{x+y}\right]_{y=e^x}^{y=2} dx$$

$$= \int_0^{\ln 2} \left(e^{x+2} - e^x e^{e^x}\right) dx = \left(e^{x+2} - e^{e^x}\right)\Big|_{x=0}^{x=\ln 2}$$

$$= e^{2+\ln 2} - e^{e^{\ln 2}} - e^2 + e = \cancel{2e^2} - \cancel{e^2} - \cancel{e^2} + e = \boxed{e}$$

Sol.

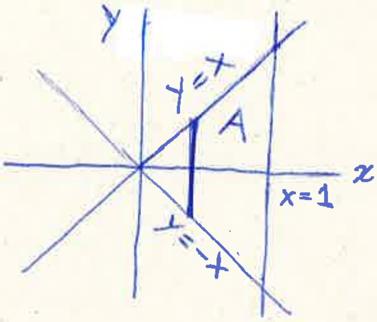
(d) $I = \iint_A e^y\,dx\,dy$, A triangle de vèrtexs $(1,0)$, $(0,1)$, $(0,-1)$.



$$I = \int_0^1 dx \int_{x-1}^{1-x} e^y\,dy = \int_0^1 \left(e^{1-x} - e^{x-1}\right) dx = \left(-e^{1-x} - e^{x-1}\right)\Big|_{x=0}^{x=1}$$

$$= -1 - 1 + e + \frac{1}{e} = \boxed{e + \frac{1}{e} - 2}$$

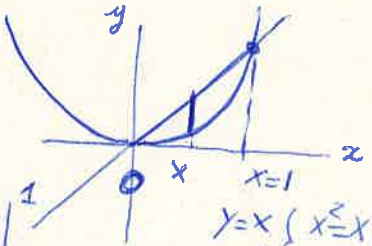
(e) $I = \iint_A xy^2\,dx\,dy$, A limitat per les rectes $x=1$, $x=y$, $x+y=0$.



Sol.

$$I = \int_0^1 dx \int_{-x}^{x} xy^2\,dy = \int_0^1 \left(x \cdot \frac{y^3}{3}\right)\Bigg|_{y=-x}^{y=x} dx = \frac{2}{3}\int_0^1 x^4\,dx = \boxed{\frac{2}{15}}$$

(f) $I = \iint_A xy\,dx\,dy$, A limitat per les corbes $x=y$, $y=x^2$



$$I = \int_0^1 dx \int_{x^2}^{x} xy\,dy = \int_0^1 \left[x\frac{y^2}{2}\right]_{y=x^2}^{y=x} dx$$

$$= \frac{1}{2}\int_0^1 (x^3 - x^5)\,dx = \left(\frac{x^4}{8} - \frac{x^6}{12}\right)\Bigg|_0^1$$

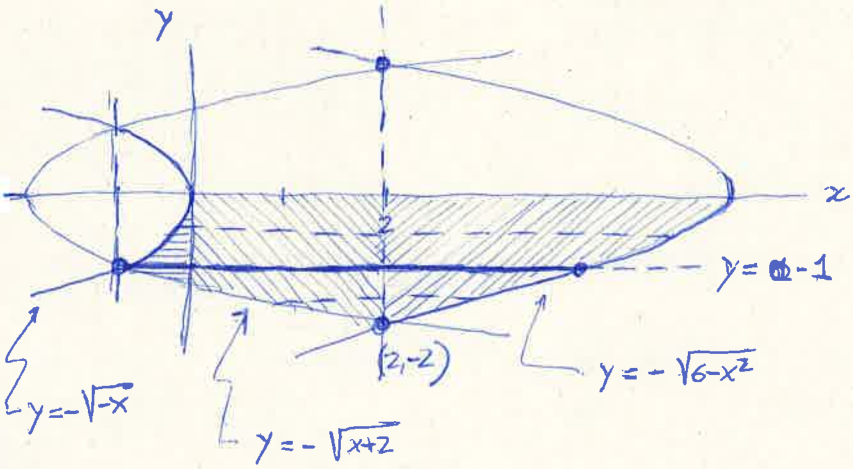
$$= \frac{1}{8} - \frac{1}{12} = \frac{3-2}{24} = \boxed{\frac{1}{24}}$$

$y=x$, $y=x^2$: $x^2=x$, $x(x-1)=0$, $x=0$, $x=1$

Gener 2005

Permuten l'ordre d'integració de la següent integral doble

$$I = \int_{-1}^{0} dx \int_{-\sqrt{x+2}}^{-\sqrt{-x}} f(x,y)\, dy + \int_{0}^{2} dx \int_{-\sqrt{x+2}}^{0} f(x,y)\, dy + \int_{2}^{6} dx \int_{-\sqrt{6-x}}^{0} f(x,y)\, dy$$



$$-\sqrt{-x} = -\sqrt{x+2}$$

$$-x = x+2 \Leftrightarrow 2x+2=0 \Leftrightarrow x=-1 ; y=-1$$

$$-\sqrt{x+2} = -\sqrt{6-x}$$

$$x+2 = 6-x$$

$$2x = 4 \Leftrightarrow x = 2$$

$$y=-2$$

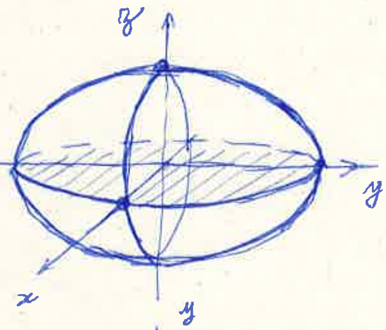
$$-y^2 \leq x \leq 6-y^2 \qquad -1 \leq y \leq 0$$

$$y^2-2 \leq x \leq 6-y^2 \qquad -2 \leq y \leq -1$$

$$I = \int_{-1}^{0} dy \int_{-y^2}^{6-y^2} f(x,y)\, dx + \int_{-2}^{-1} dy \int_{y^2-2}^{6-y^2} f(x,y)\, dx$$

15. (Col·lecció de problemes) Apartats b, c. Per a les regions de $\mathbb{R}^3$ indicades escriviu la integral triple $\iiint_A f(x,y,z)\, dx\,dy\,dz$ en termes d'integrals iterades preses en diferents ordres.

15 b) $I = \iiint_E f(x,y,z)\, dx\,dy\,dz$, $\quad E: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1 \quad (a,b,c>0)$



$$I_{1,1} = \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} f(x,y,z)\, dz$$

$$-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \leq z \leq +c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$$-b\sqrt{1-\frac{x^2}{a^2}} \leq y \leq b\sqrt{1-\frac{x^2}{a^2}}$$

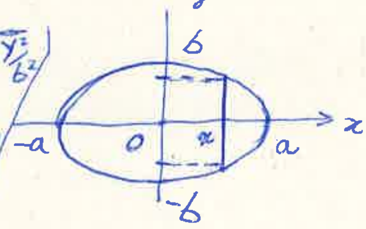
$$-a \leq x \leq a$$

$$I_{1,2} = \int_{-b}^{b} dy \int_{-a\sqrt{1-y^2/b^2}}^{+a\sqrt{1-y^2/b^2}} dx \int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} f(x,y,z)\, dz$$

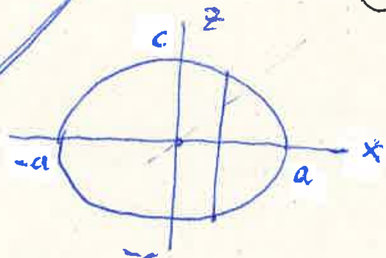
$$-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \leq z \leq c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$$-a\sqrt{1-\frac{y^2}{b^2}} \leq x \leq a\sqrt{1-\frac{y^2}{b^2}}$$

$$-b \leq y \leq b$$

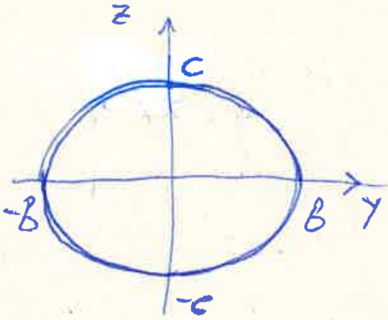


$$I_{2,1} = \int_{-a}^{a} dx \int_{-c\sqrt{1-x^2/a^2}}^{c\sqrt{1-x^2/a^2}} dz \int_{-b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}}}^{b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}}} f(x,y,z)\, dy$$



$$-b\sqrt{1-\tfrac{x^2}{a^2}-\tfrac{z^2}{c^2}} \le y \le b\sqrt{1-\tfrac{x^2}{a^2}-\tfrac{z^2}{c^2}}$$
$$-c\sqrt{1-\tfrac{x^2}{a^2}} \le z \le c\sqrt{1-\tfrac{x^2}{a^2}}$$
$$-a \le x \le a$$

$$I_{2,2} = \int_{-c}^{c} dz \int_{-a\sqrt{1-z^2/c^2}}^{a\sqrt{1-z^2/c^2}} dx \int_{-b\sqrt{1-x^2/a^2-z^2/c^2}}^{b\sqrt{1-x^2/a^2-z^2/c^2}} f(x,y,z)\, dy$$



$$I_{3,1} = \int_{-b}^{b} dy \int_{-c\sqrt{1-y^2/b^2}}^{c\sqrt{1-y^2/b^2}} dz \int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z)\, dx$$

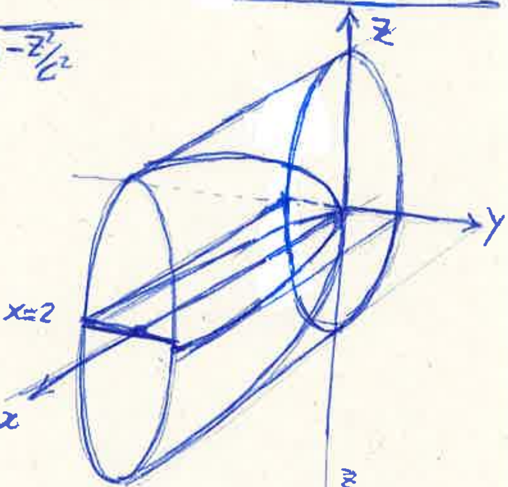
$$-a\sqrt{1-\tfrac{y^2}{b^2}-\tfrac{z^2}{c^2}} \le x \le a\sqrt{\dots}$$
$$-c\sqrt{1-\tfrac{y^2}{b^2}} \le z \le c\sqrt{1-\tfrac{y^2}{b^2}}$$
$$-b \le y \le b$$

$$I_{3,2} = \int_{-c}^{c} dz \int_{-b\sqrt{1-z^2/c^2}}^{b\sqrt{1-z^2/c^2}} dy \int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z)\, dx$$

$$-a\sqrt{1-\tfrac{y^2}{b^2}-\tfrac{z^2}{c^2}} \le x \le a\sqrt{\dots}$$
$$b\sqrt{1-\tfrac{z^2}{c^2}} \le y \le b\sqrt{1-\tfrac{z^2}{c^2}}$$
$$-c \le z \le c$$

Fubini:

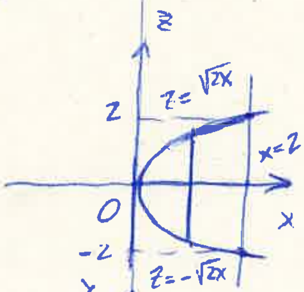
$$I_{1,1} = I_{1,2} = I_{2,1} = I_{2,2} = I_{3,1} = I_{3,2}$$



14c) $I = \iiint\limits_D f(x,y,z)\, dx\,dy\,dz$: D: $y^2+2z^2=4x,\ x=2$

$$I_{1,1} = \int_0^2 dx \int_{-\sqrt{2x}}^{\sqrt{2x}} dz \int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}} f(x,y,z)\, dy,$$

$$-\sqrt{4x-2z^2} \le y \le \sqrt{4x-2z^2}$$
$$-\sqrt{2x} \le z \le \sqrt{2x}$$
$$0 \le x \le 2$$

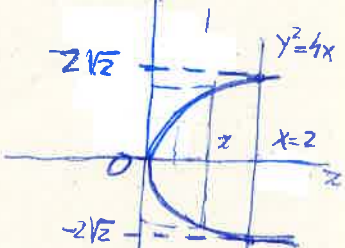


$$I_{1,2} = \int_{-2}^{2} dz \int_{z^2/2}^{2} dx \int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}} f(x,y,z)\, dy$$

$$-\sqrt{4x-2z^2} \le y \le \sqrt{4x-2z^2}$$
$$z^2/2 \le x \le 2$$
$$-2 \le z \le 2$$

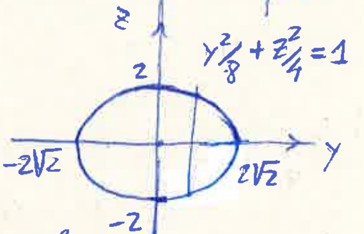
$$I_{2,1} = \int_0^2 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} dy \int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}} f(x,y,z)\, dz$$

$$-\sqrt{2x-\tfrac{y^2}{2}} \le z \le \sqrt{2x-\tfrac{y^2}{2}}$$
$$-2\sqrt{x} \le y \le 2\sqrt{x}$$
$$0 \le x \le 2$$



$$I_{2,2} = \int_{-2\sqrt2}^{2\sqrt2} dy \int_{y^2/4}^{2} dx \int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}} f(x,y,z)\, dz$$

$$-\sqrt{2x-y^2/2} \le z \le \sqrt{2x-y^2/2}$$
$$+y^2/4 \le x \le 2$$
$$-2\sqrt2 \le y \le 2\sqrt2$$



$$I_{3,1} = \int_{-2\sqrt2}^{2\sqrt2} dy \int_{-\sqrt{4-y^2/2}}^{\sqrt{4-y^2/2}} dz \int_{\frac{y^2}{4}+\frac{z^2}{2}}^{2} f(x,y,z)\, dx, \quad I_{3,2} = \int_{-2}^{2} dz \int_{-\sqrt{8-2z^2}}^{\sqrt{8-2z^2}} dy \int_{y^2/4+z^2/2}^{2} f(x,y,z)\, dx$$

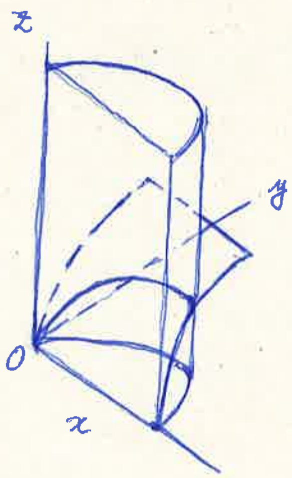
Fubini:

$$I_{1,1} = I_{1,2} = I_{2,1} = I_{2,2} = I_{3,1} = I_{3,2}.$$

15. Calculeu les integrals triples següents en les regions de $\mathbb{R}^3$ que s'indiquen,

(a) $I=\iiint_A xz\,dxdydz$. A limitat pel cilindre de base circular $x^2+y^2-2x=0$ i la superficie $z^2=2y$ $(y,z\geq 0)$

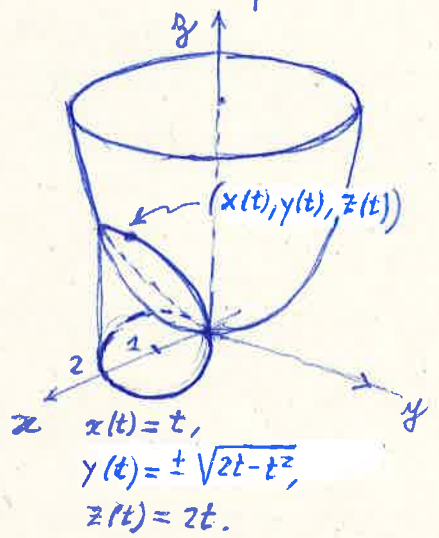
◁ Solució:



$$I=\int_0^2 x\,dx\int_0^{\sqrt{2x-x^2}} dy\int_0^{\sqrt{2y}} z\,dz=\frac{1}{2}\int_0^2 x\,dx\int_0^{\sqrt{2x-x^2}} 2y\,dy=\frac{1}{2}\int_0^2 x(2x-x^2)dx$$

$$=\frac{1}{2}\left(\frac{2}{3}x^3-\frac{x^4}{4}\right)\Bigg|_{x=0}^{x=2}=\frac{1}{2}\left(\frac{16}{3}-4\right)=\boxed{\frac{2}{3}}.\ ▷$$

(b) $I=\iiint_A zy\sqrt{x^2+y^2}\,dxdydz$ amb $A=\{(x,y,z)\in\mathbb{R}^3:\ 0\leq z\leq x^2+y^2,\ 0\leq y\leq\sqrt{2x-x^2}\}$



◁ Solució:

$$\left.\begin{array}{l} z=x^2+y^2\\ 2x=x^2+y^2\end{array}\right\}\rightarrow \begin{array}{l} z=2x\\ y=\pm\sqrt{2x-x^2}\end{array}\rightarrow\left\{\begin{array}{l} x=t,\\ y=\pm\sqrt{2t-t^2},\\ z=2t,\end{array}\right.\quad 0\leq t\leq 2.$$

Projecció sobre el pla x-y $(z=0)$:

$$\left.\begin{array}{l} x=t\\ y=\pm\sqrt{2t-t^2}\end{array}\right\}\Leftrightarrow (x-1)^2+y^2=1,$$

amb $y\geq 0$ en el nostre cas. Aleshores:

$$I=\int_0^2 dx\int_0^{\sqrt{2x-x^2}} dy\int_0^{x^2+y^2} zy\sqrt{x^2+y^2}\,dz$$

$$=\int_0^2 dx\int_0^{\sqrt{2x-x^2}}\frac{z^2}{2}y\sqrt{x^2+y^2}\Bigg|_{z=0}^{z=x^2+y^2}dy=\frac{1}{2}\int_0^2 dx\int_0^{\sqrt{2x-x^2}} y(x^2+y^2)^{5/2}dy$$

$$=\frac{1}{2}\cdot\frac{1}{7}\int_0^2 (x^2+y^2)^{7/2}\Bigg|_{y=0}^{y=\sqrt{2x-x^2}}dx=\frac{1}{14}\int_0^2\left[(\cancel{x^2}+2x-\cancel{x^2})^{7/2}-x^7\right]dx=\frac{1}{14}\left(2^{7/2}\cdot\frac{2}{9}x^{9/2}-\frac{x^8}{8}\right)\Bigg|_{x=0}^{x=2}$$

$$=\frac{1}{14}\left(2^8\cdot\frac{2}{9}-\frac{2^8}{8}\right)=\frac{2^8}{2\cdot 7}\left(\frac{16-9}{3^2\cdot 2^3}\right)=\frac{2^8\cdot\cancel{7}}{2^4\cdot 3^2\cdot\cancel{7}}=\frac{2^4}{3^2}=\boxed{\frac{16}{9}}\ ▷$$

(c) $I=\iiint_A dx\,dy\,dz$, $\quad A=\{(x,y,z)\in\mathbb{R}^3 : 1\le x\le 3,\ 1\le y\le 3,\ 0\le z\le xy\}$

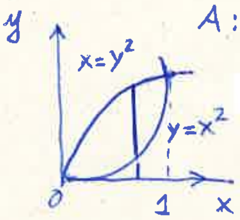
◁ Solució:

$$I=\int_1^3 dx\int_1^3 dy\int_0^{xy} dz=\int_1^3 dx\int_1^3 xy\,dy=\frac{1}{2}\int_1^3\left[xy^2\right]_{y=1}^{y=3}dx$$

$$=\frac{1}{2}\int_1^3 8x\,dx=4\int_1^3 x\,dx=4\left[\frac{x^2}{2}\right]_{x=1}^{x=3}=4\left(\frac{9}{2}-\frac{1}{2}\right)=\boxed{16}.\ ▷$$

(d) $I=\iiint_A xyz\,dx\,dy\,dz$, A limitat per les superfícies $y=x^2$, $x=y^2$, $z=xy$, $z=0$

◁ Solució: $$I=\int_0^1 x\,dx\int_{x^2}^{\sqrt{x}} dy\int_0^{xy} dz=\int_0^1 x\,dx\int_{x^2}^{\sqrt{x}} y\left[\frac{z^2}{2}\right]_0^{z=xy}dy=\frac{1}{2}\int_0^1 x\,dx\int_{x^2}^{\sqrt{x}} x^2y^3\,dy$$



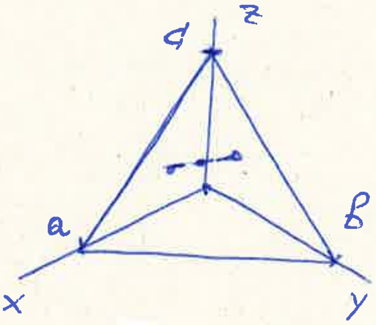
$A:$ $0\le z\le xy$, $x^2\le y\le\sqrt{x}$, $0\le x\le 1$

$$=\frac{1}{8}\int_0^1 x^3\left[y^4\right]_{y=x^2}^{y=\sqrt{x}}dx=\frac{1}{8}\int_0^1 x^3\left(x^2-x^8\right)dx$$

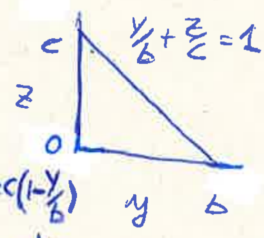
$$=\frac{1}{8}\left(-\frac{x^{12}}{12}+\frac{x^6}{6}\right)\Bigg|_{x=0}^{x=1}=\frac{1}{8}\cdot\frac{1}{12}=\boxed{\frac{1}{96}}.\ ▷$$

e) $I=\iiint_A x\,dx\,dy\,dz$, A: tetraedre limitat pels plans $x=0$, $y=0$, $z=0$, $x/a+y/b+z/c=1$, $(a,b,c>0)$.

◁ **Solució:** $$\begin{cases}0\le x\le a\left(1-\frac{y}{b}-\frac{z}{c}\right),\\ 0\le z\le c\left(1-\frac{y}{b}\right),\\ 0\le y\le b.\end{cases}\qquad A:\begin{cases}0\le y\le b\\ 0\le z\le c\left(1-\frac{y}{b}\right)\\ 0\le x\le a\left(1-\frac{y}{b}-\frac{z}{c}\right)\end{cases}$$



$$I=\iiint_A x\,dx\,dy\,dz=\int_0^b dy\int_0^{c(1-y/b)} dz\int_0^{a(1-y/b-z/c)} x\,dx$$



$$=\frac{a^2}{2}\int_0^b dy\int_0^{c(1-y/b)}\left(1-\frac{y}{b}-\frac{z}{c}\right)^2dz=-\frac{a^2}{2}\cdot\frac{c}{3}\cdot\int_0^b\left(1-\frac{y}{b}-\frac{z}{c}\right)^3\Bigg|_{z=0}^{z=c(1-y/b)}dy$$

$$=\frac{a^2}{2}\cdot\frac{c}{3}\int_0^b\left(1-\frac{y}{b}\right)^3dy=-\frac{a^2}{2}\cdot\frac{c}{3}\cdot\frac{b}{4}\left(1-\frac{y}{b}\right)\Bigg|_{y=0}^{y=b}=\boxed{\frac{a^2bc}{24}}.\ ▷$$

16) Useu coordenades polars per calcular les següents integrals dobles.

$$x = r\cos\theta \qquad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$
$$y = r\sin\theta$$

$$\iint_{T(D)} f(x,y)\,dx\,dy = \iint_D f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

a) $I = \iint_A (x^2+y^2)\,dx\,dy$, $A = \left\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 4\right\}$

◁ Solució

$$I = \int_0^{2\pi} d\theta \int_0^2 r^3\,dr = 2\pi\left[\frac{r^4}{4}\right]_0^2 = \boxed{8\pi}.$$ ▷

b) $I = \iint_A \cos(x^2+y^2)\,dx\,dy$, $A = \left\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le \frac{\pi}{2}\right\}$

◁ Solució

$$I = \int_0^{2\pi} d\theta \int_0^{\sqrt{\pi/2}} r\cos(r^2)\,dr = 2\pi\,\frac{1}{2}\left[\sin(r^2)\right]_0^{\sqrt{\pi/2}} = \boxed{\pi}.$$ ▷

c) $I = \iint_A \frac{(x+y)^2}{x^2+y^2+2}\,dx\,dy$, $A = \left\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1\right\}$

◁ Solució

$$I = \int_0^{2\pi} d\theta \int_0^1 \frac{r^3}{r^2+2}\,dr = 2\pi\int_0^1\left(1-\frac{r}{r^2+2}\right)dr = 2\pi\left[\frac{r^2}{2} - \ln(r^2+2)\right]_0^1$$

$$= 2\pi\left[\frac{1}{2} - \ln 3 + \ln 2\right] = \boxed{2\pi\left[\frac{1}{2} + \ln\left(\frac{2}{3}\right)\right]}.$$ ▷

d) $I = \iint_A \frac{dx\,dy}{(1+x^2+y^2)^2\sqrt{x^2+y^2}}$, $A = \left\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le R^2\right\}$. (Indicació: useu propietats elementals del sin i cos per veure que $\sin(\arctan(R)) = \frac{R}{\sqrt{1+R^2}}$ i $\cos(\arctan(R)) = \frac{1}{\sqrt{1+R^2}}$).

◁ Solució.

$$I=\int_0^{2\pi}d\theta\int_0^R\frac{r\,dr}{(1+r^2)^2\sqrt{r^2}}=2\pi\int_0^R\frac{dr}{(1+r^2)^2}=\left\{\begin{array}{l} r=\operatorname{tg}t \Rightarrow dr=\frac{dt}{\cos^2 t}\\ r=0 \Rightarrow t=0\\ r=R \Rightarrow t=\arctan(R)\end{array}\right.$$

$$=2\pi\int_0^{\arctan(R)}\cos^4 t\,\frac{dt}{\cos^2 t}=2\pi\int_0^{\arctan(R)}\frac{1+\cos(2t)}{2}\,dt=2\pi\left(\frac{t}{2}+\frac{\sin(2t)}{4}\right)\Bigg|_0^{\arctan(R)}$$

$$\overset{(*)}{=}\boxed{\pi\left(\arctan(R)+\frac{R}{1+R^2}\right)}\,.\triangleright$$

(*) Notem que: $\sin(2t)=2\sin t\cos t=\frac{2\sin t\cos^2 t}{\cos t}=2\operatorname{tg}t\cos^2 t=\frac{2\operatorname{tg}t}{1+\operatorname{tg}^2 t}$

e) $I=\iint_A\sqrt{x^2+y^2-9}\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2:\ 9\le x^2+y^2\le 25\}$,

◁ Solució.

$$I=\int_0^{2\pi}d\theta\int_3^5\sqrt{r^2-9}\,r\,dr=\frac{2\pi}{3}(r^2-9)^{3/2}\Big|_3^5=\frac{2\pi}{3}4^3=\boxed{\frac{128\pi}{3}}\,.\triangleright$$

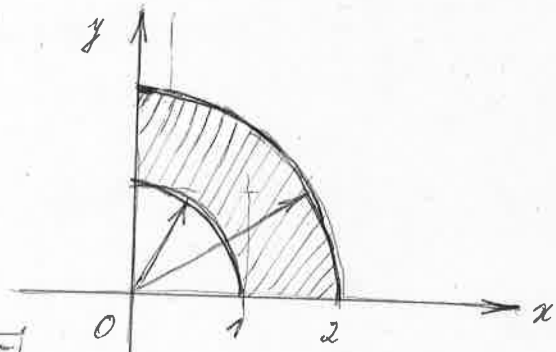
f) $I=\iint_A xy\,dx\,dy$, A intersecció amb el 1er quadrant de la corona circular de centre $(0,0)$, radi interior 1 i radi exterior 2

◁ Solució.

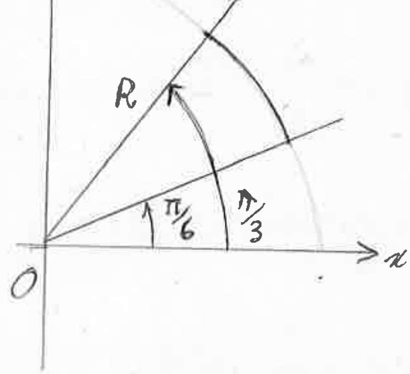
$$I=\int_0^{\pi/2}d\theta\int_1^2 r^3\cos\theta\sin\theta\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\sin(2\theta)\,d\theta\int_1^2 r^3\,dr$$

$$=\left[-\frac{1}{2}\cos(2\theta)\right]_0^{\pi/2}\cdot\left[\frac{r^4}{4}\right]_1^2=\frac{1}{2}\left(4-\frac{1}{4}\right)=\boxed{\frac{15}{8}}\,.\triangleright$$



g) $I=\iint_A x(x^2+y^2)\,dx\,dy$, A sector circular de centre $(0,0)$ i radi R format per angles entre $\frac{\pi}{3}$ i $\frac{\pi}{6}$ amb l'eix x positiu.



◁ Solució.

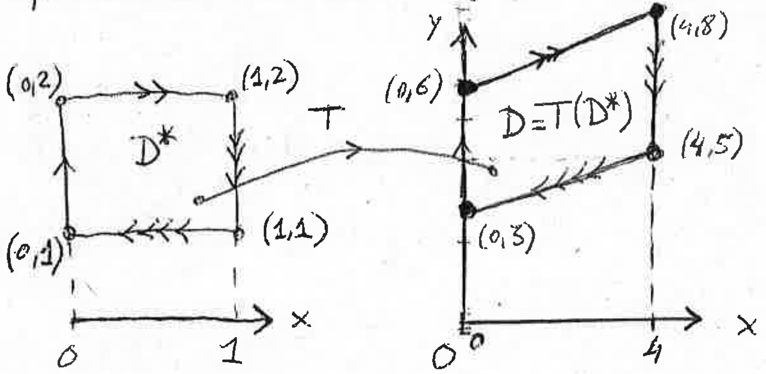
$$I=\int_{\pi/6}^{\pi/3}d\theta\int_0^R r^4\cos\theta\,dr=\int_{\pi/6}^{\pi/3}\cos\theta\,d\theta\int_0^R r^4\,dr$$

$$=\left[\sin\theta\right]_{\pi/6}^{\pi/3}\cdot\left[\frac{r^5}{5}\right]_0^R=\boxed{\frac{R^5}{10}(\sqrt{3}-1)}\,.\triangleright$$

18. Calculeu les integrals següents mitjançant el canvi de variable que s'indica a cada cas

(a) $I = \iint_D xy\,dxdy$, $D = \{(x,y) \in \mathbb{R}^2 ; 6 \le 2y - x,\ 0 \le x \le 4\}$, fent $x = 4u$ i $y = 2u + 3v$.

◁ Solució. $(x(u,v), y(u,v)) = T(u,v) = (4u, 2u+3v)$ es una aplicació afí al pla. Es comprova que la seva inversa $(u(x,y), v(x,y)) = T^{-1}(x,y) = (x/4, \frac{1}{3}(y - x/2))$ transforma el quadrilàter D en el rectangle $D^* = [0,1] \times [1,2]$ (veure figura).



$$I = \iint_{D=T(D^*)} xy\,dxdy = \iint_{D^*} x(u,v) \cdot y(u,v)\, |\det J_T(u,v)|\, du\,dv = \iint_{D^* = [0,1]\times[1,2]} 4u \left|\det\begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix}\right| du\,dv$$

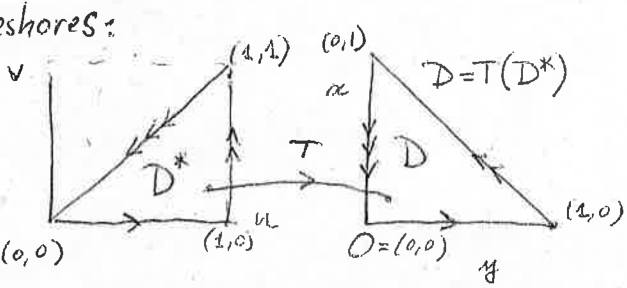
$$= \int_0^1 du \int_1^2 4u(2u+3v)\, 12\,dv = 48 \int_0^1 \left(2u^2 v + \tfrac{3}{2} u v^2\right)\Big|_{v=1}^{v=2} du$$

$$= 48 \int_0^1 \left(2u^2 + \tfrac{9}{2} u\right) du = 48 \left(\tfrac{2}{3} u^3 + \tfrac{9}{4} u^2\right)\Big|_{u=0}^{u=1} = 48\left(\tfrac{2}{3} + \tfrac{9}{4}\right)$$

$$= \frac{48}{12}(8+27) = 4 \cdot 35 = \boxed{140}. \triangleright$$

(b) $I = \iint_D \frac{dxdy}{(1+x+y)^5}$, $D = \{(x,y) \in \mathbb{R}^2 ; x \ge 0, y \ge 0, x+y \le 1\}$, fent $u = x+y$, $v = y$.

◁ Solució. La transformació $(u,v) = f(x,y) = (x+y, y)$ és una transformació afí del pla, per tant porta el triangle D de vèrtexs $(0,0)$, $(1,0)$ i $(0,1)$ en el triangle D^* de vèrtexs $(0,0)$, $(1,0)$ i $(1,1)$. Sigui $T(u,v) = f^{-1}(u,v) = (u-v, v)$. Aleshores:



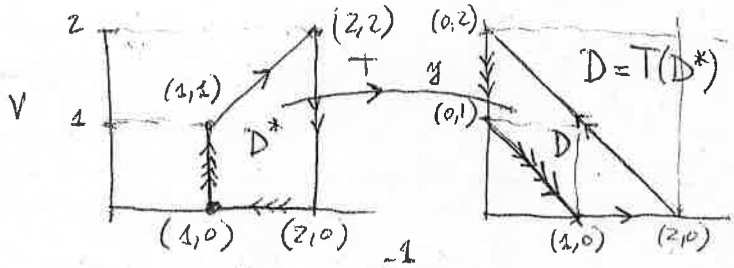
$$I = \iint_{D=T(D^*)} \frac{dxdy}{(1+x+y)^5} = \iint_{D^*} \frac{1}{(1+u)^5} |\det J_T(u,v)|\, du\,dv$$

$$= \int_0^1 \frac{du}{(1+u)^5} \int_0^u dv = \int_0^1 \frac{1+u}{(1+u)^5} du - \int_0^1 \frac{du}{(1+u)^5}$$

$$= \int_0^1 \frac{du}{(1+u)^4} - \int_0^1 \frac{du}{(1+u)^5} = -\frac{1}{3}(1+u)^{-3} + \frac{1}{4}(1+u)^{-4}\Big|_{u=0}^{u=1} = -\frac{1}{3} 2^{-3} + \frac{1}{4} 2^{-4} + \frac{1}{3} - \frac{1}{4} = \boxed{\frac{11}{192}}. \triangleright$$

(c) $\displaystyle\iint_D \frac{dx\,dy}{(x+y)^{n+1}}$, $D=\{(x,y)\in\mathbb{R}^2 : 1\le x+y\le 2,\ x\ge 0\}$, fent $u=x+y$, $v=x$.

◁ Solució. La transformació afí del pla $(u,v)=f(x,y)=(x+y,x)$ transforma el quadrilàter D en el quadrilàter D^* de vèrtexs $(u,v)=(1,0),(1,1),(2,2),(2,0)$ (veure figura).



Signi: $T(u,v)=f^{-1}(u,v)=(v,u-v)$. Aleshores $D=T(D^*)$, i

$$I=\iint_{D=T(D^*)}\frac{dx\,dy}{(x+y)^{n+1}}=\iint_{D^*}\frac{1}{u^{n+1}}\left|\det J_T(u,v)\right|du\,dv$$

$$=\iint_{D^*}\overset{u}{\frac{1}{u^{n+1}}}\left|\overbrace{\det\begin{pmatrix}0&1\\1&-1\end{pmatrix}}^{-1}\right|du\,dv=\int_1^2\frac{du}{u^{n+1}}\int_0^u dv=\int_0^1\frac{du}{u^n}=\frac{1}{1-n}u^{1-n}\Big|_1^2=\frac{1}{1-n}\left(2^{1-n}-1\right)$$

$$=\boxed{\frac{1}{n-1}\left(1-\frac{1}{2^{n+1}}\right)}\ .▷$$

(e) $\displaystyle I=\iint_D \arctan\left(x^2+y^2/2\right)dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2/2\le 1;\ x\ge 0, y\ge 0\}$, fent $x=r\cos\theta$, $y=r\sqrt{2}\cos\theta$.

◁ Solució. $0\le x^2+y^2/2=r^2\le 1$. La transformació $x=r\cos\theta$, $y=r\sqrt{2}\sin\theta$ porta la regió $x^2+y^2/2\le 1$ a disc $\bar{B}_1(0,0)$, amb centre l'origen i radi $r=1$. Com que d'altra banda $x\ge 0; y\ge 0$ en D, llavors $0\le\theta\le\pi/2$, amb la qual cosa, en coordenades polars, el domini D s'expressa com D^*: $0\le r\le 1$, $0\le\theta\le\pi/2$. Aleshores:

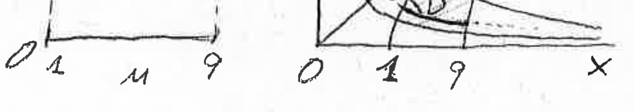
$$I=\iint_{D=T_P(D^*)}\arctan\left(x^2+y^2/2\right)dx\,dy=\iint_{D^*=[0,1]\times[0,\pi/2]}\arctan(r^2)\left|\det J_{T_P}(r,\theta)\right|dr\,d\theta=\iint_{D^*=[0,1]\times[0,\pi/2]}\arctan(r^2)\underbrace{\begin{vmatrix}\cos\theta & -\sin\theta\\ \sqrt{2}\sin\theta & -\sqrt{2}r\cos\theta\end{vmatrix}}_{=\sqrt{2}\,r(\cos^2\theta+\sin^2\theta)=\sqrt{2}\,r}dr$$

$$=\sqrt{2}\int_0^{\pi/2}d\theta\int_0^1\arctan(r^2)\,dr=\frac{\pi\sqrt{2}}{2}\left(\frac{r^2}{2}\arctan(r^2)\Big|_{r=0}^{r=1}-\int_0^1\frac{r^3}{1+r^4}dr\right)$$

$$=\frac{\pi\sqrt{2}}{2}\left(\frac{\pi}{8}-\left\{\frac{1}{4}\ln(1+r^4)\right\}_{r=0}^{r=1}\right)=\frac{\pi\sqrt{2}}{2}\left(\frac{\pi}{8}-\frac{1}{4}\ln 2\right)=\boxed{\frac{\sqrt{2}\pi}{8}\left(\frac{\pi}{2}-\ln 2\right)}\ .▷$$

(f) $\displaystyle I=\iint_D (x^2+y^2)\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2 : 1\le x^2-y^2\le 9,\ 2\le xy\le 4,\ x\ge 0, y\ge 0\}$, fent $u=x^2-y^2$, $v=2xy$.

◁ Solució. Signi $(u,v)=f(x,y)=(x^2-y^2,xy)$. Aquesta transformació estableix un difeomorfisme entre la regió $D\subseteq\mathbb{R}^2$ i el rectangle



$D^* = [1,9]\times[4,8]$. Sigui $T = f^{-1}$ l'aplicació inversa. Aleshores escrivim $D=T(D^*)$.

Nota: sigui $(x,y)\in D$. Llavors: $1\le u = x^2-y^2 \le 9$, $2\le v = xy \le 4$, $x>0$; $y>0$. Per tant: $D^* \supseteq T^{-1}(D)$. D'altra banda, per a cada $(u,v)\in D^* = [1,9]\times[4,8]$ $\exists!\,(x,y)\in D$ t.q. $T^{-1}(x,y)=(u,v)$; la qual cosa implica també que $T^{-1}(D)\supseteq D^*$ i, per tant, $D^* = T^{-1}(D)$. $D^*=T^{-1}(D)$. Es comprova, fent els càlculs, que

$$x(u,v) = \sqrt{\frac{u+\sqrt{u^2+4v^2}}{2}}, \qquad y(u,v) = \frac{\sqrt{2}\,v}{\sqrt{u+\sqrt{u^2+4v^2}}}, \quad (u,v) = [1,9]\times[4,8]$$

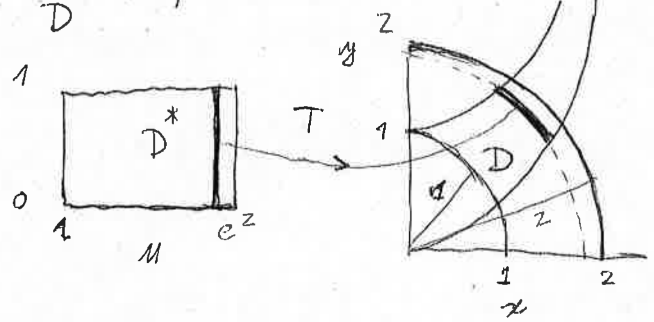
Així:

$$I = \iint_{D=T(D^*)} (x^2+y^2)\,dx\,dy = \iint_{D^*=[1,9]\times[4,8]} \left(x^2(u,v)+y^2(u,v)\right)\left|\det{}_T(u,v)\right| du\,dv$$

$$= \iint_{D^*=[1,9]\times[4,8]} \left(x^2(u,v)+y^2(u,v)\right)\frac{du\,dv}{\left|\det \underset{T^{-1}}{J}\,(x(u,v),y(u,v))\right|} = \iint_{D^*=[1,9]\times[2,4]} \left(x^2(u,v)+y^2(u,v)\right)\cdot\frac{du\,dv}{\left|\det\begin{pmatrix} 2x(u,v) & -2y(u,v)\\ 2y(u,v) & 2x(u,v)\end{pmatrix}\right|}$$

$$= \iint_{D^*=[1,9]\times[4,8]} \frac{x^2(u,v)+y^2(u,v)}{4\left(x^2(u,v)+y^2(u,v)\right)}\,du\,dv = \frac{1}{4}\iint_{D^*=[1,9]\times[4,8]} du\,dv = \frac{1}{4}(9-1)\cdot(8-4) = \boxed{8}\,.\triangleright$$

(g) $I = \displaystyle\iint_D \frac{x+2xy}{x^2+y^2}\,dx\,dy$, $D = \{(x,y)\in\mathbb{R}^2 : x^2\le y\le x^2+1,\ 1\le x^2+y^2\le e^2,\ x\ge 0\}$, fent $u=x^2+y^2$ i $v=y-x^2$.



◁ **Solució.** L'aplicació $(u,v) = f(x,y) = (x^2+y^2,\, y-x^2)$ estableix un difeomorfisme de la regió D en el rectangle $D^* = [1,e^2]\times[0,1]$. Sigui $f^{-1} = T$. Escrivim $D=T(D^*)$. Aleshores:

$$I = \iint_{D=T(D^*)} \frac{x(u,v)+2x(u,v)\cdot y(u,v)}{x^2(u,v)+y^2(u,v)}\left|\det J_T(u,v)\right| du\,dv$$

$$= \iint_{D^*=[1,e^2]\times[0,1]} \frac{x(u,v)+2x(u,v)\,y(u,v)}{x^2(u,v)+y^2(u,v)}\cdot\frac{du\,dv}{\left|\det J_{T^{-1}}(x(u,v),y(u,v))\right|} = \iint_{D^*=[1,e^2]\times[0,1]} \frac{x(u,v)+2x(u,v)\,y(u,v)}{x^2(u,v)+y^2(u,v)}\cdot\frac{du\,dv}{\begin{vmatrix} 2x(u,v) & 2y(u,v)\\ -2x(u,v) & 1\end{vmatrix}}$$

$$= \iint_{D^*=[1,e^2]\times[0,1]} \frac{x(u,v)+2x(u,v)\,y(u,v)}{x^2(u,v)+y^2(u,v)}\cdot\frac{du\,dv}{2\left(x(u,v)+2x(u,v)\,y(u,v)\right)} = \frac{1}{2}\iint_{D^*=[1,e^2]\times[0,1]} \frac{du\,dv}{u} = \frac{1}{2}\int_1^{e^2}\frac{du}{u}\int_0^1 dv$$

$$= \frac{1}{2}\ln u\,\Big|_{u=1}^{u=e^2} = \boxed{1}\,.\triangleright$$

Examen final. Gener 2019

1 [2 punts] Troben el valor de les integrals dobles següents als dominis que s'indiquen

(a) $\iint_C \frac{|y|}{x^2+y^2}\,dx\,dy$. $C=\{(x,y)\in\mathbb{R}^2: x^2+y^2\le 2,\ x^2+y^2\le 2x\}$. [1 punt]

(b) $\iint_C \frac{xy(x^2+y^2)}{xy+1}(x^2-y^2)^{xy-1}$. $C=\{(x,y)\in\mathbb{R}^2;\ y\geq 0,\ x\geq y,\ xy\le 1,\ x^2-y^2\le 1\}$.

En aquest cas, considereu el canvi $u=x^2-y^2$, $v=xy$. Llavors:

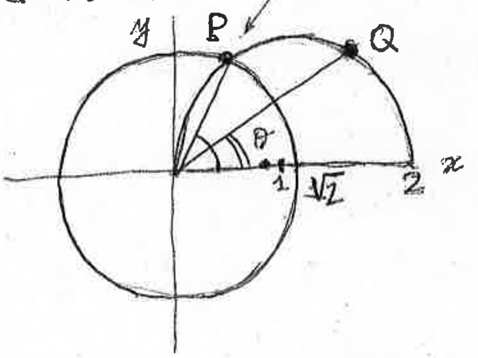
(i) Calculeu el determinant de la seva matriu Jacobiana (0.25 punts)

(ii) Expresseu el domini C en les noves variables (u,v) (0.25 punts)

(iii) Finalment, doneu el valor de la integral (0.5 punts)

◁ Solució (a)

$2\cos\theta^* = \sqrt{2} \Leftrightarrow \theta^* = \pi/4$



En coordenades polars, el domini C ve determinat per C': $\sqrt{2}\le r\le 2\cos\theta$, $0\le\theta\le\pi/4$. Amb això:

$$I=\iint_C \frac{|y|}{x^2+y^2}\,dx\,dy=\int_{-\pi/4}^{\pi/4}d\theta\int_{\sqrt{2}}^{2\cos\theta}\frac{r|\sin\theta|}{r^2}\,r\,dr$$

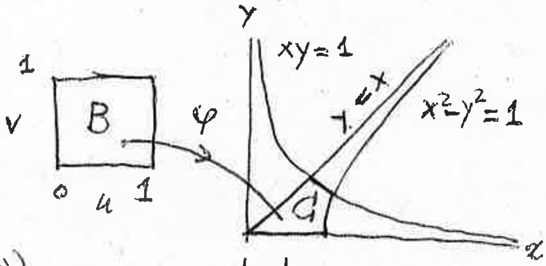
$$=\int_{-\pi/4}^{\pi/4}|\sin\theta|\,d\theta\int_{\sqrt{2}}^{2\cos\theta}dr=\int_{-\pi/4}^{\pi/4}|\sin\theta|(2\cos\theta-\sqrt{2})\,d\theta$$

$$=2\int_0^{\frac{\pi}{4}}(2\cos\theta\sin\theta-\sqrt{2}\sin\theta)\,d\theta=2\int_0^{\pi/4}\sin(2\theta)\,d\theta-2\sqrt{2}\int_0^{\frac{\pi}{4}}\sin\theta\,d\theta=\left(-\cos(2\theta)+2\sqrt{2}\cos\theta\right)\Big|_0^{\frac{\pi}{2}}$$

$$=1-\cos\left(\frac{\pi}{2}\right)+2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)-1\right)=1-0+2\sqrt{2}\left(\frac{1}{\sqrt{2}}-1\right)=1+2-2\sqrt{2}=\boxed{3-2\sqrt{2}}\ .\triangleright$$

◁ Solució (b).

Es comprova que $(u,v)=\varphi^{-1}(x,y)=(xy,x^2-y^2)$ estableix un difeomorfisme entre la regió $C\subseteq\mathbb{R}^2$ i el rectangle $B=[0,1]^2=[0,1]\times[0,1]$. Aplicant aquest canvi tenim:



$$I=\iint_{C=\varphi(B)}\frac{xy(x^2+y^2)}{xy+1}(x^2-y^2)^{xy-1}\,dx\,dy=\iint_{B=[0,1]^2}\frac{v\cdot(x^2(u,v)+y^2(u,v))}{v+1}u^{v-1}\cdot\frac{du\,dv}{|\det J_{\varphi^{-1}}(x(u,v),y(u,v))|}=$$

$$=\iint_{B=[0,1]^2}\frac{v(x^2(u,v)+y^2(u,v))}{v+1}\cdot u^{v-1}\frac{du\,dv}{2\begin{vmatrix}2x(u,v) & -2y(u,v)\\ y(u,v) & x(u,v)\end{vmatrix}}=\frac{1}{2}\iint_{B=[0,1]^2}\frac{v\cancel{(x^2(u,v)+y^2(u,v))}}{v+1}\cdot\frac{u^{v-1}\,du\,dv}{\cancel{x^2(u,v)+y^2(u,v)}}$$

$$=\frac{1}{2}\int_0^1\frac{dv}{v+1}\,du\int_0^1 u^{v-1}\,du=\frac{1}{2}\int_0^1\frac{v}{v+1}\left[\frac{u^v}{v}\right]_{u=0}^{u=1}dv=\frac{1}{2}\int_0^1\frac{dv}{1+v}=\frac{1}{2}\ln(1+v)\Big|_{v=0}^{v=1}=\boxed{\frac{\ln 2}{2}}\ .\triangleright$$

20. Useu coordenades esfèriques per calcular les següents integrals triples

(a) $I=\iiint_B x^4y^2z^3\,dx\,dy\,dz$; $B=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le a^2\}=B_a(0,0,0)$ $(a>0)$

◁ Solució. $I=\int_0^{2\pi}\cos^4\theta\sin^2\theta\,d\theta\int_{-\pi/2}^{\pi/2}\cos^2\varphi\sin^3\varphi\,d\varphi\int_0^a r^{11}\,dr=0$. ▷

(b) $I=\iiint_B z(x^2+y^2)\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3 ; x^2+y^2+z^2\le a^2, z\ge 0\}$

◁ Solució. $I=\int_0^{2\pi}d\theta\int_0^{\pi/2}\cos^3\varphi\sin\varphi\int_0^a r^5\,dr=2\pi\left[-\frac{\cos^4\varphi}{4}\right]_0^{\pi/2}\cdot\left[\frac{r^6}{6}\right]_0^a=\boxed{\frac{\pi a^6}{12}}$. ▷

(c) $I=\iiint_B\frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$, $B=\{(x,y,z)\in\mathbb{R}^3 : a^2\le x^2+y^2+z^2\le b^2\}$

◁ Solució. $I=\int_0^{2\pi}d\theta\int_{-\pi/2}^{\pi/2}\cos\varphi\,d\varphi\int_a^b r^2\frac{dr}{r^3}=\boxed{4\pi\ln\left(\frac{b}{a}\right)}$. ▷

(d) $I=\iiint_B\sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz$, B el domini de l'apartat anterior.

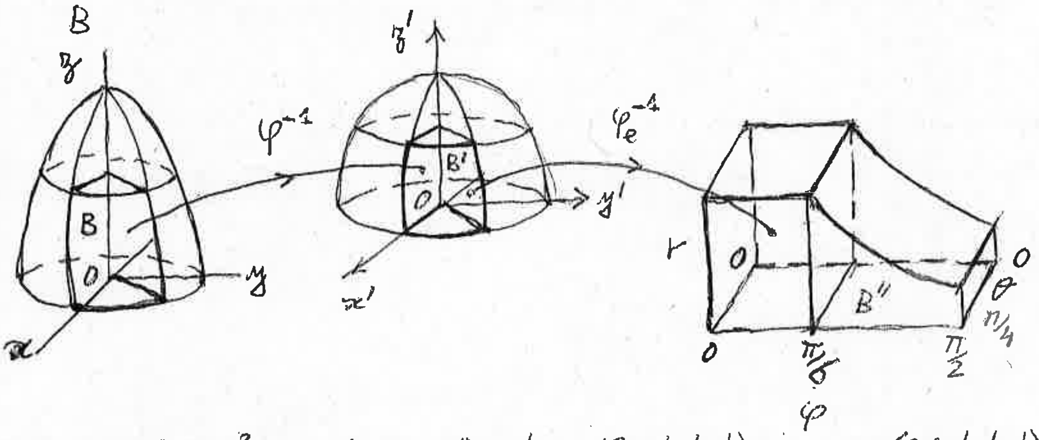
◁ Solució. $I=\int_0^{2\pi}d\theta\int_{-\pi/2}^{\pi/2}\cos\varphi\,d\varphi\int_a^b r^3e^{-r^2}\,dr=4\pi\int_a^b r^3e^{-r^2}\,dr\overset{(*)}{=}\boxed{2\pi\left(e^{-a^2}(a^2+1)-e^{-b^2}(b^2+1)\right)}$. ▷

(*) Parts: $\int_a^b r^3e^{-r^2}\,dr=-\frac{1}{2}\int_a^b r^2\,d\left(e^{-r^2}\right)=\left[-\frac{r^2e^{-r^2}}{2}\right]_a^b+\int_a^b re^{-r^2}\,dr$ $=\left[-\frac{1}{2}e^{-r^2}\right]_a^b$

$$=\frac{a^2e^{-a^2}}{2}-\frac{b^2e^{-b^2}}{2}+\frac{e^{-a^2}}{2}-\frac{e^{-b^2}}{2}=\frac{1}{2}(a^2+1)e^{-a^2}-(b^2+1)\frac{e^{-b^2}}{2}$$

22. Adapteu les coordenades esfèriques per calcular les següents integrals triples

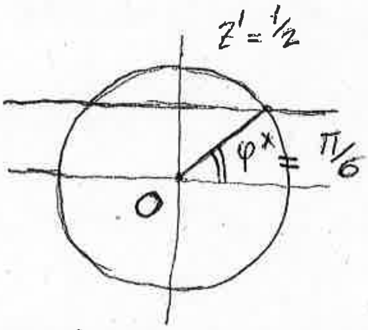
(a) $\iiint_B 16z\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+\frac{z^2}{4}\le 1, 0\le z\le 1, 0\le y\le x\}$.



- $B=\varphi(B')$ amb $x=\varphi_1(x',y',z')=x'$, $y=\varphi_2(x',y',z')=y'$, $z=\varphi_3(x',y',z')=2z'$. Llavors $B'=\{(x',y',z')\in\mathbb{R}^3 : (x')^2+(y')^2+(z')^2\le 1, 0\le z'\le \frac{1}{2}, 0\le y'\le x'\}$.

$$I=\iiint_{B=\varphi(B')}16z\,dx\,dy\,dz=\iiint_{B'}32z'\left|\det J_\varphi(x',y',z')\right|dx'dy'dz'=64\iiint_{B'}z'\,dx'dy'dz'$$

A continuació canviem a coord. esfèriques



$$x' = r\cos\theta\cos\varphi \qquad \left\{ \begin{array}{l} r = 1,\ z = 1/2 \\ \sin\varphi^* = 1/2 \text{ amb } 0 \leq \varphi^* \leq \pi/2 \\ \iff \varphi^* = \frac{\pi}{6} \end{array} \right\}$$
$$y' = r\sin\theta\cos\varphi$$
$$z' = r\sin\varphi$$

$$B' = \varphi_e(B''),\quad B'' = \left\{ (r,\theta,\varphi) \in \mathbb{R}^+ \times [0,2\pi) \times (-\pi/2, \pi/2) : 0 \leq \theta \leq \pi/4,\ 0 \leq \varphi \leq \frac{\pi}{6},\ 0 < r \leq 1 \right\}$$
$$\cup \left\{ (r,\theta,\varphi) \in \mathbb{R}^+ \times [0,2\pi) \times (-\pi/2, \pi/2) : 0 \leq \theta \leq \pi/4,\ \pi/2 \leq \varphi \leq \frac{\pi}{6},\ 0 < r \leq \frac{1/2}{\sin\varphi} \right\}$$

$$I = 64 \int_0^{\pi/4} d\theta \int_0^{\pi/6} \cos\varphi \sin\varphi \, d\varphi \int_0^1 r^3 dr + 64 \int_0^{\pi/4} d\theta \int_{\pi/6}^{\pi/4} \sin\varphi \cos\varphi \, d\varphi \int_0^{\frac{1/2}{\sin\varphi}} r^3 dr$$
$$= 64 \cdot \frac{\pi}{4} \cdot \frac{1}{8} \sin^2\left(\frac{\pi}{6}\right) + \frac{\pi}{4} \cdot \frac{64}{64} \int_{\pi/6}^{\pi/2} \frac{\cos\varphi \, d\varphi}{\sin^3\varphi} = \frac{\pi}{2} + \frac{\pi}{4}\left(\frac{-1}{\sin^2\varphi}\right)\Bigg|_{\pi/6}^{\pi/2} = \frac{\pi}{2} + \frac{\pi}{4}(4-1) = \boxed{\frac{7\pi}{8}} . \triangleright \ (\&_1)$$

(b) $\displaystyle\iiint_B \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dx\,dy\,dz,\ B = \left\{ (x,y,z) \in \mathbb{R}^3 : x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1 \right\}$

$\triangleleft$ Solució. Apliquem coordenades esfèriques adaptades $\left\{ \begin{array}{l} x = ar\cos\theta\cos\varphi \\ y = br\sin\theta\cos\varphi \\ z = cr\sin\varphi \end{array} \right\}$. En aquestes coordenades el domini B es transforma en:

$$B' : 0 < r \leq 1,\ -\pi/2 \leq \varphi \leq \pi/2,\ 0 \leq \theta < 2\pi.$$

$$I = \iiint_{B = \hat{\varphi}_e(B')} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dx\,dy\,dz = \iiint_{B'} r^2 \left| \det J_{\hat{\varphi}_e}(r,\theta,\varphi) \right| dr\,d\theta\,d\varphi$$
$$= abc \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi \, d\varphi \int_0^1 r^4 dr = abc\,(2\pi)\,2\left(\frac{r^5}{5}\right)\Bigg|_{r=0}^{r=1} = \boxed{\frac{4}{5}\pi abc} . \triangleright$$

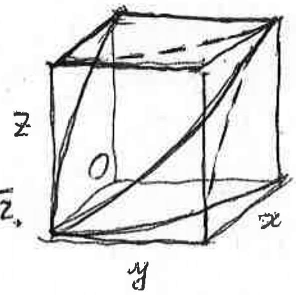
($\&_1$) En coordenades cilíndriques:

$$I = \int_0^{\frac{\pi}{4}} d\theta \int_0^1 16 z \, dz \int_0^{\sqrt{1 - z^2/4}} r \, dr = 16 \frac{\pi}{8} \int_0^1 z\left(1 - \frac{z^2}{4}\right) dz = 16 \frac{\pi}{8} \left(\frac{z^2}{2} - \frac{z^4}{16}\right)\Bigg|_{z=0}^{z=1}$$
$$= 2\pi \cdot \frac{7}{16} = \boxed{\frac{7\pi}{8}}.$$

... Veiem que el càlcul és més "directe".

23. Useu coordenades cartesianes, cilíndriques o esfèriques (o bé el principi de Cavalieri) per calcular el volum dels dominis de $\mathbb{R}^3$ limitats per les superfícies que s'indiquen.

a) $x^2+z^2=1,\ x^2+y^2=1.$



◁ Solució:

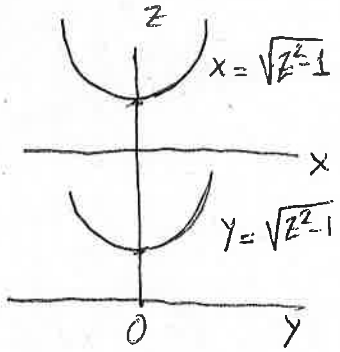
$$\frac{V}{8}=\int_0^1 dx\int_0^{\sqrt{1-x^2}} dy\int_0^{\sqrt{1-x^2}} dz=\int_0^1 dx\int_0^{\sqrt{1-x^2}}\sqrt{1-x^2}\,dy$$

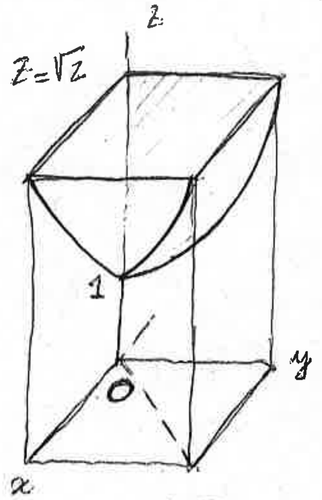
$$=\int_0^1(1-x^2)\,dx=\left(x-\tfrac{x^3}{3}\right)\Big|_{x=0}^{x=1}$$

$$=1-\tfrac{1}{3}=\tfrac{2}{3}\Rightarrow \boxed{V=\tfrac{16}{3}}\ .\triangleright$$

b) $z^2-x^2=1,\ z^2-y^2=1,\ z=\sqrt{2}.$

◁ Solució:





$$\frac{V}{4}=\int_1^{\sqrt{2}} dz\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dy\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dx=\int_1^{\sqrt{2}} dz\int_0^{\sqrt{z^2-1}}\sqrt{z^2-1}\,dx=\int_1^{\sqrt{2}}(z^2-1)\,dz=\left(\tfrac{z^3}{3}-z\right)\Big|_{z=1}^{z=\sqrt{2}}$$

$$=\frac{2\sqrt{2}}{3}-\sqrt{2}-\frac{1}{3}+1=\tfrac{2}{3}-\tfrac{\sqrt{2}}{3}=\frac{1}{3}(2-\sqrt{2})\Leftrightarrow \boxed{V=\tfrac{4}{3}(2-\sqrt{2})}\ .\triangleright$$

c) $z^2=x^2+y^2,\ z=x^2+y^2\ (z\geq 0).$

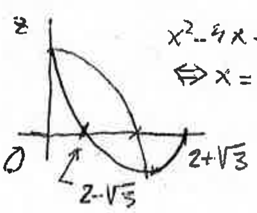
◁ Solució: $I=\int_0^{2\pi}d\theta\int_0^1 r\,dr\int_{r^2}^{r} dz=2\pi\int_0^1 r(r-r^2)\,dr=2\pi\left(\tfrac{r^3}{3}-\tfrac{r^4}{4}\right)\Big|_0^1=2\pi\,\frac{4-3}{12}=\boxed{\frac{\pi}{6}}\ .\triangleright$

d) Part de l'esfera $x^2+y^2+z^2=a^2$ que és exterior al cilindre $x^2+y^2=b^2$ $(a>b>0)$.

◁ Solució: $V=\int_0^{2\pi}d\theta\int_b^a r\,dr\int_{z=-\sqrt{a^2-r^2}}^{z=\sqrt{a^2-r^2}} dz=4\pi\int_b^a r\sqrt{a^2-r^2}\,dr=-\frac{4\pi}{3}(a^2-r^2)^{3/2}\Big|_b^a=\boxed{\frac{4\pi}{3}(a^2-b^2)^{3/2}}$

e) $z=x^2-4x+1,\ 1-z=x^2+y^2$

◁ Solució:



$x^2-4x+1=0$

$\Leftrightarrow x=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}$

$z=1-x^2-y^2=x^2-4x+1\Leftrightarrow 2x^2-4x+y^2=2(x-1)^2+y^2-2=0$

$\Leftrightarrow (x-1)^2+\frac{y^2}{2}=1$

$$V = \iiint_{C} 1\, dx\,dy\,dz = \iint_{\substack{\mathcal{E}:\ (x,y)\in\mathbb{R}^2 \\ (x-1)^2+y^2/2\le 1}} dx\,dy \int_{x^2-4x+1}^{1-x^2-y^2} dz = 2\iint_{\mathcal{E}} \left[1-(x-1)^2-y^2/2\right] dx\,dy = (\&\&)$$

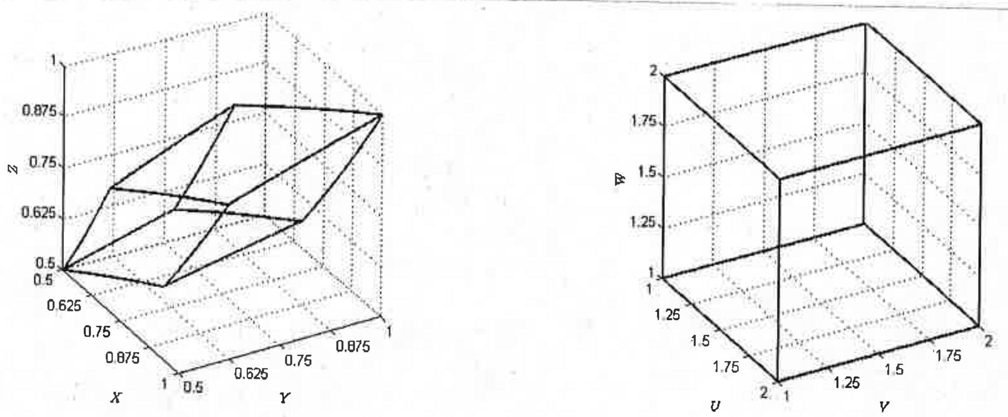
Coordenades polars adaptades: $\varphi: \left\{ \begin{array}{l} x-1 = r\cos\varphi \\ y=\sqrt{2}\, r\sin\theta,\ 0\le\theta\le 2\pi,\ 0<r<1, \end{array}\right\}$

$$\det J_{\varphi}(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sqrt{2}\sin\theta & \sqrt{2}\,r\cos\theta \end{vmatrix} = \sqrt{2}\,r.$$

$$(\&\&) = 2\sqrt{2}\int_0^{2\pi} d\theta \int_0^1 (1-r^2)\,r\,dr = 2\sqrt{2}\,(2\pi)\int_0^1 (r-r^3)\,dr = 4\pi\sqrt{2}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Bigg|_{r=0}^{r=1} = 4\sqrt{2}\,\frac{\pi}{4}$$

$$= \boxed{\pi\sqrt{2}}\ .\ \triangleright$$

f) $x^2=z,\ y^2=x,\ z^2=y,\ x^2=az,\ y^2=ax,\ z^2=ay\ (a>1)$.



(a) $\mathcal{D}$: Domini definit per les superfícies,
$x^2=z,\quad y^2=x,\quad z^2=y,$
$x^2=az,\quad y^2=ax,\quad z^2=ay.$

(b) $\mathcal{D}': 1\le u\le a,\ 1\le v\le a,\ 1\le w\le a.$

FIGURA 1. Transformació del domini $\mathcal{D}$ en $\mathcal{D}'$ pel canvi (1).

◁ *Solució:* es comprova d'immediat que el canvi

$$u=\frac{y^2}{x},\qquad v=\frac{z^2}{y},\qquad w=\frac{x^2}{z},\tag{1}$$

transforma el domini original $\mathcal{D}$, en $\mathcal{D}'=[1,a]\times[1,a]\times[1,a]$. És a dir, en un cub d'aresta $a-1$ (veure Figura 1). Aleshores el Jacobià corresponent surt més senzill. En efecte:

$$\det\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -\dfrac{y^2}{x^2} & 2\dfrac{y}{x} & 0 \\ 0 & -\dfrac{z^2}{y^2} & 2\dfrac{z}{y} \\ 2\dfrac{x}{z} & 0 & -\dfrac{x^2}{z^2} \end{vmatrix} = -7,$$

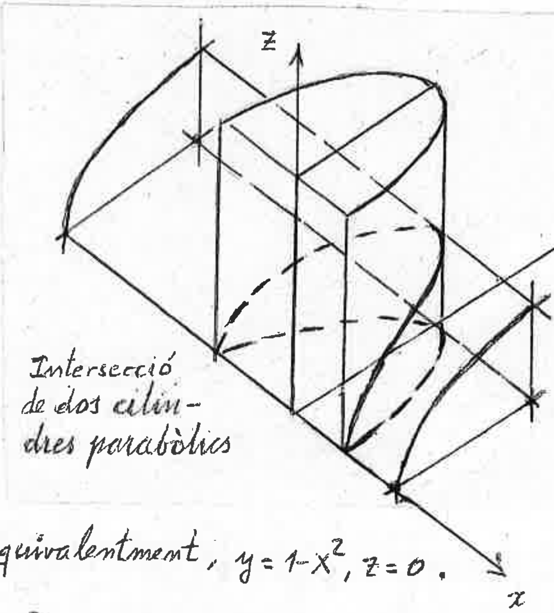
d'on:

$$\left|\det\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{\left|\det\dfrac{\partial(u,v,w)}{\partial(x,y,z)}\right|} = \frac{1}{7}.$$

Llavors el càlcul del volum es simplifica encara més:

$$V=\iiint_{\mathcal{D}} dx\,dy\,dz = \iiint_{\mathcal{D}'}\left|\det\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$
$$=\frac{1}{7}\int_1^a du\int_1^a dv\int_1^a dw = \frac{1}{7}\left(\int_1^a du\right)^3 = \frac{(a-1)^3}{7}.\ \triangleright$$

g) $z^2 = y, \quad x^2 = 1-y$



◁ Solució.

parametrització de la intersecció per a $x \geq 0, z \geq 0$

$$\left.\begin{aligned} x &= \sqrt{1-t} \\ y &= t \\ z &= \sqrt{t} \end{aligned}\right\} : \quad 0 \leq t \leq 1$$

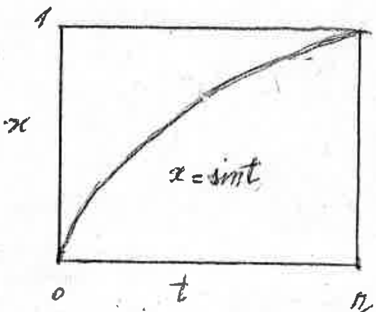
Projecció sobre el pla $z=0$:

$x = \sqrt{1-t}$, $y = t$, $z = 0$; o, equivalentment, $y = 1-x^2$, $z = 0$.

Aleshores:

$$V/4 = \int_0^1 dx \int_0^{1-x^2} dy \int_0^{\sqrt{y}} dz = \int_0^1 dx \int_0^{1-x^2} dy\, \sqrt{y} = \frac{2}{3} \int_0^1 dx \left[y^{3/2} \right]_0^{1-x^2}$$

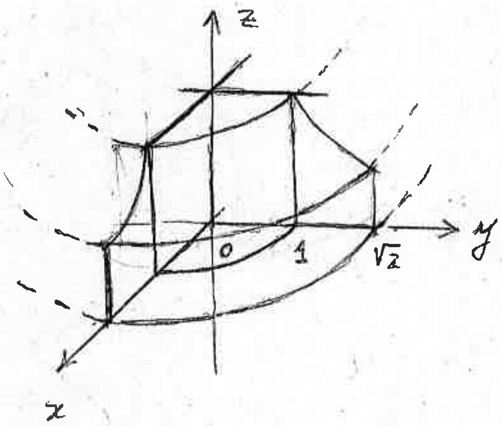
$$= \frac{2}{3} \int_0^1 \left(1-x^2\right)^{3/2} dx = \overset{\text{c.v } (*)}{\left\{ \begin{aligned} x &= \sin t \\ dx &= \cos t\, dt \end{aligned} \right\}} = \frac{2}{3} \underbrace{\int_0^{\pi/2} \cos^4 t\, dt}_{= 3\pi/16} = \frac{\pi}{8} \Rightarrow \boxed{V = \frac{\pi}{2}} . \triangleright$$



(*) Canvi de variables a la integral

h) $x^2+y^2 = 1, \quad x^2+y^2 = 2, \quad z(x^2+y^2) = 1, \quad z = 0.$

◁ Solució. En coordenades cilíndriques: $1 \leq r \leq \sqrt{2}, \quad 0 \leq z \leq \frac{1}{r^2}$

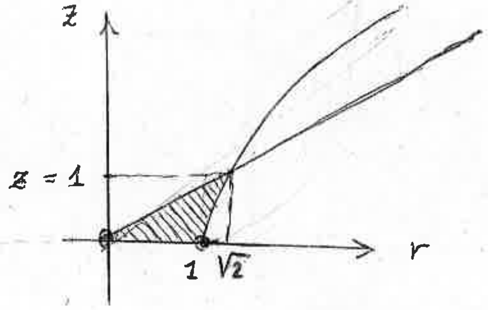


$$V = \int_0^{2\pi} d\theta \int_1^{\sqrt{2}} r\, dr \int_0^{1/r^2} dz = 2\pi \int_1^{\sqrt{2}} \frac{dr}{r}$$

$$= 2\pi \left[\ln r \right|_1^{\sqrt{2}} = \boxed{\pi \ln 2} . \triangleright$$

i) $x^2+y^2 = 2z^2, \quad x^2+y^2 = z^2+1 \quad (x \geq 0, y \geq 0, z \geq 0)$

◁ Solució Apliquem coordenades cilíndriques (pàg. següent)



$$V = \int_0^{\pi/2} d\theta \int_0^1 r\,dr \int_0^{r/\sqrt{2}} dz + \int_0^{\pi/2} d\theta \int_1^{\sqrt{2}} r\,dr \int_{\sqrt{r^2-1}}^{r/\sqrt{2}} dz$$

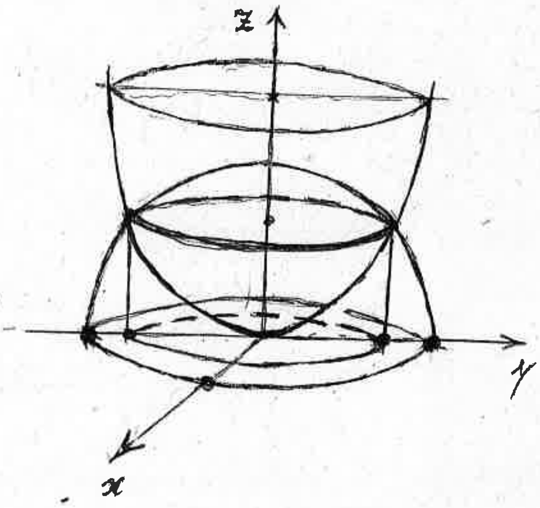
$$= \frac{\pi}{2}\int_0^1 \frac{r^2}{\sqrt{2}}\,dr + \frac{\pi}{2}\int_1^{\sqrt{2}}\left(\frac{r^2}{\sqrt{2}} - r\sqrt{r^2-1}\right)dr$$

$$= \frac{\pi}{4}\sqrt{2}\left[\frac{r^3}{3}\right]_0^1 + \frac{\pi}{2}\left[\frac{\sqrt{2}}{2}\cdot\frac{r^3}{3} - \frac{1}{3}(r^2-1)^{3/2}\right]_1^{\sqrt{2}} = \frac{\sqrt{2}\pi}{12} + \frac{\pi}{2}\left[\frac{2}{3} - \frac{1}{3} - \frac{\sqrt{2}}{6}\right] = \boxed{\frac{\pi}{6}}.$$

Alternativament (més 'fàcil'!):

$$V = \int_0^{\pi/2} d\theta \int_0^1 dz \int_{\sqrt{2}\,z}^{\sqrt{z^2+1}} r\,dr = \frac{\pi}{4}\int_0^1 (z^2+1-2z^2)\,dz = \frac{\pi}{4}\int_0^1 (1-z^2)\,dz$$

$$= \frac{\pi}{4}\left[z - \frac{z^3}{3}\right]_0^1 = \frac{\pi}{4}\left(1-\frac{1}{3}\right) = \boxed{\frac{\pi}{6}} \cdot \triangleright$$

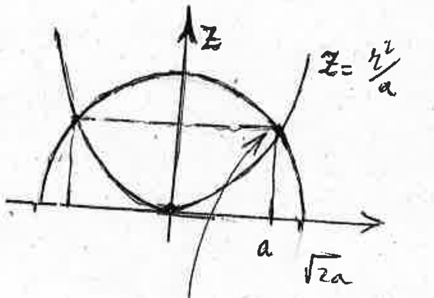


(j) $x^2+y^2+z^2 \le 2a^2$, $z \ge \frac{x^2+y^2}{a}$ $(a>0)$. En cilíndriques:

$$V = \int_0^{2\pi} d\theta \int_0^a r\,dr \int_{r^2/a}^{\sqrt{2a^2-r^2}} dz = 2\pi\int_0^a \left(r\sqrt{2a^2-r^2} - \frac{r^3}{a}\right)dr$$

$$= 2\pi\left[-\frac{1}{3}(2a^2-r^2)^{3/2} - \frac{r^4}{4a}\right]_0^a$$

$$= 2\pi\left[\frac{1}{3}(2\sqrt{2}-1) - \frac{1}{4}\right]a^3 = \boxed{2\pi a^3\left(\frac{2^{3/2}}{3} - \frac{7}{12}\right)}$$



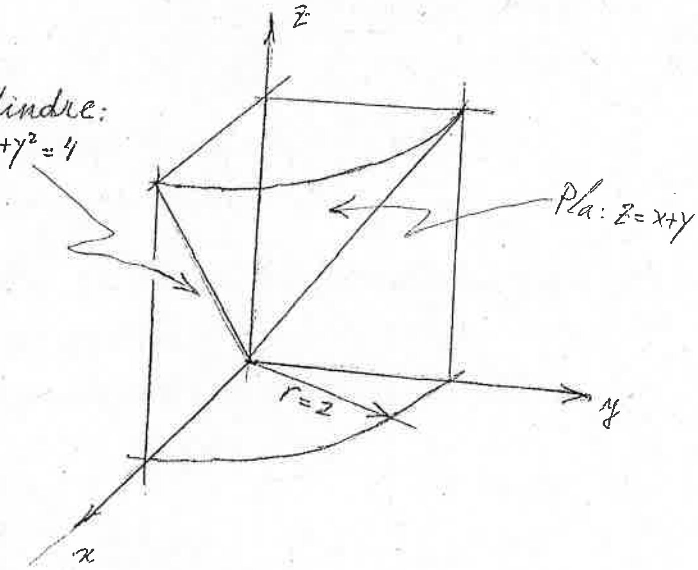
$az + z^2 = 2a^2$

$\Leftrightarrow z^2 + az - 2a^2 = 0$

d'on:

$$z = \frac{-a \pm \sqrt{a^2+8a^2}}{2} = \frac{-a\pm 3a}{2} = \begin{cases} a \to r = a \\ -2a \ (\text{NO}) \end{cases}$$

(k) $x^2+y^2=4,\ z=x+y\ (x\geq 0, y\geq 0, z\geq 0)$



$$V=\int_0^{\pi/2}d\theta\int_0^2 r\,dr\int_0^{r\cos\theta+r\sin\theta}dz$$

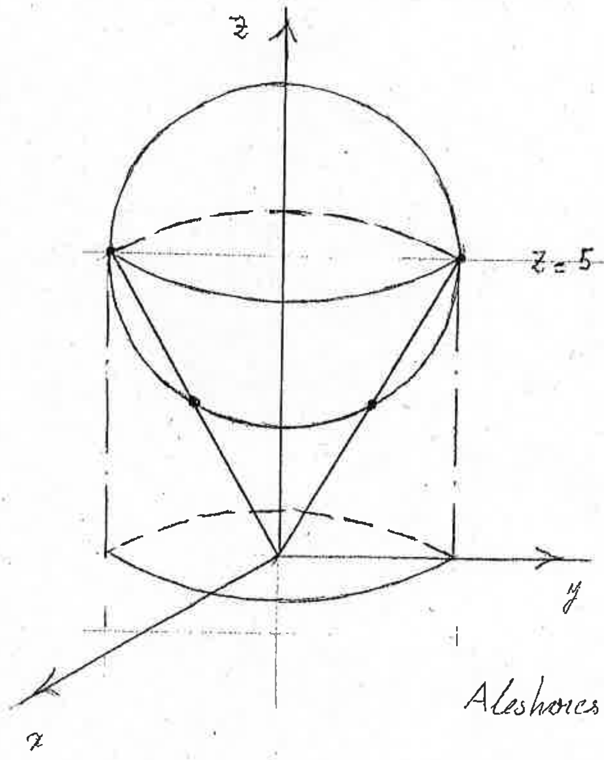
$$=\int_0^{\pi/2}d\theta\int_0^2 r^2(\cos\theta+\sin\theta)\,dr$$

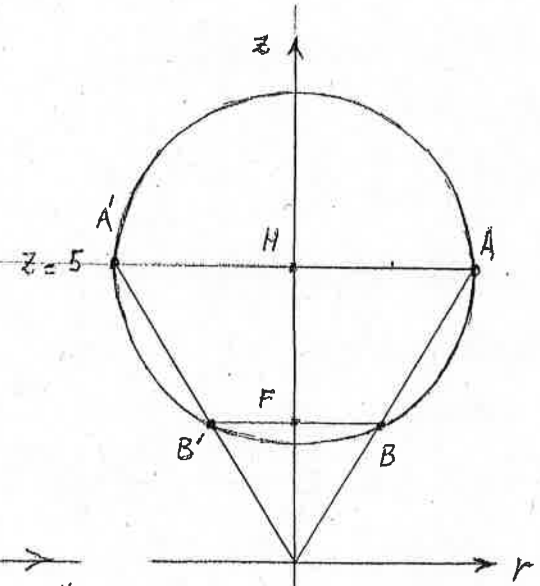
$$=\int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta\int_0^2 r^2\,dr$$

$$=\Big[\sin\theta-\cos\theta\Big]_0^{\pi/2}\cdot\left[\frac{r^3}{3}\right]_0^2=\boxed{\frac{16}{3}}\ \square$$

Exercici: substituir $z=x+y\to z=x+y-1$ i repetir el càlcul...

(l) Con de gelat definit per: $x^2+y^2\leq \frac{z^2}{5}$; $0\leq z\leq 5+\sqrt{5-x^2-y^2}$.





Alçada dels punts d'intersecció A, B, A', B'

$$(z-5)^2=5-\frac{z^2}{5}$$

$$\Longleftrightarrow\ 6z^2-50z+100=0.$$

d'on, $z=5$: alçada de A, A'

$z=\frac{10}{3}$: " " B, B'

i els 'radis' respectius són:

$r=\sqrt{5}$: distància $\overline{HA}=\overline{A'H}$

$r=\frac{2\sqrt{5}}{3}$: " $\overline{FB}=\overline{B'F}$

Aleshores el volum del cos ve donat per

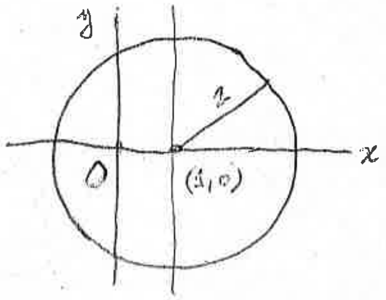
$$V=\underbrace{\frac{2}{3}\pi\sqrt{5}^3}_{\text{Volum de la semiesfera}}+\underbrace{\frac{1}{3}\pi\sqrt{5}^2\,5}_{\text{Volum del con.}}=\boxed{\left(\frac{10\sqrt{5}}{3}+\frac{25}{3}\right)\pi}$$

Gener 2017

(2a) Calculeu la integral doble $\iint_D (x^2+y^2)\,dx\,dy$ essent D —en coordenades cartesianes— la regió de la circumferència de radi 2 centrada en el punt $(x,y)=(1,0)$ i delimitada per les desigualtats $x>1, y>0$ [1·5 punts].

(2b) Calculeu el volum del cos determinat per la regió que està sobre el con $z=\sqrt{\frac{x^2+y^2}{3}}$ i dintre de l'esfera de radi $\frac{1}{2}$ centrada en $(x,y,z)=(0,0,\frac{1}{2})$. Ajut: considereu l'ús de coordenades cilíndriques o, alternativament, el principi de Cavalieri

◁ Solució (2a)



$$(x-1)^2+y^2=4$$

$$x-1=2r\cos\theta \qquad 0\le r\le 1$$

$$y=2r\sin\theta \qquad 0\le\theta\le\pi$$

$$I=4\int_0^{\pi/2}d\theta\int_0^1 r\left[(1+2r\cos\theta)^2+4r^2\sin^2\theta\right]dr$$

$$=4\int_0^{\pi/2}d\theta\int_0^1\left(r+4r^2\cos\theta+4r^3\right)dr=4\int_0^{\pi/2}\left(\frac{r^2}{2}+\frac{4}{3}r^3\cos\theta+r^4\right)\Big|_{r=0}^{r=1}d\theta$$

$$=4\int_0^{\pi/2}\left(\frac{1}{2}+\frac{4}{3}\cos\theta+1\right)d\theta=4\cdot\frac{3\pi}{4}+4\cdot\frac{4}{3}=\boxed{3\pi+\frac{16}{3}}.\ \triangleright$$

◁ Solució (2b). Fent servir coordenades esfèriques

$$x^2+y^2+\left(z-\tfrac{1}{2}\right)^2\le\frac{1}{4}$$

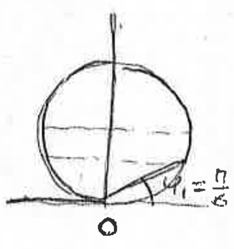
$$z\ge\sqrt{\frac{x^2+y^2}{3}}\iff z^2\ge\frac{x^2+y^2}{3}, \text{ amb } z\ge 0$$

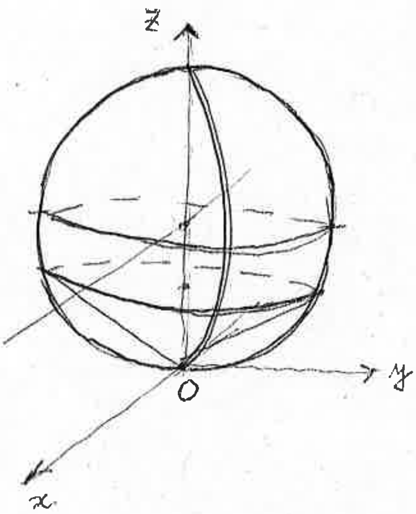
$$r^2\cos^2\theta\cos^2\varphi+r^2\sin^2\theta\cos^2\varphi+\left(r\sin\varphi-\tfrac{1}{2}\right)^2$$

$$=r^2\cos^2\varphi+r^2\sin^2\varphi-r\sin\varphi+\frac{1}{4}\le\frac{1}{4}$$

$$\iff r^2\le r\sin\varphi\iff r\le\sin\varphi$$

$$r^2\sin^2\varphi\ge\frac{r^2\cos^2\varphi}{3}\iff \operatorname{tg}\varphi\ge\frac{1}{\sqrt{3}}$$
$$0\le\varphi\le\frac{\pi}{2}$$





d'on: $\frac{\pi}{6}\le\varphi\le\frac{\pi}{2}$.

$$V=\int_0^{2\pi}d\theta\int_{\pi/6}^{\pi/2}\cos\varphi\,d\varphi\int_0^{\sin\varphi}r^2\,dr=\frac{1}{3}\int_0^{2\pi}d\theta\int_{\pi/6}^{\pi/2}\sin^3\varphi\cos\varphi\,d\varphi=\frac{2\pi}{3}\left[\frac{\sin^4\varphi}{4}\right]_{\varphi=\pi/6}^{\varphi=\pi/2}$$

$$=\frac{\pi}{6}\left(1-\frac{1}{16}\right)=\frac{15\pi}{3\cdot 2^5}=\boxed{\frac{5\pi}{32}}$$

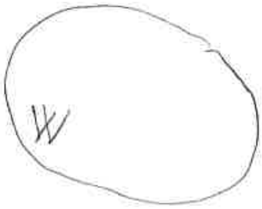
En coordenades cilíndriques:

$$V=\int_0^{2\pi}d\varphi\int_0^{1/4}dz\int_0^{\sqrt{3}z}r\,dr+\int_0^{2\pi}d\varphi\int_{1/4}^{1}dz\int_0^{\sqrt{z-z^2}}r\,dr=2\pi\int_0^{1/4}\frac{3}{2}z^2\,dz+2\pi\int_{1/4}^{1}\frac{1}{2}(z-z^2)\,dz$$

$$=\pi z^3\Big|_0^{1/4}+\left(\frac{z^2}{2}-\frac{z^3}{3}\right)\Big|_{1/4}^{1}=\pi\,\frac{3+96-64-6+1}{192}=\boxed{\frac{5\pi}{32}}.$$

CM/CG

$$X = \frac{1}{m(W)} \iiint_W x\,\rho(x,y,z)\,dx\,dy\,dz \text{ , on } m(W) = \iiint_W \rho(x,y,z)\,dx\,dy\,dz$$

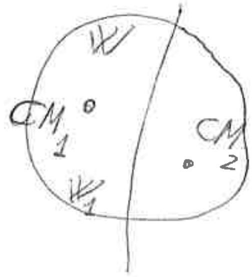


id. per Y i per Z, Id en el cas 2D. / CM amb $\rho(x,y,z) \equiv 1 \Rightarrow$ CG.

Propietats del CM i del CG.:

1. Si dividim un cos en dues o més parts, el seu CM és el mateix que s'obtindria si les masses fossin puntuals i estiguessin situades en els CM corresponents.

$$X_{CM} = \frac{1}{m(W)} \iiint_W x\,\rho(x,y,z)\,dx\,dy\,dz$$

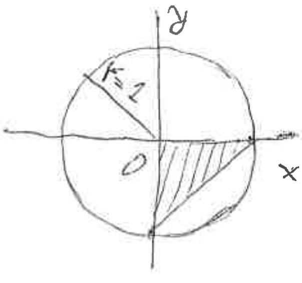


$$= \frac{m_1(W)}{m(W)} \underbrace{\frac{1}{m_1(W)} \iiint_{W_1} x\,\rho(x,y,z)\,dx\,dy\,dz}_{= X_{CM}(W_1)}$$

$$+ \frac{m_2(W)}{m(W)} \cdot \underbrace{\frac{1}{m_2(W)} \iiint_{W_2} x\,\rho(x,y,z)\,dx\,dy\,dz}_{= X_{CM}(W_2)} = \frac{m(W_1)\,X_{CM}(W_1) + m(W_2)\cdot X_{CM}(W_2)}{m(W)}$$

Id. per Y_{CM} i Z_{CM} id en el cas 2D.

Juny 2014: Troben el centre geomètric del conjunt que resulta de treure el triangle de vèrtexs $(0,0)$, $(1,0)$ i $(0,-1)$ del disc $B_1^2(0,0)$.



$$X_D = \frac{m(C)\,X_C + m(T)\,X_T}{m(D)} = 0 \Rightarrow X_C = -\frac{m(T)}{m(C)} X_T = -\frac{1/2}{\pi - 1/2} \frac{1}{3}(0+1)$$

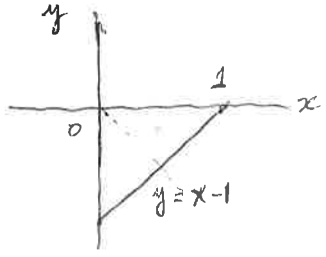
$$m(T) = A(T) = \tfrac{1}{2} \qquad Y_C = -\frac{m(T)}{m(C)} Y_T = -\frac{1/2}{\pi - 1/2} \cdot \frac{1}{3}(0-1)$$

$$m(C) = A(C) = \pi - \tfrac{1}{2}$$

$$(X_T, Y_T) = \tfrac{1}{3}\left(X_1 + X_2 + X_3,\ Y_1 + Y_2 + Y_3\right)$$

on (X_i, Y_i), $i = 1,2,3$ són les coordenades dels vèrtexs del triangle.

Aleshores: $(X_C, Y_C) = \left(-\frac{1}{6\pi - 3}, \frac{1}{6\pi - 3}\right)$.



$A(T)=\frac{1}{2}$

$$A(T)\,X_T = \int_0^1 x\,dx \int_{x-1}^0 dy = \int_0^1 x(1-x)\,dx = \left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_{x=0}^{x=1} = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}$$

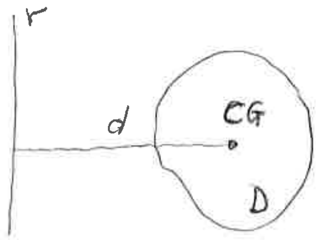
$$X_T = \frac{1}{A(T)}\cdot\frac{1}{6} = \boxed{\frac{1}{3}}, \quad Y_T = -X_T \text{ (per simetria)}$$

2on Teorema de Pappus-Guldin

El volum de revolució d'un cos generat per una placa uniforme al girar al voltant d'un eix contingut en el mateix pla que conté la placa i que no talla la placa, és igual al producte de l'àrea de la placa per la longitud de la circumferència descrita pel seu CG al girar al voltant de l'eix de revolució

$$d = d(r, CG)$$

$$V(W) = 2\pi\, d(r, CG)\cdot A(D)$$



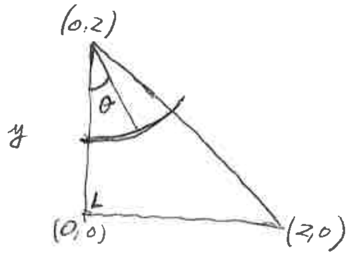
Exercici. Calculen el CG d'un quart de disc de radi R centrat a l'origen.

$$\cancel{2}\pi\bar{x}\cdot\frac{1}{4}\cancel{\pi}R^{\cancel{2}} = \frac{\cancel{2}}{3}\cancel{\pi}R^3 \iff \bar{x} = \frac{4R}{3\pi}\ (=\bar{y}, \text{ per simetria}).$$

Alesbores: $(\bar{x},\bar{y}) = \left(\frac{4R}{3\pi}, \frac{4R}{3\pi}\right)$.

Gener 2016

1 (1 punt) Signi Ω el triangle de vèrtexs $(0,0), (2,0), (0,2)$, on se li ha extret el sector circular $C = \{x^2+(y-2)^2 \le 1, x\ge 0, x+y\le 2\}$.



Solució:

$$x = r\sin\theta, \quad 0\le\theta\le \frac{\pi}{4}, \quad 0\le r\le 1$$
$$y = 2-r\cos\theta$$

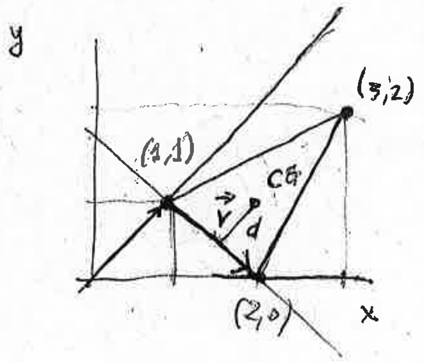
$$A(S)\,X_S = \int_0^{\pi/4}\sin\theta\,d\theta\int_0^1 r^2\,dr = (-\cos\theta)\Big|_{\theta=0}^{\theta=\pi/4}\times\left(\frac{r^3}{3}\right)\Big|_{r=0}^{r=1} = \left(1-\cos\left(\frac{\pi}{4}\right)\right)\cdot\frac{1}{3} = \frac{2-\sqrt{2}}{6}$$

$$A(T)\cdot X_T = \frac{2}{3}(0+2+0) = \frac{4}{3} = A(\Omega)\cdot X_\Omega + A(S)X_S = A(\Omega)X_\Omega + \frac{2-\sqrt{2}}{6}$$

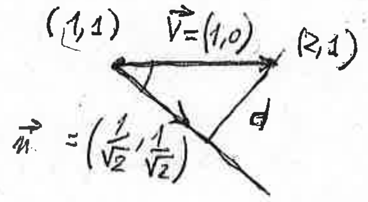
$$A(\Omega) = 2-\frac{\pi}{8}; \quad X_\Omega = \frac{1}{A(\Omega)}\left(\frac{4}{3}-\frac{2-\sqrt{2}}{6}\right) = \frac{\cancel{8}\,4}{16-\pi}\cdot\frac{8-2+\sqrt{2}}{\cancel{6}\,3} = \boxed{\frac{4(6+\sqrt{2})}{3(16-\pi)}}.$$

Gener 2015. (1p) Quin és el volum del cos generat pel triangle de vèrtexs (1,1), (2,0) i (3,2), al girar al voltant de l'eix que passa pel punt (2,0) i és perpendicular a la bisectriu del 1er quadrant?

Solució:



$$(\bar{x},\bar{y}) = \frac{1}{3}\left((1,1)+(2,0)+(3,2)\right) = \frac{1}{3}(6,3) = (2,1)$$



$$a = \langle \vec{v}, \vec{n} \rangle = \frac{1}{\sqrt{2}}$$

$$d = \sqrt{\|v\|^2 - a^2} = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$\vec{v} = (2,0)-(1,1) = (1,-1) \perp$ bisectriu del 1er quadrant.

$$A(T) = \left| \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix} \right| = \left| -\frac{2}{2}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \right|$$

$$= \left| -1 - \frac{1}{2} \right| = \boxed{\frac{3}{2}}$$

Aplicant el 2on teorema de Pappus-Guldin

$$V(T) = 2\pi d \cdot A(T) = 2\pi \cdot \frac{1}{\sqrt{2}} \cdot \frac{3}{2} = \boxed{\frac{3\pi}{\sqrt{2}}}$$

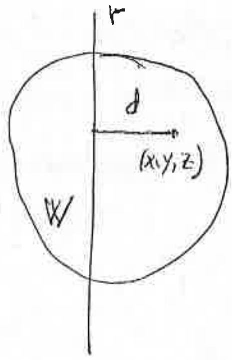
Fer el problema 28.

" " problema 34 (e)

" " " 26 (e)

Moments d'inèrcia respecte d'un eix.

$$I_r = \iiint_W d^2(r,(x,y,z))\, \rho(x,y,z)\, dx\,dy\,dz$$

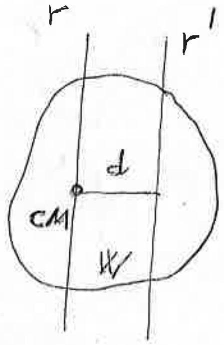


Moments d'inèrcia respecte dels eixos principals.

$$I_x = \iiint_W (y^2+z^2)\, \rho(x,y,z)\, dx\,dy\,dz,$$

$$I_y = \iiint_W (x^2+z^2)\, \rho(x,y,z)\, dx\,dy\,dz,$$

$$I_z = \iiint_W (x^2+y^2)\, \rho(x,y,z)\, dx\,dy\,dz.$$



Teorema de Steiner.

$$I_{r'} = I_r + M(W)\, d^2$$

Gener 2017.

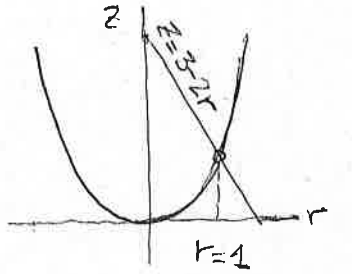
1 (a) [1 punt] Calculeu la massa del sòlid limitat pel paraboloide $z=x^2+y^2$ i el con $z+2\sqrt{x^2+y^2}=3$ si es suposa que la seva densitat és constant igual a 1

(b) [1 punt] Calculeu el moment d'inèrcia del cos de l'apartat anterior respecte de l'eix z.

(c) [1/2 punt] Trobeu el moment d'inèrcia del mateix cos respecte de l'eix perpendicular al pla $z=0$ que passa pel punt $(x,y,z)=\left(\sqrt{2}/2, \sqrt{2}/2, 0\right)$

Solució

$z=r^2$, $z=3-2r$



$3-2r=r^2$

$\Leftrightarrow r^2+2r-3=0$

$$r=\frac{-2\pm\sqrt{4+12}}{2}=\frac{-2\pm 4}{2}$$

$= \begin{cases} \boxed{-3} \text{ No} \\ 1 \end{cases}$

(a) $$M=\int_0^{2\pi} d\theta \int_0^1 r\,dr \int_{r^2}^{3-2r} dz = 2\pi\int_0^1 r(3-2r-r^2)\,dr = 2\pi\int_0^1 (3r-2r^2-r^3)\,dr$$

$$= 2\pi\left(\tfrac{3}{2}r^2-\tfrac{2}{3}r^3-\tfrac{r^4}{4}\right)\Big|_{r=0}^{r=1} = 2\pi\left(\tfrac{3}{2}-\tfrac{2}{3}-\tfrac{1}{4}\right)$$

$$= \frac{\pi}{6}(18-8-3) = \boxed{\frac{7\pi}{6}}$$

(b) $$I_z=\int_0^{2\pi} d\theta\int_0^1 r^3\,dr\int_{r^2}^{3-2r} dz = 2\pi\int_0^1 (3r^3-2r^4-r^5)\,dr$$

$$= 2\pi\left(\tfrac{3}{4}r^4-\tfrac{2}{5}r^5-\tfrac{1}{6}r^6\right)\Big|_{r=0}^{r=1} = \frac{\pi}{30}(45-24-10)$$

$$= \boxed{\frac{11\pi}{30}}$$

(c) $$I_{z'} = I_z + d^2 M = \frac{11\pi}{30} + 1\cdot\frac{7\pi}{6} = \frac{11\pi+35\pi}{30} = \frac{46\pi}{30} = \boxed{\frac{23\pi}{15}}$$

$$d^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}+\frac{1}{2} = 1$$

2 [1p] Trobeu el moment d'inèrcia del cos homogeni $W=\{(x,y,z)\in\mathbb{R}^3:\ x^2+y^2\le 1,\ x^2+z^2\le 1\}$ amb densitat k, respecte de l'eix $y=\frac{2}{5\sqrt{3}}$, $z=\frac{1}{5\sqrt{3}}$

Solució:

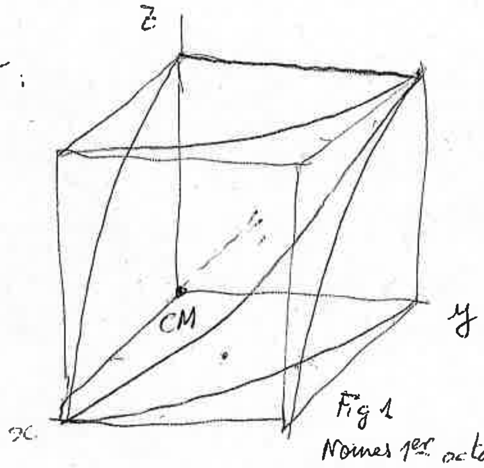


Fig 1
Només 1er octant!

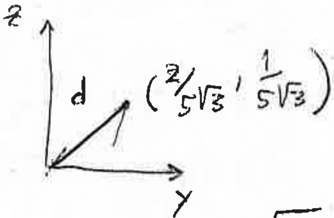
$$I_x = k\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (y^2+z^2)\,dz$$

$$= k\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(y^2 z + \tfrac{z^3}{3}\right)\Big|_{z=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy$$

$$= 2k\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[y^2\sqrt{1-x^2} + \tfrac{1}{3}(1-x^2)^{3/2}\right] dy = 4k\int_{-1}^{1}\left[\tfrac{1}{3}(1-x^2)^2 + \tfrac{1}{3}(1-x^2)^2\right]dx$$

$$= \frac{16}{3}k\int_{0}^{1}(1-2x^2+x^4)\,dx = \frac{16}{3}k\left(x-\tfrac{2}{3}x^3+\tfrac{x^5}{5}\right)\Big|_0^1 = \frac{16}{3}k\,\frac{15-10+3}{15} =$$

$$= \frac{16}{3}k\cdot\frac{8}{15} = \frac{2^7}{3^2\cdot 5}k = \boxed{\frac{128k}{45}}$$



$$I_r = I_x + Md^2 = \frac{128k}{45} + \frac{16k}{3}\left(\left(\frac{2}{5\sqrt{3}}\right)^2 + \left(\frac{1}{5\sqrt{3}}\right)^2\right)$$

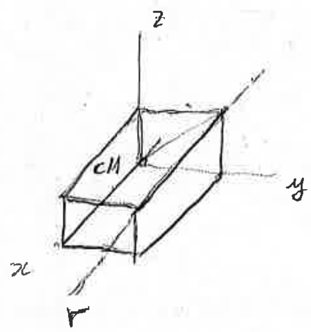
$$= \frac{128k}{45} + \frac{16k}{45} = \frac{144k}{45} = \frac{\cancel{3^2}\cdot 2^4 k}{\cancel{3^2}\cdot 5} = \boxed{\frac{16k}{5}}$$

$$\frac{M}{8} = k\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2}} dz$$

$$= k\int_0^1 (1-x^2)\,dx$$

$$= k\left(x-\tfrac{x^3}{3}\right)\Big|_{x=0}^{x=1}$$

$$= k\left(1-\tfrac{1}{3}\right) = \frac{2k}{3} \Rightarrow M = \frac{16k}{3}$$



$$d^2 = \left(\frac{2}{5\sqrt{3}}\right)^2 + \left(\frac{1}{5\sqrt{3}}\right)^2 = \frac{4}{75} + \frac{1}{75}$$

$$= \frac{5}{25\cdot 3} = \boxed{\frac{1}{15}}$$

Gener 2018

4) Signi $B\subseteq\mathbb{R}^2$ el cos homogeni de densitat 1 limitat per les superfícies $y^2=z^2-1$, $x^2=z^2-1$ i $z=2$

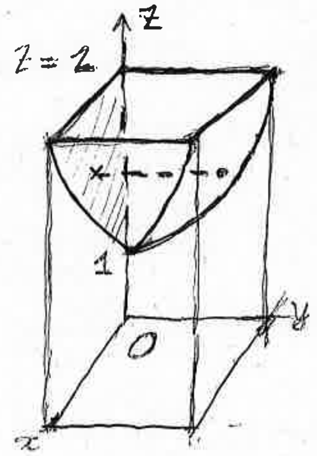
Calculeu

(a) [1/2 punt] La massa del cos $\boxed{\text{Sol. } M=16/3}$

(b) [1/2 punt] Les coordenades del centre geomètric $\boxed{\text{Sol. } (x,y,z)=(0,0,\tfrac{27}{16})}$

(c) [1 punt] Els seus moments d'inèrcia respecte dels eixos x i y

$\boxed{\text{Solució: } I_x=I_y=\frac{848}{45}}$



Solució:

(a) $y^2=z^2-1 \iff y=\pm\sqrt{z^2-1}$

$x^2=z^2-1 \iff x=\pm\sqrt{z^2-1}$

$B:\ 1\le z\le 2,\ -\sqrt{z^2-1}\le x\le\sqrt{z^2-1},\ -\sqrt{z^2-1}\le y\le\sqrt{z^2-1}.$

$$M=\int_1^2 dz\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}}dx\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}}dy=\int_1^2 dz\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}}2\sqrt{z^2-1}\,dx=4\int_1^2(z^2-1)\,dz=4\left(z^3/3-z\right)\Big|_{z=1}^{z=2}$$

$$=4\left(8/3-2-\tfrac{1}{3}+1\right)=4\left(7/3-1\right)=\boxed{\frac{16}{3}}$$

(b) Per simetria: $X_{CG}=0=Y_{CG}$

$$M\cdot Z_{CG}=\int_1^2 z\,dz\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}}dx\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}}dy=2\int_1^2 z\,dz\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}}\sqrt{z^2-1}\,dx=4\int_1^2 z(z^2-1)\,dz$$

$$=4\left(z^4/4-z^2/2\right)\Big|_{z=1}^{z=2}=4\left(4-2-\tfrac{1}{4}+\tfrac{1}{2}\right)=4\left(2+\tfrac{1}{4}\right)=\frac{36}{4}=9$$

$$Z_{CG}=\frac{36/4}{16/3}=\frac{9\cdot 3}{16}=\boxed{\frac{27}{16}}$$ Aleshores $\boxed{(X_{CG},Y_{CG},Z_{CG})=(0,0,\tfrac{27}{16})}$

(c) Per simetria $I_x=I_y=\int_1^2 dz\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}}dx\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}}(y^2+z^2)\,dy=\int_1^2 dz\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}}\left[2/3\,(z^2-1)^{3/2}+2z^2(z^2-1)^{1/2}\right]dx$

$$=\int_1^2\left[4/3\,(z^2-1)^2+4z^2(z^2-1)\right]dz=4\int_1^2\left(\tfrac{4}{3}z^4-\tfrac{5}{3}z^2+\tfrac{1}{3}\right)dz=4\left(\tfrac{4}{15}z^5-\tfrac{5}{9}z^3+\tfrac{z}{3}\right)\Big|_{z=1}^{z=2}$$

$4/3(z^4-2z^2+1)+4z^4-4z^2$

$=\frac{16}{3}z^4-\frac{20}{3}z^2+\frac{4}{3}$

$$=\frac{4}{45}\left(12z^5-25z^3+15z\right)\Big|_{z=1}^{z=2}=\frac{4}{45}\left(12\times32-25\times8+30-12+25-15\right)$$

$$=\boxed{\frac{848}{45}}$$