PROBLEMA 23. APARTAT F

Es comprova que el canvi

$$T: \ u = \frac{y^2}{x}, \qquad v = \frac{z^2}{y}, \qquad w = \frac{x^2}{z}, \tag{1}$$

transforma el domini original $\mathcal{D}$, en $\mathcal{D}' = [1,a] \times [1,a] \times [1,a]$. És a dir, en un cub d'aresta $a-1$ (veure Figura 1). Aleshores el Jacobià corresponent surt més senzill. En efecte:

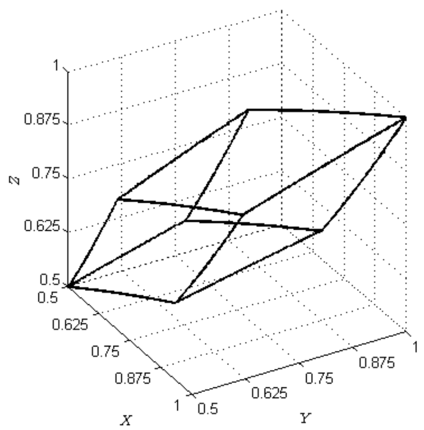
$$\det J_T(x,y,z) = \begin{vmatrix} -\dfrac{y^2}{x^2} & 2\dfrac{y}{x} & 0 \\ 0 & -\dfrac{z^2}{y^2} & 2\dfrac{z}{y} \\ 2\dfrac{x}{z} & 0 & -\dfrac{x^2}{z^2} \end{vmatrix} = -7,$$

d'on:

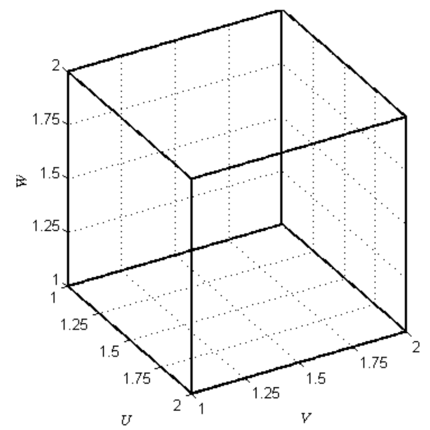
$$|\det J_{T^{-1}}(u,v,w)| = \frac{1}{|\det J_T\left(x(u,v,w), y(u,v,w), z(u,v,w)\right)|} = \frac{1}{7}.$$

Així doncs, aplicant el canvi (1) resulta el volum:

$$\begin{aligned} V = \iiint_{\mathcal{D}} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z &= \iiint_{\mathcal{D}'} |\det J_T(u,v,w)|\, \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}w \\ &= \frac{1}{7} \int_1^a \mathrm{d}u \int_1^a \mathrm{d}v \int_1^a \mathrm{d}w = \frac{1}{7} \left(\int_1^a \mathrm{d}u \right)^3 = \frac{(a-1)^3}{7}. \end{aligned}$$



(a) $\mathcal{D}$: Domini definit per les superfícies,
$x^2 = z, \quad y^2 = x, \quad z^2 = y,$
$x^2 = az, \quad y^2 = ax, \quad z^2 = ay.$



(b) $\mathcal{D}' : 1 \le u \le a,\ 1 \le v \le a,\ 1 \le w \le a.$

FIGURA 1. Transformació del domini $\mathcal{D}$ en $\mathcal{D}'$ pel canvi (1).