Integration of functions of several variables

Riemann sums and Mean Value Theorem

**1. For a *bounded* real function $f\colon R \subset \mathbb{R}^2 \to \mathbb{R}$ defined on a *rectangle* $R = [a,b] \times [c,d]$ of the plane, take a uniform partition of $R = \cup_{i,j=1}^{N} R_{ij}$ in N^2 *subrectangles* $R_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i]$, $i,j = 1, \ldots, N$, with $x_i = a + ih$, $h = \dfrac{b-a}{N}$ and $y_j = c + jk$, $k = \dfrac{d-c}{N}$, $i,j = 0,1,\ldots,N$. On each subrectangle R_{ij} take $M_{ij} = \sup_{(x,y)\in R_{ij}} f(x,y)$ and $m_{ij} = \inf_{(x,y)\in R_{ij}} f(x,y)$, $i,j = 1,\ldots,N$ and consider the *upper Riemann sum* $S_N = \displaystyle\sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) = \frac{(b-a)(d-c)}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}$ and the *lower Riemann sum* $s_N = \displaystyle\frac{(b-a)(d-c)}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij}$. If f is continuous, $\lim_{N\to\infty} S_n = \lim_{N\to\infty} s_n = I$ is called the *(Riemann) integral* of f and is denoted by $I = \iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ or simply $I = \int_R f$. Compute $\displaystyle\iint_R 1 - x\,\mathrm{d}x\,\mathrm{d}y$, where $R = [0,1] \times [0,1]$, using Riemann sums. (Sol.: $\frac{1}{2}$).

**2. The integral of a continuous real function $f\colon \Omega \to \mathbb{R}$ defined on a bounded subset Ω of $\mathbb{R}^2$ (or $\mathbb{R}^3$) can be defined through partitions $\Omega = \cup_{i,j=1}^{N} \Omega_{ij}$ (or $\Omega = \cup_{i,j,k=1}^{N} \Omega_{ijk}$) and is denoted by $I = \iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ (or $I = \iiint_R f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$) or simply $I = \int_\Omega f$. It satisfies several properties for any $f,g\colon \Omega \to \mathbb{R}$, $a,b \in \mathbb{R}$, like *Linearity:* $\int_\Omega af + bg = a\int_\Omega f + b\int_\Omega g$; *Additivity:* If $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, then $\int_\Omega f = \int_{\Omega_1} f + \int_{\Omega_2} f$; *Monotonicity:* If $f \le g$ then $\int_\Omega f \le \int_\Omega g$; (in particular $\left|\int_\Omega f\right| \le \int_\Omega |f|$); *Normalization:* $\int_\Omega 1 = \mathrm{measure}(\Omega)$, where $\mathrm{measure}(\Omega) = \mathrm{m}(\Omega)$ is $\mathrm{area}(\Omega)$ if $\Omega \subset \mathbb{R}^2$, and $\mathrm{m}(\Omega) = \mathrm{volume}(\Omega) = V(\Omega)$ if $\Omega \subset \mathbb{R}^3$; *The mean value theorem for integrals:* If $m \le f \le g$ then $m \cdot \mathrm{m}(\Omega) \le \int_\Omega f \le M \cdot \mathrm{m}(\Omega)$. Application: Prove that $4\mathrm{e}^5 \le \displaystyle\iint_\Omega \mathrm{e}^{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y \le 4\mathrm{e}^{25}$, where $\Omega = [1,3] \times [2,4]$.

3. Using the mean value theorem for integrals prove the following inequalities.

(a) $\dfrac{1}{\mathrm{e}} \le \dfrac{1}{4\pi^2}\displaystyle\iint_\Omega \mathrm{e}^{\sin(x+y)}\,\mathrm{d}x\,\mathrm{d}y \le \mathrm{e}$, where $\Omega = [-\pi,\pi] \times [-\pi,\pi]$.

*(b) $\dfrac{1}{6} \le \displaystyle\iint_\Omega \frac{\mathrm{d}x\,\mathrm{d}y}{y - x + 3} \le \frac{1}{4}$, where Ω is the triangle with vertices $(0,0)$, $(1,1)$ and $(1,0)$.

4. Let $f(x,y,z)$ be a continuous function and B_ε the ball of center (x_0,y_0,z_0) and radius ε. Prove that $f(x_0,y_0,z_0) = \lim_{\varepsilon\to 0^+} \dfrac{1}{\mathrm{volume}(B_\varepsilon)}\displaystyle\iiint_{B_\varepsilon} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

Cavalieri's principle

**5. We have already used the *Cavalieri's principle* for functions of one variable: For $f \ge g\colon [a,b] \to \mathbb{R}$ and $\Omega = \{(x,y) \in \mathbb{R}^2 : a \le x \le b, g(x) \le y \le f(x)\}$ the planar domain between the graphs of f and g, then $A(\Omega) = \int_a^b \ell(x)\,\mathrm{d}x$, where $\ell(x) = f(x) - g(x)$ is the length of the *cross section* $\Omega_x = \{y : g(x) \le y \le f(x)\}$. Example: Compute the area enclosed by an ellipse with semi-axis $a, b > 0$.

**6. The *Cavalieri's principle* for solids $W \subset \mathbb{R}^3$ is $V(W) = \int_a^b A(z)\,\mathrm{d}z$, where $A(z)$ is the area of the cross section $W_z = \{(x,y) \in \mathbb{R}^3 : (x,y,z) \in W\}$, and $[a,b]$ is the segment $\{z \in W : W_z \text{ is not empty}\}$. Application: Compute the volume bounded by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. (Sol.: $\frac{4}{3}\pi abc$).

7. Apply Cavalieri's principle to compute the following volumes from the areas of the sections with parallel planes to the coordinate planes.

***(a)** Volume bounded by the inverted cone with elliptic base $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=z^2$, with $0\le z\le h$. (Sol.: $\frac{1}{3}\pi abh^3$).

***(b)** Volume of the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$, $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ $(a,b,c>0)$. (Sol.: $\frac{abc}{6}$).

***(c)** Volume of the pyramid with rectangular base with sides a, b and height h. (Sol.: $\frac{abh}{3}$).

***(d)** Volume bounded by the spherical cap determined by the sphere $x^2+y^2+z^2=R^2$ and the condition $R-h\le z\le R$. (Sol.: $\frac{1}{3}\pi h^2(3R-h)$).

Double and triple iterated integrals and Fubini-Tonelli theorem

8. Generalize Cavalieri's principle to the computation of volumes in $\mathbb{R}^4$ and compute the volume of the 4D ball $B=\{(x,y,z,t)\in\mathbb{R}^4 : x^2+y^2+z^2+t^2\le R^2\}$. (Sol.: $\frac{\pi^2}{2}R^4$).

****9.** Given a real function $f\colon R\subset\mathbb{R}^2\to\mathbb{R}$ defined on a rectangle $R=[a,b]\times[c,d]$ of the plane, assuming for the moment that f is continuous and $f\ge 0$, apply Cavalieri's principle to prove the *Fubini-Tonelli theorem*: $\iint_{R=[a,b]\times[c,d]} f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_a^b \mathrm{d}x\int_c^d f(x,y)\,\mathrm{d}y=\int_c^d \mathrm{d}y\int_a^b f(x,y)\,\mathrm{d}x$. Application: Compute $\displaystyle\iint_R y\ln x\,\mathrm{d}x\,\mathrm{d}y$ where $R=[1,\mathrm{e}]\times[1,\mathrm{e}]$. (Sol.: $\frac{1}{2}(\mathrm{e}^2-1)$).

10. Find the following double integrals in the specified rectangles

(a) $\displaystyle\iint_R x^2y\,\mathrm{d}x\,\mathrm{d}y$, $R=[0,1]\times[0,1]$. (Sol.: $\frac{1}{6}$).

(b) $\displaystyle\iint_R \frac{x^2}{1+y^2}\,\mathrm{d}x\,\mathrm{d}y$, $R=[0,1]\times[0,1]$. (Sol.: $\frac{\pi}{12}$).

(c) $\displaystyle\iint_R (x^2+y)\,\mathrm{d}x\,\mathrm{d}y$, $R=[0,1]\times[0,2]$. (Sol.: $\frac{8}{3}$).

(d) $\displaystyle\iint_R \frac{1}{(x+2y)^2}\,\mathrm{d}x\,\mathrm{d}y$, $R=[2,5]\times[1,3]$. (Sol.: $\frac{1}{2}\ln(\frac{14}{11})$).

***(e)** $\displaystyle\iint_R \mathrm{e}^y\sin\left(\frac{x}{y}\right)\,\mathrm{d}x\,\mathrm{d}y$, $R=[-\pi/2,\pi/2]\times[1,2]$. (Sol.: 0).

***(f)** $\displaystyle\iint_R (x+y)^{27}\,\mathrm{d}x\,\mathrm{d}y$, $R=[-1,1]\times[-1,1]$. (Sol.: 0).

11. Compute $\displaystyle\iint_R x^y\,\mathrm{d}x\,\mathrm{d}y$ in $R=[0,1]\times[a,b]$, where $0<a<b$, and deduce the value of the integral $\displaystyle\int_0^1 \frac{x^b-x^a}{\ln x}\,\mathrm{d}x$. (Sol.: $\ln\left(\dfrac{b+1}{a+1}\right)$).

12. Prove that $\displaystyle 2\int_a^b\int_x^b f(x)f(y)\,\mathrm{d}y\,\mathrm{d}x=\left(\int_a^b f(x)\,\mathrm{d}x\right)^2$. *Hint*:

$$\left(\int_a^b f(x)\,\mathrm{d}x\right)^2=\iint_{[a,b]\times[a,b]} f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y.$$

****13.** Fubini-Tonelli theorem is also valid for continuous functions $f\colon \Omega \to \mathbb{R}$ not necessarily positive, and for planar domains $\Omega \subset \mathbb{R}^2$ not necessary rectangles, but the limits of integration of the iterated integrals have to be worked out. The same is true for solids $W \subset \mathbb{R}^3$. For the following regions $\Omega \subset \mathbb{R}^2$ write the double integral $\iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ in terms of iterated integrals taken in different order, $\int \mathrm{d}y \int f(x,y)\,\mathrm{d}x$ and $\int \mathrm{d}x \int f(x,y)\,\mathrm{d}y$, finding the limits of integration for x and y in each case.

(a) Ω rectangle with vertices $(1,2)$, $(5,2)$, $(5,4)$ and $(1,4)$.

***(b)** Ω parallelogram bounded by the straight lines $y = x$, $y = x - 3$, $y = 2$, $y = 4$.

***(c)** Ω region bounded by the curves $x^2 + y^2 = 2a^2$, $x^2 = ay$ ($y \geq 0$, $a > 0$).

(d) Ω region bounded by the curves $y^2 = ax$, $x^2 + y^2 = 2ax$, $y = 0$ ($y \geq 0$, $a > 0$).

***(e)** Ω region bounded by the curves $x^2 + y^2 = ax$, $x^2 + y^2 = 2ax$, $y = 0$ ($y \geq 0$, $a > 0$).

14. Compute the following double integrals on the specified domains of $\mathbb{R}^2$.

(a) $\iint_\Omega y^3\,\mathrm{d}x\,\mathrm{d}y$, $\Omega = \{(x,y) \in \mathbb{R}^2 : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2},\ 0 \leq y \leq 2\cos x\}$. (Sol.: $\frac{3}{2}\pi$).

(b) $\iint_\Omega x\,\mathrm{d}x\,\mathrm{d}y$, $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq \mathrm{e}^x\}$. (Sol.: 1).

(c) $\iint_\Omega xy\,\mathrm{d}x\,\mathrm{d}y$, $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2,\ 0 \leq y \leq \frac{x}{2}\}$. (Sol.: $\frac{1}{2}$).

(d) $\iint_\Omega \frac{x^2}{y^2}\,\mathrm{d}x\,\mathrm{d}y$, $\Omega = \{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 2,\ \frac{1}{x} \leq y \leq x\}$. (Solució: $\frac{9}{4}$).

(e) $\iint_\Omega (x + 2y)\,\mathrm{d}x\,\mathrm{d}y$, $\Omega = \{(x,y) \in \mathbb{R}^2 : -3 \leq y \leq 3,\ y^2 - 4 \leq x \leq 5\}$. (Sol.: $\frac{252}{5}$).

(f) $\iint_\Omega y^3\,\mathrm{d}x\,\mathrm{d}y$, $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq a,\ y^2 \leq 2px\}$, $a > 0$, $p > 0$. (Sol.: 0).

(g) $\iint_\Omega \frac{y}{1+x^3}\,\mathrm{d}x\,\mathrm{d}y$, $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x^2 + y^2 \geq 1\}$. (Sol.: $\frac{\ln(2)}{6}$).

(h) $\iint_\Omega x^2 \sin(xy)\,\mathrm{d}x\,\mathrm{d}y$, $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 1,\ y \leq x \leq 1\}$. (Sol.: $\frac{1}{2}(1 - \sin(1))$).

15. For the following iterated integrals write the equations of the curves limiting the regions of integration and draw these regions.

(a) $\int_1^2 \left(\int_x^{x+3} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x$. (Sol.: $y = x$, $y = x + 3$, $x = 1$, $x = 2$).

(b) $\int_{-1}^1 \left(\int_{x^2}^{2-x^2} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x$. (Sol.: $y = x^2$, $y = 2 - x^2$, $x = 1$, $x = -1$).

(c) $\int_0^2 \left(\int_{2-y}^{\sqrt{4-y^2}} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y$. (Sol.: $x + y = 2$, $x^2 + y^2 = 4$, $y = 0$, $y = 2$).

(d) $\int_0^1 \left(\int_{\sqrt{x}}^{\sqrt{2-x^2}} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x$. (Sol.: $y^2 = x$, $x^2 + y^2 = 2$, $x = 0$, $x = 1$).

16. Invert the order of integration in the following iterated integrals.

(a) $\displaystyle\int_0^{\frac{1}{2}} \left(\int_0^{1-2x} f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x.$

(b) $\displaystyle\int_0^{1} \left(\int_1^{\mathrm{e}^x} f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x.$

(c) $\displaystyle\int_0^{4} \left(\int_{3x^2}^{12x} f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x.$

(d) $\displaystyle\int_0^{\frac{\pi}{2}} \left(\int_0^{\cos x} f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x.$

*__(e)__ $\displaystyle\int_0^{\pi} \left(\int_0^{\sin x} f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x.$

(f) $\displaystyle\int_0^{1} \left(\int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\,\mathrm{d}x\right)\,\mathrm{d}y.$

(g) $\displaystyle\int_0^{2a} \left(\int_{\sqrt{2ax-x^2}}^{\sqrt{4ax}} f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x,\ a>0.$

(h) $\displaystyle\int_0^{3} \left(\int_{\frac{x}{3}}^{1} f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x.$

(i) $\displaystyle\int_0^{1} \left(\int_{-x}^{x^2} f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x.$ (Sol.: $\displaystyle\int_{-1}^{0}\int_{-y}^{1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_0^1\int_{\sqrt{y}}^{1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$).

17. Compute the following double integrals in the specified domains of $\mathbb{R}^2$.

(a) $\displaystyle\iint_\Omega (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$, Ω bounded by the straight lines $y=x$, $x+y=2a$, $x=0$ $(a>0)$. (Sol.: $\frac{4}{3}a^4$).

(b) $\displaystyle\iint_\Omega (x+2y)\,\mathrm{d}x\,\mathrm{d}y$, Ω bounded by the curves $y=x^2$, $y^2=x$. (Sol.: $\frac{9}{20}$).

*__(c)__ $\displaystyle\iint_\Omega \mathrm{e}^{x+y}\,\mathrm{d}x\,\mathrm{d}y$, Ω bounded by the curves $y=\mathrm{e}^x$, $x=0$, $y=2$. (Sol.: e).

*__(d)__ $\displaystyle\iint_\Omega \mathrm{e}^{y}\,\mathrm{d}x\,\mathrm{d}y$, Ω triangle with vertices $(1,0)$, $(0,1)$, $(0,-1)$. (Sol.: $\mathrm{e}+\frac{1}{\mathrm{e}}-2$).

(e) $\displaystyle\iint_\Omega xy^2\,\mathrm{d}x\,\mathrm{d}y$, Ω bounded by the straight lines $x=1$, $x=y$, $x+y=0$. (Sol.: $\frac{2}{15}$).

(f) $\displaystyle\iint_\Omega xy\,\mathrm{d}x\,\mathrm{d}y$, Ω bounded by the curves $x=y$, $y=x^2$. (Sol.: $\frac{1}{24}$).

18. Compute the following iterated triple integrals.

*__(a)__ $\displaystyle\int_1^2\int_0^1\int_0^{\frac{\pi}{2}} x^2y^3\sin z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$ (Sol.: $\frac{7}{12}$).

(b) $\displaystyle\int_0^1\int_0^x\int_0^{\sqrt{x^2+y^2}} z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$ (Sol.: $\frac{1}{6}$).

*__(c)__ $\displaystyle\int_0^3\int_0^{2x}\int_0^{\sqrt{xy}} z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$ (Sol.: $\frac{81}{4}$).

19. For the following regions of $\mathbb{R}^3$ write the triple integral $\displaystyle\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in terms of iterated integrals taken in different order.

(a) W tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $2x + 3y + 4z = 12$.

(b) W interior of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

(c) W body bounded by the surfaces $y^2 + 2z^2 = 4x$, $x = 2$.

20. Compute the following triple integrals in the specified regions of $\mathbb{R}^3$.

(a) $\displaystyle\iiint_W xz \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, W bounded by the cylinder with circular base $x^2+y^2-2x = 0$ and the surface $z^2 = 2y$ $(y, z \geq 0)$. (Sol.: $\frac{2}{3}$).

(b) $\displaystyle\iiint_W zy\sqrt{x^2+y^2} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq x^2 + y^2,\ 0 \leq y \leq \sqrt{2x - x^2}\}$. (Sol.: $\frac{16}{9}$).

(c) $\displaystyle\iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x \leq 3,\ 1 \leq y \leq 3,\ 0 \leq z \leq xy\}$. (Sol.: 16).

(d) $\displaystyle\iiint_W xyz \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, W bounded by the surfaces $y = x^2$, $x = y^2$, $z = xy$, $z = 0$. (Sol.: $\frac{1}{96}$).

*(**e**) $\displaystyle\iiint_W x \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, W tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ $(a, b, c > 0)$. (Sol.: $\frac{a^2bc}{24}$).

Changes of variables

****21 .** The <u>*formula for a change of variables*</u> $(x, y) = T(u, v)$, for (u, v) in a bounded domain $D \subset \mathbb{R}^2$ is

$$\iint_{\Omega=T(D)} f(x, y) \,\mathrm{d}x\,\mathrm{d}y = \iint_D f(T(u, v))\, |\det (DT(u, v))| \,\mathrm{d}u\,\mathrm{d}v, \text{ where } DT(u, v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} (u, v).$$

Application: Compute the double integral $\displaystyle\iint_\Omega (x^2 + y^2)^2 \,\mathrm{d}x\,\mathrm{d}y$ on the domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^3 - 3xy^2 \leq 4,\ 2 \leq 3x^2y - y^3 \leq 5,\ x \geq 0,\ y \geq 0\}.$$

****22 .** The change to <u>*Polar coordinates*</u> is $(x, y) = T(r, \theta) := (r \cos\theta, r \sin\theta)$. Check that $|\det (DT(x, y))| = r$ and use this change to compute the following double integrals.

(a) $\displaystyle\iint_\Omega (x^2 + y^2) \,\mathrm{d}x\,\mathrm{d}y$, $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$. (Sol.: 8π).

(b) $\displaystyle\iint_\Omega \cos(x^2 + y^2) \,\mathrm{d}x\,\mathrm{d}y$, $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq \dfrac{\pi}{2}\}$. (Sol.: π).

(c) $\displaystyle\iint_\Omega \frac{(x + y)^2}{x^2 + y^2 + 2} \,\mathrm{d}x\,\mathrm{d}y$, $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. (Sol.: $2\pi\left(\frac{1}{2} + \ln(\frac{2}{3})\right)$).

(d) $\displaystyle\iint_\Omega \frac{\mathrm{d}x\,\mathrm{d}y}{(1 + x^2 + y^2)^2\sqrt{x^2 + y^2}}$, $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$. *Hint*: Use the elementary properties of sin and cos in order to see that $\sin(\arctan R) = R/\sqrt{1 + R^2}$ and $\cos(\arctan R) = 1/\sqrt{1 + R^2}$. (Sol.: $\pi\left(\arctan R + R/(1 + R^2)\right)$).

(e) $\displaystyle\iint_\Omega \sqrt{x^2 + y^2 - 9} \,\mathrm{d}x\,\mathrm{d}y$, $\Omega = \{(x, y) \in \mathbb{R}^2 : 9 \leq x^2 + y^2 \leq 25\}$. (Sol.: $\frac{128}{3}\pi$).

***(f)** $\displaystyle\iint_\Omega xy\,\mathrm{d}x\,\mathrm{d}y$, Ω intersection with the first quadrant of the annulus with center in $(0,0)$, interior radius 1 and external radius 2. (Sol.: $\frac{15}{8}$).

(g) $\displaystyle\iint_\Omega x(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$, Ω circular sector of center $(0,0)$ and radius R, forming angles between $\pi/3$ and $\pi/6$ with the positive x axis. (Sol.: $\frac{\sqrt{3}-1}{10}R^5$).

23. Compute the areas of the following domains $\Omega \subset \mathbb{R}^2$ defined in polar coordinates.

(a) Region Ω defined by $a\cos\theta \le r \le a(1+\cos\theta)$ $(a>0)$. *Hint*: Notice that the left expression only makes sense when $\cos\theta \ge 0$. Drawing graphs of $a\cos\theta$ and $a(1+\cos\theta)$ can help you to find the admissible values of r. (Sol.: $\frac{5}{4}\pi a^2$).

***(b)** Region Ω bounded by a rose petal defined by $r = a\sin 3\theta$ $(0 \le \theta \le \dfrac{\pi}{3},\ a>0)$. (Sol.: $\frac{\pi a^2}{12}$).

(c) Region Ω defined by $\dfrac{1}{2} \le r \le |\sin(2\theta)|$. *Hint*: $|\sin(2\theta)| \ge \frac{1}{2}$ is needed so as to the expression makes sense. (Sol.: $\frac{\pi}{6}+\frac{\sqrt{3}}{4}$).

(d) Compute the double integral $\displaystyle\iint_\Omega \arcsin(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$, where Ω is the region bounded by the curve $r=\sqrt{\sin(\theta)}$ $(0\le\theta\le\dfrac{\pi}{2})$. (Sol.: $1-\frac{\pi}{4}$).

24. Compute the following double integrals using the change of variables indicated in each case.

***(a)** $\displaystyle\iint_\Omega xy\,\mathrm{d}x\,\mathrm{d}y$, $\Omega=\{(x,y)\in\mathbb{R}^2 : 6\le 2y-x\le 12,\ 0\le x\le 4\}$, taking $x=4u$ and $y=2u+3v$. (Sol.: 140).

(b) $\displaystyle\iint_\Omega \frac{1}{(1+x+y)^5}\,\mathrm{d}x\,\mathrm{d}y$, $\Omega=\{(x,y)\in\mathbb{R}^2 : x\ge0,\ y\ge0,\ x+y\le1\}$, taking $u=x+y$ and $v=y$. (Sol.: $\frac{11}{192}$).

(c) $\displaystyle\iint_\Omega \frac{\mathrm{d}x\,\mathrm{d}y}{(x+y)^{n+1}}$, $\Omega=\{(x,y)\in\mathbb{R}^2 : 1\le x+y\le 2,\ x\ge0,\ y\ge0\}$, taking $u=x+y$ and $v=x$. (Sol.: $\frac{1}{n-1}(1-\frac{1}{2^{n-1}})$).

***(d)** $\displaystyle\iint_\Omega \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{3/2}\mathrm{d}x\,\mathrm{d}y$, $\Omega=\{(x,y)\in\mathbb{R}^2 : \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\le1\}$, taking $x=ar\cos\theta$ and $y=br\sin\theta$. (Sol.: $\frac{2}{5}\pi ab$).

(e) $\displaystyle\iint_\Omega \arctan\left(x^2+\frac{y^2}{2}\right)\mathrm{d}x\,\mathrm{d}y$, $\Omega=\{(x,y)\in\mathbb{R}^2 : x^2+\dfrac{y^2}{2}\le1,\ x\ge0,\ y\ge0\}$, taking $x=r\cos\theta$ and $y=\sqrt{2}r\sin\theta$. (Sol.: $\frac{\pi\sqrt{2}}{8}(\frac{\pi}{2}-\ln(2))$).

***(f)** $\displaystyle\iint_\Omega (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$, $\Omega=\{(x,y)\in\mathbb{R}^2 : 1\le x^2-y^2\le 9,\ 2\le xy\le 4,\ x\ge0,\ y\ge0\}$, taking $u=x^2-y^2$ and $v=2xy$. (Sol.: 8).

***(g)** $\displaystyle\iint_\Omega \frac{x+2xy}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$, $\Omega=\{(x,y)\in\mathbb{R}^2 : x^2\le y\le x^2+1,\ 1\le x^2+y^2\le \mathrm{e}^2,\ x\ge0\}$, taking $u=x^2+y^2$ and $v=y-x^2$. (Sol.: 1).

****25.** The change to <u>*Cylindrical coordinates*</u> is given by $(x,y,z)=T(r,\theta,z):=(r\cos\theta, r\sin\theta, z)$. Check that $|\det(DT(x,y))| = r$ and use this change to compute the volume of the following solids.

***(a)** Solid bounded by the cone $z^2=x^2+y^2$ and the paraboloid $z=x^2+y^2$, for $z>0$. (Sol.: $\pi/6$.)

***(b)** Solid bounded by the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=b^2$ ($a>b>0$). (Sol.: $\frac{4\pi}{3}(a^2-b^2)^{3/2}$.)

26. Compute the following triple integrals using cylindrical coordinates.

(a) $\iiint_W \sqrt{x^2+y^2+z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W=\{(x,y,z)\in\mathbb{R}^3 \,:\, \sqrt{x^2+y^2}\le z\le 4\}$.
(Sol.: $\frac{128\pi}{3}(2\sqrt{2}-1)$).

(b) $\iiint_W z\mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W=\{(x,y,z)\in\mathbb{R}^3 \,:\, z^2-1\le x^2+y^2\le \dfrac{z^2}{2},\ z\ge 0\}$.
(Sol.: $\frac{\pi}{2}+\frac{3\pi}{2\mathrm{e}}-\frac{\pi}{\sqrt{\mathrm{e}}}$).

(c) $\iiint_W (x+y-2z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W=\{(x,y,z)\in\mathbb{R}^3 \,:\, x^2+y^2\le z^2,\ 0\le z\le 3\}$. (Sol.: $-\frac{81\pi}{2}$).

(d) $\iiint_W (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W=\{(x,y,z)\in\mathbb{R}^3 \,:\, x^2+y^2\le 3z\le 9\}$. (Sol.: $\frac{81\pi}{2}$).

(e) $\iiint_W z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W=\{(x,y,z)\in\mathbb{R}^3 \,:\, x^2+y^2+z^2\le 6,\ x^2+y^2\le z,\ z\ge 0\}$. (Sol.: $11\pi/3$).

****27 .** The change to <u>*Spherical coordinates*</u> is $(x,y,z)=T(r,\theta,\varphi):=(r\cos\varphi\cos\theta, r\cos\varphi\sin\theta, r\sin\varphi)$. Check that $|\det(DT(x,y))|=r^2\cos\varphi$ and use this change to the volume of the following solids.

***(a)** The ball $B_R(0)$ in $\mathbb{R}^3$ of radius $R>0$ and center at the origin. (Sol.: $4\pi R^3/3$.)

***(b)** The solid W in the ball $r\le a$ cut by the cone $\alpha\le\varphi\le\dfrac{\pi}{2}$ ($a>0$, $0<\alpha<\dfrac{\pi}{2}$). (Sol.: $\frac{2\pi a^3}{3}(1-\sin\alpha)$.)

28. Compute the following triple integrals using spherical coordinates.

(a) $\iiint_W x^4y^2z^3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W=\{(x,y,z)\in\mathbb{R}^3 \,:\, x^2+y^2+z^2\le a^2\}$. (Sol.: 0).

(b) $\iiint_W z(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W=\{(x,y,z)\in\mathbb{R}^3 \,:\, x^2+y^2+z^2\le a^2,\ z\ge 0\}$. (Sol.: $\frac{1}{12}\pi a^6$).

***(c)** $\iiint_W \dfrac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(x^2+y^2+z^2)^{3/2}}$, $W=\{(x,y,z)\in\mathbb{R}^3 \,:\, a^2\le x^2+y^2+z^2\le b^2\}$. (Sol.: $4\pi\ln(b/a)$).

***(d)** $\iiint_W \sqrt{x^2+y^2+z^2}\mathrm{e}^{-(x^2+y^2+z^2)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, W the domain of the previous item.
(Sol.: $2\pi(\mathrm{e}^{-a^2}(a^2+1)-\mathrm{e}^{-b^2}(b^2+1))$).

29. Compute the volumes of the following domains $W\subset\mathbb{R}^3$ defined in spherical coordinates.

(a) W domain on the sphere $r\le a$ cut by the cone $\alpha\le\varphi\le\dfrac{\pi}{2}$ ($a>0$, $0<\alpha<\dfrac{\pi}{2}$).
(Sol.: $\frac{2\pi a^3}{3}(1-\sin\alpha)$.)

(b) W volume closed at the deformed sphere which is defined by $r=1+0.2\sin(8\theta)\sin(\varphi)$. (This type of solids are used as models of tumours, unfortunately). (Sol.: 1.36π).

(c) Compute the triple integral $\iiint_W \dfrac{1}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where W is the region in the first octant of $\mathbb{R}^3$ bounded by the cones $\varphi=\dfrac{\pi}{4}$ and $\varphi=\arctan(2)$, and the sphere $r=\sqrt{6}$. (Recall: $\sin(\arctan(a))=\frac{a}{\sqrt{1+a^2}}$). (Sol.: $6\pi(\frac{2}{\sqrt{5}}-\frac{\sqrt{2}}{2})$).

30. Adapt the spherical coordinates in order to compute the following triple integrals.

(a) $\iiint_W 16z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + \frac{z^2}{4} \le 1,\ 0 \le z \le 1,\ 0 \le y \le x\}$. (Sol.: $\frac{7\pi}{8}$).

(b) $\iiint_W \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W = \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$.

31. Use cartesian, cylindrical or spherical coordinates (or Cavalieri's principle if needed) so as to compute the volume of the domains of $\mathbb{R}^3$ bounded by the surfaces indicated below.

***(a)** $x^2 + z^2 = 1$, $x^2 + y^2 = 1$. (Sol.: $\frac{16}{3}$).

***(b)** $z^2 - x^2 = 1$, $z^2 - y^2 = 1$, $z = \sqrt{2}$. (Sol.: $\frac{4}{3}(2 - \sqrt{2})$).

(c) $z^2 = x^2 + y^2$, $z = x^2 + y^2$ $(z \ge 0)$. (Sol.: $\frac{\pi}{6}$).

(d) A part of the sphere $x^2 + y^2 + z^2 = a^2$ that is external to the cylinder $x^2 + y^2 = b^2$ $(a > b > 0)$. (Sol.: $\frac{4\pi}{3}(a^2 - b^2)^{3/2}$).

(e) $z = x^2 - 4x + 1$, $1 - z = x^2 + y^2$. (Sol.: $\pi\sqrt{2}$).

***(f)** $x^2 = z$, $y^2 = x$, $z^2 = y$, $x^2 = az$, $y^2 = ax$, $z^2 = ay$ $(a > 1)$. (Sol.: $\frac{(a-1)^3}{7}$).

***(g)** $z^2 = y$, $x^2 = 1 - y$. (Sol.: $\frac{\pi}{2}$).

***(h)** $x^2 + y^2 = 1$, $x^2 + y^2 = 2$, $z(x^2 + y^2) = 1$, $z = 0$. (Sol.: $\pi \ln(2)$).

***(i)** $x^2 + y^2 = 2z^2$, $x^2 + y^2 = z^2 + 1$ $(x \ge 0,\ y \ge 0,\ z \ge 0)$. (Sol.: $\frac{\pi}{6}$).

***(j)** $x^2 + y^2 + z^2 = 2a^2$, $z = \dfrac{x^2 + y^2}{a}$ $(z \ge 0,\ a > 0)$. (Sol.: $2\pi a^3(\frac{2^{3/2}}{3} - \frac{7}{12})$).

(k) $x^2 + y^2 = 4$, $z = x + y$ $(x \ge 0,\ y \ge 0,\ z \ge 0)$. (Sol.: $\frac{16}{3}$).

***(l)** Ice cream cone defined by $x^2 + y^2 \le \frac{1}{5}z^2$, $0 \le z \le 5 + \sqrt{5 - x^2 - y^2}$.

32. $B = B_R(0)$ is the ball in $\mathbb{R}^3$ of radius $R > 0$ and center at the origin. We consider $T(x,y,z)$ as the temperature in the point (x,y,z) and we suppose that it is proportional to the distance between the point and the origin. In what points of B is the temperature equal to the average one? (Sol.: in $S_{3R/4}(0)$, the sphere in $\mathbb{R}^3$ of radius $3R/4$ and center at the origin.)

33. Find the <u>*center of mass*</u> of the following planar regions using the densities indicated below.

(a) Circular sector defined by an annulus with internal radius a, external radius A and central angle 2α, that is symmetric with respect to the positive x axis, assuming constant density $\rho(x,y) = 1$. (Sol.: $(\frac{2}{3}\frac{\sin\alpha}{\alpha}\frac{A^3 - a^3}{A^2 - a^2}, 0)$).

(b) Region between $y = x^2$ and $y = x$ with $\rho(x,y) = x + y$.

(c) Region between $y = 0$ and $y = x^2$ $(0 \le x \le \frac{1}{2})$ with $\rho(x,y) = 1$.

34. Find the center of mass of the solid W in each case considering homogeneous mass distribution.

(a) $W = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le R^2,\ x \ge 0,\ y \ge 0, z \ge 0\}$. (Sol.: $(\frac{3R}{8}, \frac{3R}{8}, \frac{3R}{8})$).

(b) $W = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 2,\ 0 \le y \le 6,\ 0 \le z \le 4 - x^2\}$. (Sol.: $(\frac{3}{4}, 3, \frac{8}{5})$).

(c) $W = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 4a^2,\ (x-a)^2 + y^2 + z^2 \ge a^2\}$. (Sol.: $(-\frac{a}{7}, 0, 0)$).

35. Find the total mass of the cylinder $W = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \le 2,\ 0 \le z \le 3\}$ if its density is $\rho(x,y,z) = z\mathrm{e}^{-z^2}(x^2 + y^2)$. (Sol.: $\pi(1 - \mathrm{e}^{-9})$).

****36.** Find the <u>*center of mass*</u> of the semi-sphere defined by $x^2 + y^2 + z^2 \le R^2$ and $z \ge 0$, if the density at each point is proportional to the distance to the center. (Sol.: $(0, 0, \frac{2R}{5})$).

37. $W \subset \mathbb{R}^3$ is a solid object with density $\rho(x, y, z)$. If we divide W in two equal parts, $W = W_1 \cup W_2$, and we consider $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$ and $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$ as the centers of mass of W_1 and W_2, respectively, prove that the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of W is the same as if we assume that all the mass of W_1 is concentrated in $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$ and all the mass of W_2 is in $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$.

38. Ω is a planar domain that is contained in the half-plane $\{y = 0,\ x > 0\}$ of $\mathbb{R}^3$. If we consider $(\bar{x}, 0, \bar{z})$ as the geometric center of Ω (equal to its center of mass if density is constantly equal to 1), prove that the volume of the solid of revolution W obtained by rotating Ω around the z axis is $V(W){=}2\pi\bar{x}A(\Omega)$, where $2\pi\bar{x}$ is the the length of the circle obtained by rotating $(\bar{x}, 0, \bar{z})$. *Hint*: Use cylindrical coordinates and check that $(\theta, r, z) \in \Omega^* = [0, 2\pi] \times \Omega$.

39. Apply the previous result to compute the volume of W if $\Omega = \{(x, z) \in \mathbb{R}^2 : z \leq x^2,\ x + z \leq 2,\ x \geq 0,\ z \geq 0\}$. (Sol.: $\frac{8\pi}{15}$).

40. The density of water is 1 kg/l and the density of ice is approximately 0.9 kg/l. If we immerse an ice block into water, according to *Archimedes' principle*, the upward buoyant force is equal to the weight of the displaced water.

(a) Prove that the emerging ice is the 10% of the total volume.

(b) If we have an spherical ice block, find the immersed and the emerged part.

41. The population density of a given city can be approximated by the function $\rho(x, y) = 4000\mathrm{e}^{-0.01(x^2+y^2)}$ if $x^2 + y^2 \leq 49$ or $\rho(x, y) = 0$ otherwise, where x, y are measured in km.

(a) What is the total population of the city?

(b) What is the distance $0 < R < 7$ to the geographical center of the city which allows 50% of population to live from a distance smaller or equal to R regarding to the city center?

42. Compute the *Moment of inertia* of the following solids, considering homogeneous densities equal to one.

(a) Compute I_z for the solid bounded by the cone with height H and radius of the base R given by $x^2 + y^2 \leq \dfrac{R^2}{H^2}z^2$ $(0 \leq z \leq H)$. (Sol.: $\frac{1}{10}\pi H R^4$).

(b) Compute I_z for the solid bounded by two cylinders with height h, $a^2 \leq x^2 + y^2 \leq b^2$, $0 \leq z \leq h$.

(c) Compute I_z for the solid bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$ $(z \geq 0)$. (Sol.: $I_z = \frac{\pi a^6}{3}$).

(d) Compute I_x, I_y and I_z for the solid closed by the paraboloid $z = x^2 + y^2$ and the plane $z = a$ $(a > 0)$. (Sol.: $I_x = I_y = \frac{\pi}{12}a^3(1 + 3a)$, $I_z = \frac{\pi}{6}a^3$).

(e) Compute I_x, I_y and I_z for the cylinder $x^2 + y^2 \leq R^2$, $-h \leq z \leq h$.

43. Find the gravitational potential $V(0, 0, z_0)$ generated by the solid W between two concentric spheres with radius $a < b$, $W = \{(x, y, z) \in \mathbb{R}^3 : a^2 \leq x^2 + y^2 + z^2 \leq b^2\}$, assuming constant densities equal to ρ. To simplify the problem, consider only the values of z_0 with $0 < z_0 < a$ or $z_0 > b$. (Sol.: $V(0, 0, z_0) = -2\pi G\rho(b^2 - a^2)$, if $0 < z_0 < a$; $V(0, 0, z_0) = -\frac{4\pi}{3}\frac{G\rho}{z_0}(b^3 - a^3) = -\frac{G}{z_0}\mathrm{mass}(W)$, if $z_0 > b$).

SOLVED INTEGRATION EXERCISES

CALCULUS 2, 2020 CLASS

Exercise 1

Exercise. *For a* bounded *real function* $f\colon R \subset \mathbb{R}^2 \to \mathbb{R}$ *defined on a* rectangle $R = [a, b] \times [c, d]$ *of the plane, take a uniform partition of* $R = \cup_{i,j=1}^{N} R_{ij}$ *in* N^2 subrectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i]$, $i, j = 1, \ldots, N$, *with* $x_i = a + ih$, $h = \dfrac{b-a}{N}$ *and* $y_j = c + jk$, $k = \dfrac{d-c}{N}$, $i, j = 0, 1, \ldots, N$. *On each subrectangle* R_{ij} *take* $M_{ij} = \sup_{(x,y)\in R_{ij}} f(x, y)$ *and* $m_{ij} = \inf_{(x,y)\in R_{ij}} f(x, y)$, $i, j = 1, \ldots, N$, *and consider the* upper Riemann sum $S_N = \displaystyle\sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) = \frac{(b-a)(d-c)}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}$ *and the* lower Riemann sum $s_N = \dfrac{(b-a)(d-c)}{N^2}\displaystyle\sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij}$. *If* f *is continuous,* $\lim_{N\to\infty} S_n = \lim_{N\to\infty} s_n = I$ *is called the* Riemann integral $I = \int_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ *or simply* $I = \int_R f$. *Compute* $\displaystyle\iint_R (1 - x)\,\mathrm{d}x\,\mathrm{d}y$, *where* $R = [0, 1] \times [0, 1]$, *using Riemann sums.*

Resolution: We want to compute $I = \displaystyle\iint_R (1 - x)\,\mathrm{d}x\,\mathrm{d}y$. As long as f is a continuous function on a rectangle R, we know that

$$\lim_{N\to+\infty} S_N = \lim_{N\to+\infty} s_N = \iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

We take a uniform partition of $R = [a, b] \times [c, d] = [0, 1] \times [0, 1] = \cup_{i,j=1}^{N} R_{ij}$ in N^2 *subrectangles* $R_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i]$, $i, j = 1, \ldots, N$, with $x_i = a + ih = ih$, $h = \dfrac{b-a}{N} = \dfrac{1}{N}$ and $y_j = c + jk = jk$, $k = \dfrac{d-c}{N} = \dfrac{1}{N}$, $i, j = 0, 1, \ldots, N$. On each subrectangle R_{ij} take $M_{ij} = \sup_{(x,y)\in R_{ij}} f(x, y)$ and $m_{ij} = \inf_{(x,y)\in R_{ij}} f(x, y)$, $i, j = 1, \ldots, N$, for $f(x, y) = 1 - x$, so that

$$x_i = \frac{i}{N}, \qquad M_{ij} = 1 - x_{i-1} = 1 - \frac{i-1}{N}, \quad m_{ij} = 1 - x_i = 1 - \frac{i}{N}.$$

We compute now the upper and lower Riemann sums and their limits for $N \to \infty$:

$$S_N = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \underbrace{1 - \frac{i-1}{N}}_{\text{indep. of } j} = \frac{1}{N^2}(N + N - 1 + \cdots + 1) = \frac{N+1}{2N} \xrightarrow[N\to\infty]{} \frac{1}{2},$$

$$s_N = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \underbrace{1 - \frac{i}{N}}_{\text{indep. of } j} = \frac{1}{N^2}(N - 1 + \cdots + 1 + 0) = \frac{N-1}{2N} \xrightarrow[N\to\infty]{} \frac{1}{2},$$

so $I = \lim_{N\to\infty} S_N = \lim_{N\to\infty} s_N = 1/2$.

Exercise 2

Exercise. *The integral of a continuous real function* $f\colon \Omega \to \mathbb{R}$ *defined on a bounded subset* Ω *of* $\mathbb{R}^2$ *(or* $\mathbb{R}^3$*) can be defined through partitions* $\Omega = \cup_{i,j=1}^{N} \Omega_{ij}$ *(or* $\Omega = \cup_{i,j,k=1}^{N} \Omega_{ijk}$*) and is denoted by* $I = \iint_\Omega f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ *(or* $I = \iiint_R f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$*) or simply* $I = \int_\Omega f$. *It satisfies several properties for any* $f, g\colon \Omega \to \mathbb{R}$, $a, b \in \mathbb{R}$, *like* Linearity: $\int_\Omega af + bg = a\int_\Omega f + b\int_\Omega g$; Additivity: *If* $\Omega = \Omega_1 \cup \Omega_2$,

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$\Omega_1 \cap \Omega_2 = \emptyset$, *then* $\int_\Omega f = \int_{\Omega_1} f + \int_{\Omega_2} f$; Monotonicity: *If* $f \le g$ *then* $\int_\Omega f \le \int_\Omega g$; *(in particular* $\left|\int_\Omega f\right| \le \int_\Omega |f|$*);* Normalization: $\int_\Omega 1 = measure(\Omega)$, *where* $measure(\Omega) = m(\Omega)$ *is* $area(\Omega) = A(\Omega)$ *if* $\Omega \subset \mathbb{R}^2$, *and* $m(\Omega) = volume(\Omega) = V(\Omega)$ *if* $\Omega \subset \mathbb{R}^3$*;* The mean value theorem for integrals: *If* $m \le f \le g$ *then* $m \cdot m(\Omega) \le \int_\Omega f \le M \cdot m(\Omega)$. *Application: Prove that* $4\,\mathrm{e}^5 \le \iint_\Omega \mathrm{e}^{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y \le 4\,\mathrm{e}^{25}$, *where* $\Omega = [1,3] \times [2,4]$.

Resolution: For (x,y) in the rectangle Ω, it is clear than $\mathrm{e}^5 = \mathrm{e}^{1^2+2^2} \le \mathrm{e}^{x^2+y^2} \le \mathrm{e}^{3^2+4^2} = \mathrm{e}^{25}$, so applying the Mean Value Theorem for integrals (MVT), we have

$$\mathrm{e}^5 A(\Omega) \le \iint_\Omega \mathrm{e}^{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y \le \mathrm{e}^{25} A(\Omega).$$

Since $A(\Omega) = \iint_\Omega \mathrm{d}y\,\mathrm{d}x = \int_1^3 \int_2^4 \mathrm{d}y\,\mathrm{d}x = \int_1^3 \mathrm{d}x \int_2^4 \mathrm{d}y = [x]_{x=1}^{x=3} \cdot [y]_{y=2}^{y=4} = 2 \cdot 2 = 4$, we get finally

$$4\,\mathrm{e}^5 \le \iint_\Omega \mathrm{e}^{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y \le 4\,\mathrm{e}^{25}.$$

EXERCISE 3A

Exercise. *Using mean value theorem for integrals prove the following inequalities.*
$I = \dfrac{1}{\mathrm{e}} \le \dfrac{1}{4\pi^2} \iint_\Omega \mathrm{e}^{\sin(x+y)}\,\mathrm{d}x\,\mathrm{d}y \le \mathrm{e}$, *where* $\Omega = [-\pi, \pi] \times [-\pi, \pi]$

Resolution:

First method:

Let us compute the area of the domain:

$$A(\Omega) = A(\text{rectangle}) = 2\pi \times 2\pi = 4\pi^2$$

We know that x and y are between:

$$-\pi \le \mathbf{x} \le \pi / -\pi \le \mathbf{y} \le \pi$$

If we sum them we get: $-2\pi \le x + y \le 2\pi$. If we introduce the sinus we get: $\sin(-2\pi) \le \sin(x+y) \le \sin(2\pi)$.

However we know that sinus is **bounded** and goes from -1 to 1, so we can say that our function is between those values:

$$-1 \le \sin(x+y) \le 1$$

Now we write down the exponential:

$$\mathrm{e}^{-1} \le \mathrm{e}^{\sin(x+y)} \le \mathrm{e}$$

If we simplify we get:

$$\frac{1}{\mathrm{e}} \le \mathrm{e}^{\sin(x+y)} \le \mathrm{e}$$

Now we write down the integral:

$$\frac{1}{\mathrm{e}} \iint_\Omega 1\,\mathrm{d}x\,\mathrm{d}y \le \iint_\Omega \mathrm{e}^{\sin(x+y)}\,\mathrm{d}x\,\mathrm{d}y \le \mathrm{e} \iint_\Omega 1\,\mathrm{d}x\,\mathrm{d}y$$

Both side integrals, the one in the right and in the left, have the same value: $\mathrm{A}(\Omega) = 4\pi^2$. So we have:

$$4\pi^2 \frac{1}{\mathrm{e}} \le \iint_\Omega \mathrm{e}^{\sin(x+y)}\,\mathrm{d}x\,\mathrm{d}y \le 4\pi^2 \mathrm{e}$$

Now if we divide by the area:

$$\boxed{\frac{1}{\mathrm{e}} \le \frac{1}{4\pi^2} \iint_\Omega \mathrm{e}^{\sin(x+y)}\,\mathrm{d}x\,\mathrm{d}y \le \mathrm{e}}$$

We have proved through the mean value theorem (for integrals) the inequalities.

Second method:

The mean value theorem says the following:

$$m \le f(x,y) \le M, \implies mA(\Omega) \le I = \iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y \le MA(\Omega)$$

Following this theorem we see that $\mathrm{mA}(\Omega) = \dfrac{1}{\mathrm{e}}$, and that $\mathrm{MA}(\Omega) = \mathrm{e}$.
Recall that $\mathrm{A}(\Omega) = 4\pi^2$.
With this data we can compute the possible maximum and minimum values of the function in our domain.

$$mA(\Omega) = \frac{1}{\mathrm{e}} \to m = \frac{\frac{1}{\mathrm{e}}}{4\pi^2} \to m = \frac{1}{4\mathrm{e}\pi^2}; MA(\Omega) = \mathrm{e} \to M = \frac{\mathrm{e}}{4\pi^2} \to M = \frac{\mathrm{e}}{4\pi^2}$$

Maximum of f $= \dfrac{\mathrm{e}}{4\pi^2}$; minimum of f $= \dfrac{1}{4\mathrm{e}\pi^2}$.
Then, as $\dfrac{1}{4\mathrm{e}\pi^2} \leq \dfrac{1}{4\pi^2}\mathrm{e}^{\sin(x+y)} \leq \dfrac{\mathrm{e}}{4\pi^2}$, we can claim that $\boxed{\dfrac{1}{\mathrm{e}} \leq \dfrac{1}{4\pi^2}\iint_\Omega \mathrm{e}^{\sin(x+y)} \leq \mathrm{e}}$.

EXERCISE 3B

Exercise. *Using the mean value theorem for integrals prove the following inequality:*

$$\frac{1}{6} \leq I = \iint_\Omega \frac{\mathrm{d}x\,\mathrm{d}y}{y-x+3} \leq \frac{1}{4},$$

where Ω *is a triangle with vertices* $(0,0)$*,* $(1,1)$ *and* $(1,0)$*.*

Plan: We will proceed as follows: the first step will be to find a lower and and an upper bound of our function in the domain given, and then we simply apply the mean value theorem.

Resolution: To look for lower and upper bounds of the integrand $f(x,y) = \dfrac{1}{y-x+3}$ we should first focus on determining the set given. The triangle Ω is bounded by the sides $y = 0$, $x = 1$, $y = x$ so $\Omega = \{(x,y) \colon y \geq 0, x \leq 1, y \leq x\}$. We will find lower and upper bounds by using the inequalities bounding our set $\Omega = \{(x,y) \colon 0 \leq y \leq x \leq 1\}$.

We are going first to bound the denominator $y-x+3$. Just by adding $-x+3$ to both sides of $y \leq x$ we get the upper bound $y-x+3 \leq 3$. On the other hand, $1 \geq x$ implies $-1 \leq -x$ or, adding $y+3$ to both sides of this inequality, $y+2 \leq y-x+3$ and using that $0 \leq y$ and therefore $2 \leq y+2$ we get the lowe bound $2 \leq y-x+3$. Putting together the two inequalities obtained, we get $2 \leq y-x+3 \leq 3$. Finally, inverting this expression we get a lower bound $m = \frac{1}{3}$ and an upper bound $M = \frac{1}{2}$ for $f(x,y)$ on the triangle Ω:

$$m = \frac{1}{3} \leq \frac{1}{y-x+3} \leq \frac{1}{2} = M.$$

Now let us recall the mean value theorem for a function f on a planar domain Ω:

$$m \leq f(x,y) \leq M \text{ for } (x,y) \in \Omega \implies mA(\Omega) \leq \iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq MA(\Omega).$$

As a direct application of this theorem, we get that

$$\frac{A(\Omega)}{3} \leq \iint_\Omega \frac{\mathrm{d}x\,\mathrm{d}y}{y-x+3} \leq \frac{A(\Omega)}{2}.$$

It remains to compute $A(\Omega)$. As Ω is a right triangle with side lengths equal to 1,

$$A(\Omega) = A(\text{triangle}) = \frac{\text{base} \cdot \text{height}}{2} = \frac{1}{2},$$

and finally we get the desired inequality

$$\frac{1}{6} \leq \iint_\Omega \frac{\mathrm{d}x\,\mathrm{d}y}{y-x+3} \leq \frac{1}{4}.$$

EXERCISE 4

Exercise. *Let* $f(x,y,z)$ *be a continuous function and* B_r *the ball of center* $(x_0, y_0, z_0$ *and radius* r*. Prove that* $f(x_0,y_0,z_0) = \displaystyle\lim_{r\to 0^+} \frac{1}{\text{volume}(B_r)} \iiint_{B_r} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

Resolution: Using the definition of limit, we want to prove that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < r < \delta \implies \left| \frac{1}{\text{volume}(B_r)} \iiint_{B_r} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - f(x_0,y_0,z_0) \right| \leq \varepsilon.$$

Take any $\varepsilon > 0$. As f is continuous, there exists δ such that

$$\|(x,y,z) - (x_0,y_0,z_0)\| < \delta \implies |f(x,y,z) - f(x_0,y_0,z_0)| < \varepsilon.$$

For any r satisfying $0 < r < \delta$ and any $(x,y,z) \in B_r$, $\|(x,y,z) - (x_0,y_0,z_0)\| < r < \delta$ and therefore $|f(x,y,z) - f(x_0,y_0,z_0)| < \varepsilon$. By the mean value theorem for integrals, we have that

$$\left| \iiint_{B_r} (f(x,y,z) - f(x_0,y_0,z_0))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \right| < \varepsilon \iiint_{B_r} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \varepsilon \text{ volume}\left(B_r\right),$$

and we are essentially done, since

$$\begin{aligned}\iiint_{B_r} (f(x,y,z) - f(x_0,y_0,z_0))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \iiint_{B_r} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \iiint_{B_r} f(x_0,y_0,z_0)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \iiint_{B_r} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - f(x_0,y_0,z_0) \iiint_{B_r} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \iiint_{B_r} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - f(x_0,y_0,z_0) \text{ volume}(B_r),\end{aligned}$$

so that

$$\begin{aligned}&\left| \frac{1}{\text{volume}(B_r)} \iiint_{B_r} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - f(x_0,y_0,z_0) \right| \\ &\qquad = \left| \frac{1}{\text{volume}(B_r)} \iiint_{B_r} (f(x,y,z) - f(x_0,y_0,z_0))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \right| < \varepsilon.\end{aligned}$$

EXERCISE 5

Exercise. *We have already used the Cavalieri's principle for functions of one variable: For* $f \geq g : [a,b] \to \mathbb{R}$ *and* $\Omega = \{(x,y) \in \mathbb{R}^2 \colon a \leq x \leq b,\ g(x) \leq y \leq f(x)\}$*, the planar domain between the graphs of* f *and* g*, then* $A(\Omega) = \int_a^b \ell(x)\,\mathrm{d}x$*, where* $\ell(x) = f(x) - g(x)$ *is the length of the cross section* $\Omega_x = \{y : g(x) \leq y \leq f(x)\}$*. Example: Compute the area enclosed by an ellipse with semi-axis* $a, b > 0$*.*

Resolution: The formula of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are the semi-axis. The area of the figure is defined by the domain:

$$\begin{aligned}\Omega &= \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\} = \left\{(x,y) \in \mathbb{R}^2 : \frac{y^2}{b^2} \leq 1 - \frac{x^2}{a^2}\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : y^2 \leq b^2\left(1 - \frac{x^2}{a^2}\right)\right\} = \left\{(x,y) \in \mathbb{R}^2 : -a \leq x \leq a, -b\sqrt{1 - \frac{x^2}{a^2}} \leq y \leq b\sqrt{1 - \frac{x^2}{a^2}}\right\}.\end{aligned}$$

These are the cross-sections Ω_x we are going to use to apply Cavalieri's principle:

$$\Omega_x = \left\{y : -b\sqrt{1 - \frac{x^2}{a^2}} \leq y \leq b\sqrt{1 - \frac{x^2}{a^2}}\right\}.$$

Applying the definition found in the statement, we will define $g(x) = -b\sqrt{1 - \frac{x^2}{a^2}}$, and $f(x) = b\sqrt{1 - \frac{x^2}{a^2}}$. Then, $\ell(x) = f(x) - g(x) = b\sqrt{1 - \frac{x^2}{a^2}} - \left(-b\sqrt{1 - \frac{x^2}{a^2}}\right) = 2b\sqrt{1 - \frac{x^2}{a^2}}$. Now, we can compute the area:

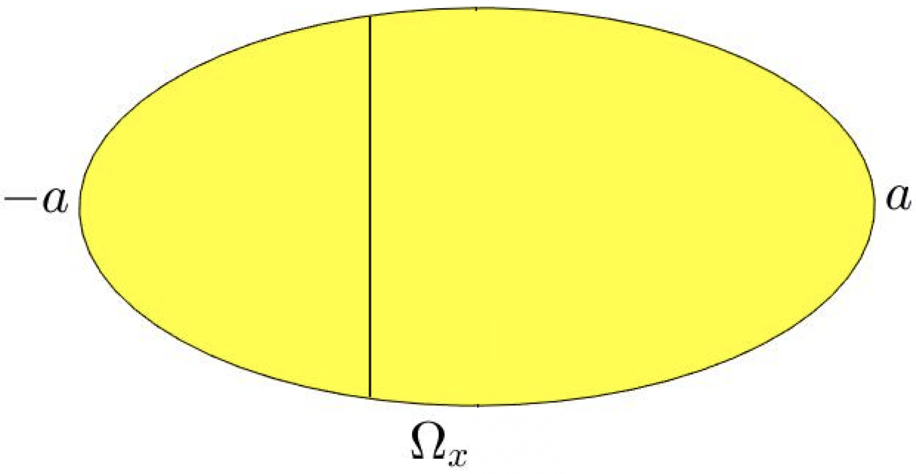


Figure 1. Domain Ω and cross-section Ω_x from exercise 5.

$$A(\Omega) = \int_{-a}^{a} \ell(x)\,\mathrm{d}x = 2b\int_{-a}^{a} \sqrt{1-\frac{x^2}{a^2}}\,\mathrm{d}x$$

where we perform the change $x = a\sin t,\ \mathrm{d}x = a\cos t\,\mathrm{d}t,$

$$\begin{aligned}
&= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\cos t\,\mathrm{d}t\\
&= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos t|\cos t\,\mathrm{d}t\\
&= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,\mathrm{d}t\\
&= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}+\frac{1}{2}\cos 2t\right)\,\mathrm{d}t\\
&= 2ab\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\,\mathrm{d}t + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos 2t\,\mathrm{d}t\right)\\
&= 2ab\left(\frac{1}{2}\,[t]_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} + \frac{1}{2}\left[\frac{\sin 2t}{2}\right]_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}}\right)\\
&= 2ab\left(\frac{1}{2}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)+\frac{1}{4}\left(\sin(\pi)-\sin(-\pi)\right)\right)\\
&= 2ab\left(\frac{\pi}{2}\right)\\
&= \pi ab.
\end{aligned}$$

$$\boldsymbol{A(\Omega) = \pi ab.}$$

Exercise 6

Exercise. *The* <u>*Cavalieri's principle*</u> *for solids* $W \subset \mathbb{R}^3$ *is* $V(W)=\int_a^b A(z)\,\mathrm{d}z$*,where* $A(z)$ *is the area of the cross section* $W_z=\{(x,y)\in\mathbb{R}^3\ :\ (x,y,z)\in W\}$*, and [a,b] is the segment* $\{z\in W : W_z\}$ *is not empty. Application: compute the volume bounded by the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

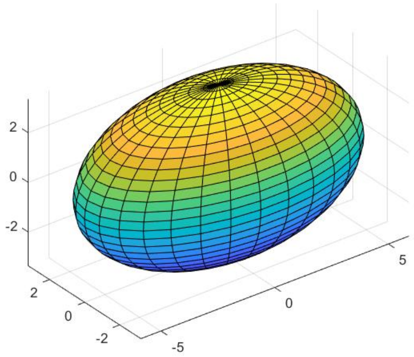


FIGURE 2. Ellipsoid W from Exercise 6.

Resolution: To apply Cavalieri's principle, first we have to find the cross section W_z. As

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \iff \frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z^2}{c^2} \iff \frac{x^2}{a^2\left(1-\frac{z^2}{c^2}\right)}+\frac{y^2}{b^2\left(1-\frac{z^2}{c^2}\right)}=1,$$

then $W_z=\left\{(x,y)\colon \frac{x^2}{\overline{a}(z)^2}+\frac{y^2}{\overline{b}(z)^2}=1\right\}$ is an ellipse with semi-axes $\overline{a}(z)$, $\overline{b}(z)$ satisfying

$$\overline{a}\,(z)^2=a^2\left(1-\frac{z^2}{c^2}\right) \quad \text{and} \quad \overline{b}\,(z)^2=b^2\left(1-\frac{z^2}{c^2}\right),$$

so its area is $A(z)=\pi\overline{a}(z)\overline{b}(z)$. We now can compute the volume:

$$\begin{aligned}V(W)&=\int_{z_{\min}}^{z_{\max}}A(z)\,\mathrm{d}z=\int_{-c}^{c}\pi\overline{a}(z)\overline{b}(z)\,\mathrm{d}z=\int_{-c}^{c}\pi\sqrt{a^2\left(1-\frac{z^2}{c^2}\right)}\sqrt{b^2\left(1-\frac{z^2}{c^2}\right)}\,\mathrm{d}z\\&=\int_{-c}^{c}\pi a\sqrt{\left(1-\frac{z^2}{c^2}\right)}\,b\sqrt{\left(1-\frac{z^2}{c^2}\right)}\,\mathrm{d}z=\int_{-c}^{c}\pi ab\left(1-\frac{z^2}{c^2}\right)\,\mathrm{d}z=\pi ab\left[z-\frac{z^3}{3c^2}\right]_{z=-c}^{z=c}\\&=\pi ab\left(c-\frac{c}{3}+c-\frac{c}{3}\right)=\pi ab\left(2c-\frac{2c}{3}\right)=\frac{4}{3}\pi abc.\end{aligned}$$

The volume bounded by the ellipsoid is $\boldsymbol{\frac{4}{3}\pi abc}$.

EXERCISE 7A

Exercise. *Apply Cavalieri's principle to compute the Volume bounded by the inverted cone with elliptic base* $z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$*, with* $0\le z\le h$*, from the areas of the sections with parallel planes to the coordinate planes.*

Resolution: We already know the characteristic area of the base, which is elliptic, so let's define our integration domain W:

$$W=\left\{(x,y,z)\colon z^2\ge\frac{x^2}{a^2}+\frac{y^2}{b^2},0\le z\le h\right\}.$$

The cross section of the volume is expressed as:

$$\begin{aligned}W_z&=\left\{(x,y),z\in[0,h]\colon \frac{x^2}{a^2}+\frac{y^2}{b^2}\le z^2\right\}=\left\{(x,y),z\in[0,h]\colon \frac{x^2}{a^2z^2}+\frac{y^2}{b^2z^2}\le 1\right\}\\&=\left\{(x,y),z\in[0,h]\colon \frac{x^2}{(az)^2}+\frac{y^2}{(bz)^2}\le 1\right\}.\end{aligned}$$

As we can see in the figure a cross section W_z of W is an ellipse. Let's now use the formula of the area of an ellipse, so we get that the area of a cross section W_z is $A(z)=\pi(az)(bz)=\pi abz^2$.

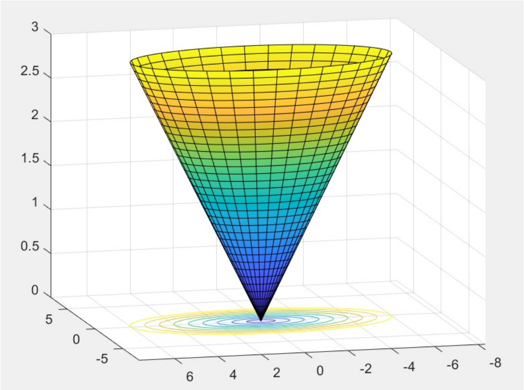


FIGURE 3. Inverted cone with elliptic base W of Exercise 7a for $a = 1$, $b = 2$ and $h = 3$.

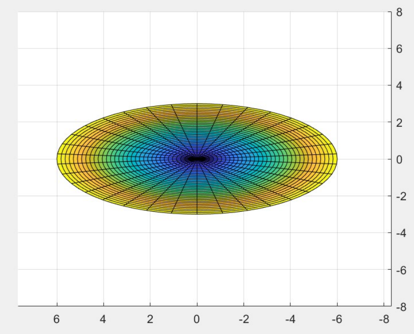


FIGURE 4. Cross section W_z of the inverted cone of Exercise 7a.

Now applying Cavalieri's principle we know that to obtain the volume of the figure we can just integrate the area of the cross section:

$$V(W) = \int_0^h \pi abz^2 \,\mathrm{d}z = \left[\frac{\pi abz^3}{3}\right]_{z=0}^{z=h} = \frac{\pi abh^3}{3}.$$

EXERCISE 7B

Exercise. *Volume of the tetrahedron bounded by the planes x=0 , y=0 , z=0 ,* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$ $(a, b, c > 0)$.

Resolution: We know that W is bounded by some inequalities to be found:

$$W = \left\{(x, y, z) \colon x \underset{<}{\geqq} 0, y \underset{<}{\geqq} 0, z \underset{<}{\geqq} 0, \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \underset{<}{\geqq} 1\right)\right\}.$$

Then

$$\begin{aligned} y = 0, z = 0 &\colon \frac{x}{a} = 1 \implies x = a > 0,\\ x = 0, z = 0 &\colon \frac{y}{b} = 1 \implies y = b > 0,\\ y = 0, x = 0 &\colon \frac{z}{c} = 1 \implies z = c > 0. \end{aligned}$$

So now we know:

$$W = \left\{(x, y, z) \colon x \geq 0, y \geq 0, z \geq 0, \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1\right)\right\}.$$

With this we now know that we have a tetrahedron.

Now we can apply Cavalieri's principle (obtaining W_z):

$$W_z = \left\{(x, y) \colon \frac{x}{a} + \frac{y}{b} \leq 1 - \frac{z}{c}, x \geq 0, y \geq 0\right\},$$

which is a right triangle with legs

$$\begin{aligned} y = 0, 0 \leq x \leq a\left(1 - \frac{z}{c}\right),\\ x = 0, 0 \leq y \leq b\left(1 - \frac{z}{c}\right). \end{aligned}$$

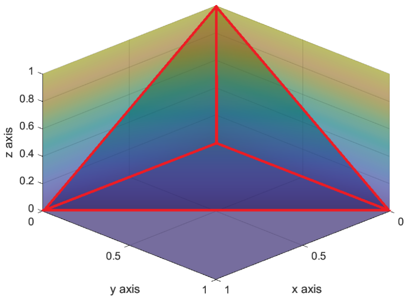


FIGURE 5. Tetrahedron W from Exercise 7b

Knowing this we can represent the each area of the sectioned tetrahedron in function of z:

$$A(z) = \frac{ab\left(1-\frac{z}{c}\right)^2}{2} = \frac{\text{base}\cdot\text{height}}{2},$$

and we can check it is right trying for 0 and c:

$$A(0) = \frac{ab}{2},$$
$$A(c) = 0.$$

Finally, the volume is going to be:

$$\begin{aligned} V(W) &= \int_0^c A(z)\,\mathrm{d}z = \frac{ab}{2}\int_0^c \left(1-\frac{z}{c}\right)^2 \mathrm{d}z \\ &= \frac{ab}{2}\left[\frac{-c}{3}(1-\frac{z}{c})^3\right]_0^c = \frac{ab}{2}\frac{c}{3} = \frac{abc}{6}. \end{aligned}$$

EXERCISE 7C

Exercise. *Find the volume of the pyramid with rectangular base with sides a, b and height h.*

Resolution: Applying Cavalieri's principle, we select a rectangular slice of the pyramid. Recall that even though all the figures represent a square base pyramid, we are looking for the volume of a rectangular base pyramid.

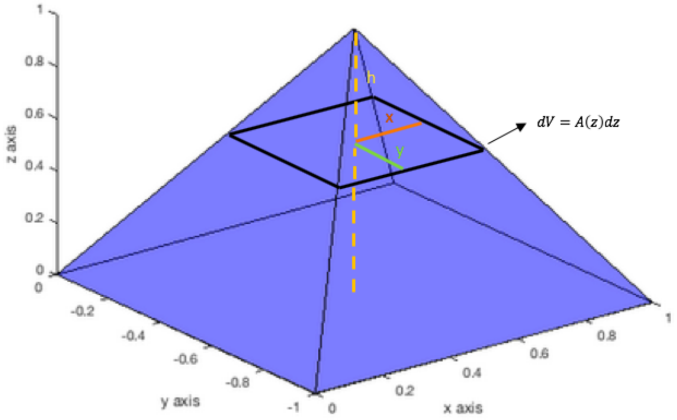


FIGURE 6. Exercise 7c, rectangular base pyramid for $a = b = h = 1$

In order to find the volume of the pyramid we have to sum all the slices that conform it, that is, all the little volumes like the one that is showed in Figure 1 ($\mathrm{d}V$). This slice is a prism (it has some height)and can be written as the area of the base (rectangle) times the height ($\mathrm{d}z$). The area of the rectangle is length times width, that is, $A(z) = 2x2y$.

$$\mathrm{d}V = A(z)\,\mathrm{d}z = 2x2y\,\mathrm{d}z$$

As we know that $z \in [0, h]$

$$V = \int_0^h 2x2y\,\mathrm{d}z$$

To compute the previous integral, we need to know the distances x and y in terms of z. Notice that both x and y depend linearly on z. So, we can draw a straight line from h to the base of the pyramid in order to find the equation that relates both x and y with z. See Figures 2 and 3.

<u>For x</u>:

$$\text{Equation straight line: } z = m_1 x + n_1\,;\ \text{Slope of the straight line}: m_1 = \frac{z_2 - z_1}{x_2 - x_1}$$

$$m_1 = \frac{0 - h}{b/2 - 0} = \frac{-2h}{b}\,;\ n_1 = h$$

$$z = \frac{-2h}{b}x + h \rightarrow x = b\left(\frac{1}{2} - \frac{z}{2h}\right)$$

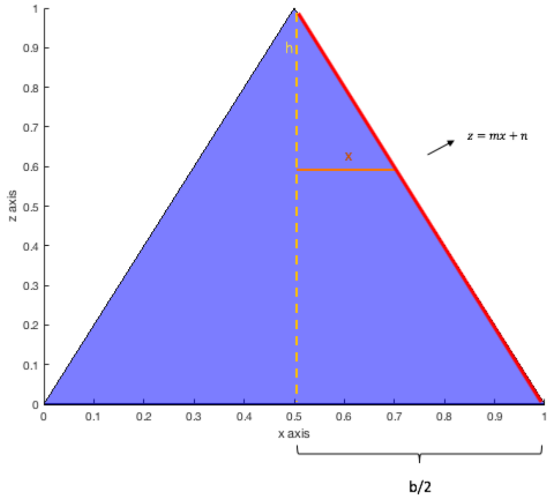


FIGURE 7. View rectangular base pyramid from the $x - z$ axis, for $h = b = 1$

<u>For y</u>:

$$\text{Equation straight line: } z = m_2 y + n_2\,;\ \text{Slope of the straight line}: m_2 = \frac{z_2 - z_1}{y_2 - y_1}$$

$$m_2 = \frac{0 - h}{a/2 - 0} = \frac{-2h}{a}\,;\ n_2 = h$$

$$z = \frac{-2h}{a}y + h \rightarrow y = a\left(\frac{1}{2} - \frac{z}{2h}\right)$$

Now we can write $A(z)$ in terms of z:

$$A(z) = 2x2y = 2b\left(\frac{1}{2} - \frac{z}{2h}\right)2a\left(\frac{1}{2} - \frac{z}{2h}\right) = ab\left(1 - \frac{z}{h}\right)^2$$

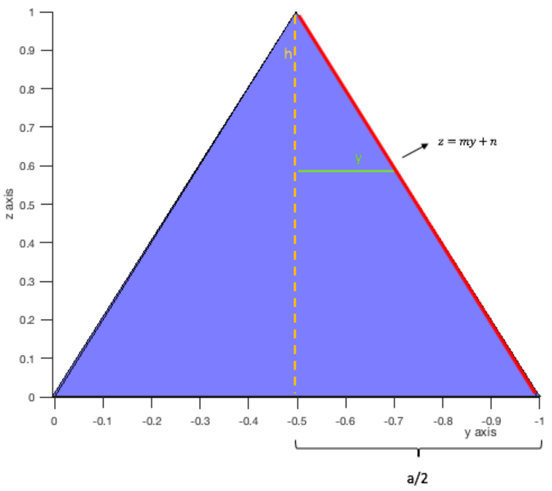


FIGURE 8. View rectangular base pyramid from the $y - z$ axis, for $h = a = 1$

At this point, we are able to solve the integral presented before in order to know the volume of a rectangle base pyramid with sides a, b and height h.

$$V = ab\int_0^h \left(1-\frac{z}{h}\right)^2 \mathrm{d}z = -abh\int_0^h \frac{-1}{h}\left(1-\frac{z}{h}\right)^2 = -abh\left[\frac{\left(1-\frac{z}{h}\right)^3}{3}\right]_0^h = -abh\left(0-\frac{1}{3}\right) = \frac{abh}{3}.$$

EXERCISE 7D

Exercise. *Apply Cavalieri's principle to compute the volume bounded by the spherical cap determined by the sphere* $x^2+y^2+z^2=R^2$ *and the condition* $R-h \le z \le R$.

Resolution: In this exercise we are given the spherical cap

$$W = \{(x,y,z) : x^2+y^2+z^2 \le R^2, R-h \le z \le R\}.$$

The condition $R-h \le z \le R$ tells us that with respect to the z axis the sphere goes from $R-h$ to R. The cross section perpendicular to the z axis is then

$$W_z = \{(x,y,z) : x^2+y^2 \le R^2-z^2\}.$$

This cross section has the shape of a circle whose area is πr^2 in our case this $r^2 = R^2 - z^2$ as it can be seen in the graph, so the area of the cross section would be $A(z) = \pi(R^2-z^2)$, such that $A(R) = 0$ and $A(R-h) = \pi(R^2-(R-h)^2)$.
To find the volume of this sphere we will integrate this area $A(z)$ as a function of z from $z = R-h$ to $z = R$.

$$\begin{aligned} V(W) &= \int_{R-h}^{R} A(z)\,\mathrm{d}z = \pi\int_{R-h}^{R}\left(R^2-z^2\right)\mathrm{d}z \\ &= \pi\left[R^2z-\frac{z^3}{3}\right]_{z=R-h}^{z=R} = \pi\left(R^3-\frac{R^3}{3}-R^2(R-h)+\frac{(R-h)^3}{3}\right) \\ &= \pi\left(R^3-\frac{R^3}{3}-R^3+R^2h+\frac{R^3-3R^2h+3Rh^2-h^3}{3}\right) \\ &= \pi\left(\frac{3Rh^2-h^3}{3}\right) = \frac{\pi h^2}{3}(3R-h). \end{aligned}$$

EXERCISE 8

Exercise. *Generalize Cavalieri's principle to the computation of volumes in* $\mathbb{R}^4$ *and compute the volume of the 4D ball* $B = \left\{(x,y,z,t) \in \mathbb{R}^4 \colon x^2+y^2+z^2+t^2 \le R^2\right\}$.

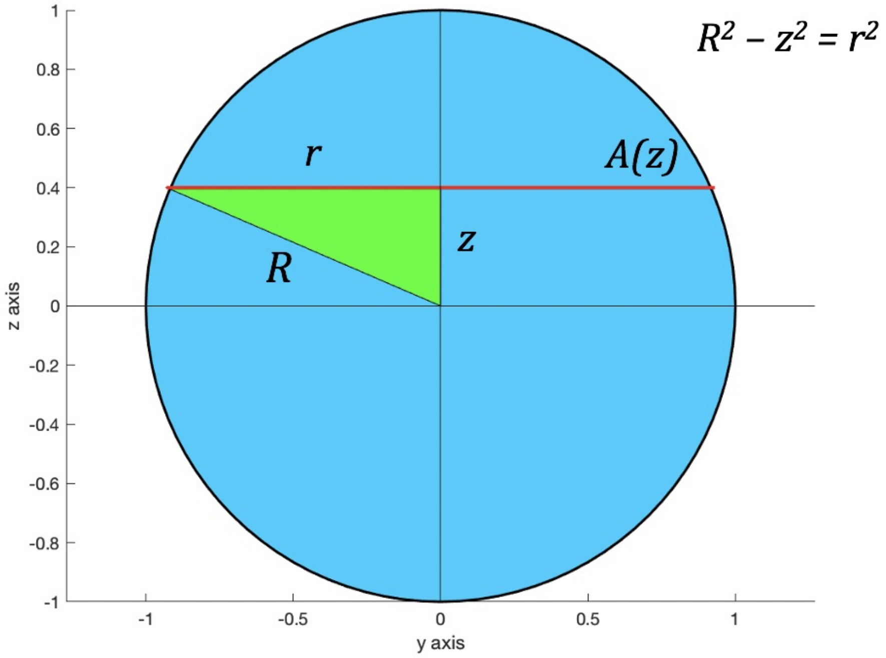


FIGURE 9. Front view of the sphere of Exercise 7d with radius R and its cross section of area $A(z)$ shown in red.

Resolution: We compute the cross-section B_t of B for a fixed t:

$$B_t = \{(x, y, z) : x^2 + y^2 + z^2 \leq R^2 - t^2\},$$

which happens to be a 3D-ball with radius $R'_t = \sqrt{R^2 - t^2}$ with 3D-volume $V(t) = \frac{4\pi}{3}\sqrt{R^2 - t^2}^3$.

Thus, by Cavalieri's principle for the computation of volumes in $\mathbb{R}^4$,

$$V(B) = \int_{-R}^{R} V(t)\,\mathrm{d}t = \int_{-R}^{R} \frac{4\pi}{3}\sqrt{R^2 - t^2}^3\,\mathrm{d}t = \frac{4\pi}{3}\int_{-R}^{R}(R^2 - t^2)^{3/2}\,\mathrm{d}t = \frac{8\pi}{3}R^3\int_0^R \left(1 - \frac{t^2}{R^2}\right)^{3/2}\mathrm{d}t$$

To compute the integral, we use the change of variable $t = R\sin\theta,\ \mathrm{d}t = R\cos\theta\,\mathrm{d}\theta$:

$$\begin{aligned}
V(W) &= \frac{8\pi}{3}R^3\int_0^{\pi/2}(1 - \sin^2\theta)^{3/2}R\cos\theta\,\mathrm{d}\theta = \frac{8\pi}{3}R^4\int_0^{\pi/2}(\cos^2\theta)^{3/2}\cos\theta\,\mathrm{d}\theta \\
&= \frac{8\pi}{3}R^4\int_0^{\pi/2}\cos^4\theta\,\mathrm{d}\theta = \frac{8\pi}{3}R^4\int_0^{\pi/2}(\cos^2\theta)^2\,\mathrm{d}\theta = \frac{8\pi}{3}R^4\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2\mathrm{d}\theta \\
&= \frac{8\pi}{12}R^4\int_0^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta\,\mathrm{d}\theta = \frac{2\pi}{3}R^4\int_0^{\pi/2}\mathrm{d}\theta + \int_0^{\pi/2} 2\cos 2\theta\,\mathrm{d}\theta + \int_0^{\pi/2}\cos^2 2\theta\,\mathrm{d}\theta \\
&= \frac{2\pi}{3}R^4\left(\frac{\pi}{2} + [\sin 2\theta]_0^{\frac{\pi}{2}} + \int_0^{\pi/2}\frac{1}{2} + \frac{1}{2}\cos 4\theta\,\mathrm{d}\theta\right) = \frac{2\pi}{3}R^4\left(\frac{\pi}{2} + \frac{\pi}{4} + \left[\frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{2}}\right) \\
&= \frac{\pi^2 R^4}{2}.
\end{aligned}$$

EXERCISE 9

Exercise. *Given a real function* $f\colon R \subset \mathbb{R}^2 \to \mathbb{R}$ *defined on a rectangle* $R = [a, b] \times [c, d]$ *of the plane, assuming for the moment that* f *is continuous and* $f \geq 0$*, apply Cavalieri's principle to prove the* Fubini-Tonelli theorem*:* $\iint_{R=[a,b]\times[c,d]} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}x \int_c^d f(x, y)\,\mathrm{d}y = \int_c^d \mathrm{d}y \int_a^b f(x, y)\,\mathrm{d}x.$

Application: Compute $\iint_R y\ln x\,\mathrm{d}x\,\mathrm{d}y$ *where* $R = [1, \mathrm{e}] \times [1, \mathrm{e}]$.

Resolution: The volume of the solid

$$W = \{(x, y, z)\colon (x, y) \in R, 0 \leq z \leq f(x, y)\}$$

is $V(W)=\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y$. Applying Cavalieri's principle, we get $V(W)=A(x)\,\mathrm{d}x$, where $A(x)$ is the area of the *cross-section*

$$W_x=\{(x,y,z)\colon (x,y,z)\in W\}=\{(x,y,z)\colon y\in[c,d],0\le z\le f(x,y)\},$$

which is therefore given by $A(x)=\int_a^b f(x,y)\,\mathrm{d}y$. So,

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_a^b\left(\int_a^b f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x=\int_a^b \mathrm{d}x\int_c^d f(x,y)\,\mathrm{d}y,$$

introducing this last notation to avoid parentheses.

In a totally analogous way, $V(W)=A(y)\,\mathrm{d}y$, where $A(y)$ is the area of the *cross-section*

$$W_y=\{(x,y,z)\colon (x,y,z)\in W\}=\{(x,y,z)\colon x\in[a,b],0\le z\le f(x,y)\},$$

which is therefore given by $A(y)=\int_c^d f(x,y)\,\mathrm{d}x$. So, we get the *Fubini-Tonelli theorem*:

$$\int_a^b \mathrm{d}x\int_c^d f(x,y)\,\mathrm{d}y=\iint_{[a,b]\times[c,d]} f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_c^d \mathrm{d}y\int_c^d f(x,y)\,\mathrm{d}x.$$

In the application we have $f(x,y)=y\ln x$ on $R=[1,\mathrm{e}]\times[1,\mathrm{e}]$.

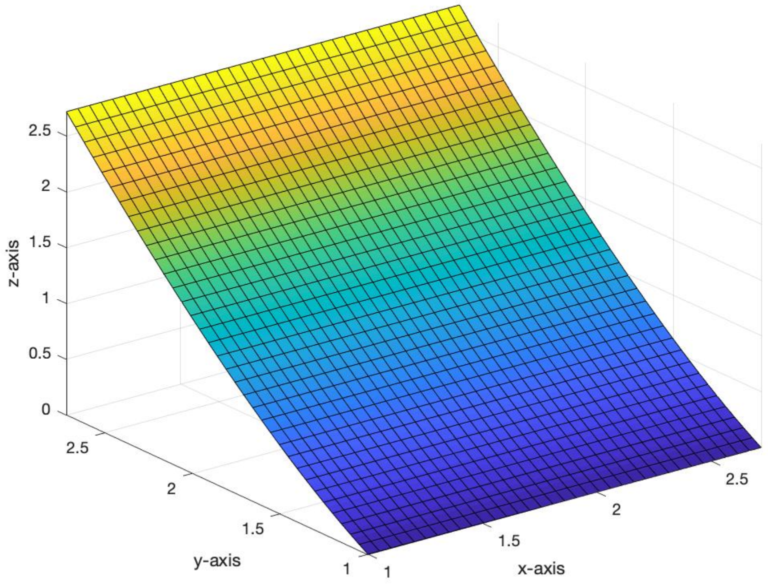


FIGURE 10. $f(x,y)=y\ln x$ on $R=[1,\mathrm{e}]\times[1,\mathrm{e}]$ from Exercise 9.

$$\begin{aligned}\iint_{[1,\mathrm{e}]\times[1,\mathrm{e}]} y\ln x\,\mathrm{d}x\,\mathrm{d}y&=\int_1^{\mathrm{e}}\mathrm{d}y\int_1^{\mathrm{e}} y\ln x\,\mathrm{d}x=\int_1^{\mathrm{e}} y\,\mathrm{d}y\int_1^{\mathrm{e}}\ln x\,\mathrm{d}x\\&=\left[\frac{y^2}{2}\right]_{y=1}^{y=\mathrm{e}}\underbrace{\left[x\,(\ln x-1)\right]_{x=1}^{x=\mathrm{e}}}_{1}=\frac{1}{2}\left(\mathrm{e}^2-1\right).\end{aligned}$$

EXERCISE 10A

Exercise. *Find the double integral* $\iint_R x^2y\,\mathrm{d}x\,\mathrm{d}y$ *in the rectangle* $R=[0,1]\times[0,1]$.

Resolution: To find this integral, we can write the iterated integrals in any order without modifying the limits of integration of each variable:

$$\iint_R x^2y\,\mathrm{d}x\,\mathrm{d}y=\int_0^1 x^2\,\mathrm{d}x\int_0^1 y\,\mathrm{d}y=\left[\tfrac{1}{3}x^3\right]_{x=0}^{x=1}\cdot\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=1}=\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{6}.$$

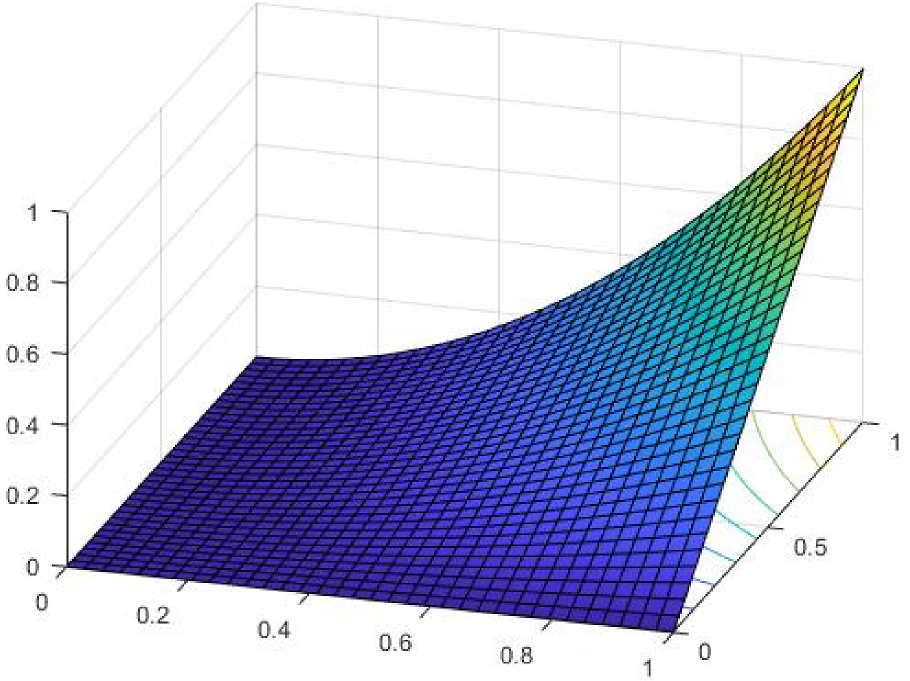


FIGURE 11. Graph of $z = x^2y$ over the rectangle R from Exercise 10a.

EXERCISE 10B

Exercise. *Find the following double integral* $I = \iint_R \frac{x^2}{1+y^2}\,\mathrm{d}x\,\mathrm{d}y$ *in the rectangle* $R = [0,1] \times [0,1]$.

Resolution:

$$I = \iint_R \frac{x^2}{1+y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_1^0 \frac{1}{1+y^2}\,\mathrm{d}y \int_0^1 x^2\,\mathrm{d}x = \int_0^1 \left[\frac{1}{1+y^2}\frac{x^3}{3}\right]_{x=0}^{x=1} \mathrm{d}y$$
$$= \int_0^1 \frac{1}{3(1+y^2)}\,\mathrm{d}y = \left[\frac{1}{3}\arctan y\right]_{y=0}^{y=1} = \frac{1}{3}\cdot\frac{\pi}{4} = \frac{\pi}{12}.$$

EXERCISE 10C

Exercise. *Find the following double integral in the specified rectangle:* $I = \iint_R (x^2 + y)\,\mathrm{d}x\,\mathrm{d}y$, $R = [0,1] \times [0,2]$.

Resolution: It is a simple double integral, so we can solve it without applying any change of variables. We just have to integrate first respect one variable and then we integrate the result respect the other one. As it is a rectangle, the order of integration does not matter. In this case we have integrated first respect to the variable y.

$$\begin{aligned}I &= \iint_R (x^2+y)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^1 \mathrm{d}x \int_0^2 (x^2+y)\,\mathrm{d}y\\ &= \int_0^1 \left[yx^2+\frac{y^2}{2}\right]_{y=0}^{y=2}\mathrm{d}x\\ &= \int_0^1 (2x^2+2)\,\mathrm{d}x\\ &= \left[\frac{2x^3}{3}+2x\right]_{x=0}^{x=1}\\ &= \frac{2}{3}+2=\boxed{\frac{8}{3}}\end{aligned}$$

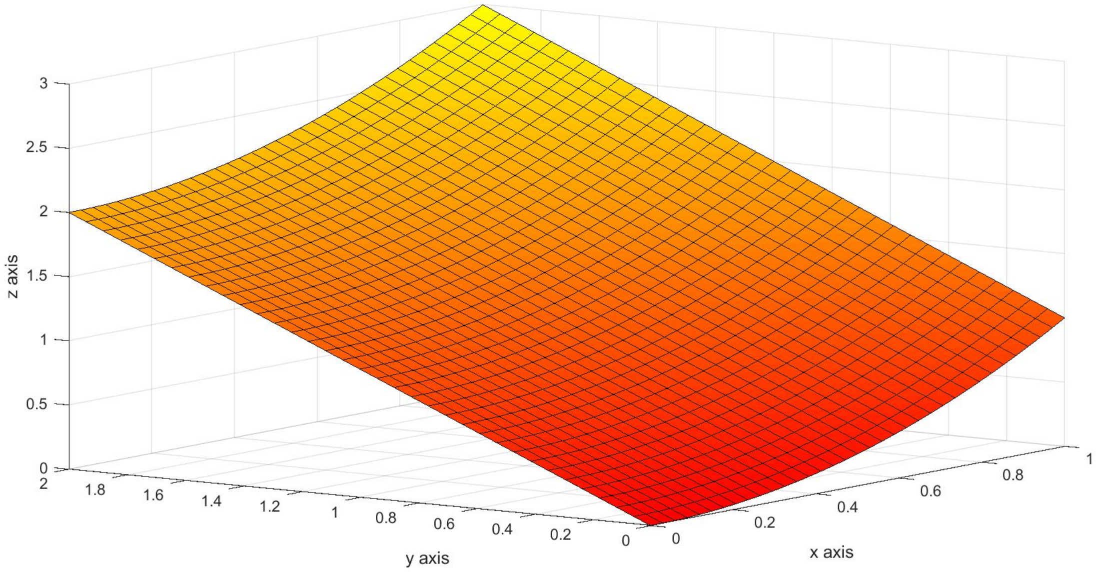


FIGURE 12. Plot of the integrand of $I=\iint_R(x^2+y)\,\mathrm{d}x\,\mathrm{d}y$ on the domain R. Exercise 10c.

EXERCISE 10D

Exercise. *Find the double integral* $I=\iint_R \frac{1}{(x+2y)^2}\,\mathrm{d}x\,\mathrm{d}y$ *in the rectangle* $R=[2,5]\times[1,3]$*:*

Resolution: First we integrate the function with respect to x:

$$I=\int_1^3 \mathrm{d}y\int_2^5 \frac{1}{(x+2y)^2}\,\mathrm{d}x=\int_1^3\left[\frac{-1}{x+2y}\right]_{x=2}^{x=5}\mathrm{d}y=\int_1^3 \frac{1}{2(1+y)}\,\mathrm{d}y-\int_1^3\frac{1}{5+2y}\,\mathrm{d}y$$

$$=\frac{1}{2}\left[\ln(1+y)\right]_{y=1}^{y=3}-\frac{1}{2}\left[\ln(5+2y)\right]_{y=1}^{y=3}=\frac{1}{2}\left[\ln(4)-\ln(2)-\ln(11)+\ln(7)\right]$$

$$=\frac{1}{2}\left[\ln\frac{\frac{4}{2}}{\frac{11}{7}}\right]=\boxed{\frac{1}{2}\ln\left(\frac{14}{11}\right)}$$

EXERCISE 10F

Exercise. *Find the double integral in the specified rectangle:* $I=\iint_R \mathrm{e}^y\sin\left(\frac{x}{y}\right)\mathrm{d}x\,\mathrm{d}y$ *in the rectangle* $R=[-\pi/2,\pi/2]\times[1,2]$.

Resolution:

$$I = \iint_R \mathrm{e}^y \sin\left(\frac{x}{y}\right) \mathrm{d}x\,\mathrm{d}y = \int_{-\pi/2}^{\pi/2} \mathrm{e}^y \,\mathrm{d}y \int_1^2 \sin\left(\frac{x}{y}\right) \mathrm{d}x$$

$$= \int_1^2 \mathrm{e}^y \left[-\cos\left(\frac{x}{y}\right) y\right]_{x=-\pi/2}^{x=\pi/2} \mathrm{d}y$$

$$= \int_1^2 -\left[\cos\left(\frac{\pi/2}{y}\right) y\,\mathrm{e}^y\right] + \left[\cos\left(\frac{-\pi/2}{y}\right) y\,\mathrm{e}^y\right] \mathrm{d}y = 0,$$

Since $-\left[\cos\left(\frac{\pi/2}{y}\right) y\,\mathrm{e}^y\right] + \left[\cos\left(\frac{-\pi/2}{y}\right) y\,\mathrm{e}^y\right]$ is the same number, as $\cos(\alpha) = \cos(-\alpha)$ and they have opposite sign, $\boxed{I = 0}$.

Note that the integrand $f(x,y) = \mathrm{e}^y \sin\left(\frac{x}{y}\right) \mathrm{d}x\,\mathrm{d}y$ is odd in the x variable: $f(-x,y) = -f(x,y)$, whereas the rectangle is symmetric with respect to the x variable, so its is clear that $I = 0$. Indeed, introducing the change of variables $(x,y) = T(u,v) = (-u,v)$, it is easy to check that $R = T(R)$, $|\det DT(u,v)| = 1$, so that

$$I = \iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_R f(-u,v)\,\mathrm{d}u\,\mathrm{d}v = -\iint_R f(u,v)\,\mathrm{d}u\,\mathrm{d}v = -I,$$

and hence $I = 0$.

EXERCISE 11

Exercise. *Compute* $\iint_R x^y \,\mathrm{d}x\,\mathrm{d}y$ *in* $R = [0,1] \times [a,b]$ *, where* $0 < a < b$*, and deduce the value of the integral* $\int_0^1 \frac{x^b - x^a}{\ln x} \mathrm{d}x$*.*

Resolution: The domain of this integral is rectangular. In other words, it doesn't matter the order when integrating. Let us start integrating with respect to x. The limits are:

$$\begin{cases} 0 < x < 1, \\ a < y < b. \end{cases}$$

Therefore

$$I = \int_a^b \int_0^1 x^y \,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left[\frac{x^{y+1}}{y+1}\right]_{x=0}^{x=1} \mathrm{d}y = \int_a^b \frac{1^{y+1}}{y+1} \mathrm{d}y.$$

Since $y > 0$ and $a, b > 0$, then:

$$I = \int_a^b \frac{1}{y+1} \mathrm{d}y = [\ln|y+1|]_{y=a}^{y=b} = \ln\left|\frac{b+1}{a+1}\right| = \ln\frac{b+1}{a+1}.$$

Now, we have to deduce the value of $\int_0^1 \frac{x^b - x^a}{\ln x} \mathrm{d}x$. We will invert the integration order of the original integral I, integrating with respect to y:

$$I = \int_0^1 \int_a^b x^y \,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left[\frac{x^y}{\ln x}\right]_{y=a}^{y=b} \mathrm{d}x = \int_0^1 \frac{x^b - x^a}{\ln x} \mathrm{d}x.$$

We have two expressions for the same integral, which have to be equal:

$$\int_0^1 \frac{x^b - x^a}{\ln x} \mathrm{d}x = \ln\frac{b+1}{a+1}.$$

EXERCISE 13A

Exercise. *Fubini-Tonelli theorem is also valid for continuous functions f:* $\Omega \longrightarrow \mathbb{R}^2$ *not necessarily positive, and for planar domains* $\Omega \subset \mathbb{R}^2$ *not necessary rectangles, but the limits of integration of the iterated integrals have to be worked out. The same is true for solids* $W \subset \mathbb{R}^2$*. For the region* $\Omega \longrightarrow \mathbb{R}^2$ *a rectangle with vertices* $(1,2)$*,* $(5,2)$*,* $(5,4)$ *and* $(1,4)$*, write the double integral*

$$\iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

in terms of iterated integrals taken in different order, $\int \mathrm{d}y \int f(x,y)\,\mathrm{d}x$ *and* $\int \mathrm{d}x \int f(x,y)\,\mathrm{d}y$*, finding the limits of integration for* x *and* y *in each case.*

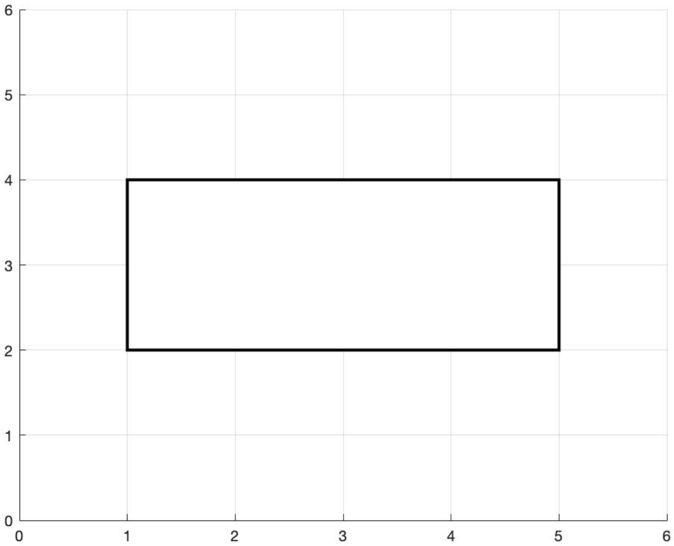


FIGURE 13. Ω domain, Rectangle with vertices (1,2), (5,2), (5,4) and (1,4) from Exercise 13a

Resolution:

$$I = \int_{x=1}^{x=5} \int_{y=2}^{y=4} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_{y=2}^{y=4} \int_{x=1}^{x=5} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Since it is a rectangular Domain, and the limits are constants, I can apply Fubini-Tonelli theorem.

EXERCISE 13B

Exercise. *For the region* $\Omega \subset \mathbb{R}^2$*, where* Ω *is a parallelogram bounded by the straight lines* $y = x$*,* $y = x - 3$*,* $y = 2$ *and* $y = 4$*, write the double integral* $\iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ *in terms of iterated integrals taken in different order,* $\int \mathrm{d}y \int f(x,y)\,\mathrm{d}x$ *and* $\int \mathrm{d}x \int f(x,y)\,\mathrm{d}y$*, finding the limits of integration for* x *and* y *in each case.*

Resolution: First, we draw a sketch of the region Ω.

Now, we look for the limits of the integral depending on the order. First, we start with $I = \int \mathrm{d}y \int f(x,y)\,\mathrm{d}x$. In this case, we need to fix y and then look for the limits of x.

Then, we find that $\Omega = \{(x,y) : 2 \le y \le 4, y \le x \le y+3\}$. Therefore, we have the following limits of integration:

$$I = \int_2^4 \mathrm{d}y \int_y^{y+3} f(x,y)\,\mathrm{d}x$$

For instance, we solve this integral for $f(x,y) = 1$.

$$I = \int_2^4 \mathrm{d}y \int_y^{y+3} 1\,\mathrm{d}x = \int_2^4 \mathrm{d}y\,[x]_{x=y}^{x=y+3} = \int_2^4 3\,\mathrm{d}y = 3\,[y]_{y=2}^{y=4} = 3(3) = 6 = A(\Omega)$$

Now, we continue with $I = \int \mathrm{d}x \int f(x,y)\,\mathrm{d}y$. In this case, we need to fix x and then look for the limits of y.

As we can see in Figure 3, in Ω_1 the x is fixed from $x = 2$ to $x = 4$, and we find that y goes from $y = 2$ to $y = x$. In Ω_2 the x is fixed from $x = 4$ to $x = 5$, and we find that y goes from $y = 2$ to $y = 4$. In Ω_3 the x is fixed from $x = 5$ to $x = 7$, and we find that y goes from $y = x - 3$ to $y = 4$.

Then, we have $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where

$$\Omega_1 = \{(x,y) : 2 \le x \le 4, 2 \le y \le x\}$$

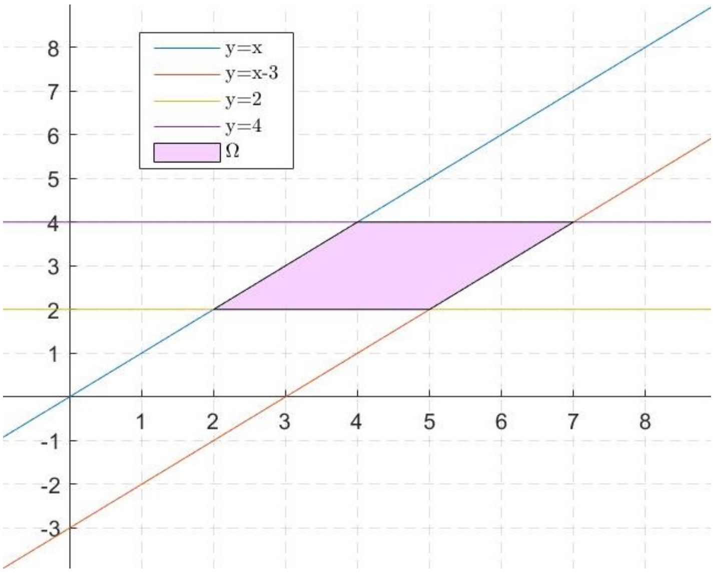


FIGURE 14. Region Ω bounded by $y = x$, $y = x - 3$, $y = 2$ and $y = 4$ in Exercise 13b.

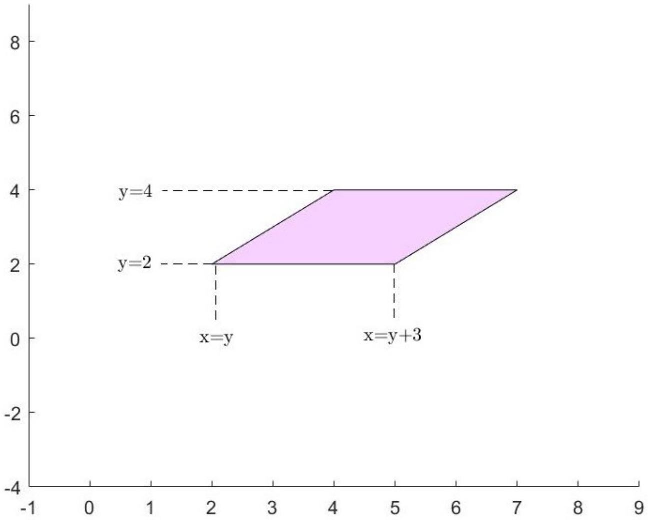


FIGURE 15. We fix y from $y = 2$ to $y = 4$. We find that x goes from $x = y$ to $x = y+3$ in Exercise 13b.

$$\Omega_2 = \{(x, y) : 4 \le x \le 5, 2 \le y \le 4\}$$
$$\Omega_3 = \{(x, y) : 5 \le x \le 7, x - 3 \le y \le 4\}$$

Therefore, we have the following limits of integration:

$$I = \iint_{\Omega_1} f(x, y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{\Omega_2} f(x, y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{\Omega_3} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

$$I = \int_2^4 \mathrm{d}x \int_2^x f(x, y)\,\mathrm{d}y + \int_4^5 \mathrm{d}x \int_2^4 f(x, y)\,\mathrm{d}y + \int_5^7 \mathrm{d}x \int_{x-3}^4 f(x, y)\,\mathrm{d}y$$

For instance, we solve this integral for $f(x, y) = 1$.

$$I = \int_2^4 \mathrm{d}x \int_2^x 1\,\mathrm{d}y + \int_4^5 \mathrm{d}x \int_2^4 1\,\mathrm{d}y + \int_5^7 \mathrm{d}x \int_{x-3}^4 1\,\mathrm{d}y$$

$$I = \int_2^4 \mathrm{d}x\,[y]_{y=2}^{y=x} + \int_4^5 \mathrm{d}x\,[y]_{y=2}^{y=4} + \int_5^7 \mathrm{d}x\,[y]_{y=x-3}^{y=4}$$

$$I = \int_2^4 (x - 2)\,\mathrm{d}x + \int_4^5 2\,\mathrm{d}x + \int_5^7 (7 - x)\,\mathrm{d}x$$

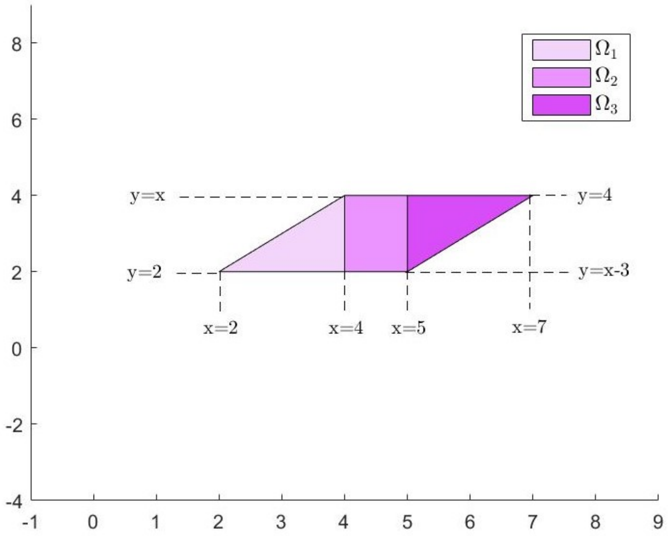


FIGURE 16. We can split the whole region Ω in three smaller regions in Exercise 13b.

$$I = \left[\frac{x^2}{2} - 2x\right]_{x=2}^{x=4} + [2x]_{x=4}^{x=5} + \left[7x - \frac{x^2}{2}\right]_{x=5}^{x=7} = 2 + 2 + 2 = 6 = A(\Omega)$$

Note that Ω_1, Ω_2, Ω_3 are of the same size. Note also that both integrals we computed give the same result (which is the area of the region Ω) independently of the order of integration.

EXERCISE 13C

Exercise. *Fubini-Tonelli theorem is also valid for continuous functions* $f\colon \Omega \longrightarrow \mathbb{R}$ *not necessarily positive, and for planar domains* $\Omega \subset \mathbb{R}^2$ *not necessarily rectangles, but the limits of integration of the iterated integrals have to be worked out. The same is true for solids* $W \subset \mathbb{R}^3$*. For the region* $\Omega \subset \mathbb{R}^2$ *bounded by the curves* $x^2+y^2 = 2a^2, x^2 = ay, (y \geq 0, a > 0)$*, write the double integral* $\iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ *in terms of iterated integrals taken in different order,* $\int \mathrm{d}y \int f(x,y)\,\mathrm{d}x$ *and* $\int \mathrm{d}x \int f(x,y)\,\mathrm{d}y$*, finding the limits of integration for* x *and* y *in each case.*

Resolution: Although the statement asks to start by the y variable, we will start by the x variable.

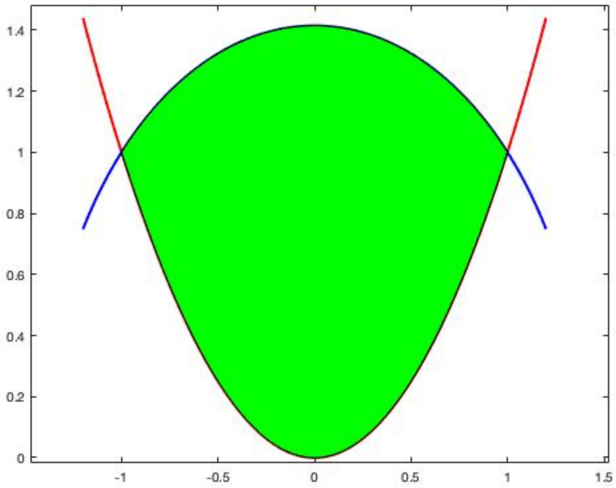


FIGURE 17. Domain Ω between the curves $x^2+y^2 = 2a^2, x^2 = ay$ of Exercise 13c, for $a = 1$.

This is a personal preference and does not affect the problem at all as we will compute first one way and then the other. We first see where the two curves $x^2+y^2=2a^2, x^2=ay$ intersect:

$$x^2+y^2=2a^2,\ x^2=ay \implies ay+y^2=2a^2 \iff y^2+ay-2a^2=0.$$

Solving for y the quadratic equation that satisfies, we find $y=a$ and $y=\cancel{-2a}$. Now we substitute the value $y=a$ in the equation of $x^2=ay$ and find that $x^2=a^2$, so that the two curves intersect at the points $(-a,a)$ and (a,a). Looking at the figure, we can write two different expressions for Ω.

Fixing first x:

$$\Omega=\left\{(x,y)\colon -a\le x\le a, \frac{x^2}{a}\le y\le\sqrt{2a^2-x^2}\right\}.$$

Now that we have Ω defined, we can write the iterated integrals:

$$\iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_{-a}^{a}\mathrm{d}x\int_{\frac{x^2}{a}}^{\sqrt{2a^2-x^2}}\mathrm{d}y f(x,y).$$

Fixing first y:

$$\Omega=\{(x,y)\colon 0\le y\le a, -\sqrt{ay}\le x\le\sqrt{ay}\}\cup\left\{(x,y)\colon a\le y\le a\sqrt{2}, -\sqrt{2a^2-y^2}\le x\le\sqrt{2a^2-y^2}\right\}.$$

Now that we have Ω defined, we can write the iterated integrals:

$$\iint_\Omega f(x,y)\,\mathrm{d}y\,\mathrm{d}x=\int_0^a\mathrm{d}y\int_{-\sqrt{ay}}^{\sqrt{ay}}\mathrm{d}x f(x,y)+\int_a^{a\sqrt{2}}\mathrm{d}y\int_{-\sqrt{2a^2-y^2}}^{\sqrt{2a^2-y^2}}\mathrm{d}x f(x,y).$$

EXERCISE 13D

Exercise. *For the region $\Omega\subset\mathbb{R}^2$ bounded by the curves $y^2=ax$, $x^2+y^2=2ax$, $y=0$ $(y\ge 0,\ a>0)$, write the double integral $I=\iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ in terms of iterated integrals taken in different order, $\int\mathrm{d}y\int f(x,y)\,\mathrm{d}x$ and $\int\mathrm{d}x\int f(x,y)\,\mathrm{d}y$, finding the limits of integration for x and y in each case.*

FIGURE 18. The circle $(x-a)^2+y^2=a^2$ is over the parabola $y^2=ax$ for $0<x<a=1$ from Exercise 13d.

Resolution: The equation $x^2+y^2=2ax$ can be written as $(x-a)^2+y^2=a^2$, so we have a circle of radius a centered in the point $(a,0)$, jointly with a parabola $y^2=ax$. Both curves intersect, on $y\ge 0$, just on the two points $(0,0)$, and (a,a). For $0<x<a$, the circle $y^2=2ax-ax^2$ is located over the parabola $y^2=ax$. This can be checked, for instance for $x=a/2$. Alternatively, $0<x<a \implies x^2<ax \implies x^2+ax<2ax \implies ax<2ax-x^2$.

Therefore the domain Ω between the circle and the parabola is inside the circle and outside the parabola, and is contained in the square $[0,a]\times[0,a]$, so it can be written as

$$\Omega=\left\{(x,y)\colon ax\le y^2, (x-a)^2+y^2\le a^2, y\ge 0\right\}=\left\{(x,y)\colon ax\le y^2, x^2+y^2\le 2ax, y\ge 0\right\},$$

and we are going to give equivalent definitions of Ω which will provide immediately the limits of integration in the iterated integrals.

Fixing first x:

$$\Omega = \{(x,y)\colon ax \le y^2 \le 2ax - ax^2, y \ge 0\} = \left\{(x,y)\colon 0 \le x \le a, \sqrt{ax} \le y \le \sqrt{2ax - ax^2}\right\},$$

our integral is

$$I = \int_0^a \mathrm{d}x \int_{\sqrt{ax}}^{\sqrt{2ax-x^2}} f(x,y)\,\mathrm{d}y.$$

Fixing first y: Inside the square $0 \le x \le a$, $0 \le y \le a$, the equation of the circle $(x-a)^2 + y^2 = a^2$ is equivalent to $x = a - \sqrt{a^2 - y^2}$ and the equation of the disk $(x-a)^2 + y^2 \le a^2$ is equivalent to $x \ge a - \sqrt{a^2 - y^2}$ so that

$$\Omega = \left\{(x,y)\colon 0 \le y \le a, a - \sqrt{a^2 - y^2} \le x \le y^2/a\right\},$$

our integral is

$$I = \int_0^a \mathrm{d}y \int_{a-\sqrt{a^2-y^2}}^{y^2/a} f(x,y)\,\mathrm{d}x.$$

EXERCISE 13E

Exercise. *For the Ω region bounded by the curves $x^2 + y^2 = ax, x^2 + y^2 = 2ax$ $(y \ge 0,\ a \ge 0)$, write the double integral $I = \iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ in terms of iterated integrals taken in different order, $\int dy \int f(x,y)\,\mathrm{d}x$ and $\int dx \int f(x,y)\,\mathrm{d}y$, finding the limits of integration for x and y in each case.*

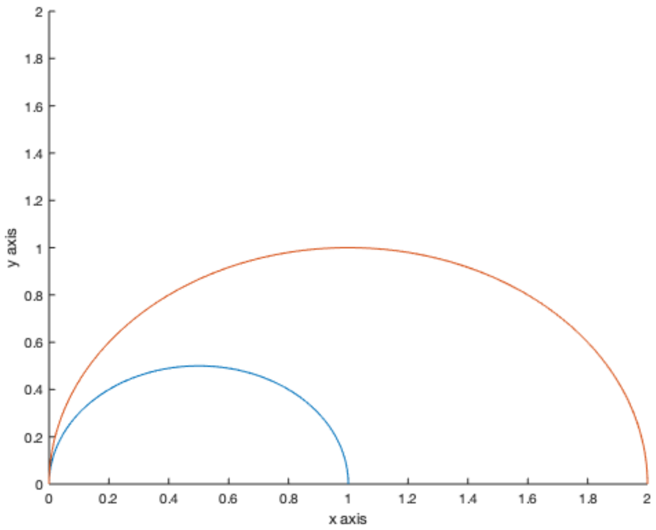


FIGURE 19. Curves that determine Ω of Exercise 13e (also recall $y \ge 0$) for $a = 1$.

Resolution: We will first work on $\int \mathrm{d}x \int f(x,y)\,\mathrm{d}y$.

We look for the correct inequalities in $\{(x,y)\colon x^2 + y^2 \underset{<}{\geqq} ax, x^2 + y^2 \underset{<}{\geqq} 2ax, y \ge 0\}$ to define Ω. We can identify that the region Ω will be between two circles and it will be above the y-axis. We can isolate y and also seek the intersections between the circles and the y-axis to determine the limits of integration.

$$x^2 + y^2 = ax \Leftrightarrow y^2 = ax - x^2 \Leftrightarrow y = \sqrt{ax - x^2} \implies \sqrt{ax - x^2} = 0 \Leftrightarrow x_1 = 0, x_2 = a,$$

$$x^2 + y^2 = 2ax \Leftrightarrow y^2 = 2ax - x^2 \Leftrightarrow y = \sqrt{2ax - x^2} \implies \sqrt{2ax - x^2} = 0 \Leftrightarrow x_1 = 0, x_2 = 2a.$$

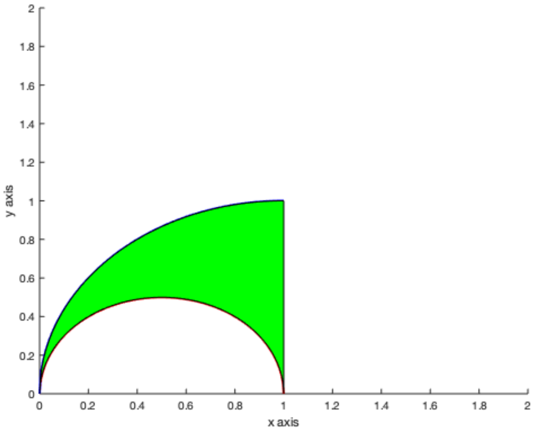


FIGURE 20. Subregion Ω_1 of Exercise 13e for $a = 1$.

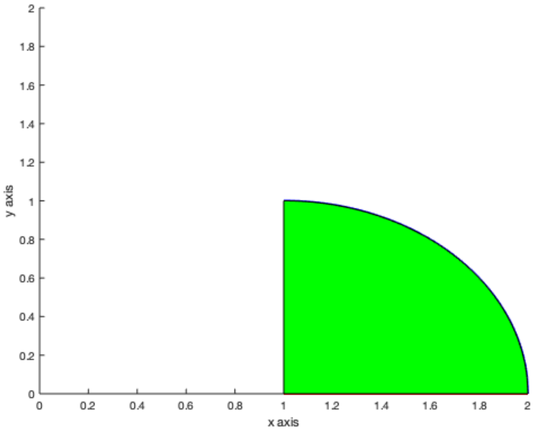


FIGURE 21. Subregion Ω_2 of Exercise 13e for $a = 1$.

We can now determine the inequalities, so $\Omega = \{(x,y)\colon x^2+y^2 \geq ax, x^2+y^2 \leq 2ax, y \geq 0\}$ and we can also determine the limits of integration, splitting $\Omega = \Omega_1 \cup \Omega_2$ in two subregions, with

$$\begin{aligned}\Omega_1 &= \{(x,y)\colon 0 \leq x \leq a, \sqrt{ax-x^2} \leq y \leq \sqrt{2ax-x^2}\},\\ \Omega_2 &= \{(x,y)\colon a \leq x \leq 2a, 0 \leq y \leq \sqrt{2ax-x^2}\},\end{aligned}$$

so that

$$\begin{aligned}I &= \iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^a \mathrm{d}x \int_{\sqrt{ax-x^2}}^{\sqrt{2ax-x^2}} f(x,y)\,\mathrm{d}y + \int_a^{2a} \mathrm{d}x \int_0^{\sqrt{2ax-x^2}} f(x,y)\,\mathrm{d}y\end{aligned}$$

Now we will work on $\displaystyle\int dy \int f(x,y)\,\mathrm{d}x$.

Since $\Omega = \{(x,y)\colon x^2+y^2 \geq ax, x^2+y^2 \leq 2ax, y \geq 0\}$, we can see that we will have some issues on the limits of x (because they will depend both on y and x), so to be able to compute them we will write the inequalities slightly different, substracting some factor of ax and adding some factor of a^2 to both sides of the inequalities to get standard equations for the circles:

$$\begin{aligned}&x^2+y^2 \geq ax \Leftrightarrow x^2-ax+y^2 \geq 0 \Leftrightarrow x^2-ax+\left(\frac{a}{2}\right)^2+y^2 \geq \left(\frac{a}{2}\right)^2 \Leftrightarrow \left(x-\left(\frac{a}{2}\right)\right)^2+y^2 \geq \frac{a^2}{4},\\ &x^2+y^2 \leq 2ax \Leftrightarrow x^2-2ax+y^2 \leq 0 \Leftrightarrow x^2-2ax+a^2+y^2 \leq a^2 \Leftrightarrow (x-a)^2+y^2 \leq a^2,\end{aligned}$$

so we now we have

$$\Omega = \left\{(x,y)\colon \left(x-\left(\frac{a}{2}\right)\right)^2+y^2 \geq \frac{a^2}{4}, (x-a)^2+y^2 \leq a^2, y \geq 0\right\}.$$

We already know the intersections with the y-axis from the previous part but we still have to isolate x on both equations to be able to find the limits of integration.

$$\begin{gathered}\left(x-\left(\frac{a}{2}\right)\right)^2+y^2 = \frac{a^2}{4} \Leftrightarrow \left(x-\left(\frac{a}{2}\right)\right)^2 = \frac{a^2}{4}-y^2 \Leftrightarrow x-\frac{a}{2} = \pm\sqrt{\frac{a^2}{4}-y^2} \Leftrightarrow x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4}-y^2},\\ (x-a^2)^2+y^2 = a^2 \Leftrightarrow (x-a^2)^2 = a^2-y^2 \Leftrightarrow x-a = \pm\sqrt{a^2-y^2} \Leftrightarrow x = a \pm \sqrt{a^2-y^2}.\end{gathered}$$

We can now determine the limits of integration partitioning $\Omega = \Omega_1' \cup \Omega_2' \cup \Omega_3'$ in three subregions:

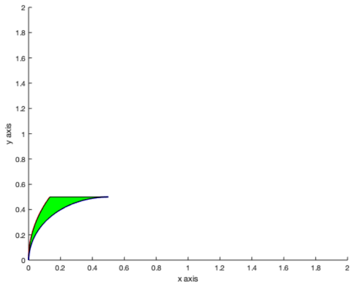


FIGURE 22. Ω_1' of Exercise 13e for $a = 1$.

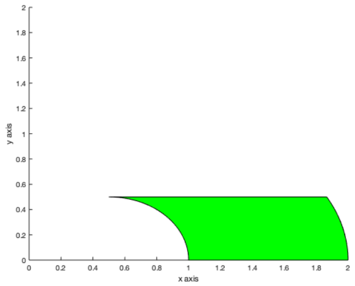


FIGURE 23. Ω_2' of Exercise 13e for $a = 1$.

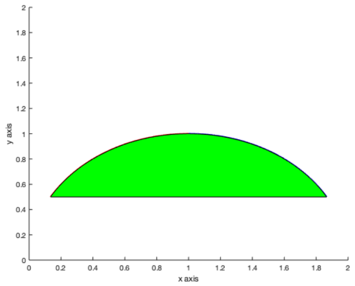


FIGURE 24. Ω_3' of Exercise 13e for $a = 1$.

$$\begin{aligned}
\Omega_1' &= \left\{(x,y)\colon 0 \le y \le \frac{a}{2}, a - \sqrt{a^2 - y^2} \le x \le \frac{a}{2} - \sqrt{\frac{a}{4} - y^2}\right\},\\
\Omega_2' &= \left\{(x,y)\colon 0 \le y \le \frac{a}{2}, \frac{a}{2} + \sqrt{\frac{a^2}{4} - y^2} \le x \le a + \sqrt{a^2 - y^2}\right\}\\
\Omega_3' &= \left\{(x,y)\colon \frac{a}{2} \le y \le a, a - \sqrt{a^2 - y^2} \le x \le a + \sqrt{a^2 - y^2}\right\},
\end{aligned}$$

so

$$\begin{aligned}
I &= \iint_\Omega f(x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&= \int_0^{\frac{a}{2}} \mathrm{d}y \int_{a-\sqrt{a^2-y^2}}^{\frac{a}{2}-\sqrt{\frac{a}{4}-y^2}} f(x,y)\,\mathrm{d}x + \int_0^{\frac{a}{2}} \mathrm{d}y \int_{\frac{a}{2}+\sqrt{\frac{a^2}{4}-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,\mathrm{d}x + \int_{\frac{a}{2}}^{a} \mathrm{d}y \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,\mathrm{d}x.
\end{aligned}$$

EXERCISE 14A

Exercise. *Compute the following double integral on the specified domain of* $\mathbb{R}^2$*:* $I = \iint_\Omega y^3\,\mathrm{d}x\,\mathrm{d}y$, $\Omega = (x,y) \in \mathbb{R}^2\colon -\frac{\pi}{2} \le x \le \frac{\pi}{2},\ 0 \le y \le 2\cos x$.

Resolution:

$$\begin{aligned}
I &= \iint_\Omega y^3\,\mathrm{d}x\,\mathrm{d}y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos x} y^3\,\mathrm{d}y\,\mathrm{d}x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{y^4}{4}\right]_{y=0}^{y=2\cos x} \mathrm{d}x\\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2\cos(x))^4}{4} - \frac{(0)^4}{4}\,\mathrm{d}x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2^4\cos^4(x)}{4}\,\mathrm{d}x = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(x)\,\mathrm{d}x
\end{aligned}$$

To solve this integral, we can use Euler's formula, which is: $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$, and from that, we can obtain that $\cos\theta = \dfrac{\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}}{2}$.

So if we have $\cos^4(x)$, its equal to $\left(\dfrac{\mathrm{e}^{ix} + \mathrm{e}^{-ix}}{2}\right)^4$, and making the proper calculations, we obtain $\dfrac{\mathrm{e}^{-4ix} + \mathrm{e}^{4ix} + 4\,\mathrm{e}^{-2ix} + 4\,\mathrm{e}^{2ix} + 6}{16}$.

Then, using the cosine formula previously mentioned, we can substitute the results obtained in that formula in order to get the following: $\dfrac{2\cos 4x + 8\cos 2x + 6}{16}$.

So we can say that $\cos^4(x) = \dfrac{3}{8} + \dfrac{\cos 2x}{2} + \dfrac{\cos 4x}{8}$, and we can continue with the integral:

$$\begin{aligned}
I &= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(x)\,\mathrm{d}x = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{8} + \frac{\cos(2x)}{2} + \frac{\cos(4x)}{8}\,\mathrm{d}x \\
&= 4\left(\frac{3}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}x + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2x)\,\mathrm{d}x + \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(4x)\,\mathrm{d}x\right) \\
&= 4\left[\frac{3}{8}x + \frac{1}{2}\frac{1}{2}\sin(2x) + \frac{1}{8}\frac{1}{4}\sin(4x)\right]_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} \\
&= 4\left(\frac{3}{8}\frac{\pi}{2} + \frac{1}{4}\sin\pi + \frac{1}{32}\sin(2\pi) - \left(\frac{3}{8}\left(-\frac{\pi}{2}\right) + \frac{1}{4}\sin(-\pi) + \frac{1}{32}\sin(-2\pi)\right)\right) \\
&= 4\left(\frac{3\pi}{16} + \frac{3\pi}{16}\right) = 4\frac{3\pi}{8} \\
&= \frac{3\pi}{2}.
\end{aligned}$$

That is, the final result of the integral is $I = \dfrac{3\pi}{2}$.

Exercise 14b

Exercise. *Compute the following double integral* $I = \iint_\Omega x\,\mathrm{d}x\,\mathrm{d}y$, *where* $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le \mathrm{e}^x\}$.

Resolution:

$$\begin{aligned}
I &= \int_0^1 \left(\int_0^{\mathrm{e}^x} x\,\mathrm{d}y\right)\mathrm{d}x \\
&= \int_0^1 x\Big[y\Big]_{y=0}^{y=\mathrm{e}^x}\,\mathrm{d}x \\
&= \int_0^1 x\,\mathrm{e}^x\,\mathrm{d}x
\end{aligned}$$

We integrate by parts doing the change $u = x$, $\mathrm{d}v = \mathrm{e}^x\,\mathrm{d}x$.

$$\begin{aligned}
&\left[x\,\mathrm{e}^x - \int \mathrm{e}^x\,\mathrm{d}x\right]_{x=0}^{x=1} \\
&= \Big[x\,\mathrm{e}^x - \mathrm{e}^x\Big]_{x=0}^{x=1}
\end{aligned}$$

and so

$$I = \Big[\mathrm{e}^x(x-1)\Big]_{x=0}^{x=1} = 1$$

Exercise 14c

Exercise. *Compute the double integral* $I = \iint_\Omega xy\,\mathrm{d}x\,\mathrm{d}y$, *on the domain* $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2,\ 0 \le y \le \frac{x}{2}\}$.

Resolution:

$$\begin{aligned}
&\int_0^2 x\left(\int_0^{x/2} y\,\mathrm{d}y\right)\mathrm{d}x\\
&=\int_0^2 x\left[\frac{y^2}{2}\right]_{y=0}^{y=x/2}\mathrm{d}x\\
&=\int_0^2 x\cdot\frac{x^2}{8}\,\mathrm{d}x\\
&=\frac{1}{8}\int_0^2 x^3\,\mathrm{d}x\\
&=\frac{1}{8}\left[\frac{x^4}{4}\right]_{x=0}^{x=2}\\
&=\frac{1}{8}\cdot\frac{16}{4}=\frac{16}{32}=\boxed{\frac{1}{2}}.
\end{aligned}$$

EXERCISE 14D

Exercise. *Compute the following double integral* $I=\iint_\Omega \frac{x^2}{y^2}\,\mathrm{d}x\,\mathrm{d}y$ *on the domain* $\Omega=\{(x,y)\in\mathbb{R}^2\colon 1\le x\le 2\ ,\frac{1}{x}\le y\le x\}$.

Resolution: In this exercise we have to compute the integral with the following limits of integration: $1\le x\le 2$ and $\frac{1}{x}\le y\le x$:

$$\begin{aligned}
I&=\iint_\Omega \frac{x^2}{y^2}\,\mathrm{d}x\,\mathrm{d}y=\int_1^2\int_{\frac{1}{x}}^{x}\frac{x^2}{y^2}\,\mathrm{d}y\,\mathrm{d}x\\
&=\int_1^2\left[-\frac{x^2}{y}\right]_{y=\frac{1}{x}}^{y=x}\mathrm{d}x=\int_1^2(-\frac{x^2}{x})-(-x^3)\,\mathrm{d}x\\
&=\int_1^2 -x+x^3\,\mathrm{d}x=\left[-\frac{x^2}{2}+\frac{x^4}{4}\right]_{x=1}^{x=2}\\
&=(-\frac{4}{2}+\frac{16}{4})-(-\frac{1}{2}+\frac{1}{4})=\frac{8}{4}+\frac{1}{4}=\boldsymbol{\frac{9}{4}}.
\end{aligned}$$

EXERCISE 14E

Exercise. *Compute the following double integral* $I=\iint_\Omega (x+2y)\,\mathrm{d}x\,\mathrm{d}y$ *on the domain* $\Omega=\{(x,y)\in\mathbb{R}^2\colon\ -3\le y\le 3, y^2-4\le x\le 5\}$.

Resolution:

$$\begin{aligned}
I&=\int_{-3}^{3}\mathrm{d}y\left[\frac{x^2}{2}+2yx\right]_{x=y^2-4}^{x=5}\\
&=\int_{-3}^{3}\frac{25}{2}+10y-\frac{(y^2-4)^2}{2}-2y(y^2-4)\,\mathrm{d}y\\
&=\left[\frac{9y}{4}+9y^2-\frac{y^5}{10}+\frac{4y^3}{3}-\frac{y^4}{2}\right]_{y=-3}^{y=3}=\frac{252}{5}.
\end{aligned}$$

Recall that, firstly we have to integrate with respect to x and afterwards with respect to y.

Exercise 14f

Exercise. *Compute* $I = \iint_\Omega y^3 \,\mathrm{d}x\,\mathrm{d}y$ *on the domain* $\Omega = \{(x,y) \in \mathbb{R}^2 \colon 0 \le x \le a, y^2 \le 2px\}$ *with* $a, p > 0$.

Resolution: The boundary curves $x = 0$ and $x = a$ are straight lines, whereas $y^2 = 2px$ is a parabola around the x-axis. Therefore $\Omega = \left\{0 \le x \le a, -\sqrt{2px} \le y \le \sqrt{2px}\right\}$ and

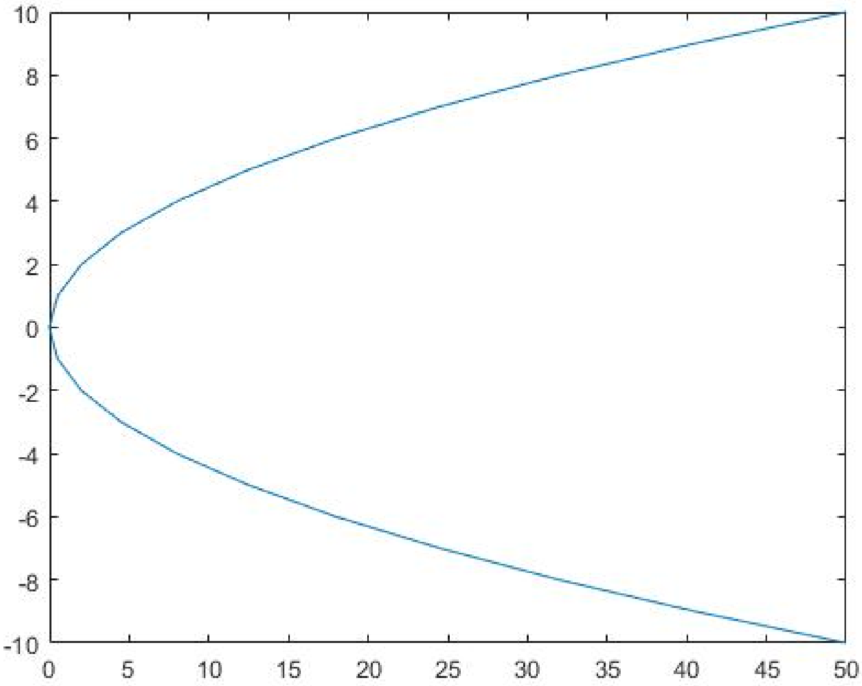


Figure 25. domain Ω from exercise 14f

$$I = \iint_\Omega y^3 \,\mathrm{d}x\,\mathrm{d}y = \int_0^a \mathrm{d}x \int_{-\sqrt{2px}}^{\sqrt{2px}} y^3 \,\mathrm{d}y = \int_0^a \left[\frac{y^4}{4}\right]_{y=-\sqrt{2px}}^{y=\sqrt{2px}} \mathrm{d}x$$
$$= \int_0^a \left(\frac{(2px)^2}{4} - \frac{(2px)^2}{4}\right) \mathrm{d}x = \frac{1}{4}\int_0^a 0 \,\mathrm{d}x = 0.$$

Indeed, this result is clear from the fact that the domain Ω is symmetric in the y variable, whereas the integrand y^3 is an odd function of y. In other words, the change of variables $(x,y) = T(u,v) = (u,-v)$ shows that $I = -I$.

Exercise 14g

Exercise. *Compute the following double integral* $I = \iint_\Omega \frac{y}{1+x^3} \,\mathrm{d}x\,\mathrm{d}y$, *where* $\Omega = \{(x,y) \in \mathbb{R}^2 \colon 0 \le x \le 1, 0 \le y \le 1, 1 \le x^2 + y^2\}$.

Resolution: Looking at the figure we conclude that the domain Ω can be written as:

$$\Omega = \{(x,y) \colon 0 \le x \le 1, \sqrt{1-x^2} \le y \le 1\}.$$

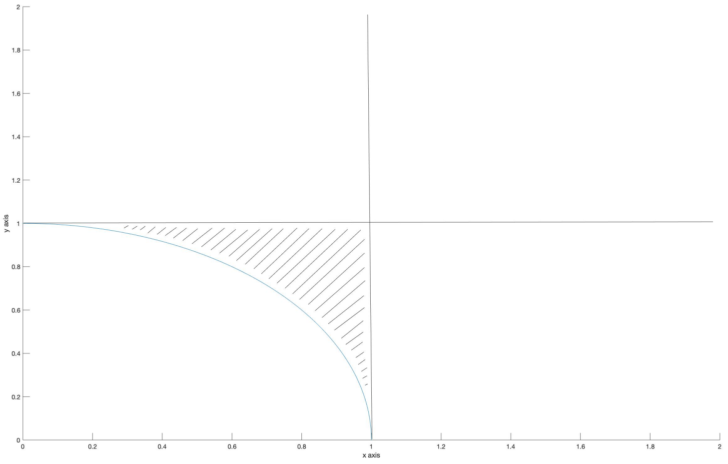


Figure 26. Domain Ω of Exercise 14g.

Then

$$I=\int_0^1\int_{\sqrt{1-x^2}}^1 \frac{y}{1+x^3}\,\mathrm{d}y\,\mathrm{d}x=\int_0^1\left[\frac{y^2}{2(1+x^3)}\right]_{y=\sqrt{1-x^2}}^{y=1}\mathrm{d}x=\int_0^1\frac{1}{2(1+x^3)}\,\mathrm{d}x-\int_0^1\frac{1-x^2}{2(1+x^3)}\,\mathrm{d}x$$
$$=\int_0^1\frac{1}{2(1+x^3)}\,\mathrm{d}x-\int_0^1\frac{1}{2(1+x^3)}\,\mathrm{d}x+\int_0^1\frac{x^2}{2(1+x^3)}\,\mathrm{d}x=\int_0^1\frac{x^2}{2(1+x^3)}\,\mathrm{d}x$$
$$=\frac{1}{6}\left[\ln(1+x^3)\right]_{x=0}^{x=1}=\frac{\ln 2}{6}-\frac{\ln 1}{6}=\frac{\ln 2}{6}.$$

Exercise14h

Exercise. *Compute the following double integral* $I=\iint_\Omega x^2\sin(xy)\,dx\,dy$, *where* $\Omega=\{(x,y)\in\mathbb{R}^2\colon 0\le y\le 1, y\le x\le 1\}$.

Resolution: We notice that if we first start integrating with respect to y, the exercise is much easier. Therefore, let us invert the order of integration, so that $I=\iint_\Theta x^2\sin(xy)\,\mathrm{d}y\,\mathrm{d}x$, where $\Theta=\{(x,y)\in\mathbb{R}^2\colon x_1(x)\le y\le x_2(x), a\le x\le b\}=\Omega$, that is, we are only expressing Ω with other inequalities.

We see that changing the integration limits for both x and y then $x_{\max}=1$; $x_{\min}=0$; $y_{\max}=x$ and $y_{\min}=0$. Therefore, $\Theta=\{(x,y)\in\mathbb{R}^2\colon 0\le y\le x, 0\le x\le 1\}$.

$$\begin{aligned}I&=\int_0^1\int_0^x x^2\sin(xy)\,\mathrm{d}y\,\mathrm{d}x=\int_0^1 x\,\mathrm{d}x\int_0^x x\cdot\sin(xy)\,\mathrm{d}y\\&=\int_0^1 x\,\mathrm{d}x\left[-\cos(xy)\right]_{y=0}^{y=x}=\int_0^1 x\left[-\cos(x^2)-(-\cos(0))\right]\,\mathrm{d}x\\&=\int_0^1 -x\cdot\cos(x^2)+x\,\mathrm{d}x=\left[\frac{-\sin(x^2)}{2}\right]_0^1+\left[\frac{x^2}{2}\right]_0^1\\&=\frac{-\sin(1)}{2}-\frac{-\sin(0)}{2}+\frac{1}{2}-\frac{0}{2}=\frac{-\sin(1)}{2}+\frac{1}{2}\\&=\frac{1}{2}\left(1-\sin(1)\right).\end{aligned}$$

Exercise 15a

Exercise. *For the following iterated integral write the equations of the curves limiting the regions of integration and draw the region:*

$$\int_1^2\left(\int_x^{x+3} f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x.$$

```
x=linspace(0,1.5);
y=linspace(0,1.5);
ymax = x;
y2 = 1 * ones(size(x));
xmax=1*ones(size(y));
y3=1*ones(size(x))

plot(x,ymax,'g','LineWidth', 2)
hold on;
plot(xmax,y,'b','LineWidth', 2)
plot(x,y2,'r','LineWidth', 2)
xlabel('X axis')
ylabel('Y axis')
X=[x,fliplr(x)];
Y=[y3,fliplr(ymax)];
fill(Y,X,'y');
hold off;

xlim([0 1.01])
ylim([0 1.01])
```

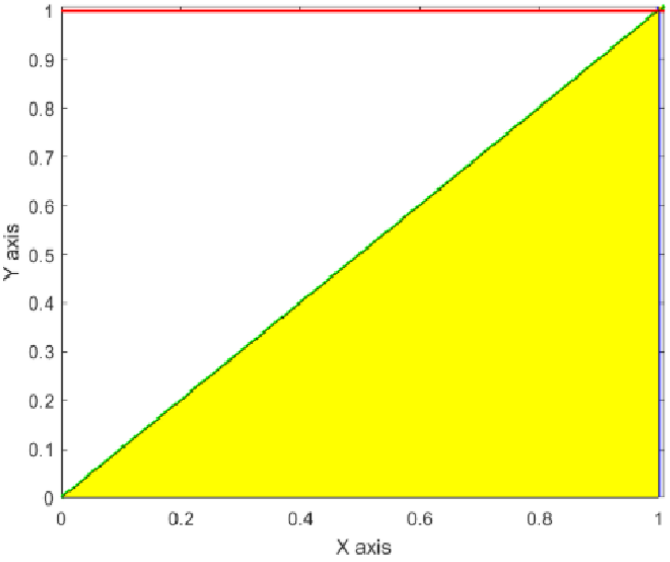


Figure 27. Domain Ω of Exercise 14h pltted with Matlab.

Resolution:

We are going to denominate the domain we must integrate as Ω.

From the limits of integration, we see that the differential of y has as limits of integration x and $x+3$. Moreover, in that sense, the differential of x has as limits of integration 1 and 2. Therefore, we can conclude that the x axis goes from 1 to 2 and the y axis from x to $x+3$.

As we find these boundaries, which consist on 4 straight lines as shown the figure

$$1 \leq x \leq 2 \implies x = 1, \; x = 2,$$

$$x \leq y \leq x+3 \implies y = x, \; y = x+3,$$

we can say that:

$$\Omega = \{(x, y) \colon 1 \leq x \leq 2, x \leq y \leq x+3\}.$$

Exercise. Exercise 15b

For the following iterated integral write the equations of the curves limiting the region of integration and draw this region.

$$\int_{-1}^{1} \left(\int_{x^2}^{2-x^2} f(x, y) \, \mathrm{d}y \right) \, \mathrm{d}x$$

```
clf
syms x y
fun=@(x,y) x^2*sin(x*y)
fsurf(fun,'ShowContour','on')
xlabel('X axis')
ylabel('Y axis')
zlabel('Z axis')
```

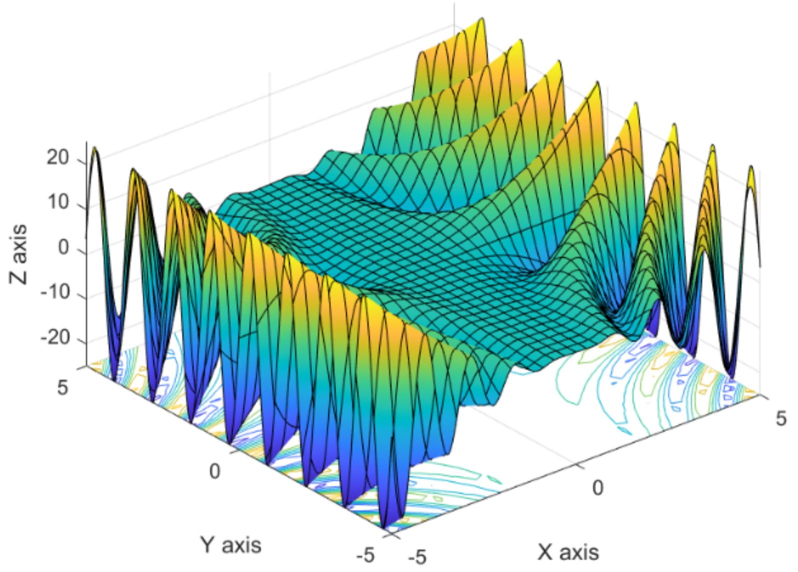


Figure 28. Integrand of Exercise 14h plotted with Matlab.

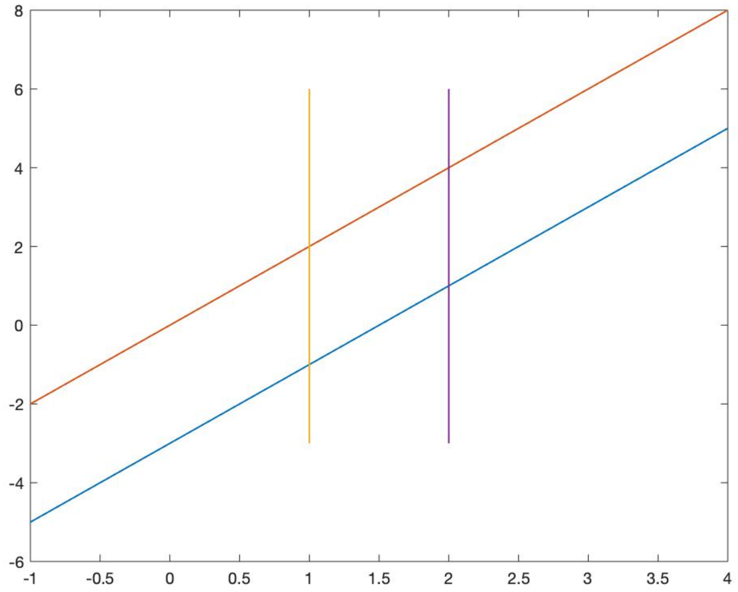


Figure 29. Domain Ω from Exercise 15a.

Resolution: The integral to be computed is really a double integral $I = \iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ over the domain $\Omega = \{(x,y) \in \mathbb{R}^2 \colon -1 \leq x \leq 1, x^2 \leq y \leq 2 - x^2\}$. From here, we get the equations from the integrating limits:

- Equation 1 is $x = -1$
- Equation 2 is $x = 1$
- Equation 3 is $y = x^2$
- Equation 4 is $y = 2 - x^2$

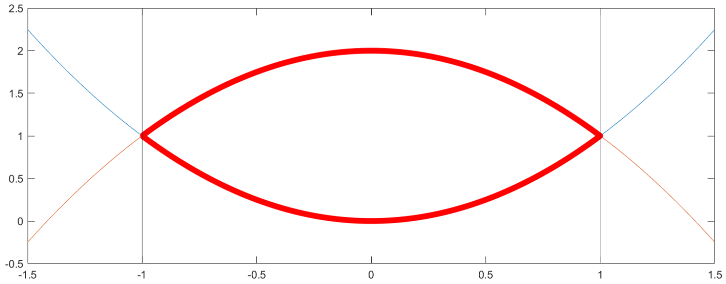


FIGURE 30. Curves in red limiting the region of integration of Exercise 15b.

From the figure we see that just two curves limite the region of integration: $y = x^2$ and $y = 2 - x^2$. The region Ω is enclosed between these two curves which intersect exactly for $x = \pm 1$, and in these two points $y = 1$.

EXERCISE 15C

Exercise. *For the iterated integral* $\int_0^2 \left(\int_{2-y}^{\sqrt{4-y^2}} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y$ *write the equations of the curves limiting the region of integration and draw this region.*

Resolution: We first take the limits of the integral to find the forms they have in cartesian coordinates. First, from the limits on the x-variable:

$$2 - y \le x \le \sqrt{4 - y^2},$$

we have $y = -x + 2$, which is a straight line, and $x^2 = 4 - y^2$ or $x^2 + y^2 = 4$, which is a circle centered at the origin of radius 2.

From the limits on the y-variable we find just two straight lines: $y = 0$ and $y = 2$. Looking at the figure, we can see that the domain Ω enclosed by the limits is $\Omega = \{(x.y) \in \mathbb{R}^2 \colon x + y \ge 2, x^2 + y^2 \le 4\}$, so the integral I is a double integral on the domain ω:

$$I = \iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

EXERCISE 15D

Exercise. *For the following iterated integral* $I = \int_0^1 \left(\int_{\sqrt{x}}^{\sqrt{2-x^2}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$ *write the equations of the curves limiting the region of integration and draw this region.*

Resolution:

(1) Integral respect to y:
  - The lower curve is $y = \sqrt{x}$. If we make the square at two sides in order to vanish the square root, then the equation looks like $y^2 = x$. This function is likely to be a parabola, but instead to be on y axis as usually, it is on x axis.
  - The upper curve is $y = \sqrt{2 - x^2}$. Following the same criteria as before, we raise to the power two the two sides of the equation: $y^2 = 2 - x^2$. Then we write the equation in the typical circumference way: $x^2 + y^2 = 2$. This function is a circumference centered at point $(0,0)$ and with radius $r = \sqrt{2}$.

(2) Integral respect to x:
  - The lower curve is $x = 0$.

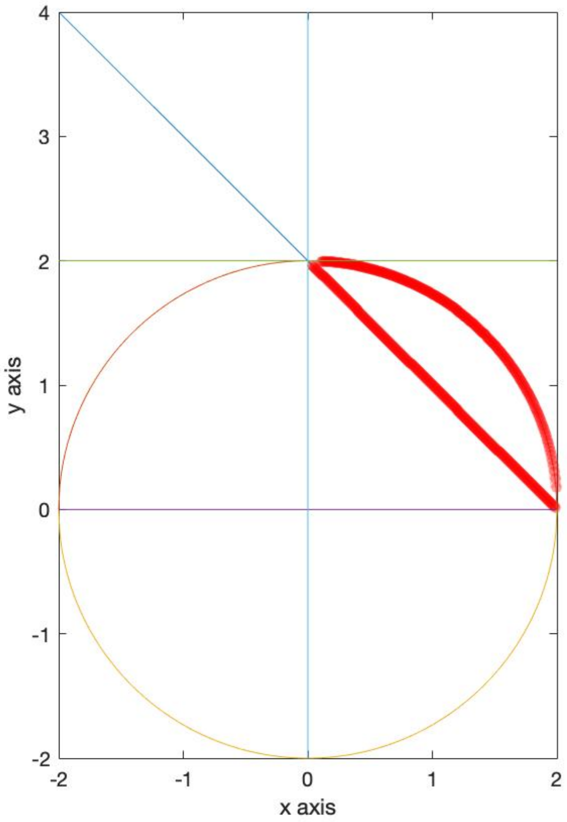


FIGURE 31. Domain of integration Ω from Exercise 15c

- The upper curve is $x = 1$.
  Both lines are completely vertical (parallel to y axis) and each one crosses the x axis at the coefficient which they are equalised (the points $(0, 0)$ and $(1, 0)$).

The picture shows the area expressed by the integral I.

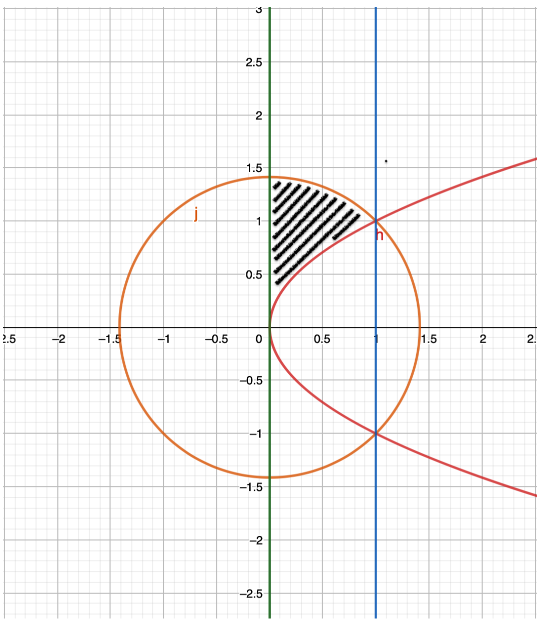


FIGURE 32. Graphic representation of exercise 15d

EXERCISE 16B

Exercise. *Invert the order of integration in the following iterated integral* $I = \int\limits_0^1 \left(\int\limits_1^{e^x} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$

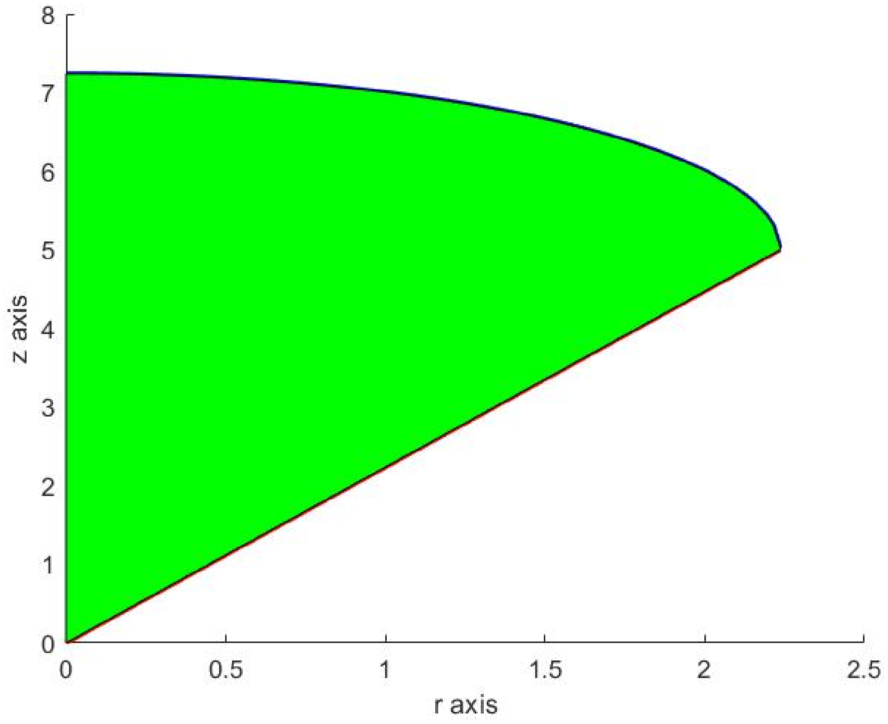


FIGURE 33. Domain Ω from Exercise 16b

Resolution: Indeed $I = \iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, where $\Omega = \{(x,y) \in \mathbb{R}^2 \colon 0 \le x \le 1 \text{ and } 1 \le y \le e^x\}$. As we have to invert the order of integration, we should be able to read the next graphic upwards instead of reading it from left to right.

- y axis, from 1 to e: $\int_1^e f(x,y)\,\mathrm{d}y$, where $J = \{y \in \mathbb{R} \colon 1 \le y \le e\}$.
- x axis, from $e^x = y$ to 1: $\int_{\ln y}^1 f(x,y)\,\mathrm{d}x$, where $I = \{x \in \mathbb{R} \colon \ln y \le x \le 1\}$,

Result: $\int\limits_1^e \left(\int\limits_{\ln y}^1 f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$

EXERCISE 16C

Exercise. *Invert the order of integration in the iterated integral* $I = \int_0^4 \left(\int_{3x^2}^{12x} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$

Resolution. By the statement we know that we have to compute $I = \iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, where $\Omega = \{(x,y) \in \mathbb{R}^2 \colon 0 \le x \le 4 \;,\; 3x^2 \le y \le 12x\}$.

In order to invert the order of integration we have to look the graph from left to right, instead from looking from bottom to top. By doing this, we are now fixing the y such that:

$$0 \le y \le 48.$$

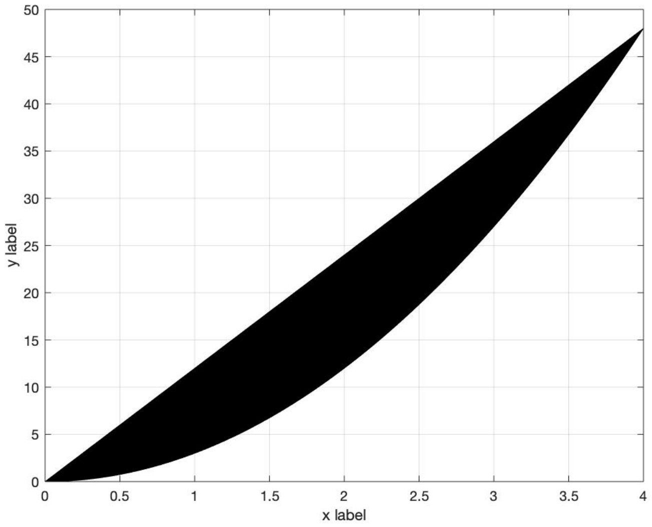


Figure 34. Domain Ω from Exercise 16c

Now we have to write x depending on y:

$$\text{Upper limit: } 12x = y \implies x = \frac{y}{12},$$

$$\text{Lower limit: } 3x^2 = y \implies x = \sqrt{\frac{y}{3}}.$$

As now we see the figure from left to right we have to change the order of the limits of x as now we go from the straight line to the parabola, so:

$$\frac{y}{12} \leq x \leq \sqrt{\frac{y}{3}}.$$

Finally, we have the following domain: $\Omega = \left\{(x,y) \in \mathbb{R}^2 \colon \frac{y}{12} \leq x \leq \sqrt{\frac{y}{3}}, 0 \leq y \leq 48\right\}$, and thus we get:

$$I = \int_0^4 \left(\int_{3x^2}^{12x} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

Exercise 16d

Exercise. *Invert the order of integration in the following iterated integral:*

$$I = \int_0^{\frac{1}{2}} \left(\int_0^{1-2x} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$$

Resolution:

We are computing this integral on a 2D domain Ω bounded by the straight lines $x = 0$, $y = 0$, $y = 1 - 2x$.

First, we have to isolate the variable x:

$$y = 1 - 2x; x = \frac{1-y}{2}$$

In the graph we can clearly see that the intersections appear in the points:

$$A = (0,0); B = (0,1); C = (1/2,0);$$

Therefore, we can conclude that y ranges from 0 to 1 and x ranges from 0 through the line:

$$x = \frac{1-y}{2}$$

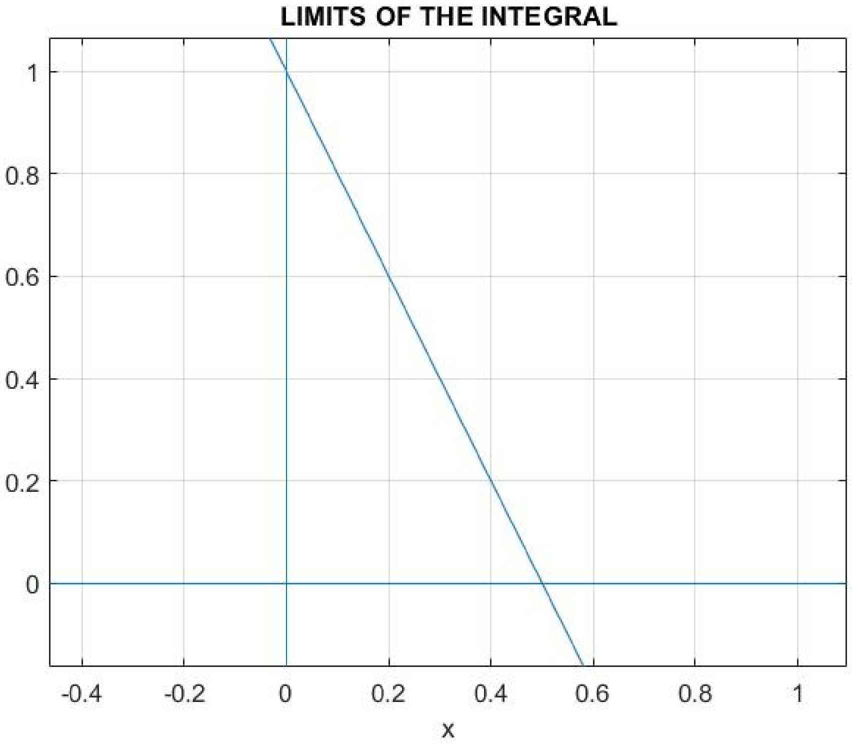


FIGURE 35. from Exercise 16a: limits on a 2D plane

After swapping the order of the differentials, the inversion of the limits goes as follows:

$$I = \int_0^1 \left(\int_0^{\frac{1-y}{2}} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y$$

EXERCISE 16D

Exercise. *Invert the order of integration in the iterated integral* $\int_0^{\frac{\pi}{2}} \left(\int_0^{\cos x} f(x,y)dy \right) \mathrm{d}x$.

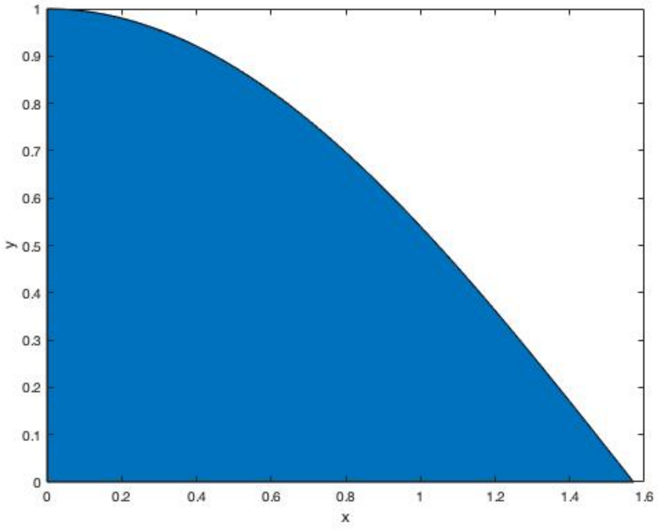


FIGURE 36. Domain D from Exercise 16d.

Resolution: The given integration domain is the following

$$D = \left\{ (x,y) : 0 \leq y \leq \cos x, 0 \leq x \leq \frac{\pi}{2} \right\}.$$

Since we want to invert the order of integration, first we will determine the values between which y ranges. By looking at the graph we can easily observe that y can take values between 0 and 1. Then, we have to determine the limits for our variable x. By again observing the figure, we see that 0 is the lower boundary of x and the curve $y = \cos x$ is its upper boundary. We must rewrite the expression of the curve in order to express it now as a function of y. Hence, isolating x from the previous expression of the curve: $y = \cos x$, we have $x = \arccos y$.

Therefore, we can now express the domain as follows:

$$D = \{(x, y) : 0 \leq y \leq 1, 0 \leq x \leq \arccos y\}\,.$$

From this, we can finally rewrite the given iterated integral with inverted order of integration

$$\int_0^1 \left(\int_0^{\arccos y} f(x, y) dx \right) dy.$$

Exercise 16e

Exercise. *Invert the order of integration in the following iterated integral:* $\int_0^{\pi} \left(\int_0^{\sin x} f(x, y)\,\mathrm{d}y \right) \mathrm{d}x.$

Resolution: We must find the order of integration such that the following equality is true:

$$\int_0^{\pi} \mathrm{d}x \int_0^{\sin x} f(x, y)\,\mathrm{d}y = \iint_{\Omega} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

Let's study the domain:

$$\Omega = \{(x, y)\colon 0 \leq x \leq \pi, 0 \leq y \leq \sin x\} = \{(x, y)\colon y_1 \leq y \leq y_2, x_1 \leq x \leq x_2\}$$

We can look at the graph of the domain from two different points of view:

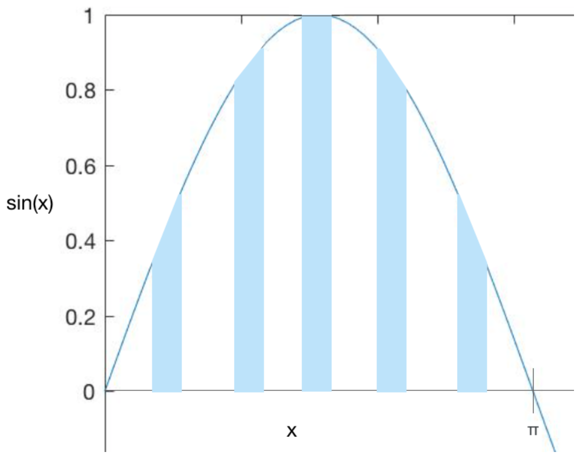


Figure 37. Fixing x-axis in Exercise 16e

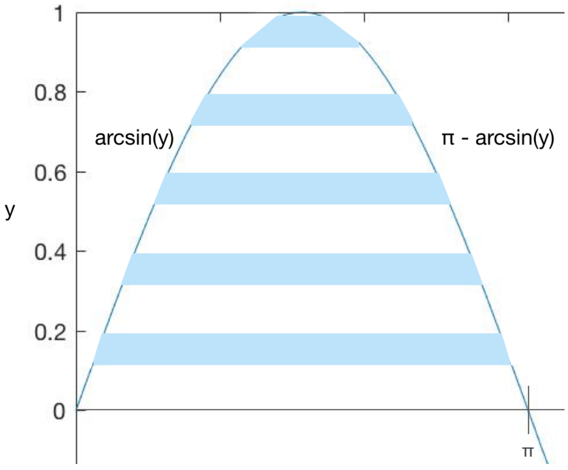


Figure 38. Fixing y-axis in Exercise 16e

The initial order of integration is given fixing the x-axis (figure 37). In order to invert the order of integration we must look at the graph fixing the y-axis (figure 38).

In this case it is clear that the minimum value of y is 0, so $y_1 = 0$, and the maximum value of y is 1, so $y_2 = 1$. We also know that $\sin x_1 = \sin x_2 = y$, therefore

$$y = \sin x \Leftrightarrow x = \arcsin y, \quad x_1 = \arcsin y \in [0, \frac{\pi}{2}] \quad \textit{because} \quad y \in [0.1]$$

In order to find x_2 we must check that it matches the upper bound of the domain, so

$$\frac{x_1 + x_2}{2} = \frac{\pi}{2}, \quad x_2 = \pi - x_1 = \pi - \arcsin y$$

Substituting,

$$\Omega = \{(x, y)\colon 0 \leq y \leq 1, \arcsin y \leq x \leq \pi - \arcsin y\}$$

Therefore the solution is

$$\boldsymbol{\int_0^1 \mathrm{d}y \int_{\arcsin y}^{\pi-\arcsin y} f(x,y)\,\mathrm{d}x}$$

EXERCISE 16F

Exercise. *Invert the order of integration in the following iterated integral:*

$$I=\int_0^1\left(\int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

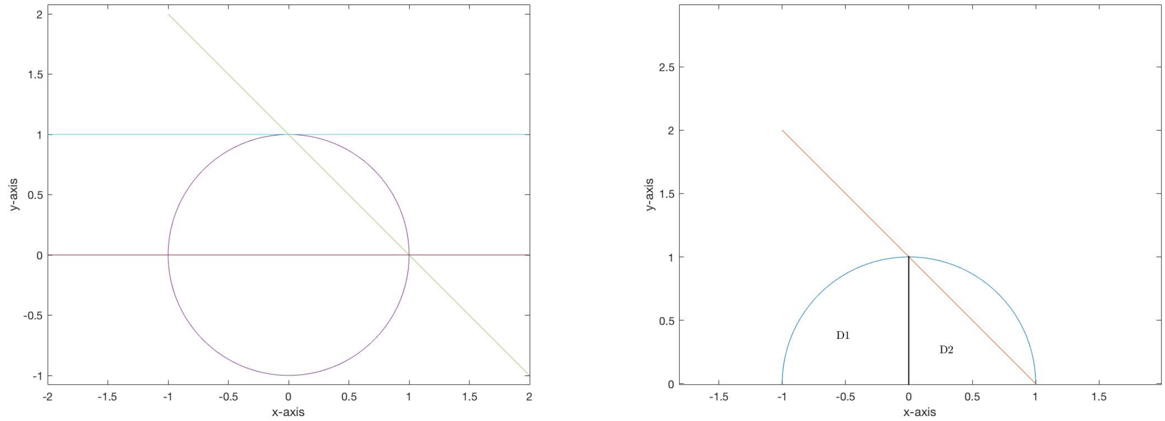


FIGURE 39. Domain D of Exercise 16f.

Resolution: The domain of integration is $D=\{(x,y,z)\colon 0\le y\le 1, -\sqrt{1-y^2}\le x\le 1-y\}$. If we represent the set D, we obtain two different regions: $D_1=\{(x,y,z): -1\le x\le 0, 0\le y\le \sqrt{1-x^2}\}$ which is a quarter of a circle and $D_2=\{(x,y,z): 0\le x\le 1, 0\le y\le 1-x\}$ which is a triangle. Therefore,

$$A(D)=\int\limits_{-1}^{0}\int\limits_{0}^{\sqrt{1-x^2}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x+\int\limits_{0}^{1}\int\limits_{0}^{1-x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

EXERCISE 16G

Exercise. *Invert the order of integration in the iterated integral* $\displaystyle\int_0^{2a}\left(\int_{\sqrt{2ax-x^2}}^{\sqrt{4ax}} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x$, $a>0$.

Resolution: We have to compute $I=\displaystyle\iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ on the domain

$$\Omega=\left\{(x,y)\in\mathbb{R}^2\colon 0\le x\le 2a, \sqrt{2ax-x^2}\le y\le\sqrt{4ax}\right\}.$$

Notice that the boundary $\partial\Omega$ of the domain Ω has three pieces, the parabola $y^2=4ax$ at the top, the circle $y^2=2ax-x^2 \iff (x-a)^2+y^2=a^2$ at the bottom, and the straight line $x=2a$ at the right. In order to invert the order of integration we have to look at the graph from left to right, instead from looking from bottom to top. By doing this, we are now fixing the y such that $0\le y\le 2\sqrt{2}a$, and we will have to consider separately the cases $a\le y\le 2\sqrt{2}a$, and $0\le y\le a$, and inside this last case, we will distinguish between $0\le x\le a$ and $a\le x\le 2a$:

(1) On $a\le y\le 2\sqrt{2}a$, Ω is bounded by the parabola $x=\dfrac{y^2}{4a}$ at the left and by the straight line $x=2a$ at the right.

(2) On $0\le y\le a$ and $0\le x\le a$, Ω is bounded by the by the parabola $x=\dfrac{y^2}{4a}$ at the left and by the circle $x=a-\sqrt{a^2-y^2}$ at the right.

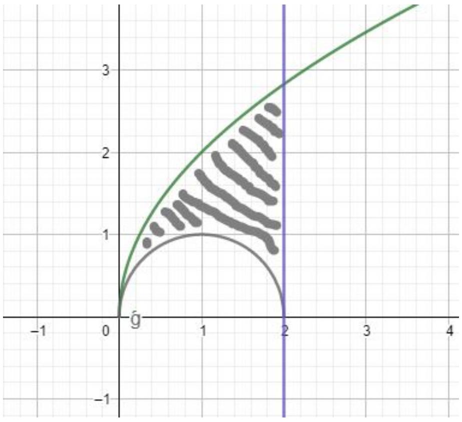


FIGURE 40. Domain Ω of Exercise 16g for $a = 1$.

(3) On $0 \le y \le a$ and $a \le x \le 2a$, Ω is bounded by the by the circle $x = a + \sqrt{a^2 - y^2}$ at the left and by the straight line $x = 2a$ at the right.

In other words, we split our domain in three subdomains $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, with

$$\Omega_1 = \left\{(x, y)\colon a \le y \le 2\sqrt{2}a,\ \frac{y^2}{4a} \le x \le 2a\right\},$$
$$\Omega_2 = \left\{(x, y)\colon 0 \le y \le a,\ \frac{y^2}{4a} \le x \le a - \sqrt{a^2 - y^2}\right\},$$
$$\Omega_3 = \left\{(x, y)\colon 0 \le y \le a,\ a + \sqrt{a^2 - y^2} \le x \le 2a\right\}.$$

From this partition of Ω the iterated integrals for the subdomains follow easily

$$I = \int_a^{2\sqrt{2}a} \mathrm{d}y \int_{\frac{y^2}{4a}}^{2a} f(x, y)\,\mathrm{d}x + \int_0^a \mathrm{d}y \int_{\frac{y^2}{4a}}^{a-\sqrt{a^2-y^2}} f(x, y)\,\mathrm{d}x + \int_0^a \mathrm{d}y \int_{a+\sqrt{a^2-y^2}}^{2a} f(x, y)\,\mathrm{d}x.$$

EXERCISE 16H

Exercise. *Invert the order of integration in the following iterated integral:*

$$I = \int_0^3 \left(\int_{\frac{x}{3}}^1 f(x, y)\,\mathrm{d}y\right) \mathrm{d}x.$$

Resolution: To solve this exercise, we plot the domain $\Omega = \{(x, y)\colon 0 \le x \le 3, x/3 \le y \le 1\}$. We see that the x variable runs in $0 \le x \le 3$ and then the y variable runs between the equations $\frac{x}{3} \le y \le 1$. Therefore, looking at the figure, and changing the order of integration by fixing first the y variable, we get $0 \le y \le 1$, and going through the x variable we obtain $0 \le x \le 3y$, so that the domain of integration can be also written as $\Omega = \{(x, y)\colon 0 \le y \le 1, 0 \le x \le 3y\}$. The solution is thus

$$I = \int_0^1 \left(\int_0^{3y} f(x, y)\ dx\right) dy.$$

EXERCISE 16I

Exercise. *Invert the order of integration in the following iterated integral:* $\int_0^1 (\int_{-x}^{x^2} f(x, y)\,\mathrm{d}y)\,\mathrm{d}x$.

Resolution: First, the domain of our integral is $\Omega = \{(x, y) \in \mathbb{R}^2\colon 0 \le x \le 1,\ -x \le y \le x^2\}$, obtained by reading the graph vertically (fixing the x-axis). We have to invert the order of integration, so we now look at the graph horizontally (from left to right) and fix the y-axis. The domain is split in 2:

$$\Omega^* = \{(x, y) \in \mathbb{R}^2\colon -1 \le y \le 0,\ -y \le x \le 1\},$$

which corresponds to the lower part of Ω, and

$$\Omega^{**} = \{(x, y) \in \mathbb{R}^2\colon 0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$$

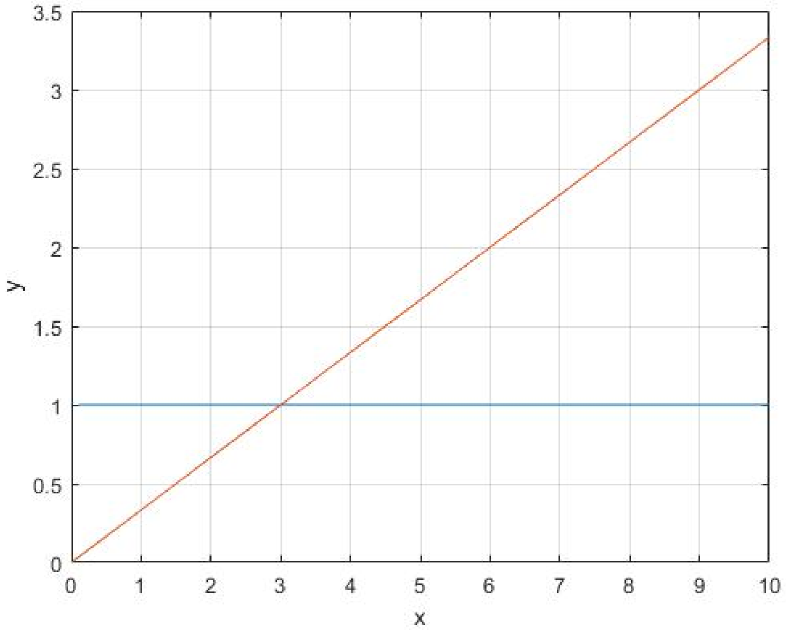


Figure 41. Domain of Exercise 16h

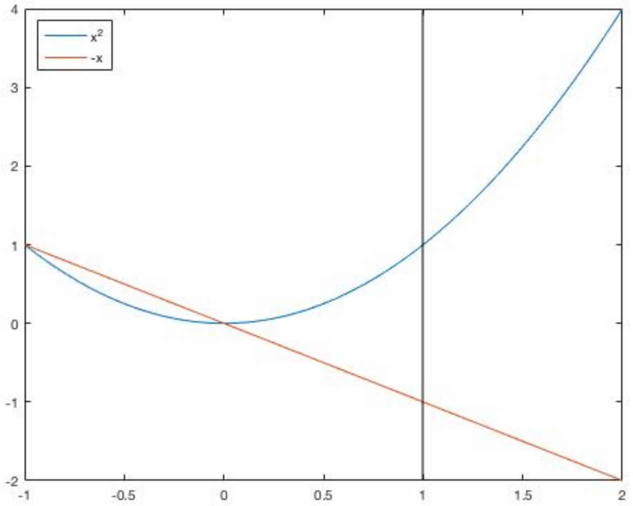


Figure 42. Domain of Exercise 16i.

which corresponds to the upper part of Ω. Hence, the solution is the following:

$$\int_{-1}^{0}\int_{-y}^{1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{0}^{1}\int_{\sqrt{y}}^{1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Exercise 17a

Exercise. *Compute* $I = \iint_{\Omega} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$*, where* Ω *is the planar domain bounded by the straight lines* $y = x$*,* $x + y = 2a$*,* $x = 0$ *(*$a > 0$*).*

Resolution: The first thing we need to do is to define Ω. We draw the three lines that we are given by the statement in a graph keeping in mind the restriction $a > 0$. The domain is restricted by the equations $y = x$ (blue line), $y = 2a - x$ (green line) and $x = 0$ (the y axis, in purple). Therefore, our domain is the left triangle. The easiest way to integrate this domain is to go up through y. Doing that we get $\Omega = \{(x,y) \in \mathbb{R}^2 \colon 0 \le x \le a, x \le y \le 2a - x\}$. Now that we have our domain, we integrate on

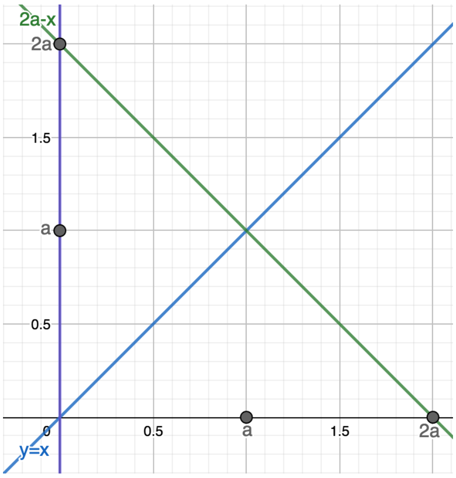


FIGURE 43. Graph for exercise 17a

it:

$$
\begin{aligned}
I &= \iint_{\Omega} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_0^a \mathrm{d}x \int_x^{2a-x} (x^2+y^2)\,\mathrm{d}y \\
&= \int_0^a x^2(2a-2x)\,\mathrm{d}x + \int_0^a \frac{8a^3-12a^2x+6ax^2-2x^3}{3}\,\mathrm{d}x \\
&= \left[\frac{2ax^3}{3}\right]_{x=0}^{x=a} - \left[\frac{2x^4}{4}\right]_{x=0}^{x=a} + \left[\frac{8a^3x}{3}\right]_{x=0}^{x=a} - \left[\frac{12a^2x^2}{6}\right]_{x=0}^{x=a} + \left[\frac{6ax^3}{9}\right]_{x=0}^{x=a} - \left[\frac{2x^4}{12}\right]_{x=0}^{x=a} \\
&= \frac{2a^4}{3} - \frac{2a^4}{4} + \frac{8a^4}{3} - \frac{12a^4}{6} + \frac{6a^4}{9} - \frac{2a^4}{12} = \frac{a^4}{6} + \frac{7a^4}{6} = \frac{4a^4}{3}.
\end{aligned}
$$

EXERCISE 17B

Exercise. *Compute the following double integral* $I = \iint_{\Omega} (x+2y)\,\mathrm{d}x\,\mathrm{d}y$, *where* Ω *is bounded by the curves* $y = x^2$, $y^2 = x$.

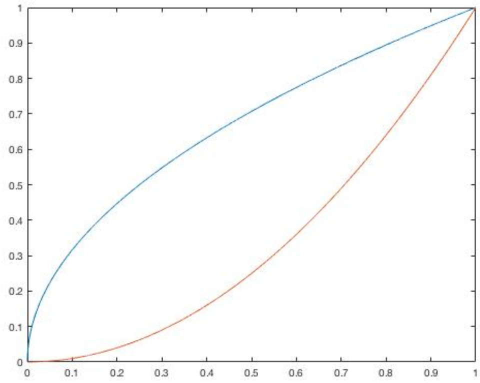


FIGURE 44. Domain Ω of Exercise 17b.

Resolution: First let us find the fixed interval of the variable x where the curves create a bounded area:

$$0 \le x \le 1,$$

where $x = 0, 1$ corresponds to the intersection between the curves. In order to proceed we will rise through the y axis starting from the equation $y = x^2$ up to $y = \sqrt{x}$:

$$x^2 \le y \le \sqrt{x},$$

so that $\Omega = \left\{(x,y)\colon 0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}\right\}$. We are left out with the follow integral to solve:

$$I = \int_0^1 \mathrm{d}x \int_{x^2}^{\sqrt{x}} (x+2y)\,\mathrm{d}y = \int_0^1 \left[xy + y^2\right]_{y=x^2}^{y=\sqrt{x}} \mathrm{d}x = \int_0^1 x\sqrt{x} - x^3 + x - x^4\,\mathrm{d}x = 9/20.$$

Exercise 17c

Exercise. *Compute the following double integral* $I = \iint_\Omega \mathrm{e}^{x+y}\,\mathrm{d}x\,\mathrm{d}y$*, with the domain* Ω *bounded by the curves* $y = \mathrm{e}^x, x = 0, y = 2$.

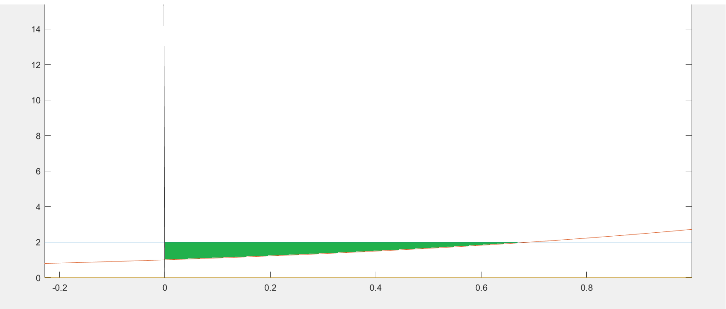


Figure 45. Domain Ω from Exercise 17c.

Resolution: To start, first we need to find the limits of the domain. We will proceed by fixing y. Therefore we need its intersection with the other function.

$$y = \mathrm{e}^x \text{ minimum for } x = 0 : y_{\text{min}} = \mathrm{e}^0 = 1.$$

Then, from the statement of the problem, we can say that these are the limits of the integral:

$$1 \leq y \leq 2, 0 \leq x \leq \ln y,$$

or $\Omega = \{(x,y)\colon 1 \leq y \leq 2, 0 \leq x \leq \ln y, \}$. To achieve this last expression we have isolated x from the equation that related y and x: $y = \mathrm{e}^x \iff \ln y = x$.

Once we have established the limits, we can proceed with the integral:

$$\begin{aligned} I &= \iint_\Omega \mathrm{e}^{(x+y)}\,\mathrm{d}x\,\mathrm{d}y = \int_1^2 \int_0^{\ln y} \mathrm{e}^{(x+y)}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_1^2 \mathrm{e}^y\,\mathrm{d}y \int_0^{\ln y} \mathrm{e}^x\,\mathrm{d}x = \int_1^2 \mathrm{e}^y (y-1)\,\mathrm{d}y. \end{aligned}$$

Now we must apply integration by parts $[u = y-1,\ \mathrm{d}u = \mathrm{d}y,\ \mathrm{d}v = \mathrm{e}^y\,\mathrm{d}y, v = \mathrm{e}^y]$ and proceed.

$$I = \int_1^2 \mathrm{e}^y (y-1)\,\mathrm{d}y = [(\mathrm{e}^y)(y-1) - \mathrm{e}^y]_1^2 = \mathrm{e}^2 - \mathrm{e}^2 + e = e.$$

Exercise 17d

Exercise. *Compute the following double integral in the specified domains of* $\mathbb{R}^2$*:* $I = \iint_\Omega \mathrm{e}^{x+y}\,\mathrm{d}x\,\mathrm{d}y$*;* $\Omega =$ *triangle with vertices* $(1,0), (0,1)$ *and* $(0,-1)$.

Resolution:

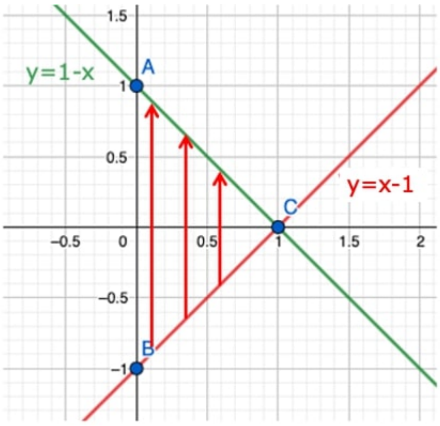


FIGURE 46. Once the variable x of Exercise 17d is fixed between 0 and 1, the variable y goes from $x-1$ to $1-x$.

$$\begin{aligned} I &= \iint_\Omega \mathrm{e}^y \,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}x \int_{x-1}^{1-x} \mathrm{e}^y \,\mathrm{d}y = \int_0^1 \left[\mathrm{e}^y\right]_{y=x-1}^{y=1-x} \mathrm{d}x \\ &= \int_0^1 \mathrm{e}^{1-x} - \mathrm{e}^{x-1} \,\mathrm{d}x = \left[-\mathrm{e}^{1-x} - \mathrm{e}^{x-1}\right]_{x=0}^{x=1} \\ &= -\mathrm{e}^{1-1} - \mathrm{e}^{1-1} + \mathrm{e}^1 + \mathrm{e}^{-1} = -1 - 1 + \mathrm{e} + \frac{1}{\mathrm{e}} \\ &= \mathrm{e} + \frac{1}{\mathrm{e}} - 2. \end{aligned}$$

EXERCISE 17F

Exercise. *Compute the double integral* $\iint_\Omega xy \,\mathrm{d}x\,\mathrm{d}y$, *where* Ω *is the planar domain bounded by the curves* $x = y$ *and* $y = x^2$.

Resolution: The domain is:

$$\Omega = \left\{(x,y) : \ 0 \le x \le 1, \ x^2 \le y \le x\right\}.$$

Then

$$I = \iint_\Omega xy \,\mathrm{d}x\,\mathrm{d}y = \int_0^1 x \,\mathrm{d}x \int_{x^2}^{x} y \,\mathrm{d}y = \int_0^1 x \left[\frac{y^2}{2}\right]_{y=x^2}^{y=x} \mathrm{d}x = \int_0^1 \frac{x^3}{2} - \frac{x^5}{2} \,\mathrm{d}x = \left[\frac{x^4}{8} - \frac{x^6}{12}\right]_{x=0}^{x=1} = \frac{1}{24}.$$

EXERCISE 18A

Exercise. *Compute the following iterated triple integral* $\int\limits_1^2 x^2 \,\mathrm{d}x \int\limits_0^1 y^3 \,\mathrm{d}y \int\limits_0^{\frac{\pi}{2}} \sin z \,\mathrm{d}z$.

Resolution: $\int\limits_1^2 x^2 \,\mathrm{d}x \int\limits_0^1 y^3 \,\mathrm{d}y \int\limits_0^{\frac{\pi}{2}} \sin z \,\mathrm{d}z = \int\limits_1^2 x^2 \,\mathrm{d}x \int\limits_0^1 y^3 \left[-\cos z\right]_{z=0}^{z=\frac{\pi}{2}} \,\mathrm{d}y = \int\limits_1^2 x^2 \left[\frac{y^4}{4}\right]_{y=0}^{y=1} \mathrm{d}x = \frac{7}{12}$.

EXERCISE 18B

Exercise. *Compute the following iterated triple integral* $I = \int_0^3 \int_0^{2x} \int_0^{\sqrt{xy}} z \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$.

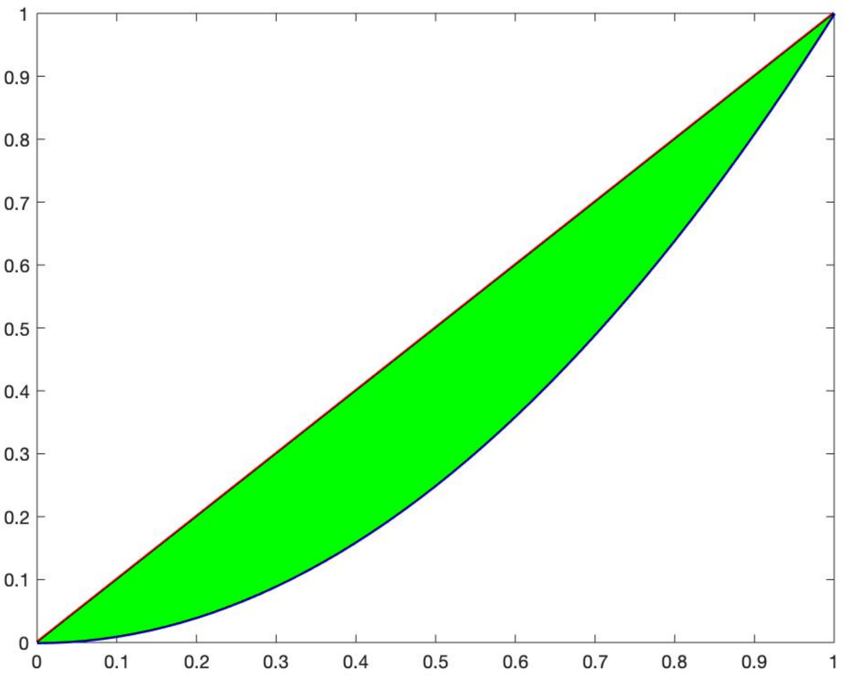


Figure 47. Domain of the double integral from Exercise 17f.

Resolution:

$$I = \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^{\sqrt{x^2+y^2}} z\,\mathrm{d}z = \int_0^1 \mathrm{d}x \int_{y=0}^{y=x} \left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{x^2+y^2}} \mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^x \frac{x^2+y^2}{2}\,\mathrm{d}y$$
$$= \frac{1}{2}\int_0^1 \mathrm{d}x \int_0^x x^2+y^2\,\mathrm{d}y = \frac{1}{2}\int_0^1 \left[x^2y+\frac{y^3}{3}\right]_{y=0}^{y=x} \mathrm{d}x = \frac{1}{2}\int_0^1 \left(x^3+\frac{x^3}{3}\right)\mathrm{d}x$$
$$= \frac{1}{2}\int_0^1 \frac{4}{3}x^3\,\mathrm{d}x = \frac{2}{3}\int_0^1 x^3\,\mathrm{d}x = \frac{2}{3}\left[\frac{x^4}{4}\right]_{x=0}^{x=1} = \frac{2}{3}\cdot\frac{1}{4} = \frac{1}{6}.$$

Exercise 18c

Exercise. *Compute the following iterated triple integral* $I = \int\limits_0^3 \int\limits_0^{2x} \int\limits_0^{\sqrt{xy}} z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$

Resolution: We have to integrate the iterated integrals following a concrete order:

$$I = \int\limits_0^3 \mathrm{d}x \int\limits_0^{2x} \mathrm{d}y \int\limits_0^{\sqrt{xy}} z\,\mathrm{d}z = \int\limits_0^3 \mathrm{d}x \int\limits_0^{2x} \left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{xy}} \mathrm{d}y = \frac{1}{2}\int\limits_0^3 \mathrm{d}x \int\limits_0^{2x} xy\,\mathrm{d}y = \frac{1}{2}\int\limits_0^3 \left[\frac{xy^2}{2}\right]_{y=0}^{y=2x} \mathrm{d}x$$
$$= \frac{1}{2}\int\limits_0^3 \frac{4x^3}{2}\,\mathrm{d}x = \int\limits_0^3 x^3\,\mathrm{d}x = \left[\frac{x^4}{4}\right]_{x=0}^{x=3} = \frac{81}{4}.$$

Exercise 19a

Exercise. *For the following regions of* $\mathbb{R}^3$ *write the triple integral* $\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *in terms of iterated integrals taken in different order. W tetrahedron bounded by the planes* $x=0, y=0, z=0, 2x+3y+4z=12$.

Resolution: We need to obtain the triple integral:

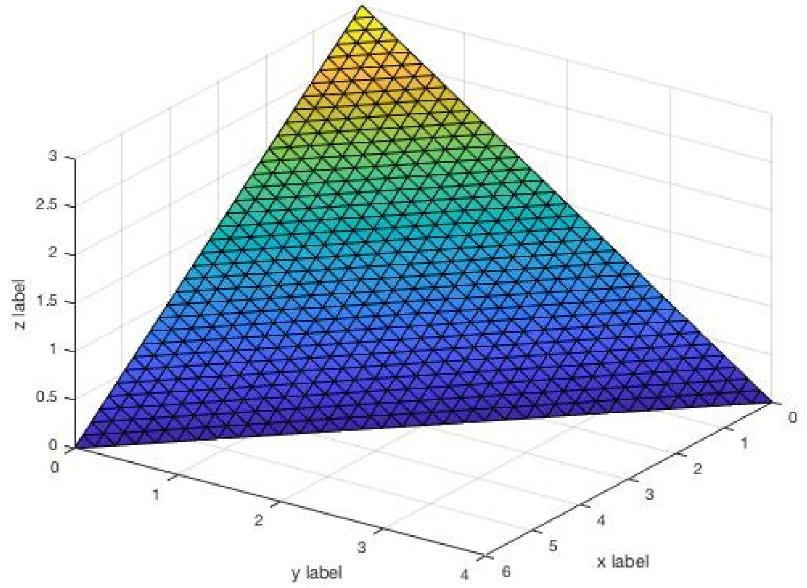


FIGURE 48. W from Exercise 19a: tetrahedron

$$I = \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$I = \int_{x_1}^{x_2} \mathrm{d}x \int_{y_1}^{y_2} \mathrm{d}y \int_{z_1}^{z_2} \mathrm{d}z$$

We know that $x_1 = y_1 = z_1 = 0$ since we have 3 planes equal to 0. We are bound by those values. We need to find the **upper bounds.**

Solving for z_2, we get:

$$12 - 2x - 3y = 4z_2 \rightarrow z_2 = 3 - \tfrac{1}{2}x - \tfrac{3}{4}y$$

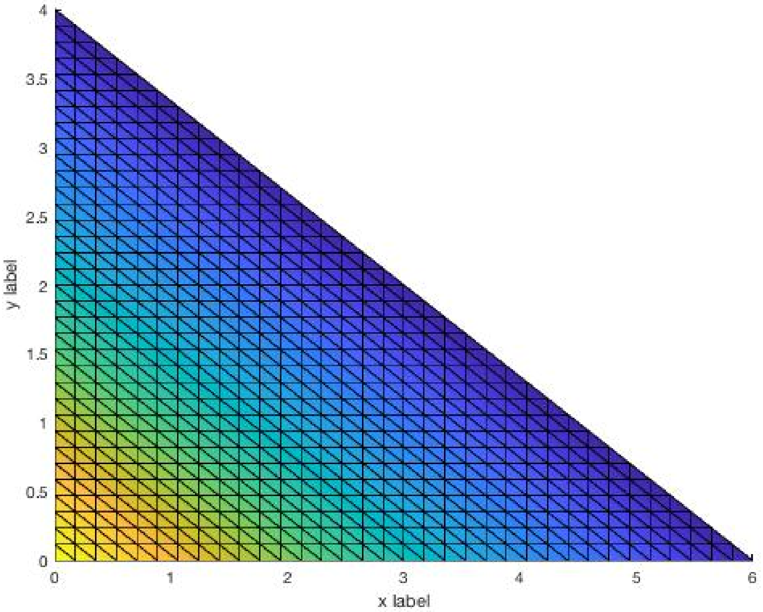


FIGURE 49. xy plane, when z=0.

Solving for y_2, we notice that in 3D, there exists 2 intersections on the xy-plane, when $x = 0$ and $y = 0$. We can include them in one 2-variable equations when $z = 0$.
If $z = 0$

$$3 - y_2 + 2x = 12 \to y_2 = 4 - \frac{2}{3}x$$

Solving for x_2, we find where $4 - \frac{2}{3}$x intersects the x-axis when $z = 0$ and $y = 0$. Therefore we get:
If $z = 0$ and $y = 0$:

$$2x_2 = 12 \to x_2 = 6$$

So our integral is:

$$\boxed{I = \int\limits_0^6 \mathrm{d}x \int\limits_0^{4-\frac{2x}{3}} \mathrm{d}y \int\limits_0^{3-\frac{3y}{4}-\frac{x}{2}} \mathrm{d}z}$$

Exercise 19b

Exercise. *For the following region of* $\mathbb{R}^3$ *write the triple integral* $I = \iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *in terms of iterated integrals taken in different order, where* W *is the interior of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Plan: The path to follow is the following. First, we need to identify the object whose volume we are asked to calculate. Then, we will be able to decide whether or not to compute any change of variables. If so, we will compute it, along with its Jacobian. Then we will set the limits of our 3 variables. Finally, we will be able to compute the triple integral.
Resolution: We are asked to compute the volume of an sphere-like object. Then, let us perform a change of variables to adapted spherical coordinates:

$$(x,y,z) = T(r,\theta,\varphi) = (a \cdot r\cos(\theta)cos(\varphi), b \cdot r\sin(\theta)cos(\varphi), c \cdot r\sin(\varphi)).$$

Let us now compute the determinant of the Jacobian matrix:

$$\begin{aligned}
\det\left(DT(x,y)\right) &= \begin{vmatrix} x_\theta & x_\varphi & x_r \\ y_\theta & y_\varphi & y_r \\ z_\theta & z_\varphi & z_r \end{vmatrix} = \begin{vmatrix} -ar\sin\theta\cos\varphi & -ar\sin\varphi\cos\theta & a\cos\theta\cos\varphi \\ br\cos\theta\cos\varphi & -br\sin\varphi\sin\theta & b\sin\theta\cos\varphi \\ 0 & cr\cos\varphi & c\sin\varphi \end{vmatrix} \\
&= abcr^2 \begin{vmatrix} -\sin\theta\cos\varphi & -\sin\varphi\cos\theta & \cos\theta\cos\varphi \\ \cos\theta\cos\varphi & -\sin\varphi\sin\theta & \sin\theta\cos\varphi \\ 0 & \cos\varphi & \sin\varphi \end{vmatrix} \\
&= abcr^2(\sin^2\varphi\sin^2\theta\cos\varphi + \cos^2\theta\cos^3\varphi + \cos^2\theta\sin^2\varphi\cos\varphi + \cos^3\varphi\sin^2\theta) \\
&= abcr^2\cos\varphi(\sin^2\theta + \cos^2\theta = abcr^2\cos\varphi.
\end{aligned}$$

The absolute value of the determinant of the Jacobian matrix is then $|\det\left(DT(x,y)\right)| = \left|abcr^2\cos\varphi\right| = abcr^2\cos\varphi$, since $\pi/2 \leq \varphi \leq \pi/2$.

Now, let us define the limits of the variables θ, φ and r. We will do this helping ourselves with the plot of the ellipsoid. The plot is made giving some values to r, a, b and c: $r = 1, a = 2, b = 4, c = \frac{1}{2}$.

We see that the longitude θ has no restriction between 0 and 2π, as the angle moves along the vertical axis a full turn. Now, looking from bottom to top, the latitude φ has no restriction either (it rotates the full 180 degrees between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ in this case). The limits of r can be slightly more complicated: r always starts at the origin $(0,0,0)$ and expands until getting to the boundary of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. But just substituting x, y, z by the adapted spherical coordinates: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = r^2$, so the maximum value for r in the interior W of the ellipsoid is 1.

Then the interior of the ellipsoid $W = \{(x,y,z)\colon \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\}$ is of the form $W = T(B)$ where $B = \{(r,\theta,\varphi)\colon 0 \leq \theta \leq 2\pi, -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}, 0 \leq r \leq 1\}$, and we can compute the volume $V(W)$ inside the ellipsoid

$$V(W) = \int_0^{2\pi} \mathrm{d}\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\varphi \int_0^1 abc \cdot r^2\cos\varphi\,\mathrm{d}r.$$

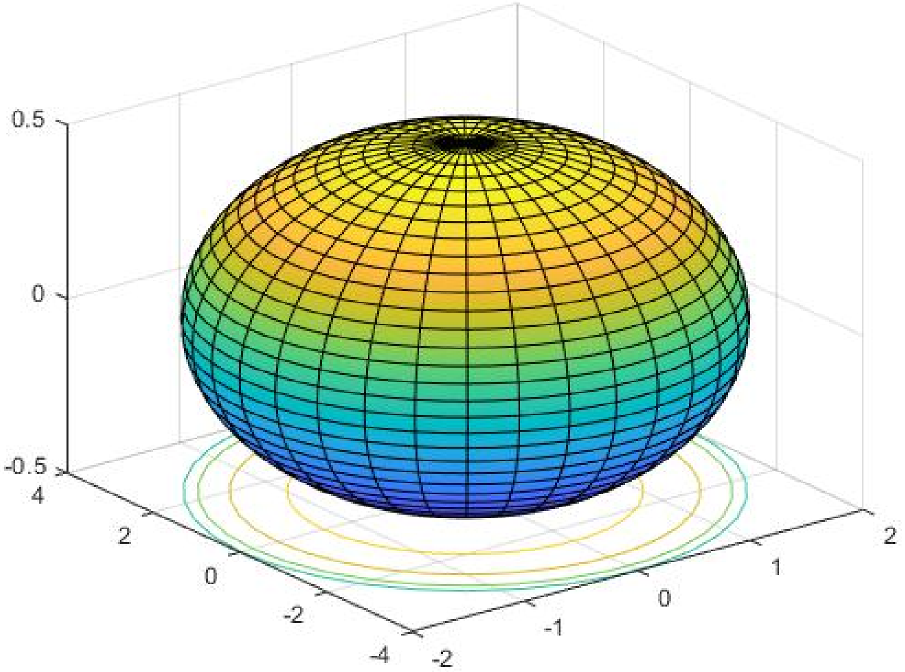


Figure 50. Ellipsoid plot from Exercise 19b.

The elements can be rearranged so that no integral depends on other variables, obtaining the desired iterable integral

$$V(W) = abc\left(\int_0^{2\pi} \mathrm{d}\theta\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\,\mathrm{d}\varphi\right)\left(\int_0^1 r^2\,\mathrm{d}r\right).$$

In this case, we can apply the Fubini theorem and compute the three integrals separately and multiply the results at the end:

$$V(W) = abc\cdot 2\pi\cdot\left(\sin\frac{\pi}{2} - \sin -\frac{\pi}{2}\right)\cdot\frac{1}{3} = \frac{4}{3}\pi abc.$$

Exercise 19c

Exercise. *For the region W of $\mathbb{R}^3$ bounded by the surfaces $y^2 + 2z^2 = 4x$, $x = 2$, write the triple integral $I = \iiint_W f(x,y,z)\,\mathrm{d}x\,dy\,dz$ in terms of iterated integrals taken in different order.*

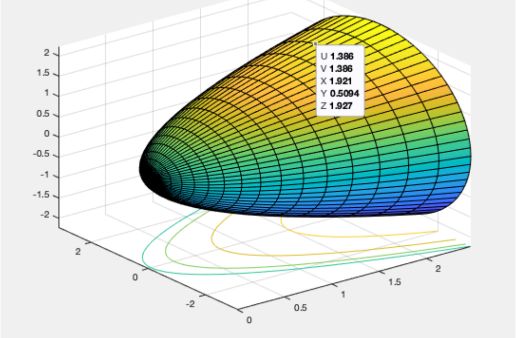


Figure 51. Domain W of Exercise 19c.

Resolution: The region bounded by the two surfaces is defined as

$$W = \left\{(x,y,z) \in \mathbb{R}^3 \colon y^2 + 2z^2 \leq 4x, x \leq 2\right\} = \left\{(x,y,z) \in \mathbb{R}^3 \colon y^2 + 2z^2 \leq 4x \leq 8\right\}.$$

Although we can choose up to 8 different orders for the iterated integrals, we will show just two.

If we first bound z, we simply get $2z^2 \leq 8 \iff z^2 \leq 4 \iff -2 \leq z \leq 2$. Once the variable z is fixed, let us bound y by $y^2 + 2z^2 \leq 8 \iff y^2 \leq 8 - 2z^2 \iff -\sqrt{8-2z^2} \leq y \leq \sqrt{8-2z^2}$. Finally x is bounded by $y^2 + 2z^2 \leq 4x \leq 8 \iff y^2/4 + z^2/2 \leq x \leq 2$. We can write W in an equivalent way as

$$W = \left\{(x,y,z) \in \mathbb{R}^3 \colon -2 \leq z \leq 2,\ -\sqrt{8-2z^2} \leq y \leq \sqrt{8-2z^2},\ y^2/4 + z^2/2 \leq x \leq 2\right\},$$

from which we can write the associated iterated integral as

$$I = \int_{-2}^{2} \mathrm{d}z \int_{-\sqrt{8-2z^2}}^{\sqrt{8-2z^2}} \mathrm{d}y \int_{y^2/4+z^2/2}^{2} f(x,y,z)\,\mathrm{d}x.$$

If we first bound x, we simply get $0 \leq x \leq 2$. Once the variable x is fixed, let us bound z by $2z^2 \leq 4x \iff z^2 \leq 2x \iff -\sqrt{2x} \leq z \leq \sqrt{2x}$. Finally y gets bounded by $y^2 + 2z^2 \leq 4x \iff y^2 \leq 4x - 2z^2 \iff -\sqrt{4x-2z^2} \leq y \leq \sqrt{4x-2z^2}$. We can write W as

$$W = \left\{(x,y,z) \in \mathbb{R}^3 \colon 0 \leq x \leq 2, -\sqrt{2x} \leq z \leq \sqrt{2x}, -\sqrt{4x-2z^2} \leq y \leq \sqrt{4x-2z^2}\right\},$$

from which we can write the associated iterated integral as

$$I = \int_{0}^{2} \mathrm{d}x \int_{-\sqrt{2x}}^{\sqrt{2x}} \mathrm{d}z \int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}} f(x,y,z)\,\mathrm{d}y.$$

Now let us analyze the problem using adapted cylindrical coordinates:

$$(x,y,z) = T(x,r,\theta) = \left(x, 2r\cos\theta, \sqrt{2}r\sin\theta\right), \qquad |\det DT(x,r,\theta)| = 2\sqrt{2}r,$$

which satisfy $\dfrac{y^2}{4} + \dfrac{z^2}{2} = r^2$, so that $W = T(B)$, with

$$B = \left\{(x,r,\theta) \colon 0 \leq \theta \leq 2\pi, r \geq 0, r^2 \leq x \leq 2\right\} = \left\{(x,r,\theta) \colon 0 \leq \theta \leq 2\pi, 0 \leq r \leq \sqrt{2}, r^2 \leq x \leq 2\right\},$$

from which we can write the associated iterated integral as

$$I = \int_{0}^{2\pi} \mathrm{d}\theta \int_{0}^{\sqrt{2}} r\,\mathrm{d}r \int_{r^2}^{2} f\left(x, 2r\cos\theta, \sqrt{2}r\sin\theta\right)\,\mathrm{d}x.$$

Exercise 20 a

Exercise. *Compute the triple integral* $\displaystyle\iiint_W xz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, *$W$ bounded by the cylinder with circular base $x^2 + y^2 - 2x = 0$ and the surface $z^2 = 2y$, $(y, z > 0)$.*

Resolution: The domain W given is bounded by some inequalities to be found:

$$\begin{aligned} W &= \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - 2x \overset{\geq}{\underset{<}{=}} 0, z^2 \overset{\geq}{\underset{<}{=}} 2y, y > 0, z > 0\right\} \\ &= \left\{(x,y,z) \in \mathbb{R}^3 : (x-1)^2 + y^2 \overset{\geq}{\underset{<}{=}} 1, z \overset{\geq}{\underset{<}{=}} \sqrt{2y}, y > 0, z > 0\right\}. \end{aligned}$$

From the figure we can define the limits of integration of the triple integral as the following:

$$W = \left\{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 2, 0 \leq y \leq \sqrt{1-(x-1)^2}, 0 \leq z \leq \sqrt{2y}\right\}.$$

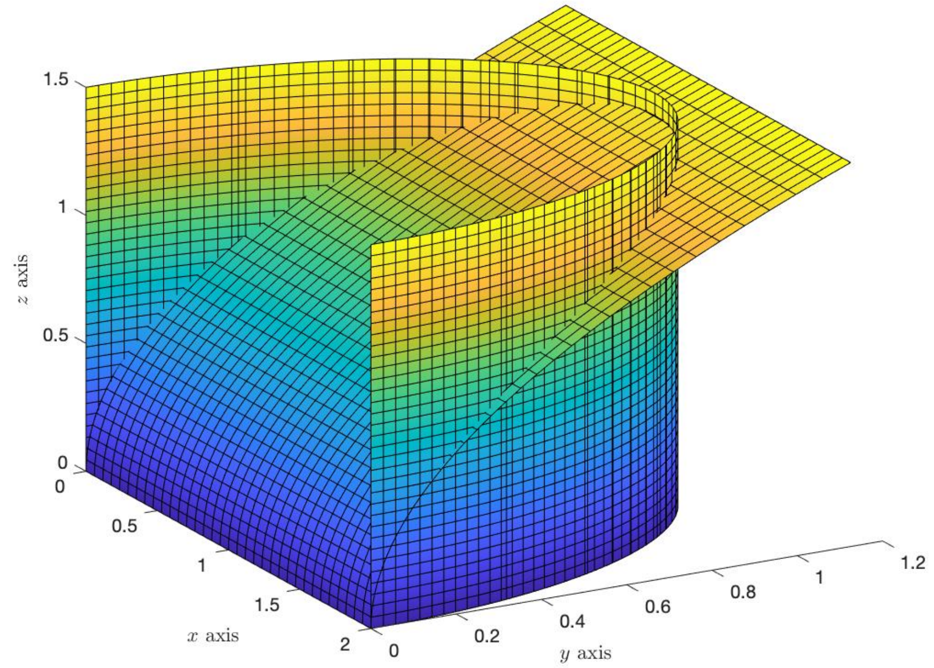


FIGURE 52. Domain W satisfying $(x-1)^2+y^2\le 1$, $0\le z\le\sqrt{2y}$ from Exercise 20a.

Now, we can compute the triple integral:

$$\begin{aligned}
V(W) &= \int_0^2 x\,\mathrm{d}x \int_0^{\sqrt{1-(x-1)^2}}\,\mathrm{d}y \int_0^{\sqrt{2y}} z\,\mathrm{d}z \\
&= \int_0^2 x\,\mathrm{d}x \int_0^{\sqrt{1-(x-1)^2}} \frac{1}{2}\left[z^2\right]_{z=0}^{z=\sqrt{2y}}\,\mathrm{d}y \\
&= \int_0^2 x\,\mathrm{d}x \int_0^{\sqrt{1-(x-1)^2}} y\,\mathrm{d}y \\
&= \int_0^2 \frac{x}{2}\left[y^2\right]_{y=0}^{y=\sqrt{1-(x-1)^2}}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^2 x\left(1-(x-1)^2\right)\mathrm{d}x \\
&= \frac{1}{2}\int_0^2 \left(x-x^3+2x^2-x\right)\mathrm{d}x \\
&= \frac{1}{2}\int_0^2 \left(2x^2-x^3\right)\mathrm{d}x \\
&= \int_0^2 x^2\,\mathrm{d}x - \frac{1}{2}\int_0^2 x^3\,\mathrm{d}x \\
&= \frac{1}{3}\left[x^3\right]_0^2 - \frac{1}{2}\frac{1}{4}\left[x^4\right]_{x=0}^{x=2} \\
&= \frac{8}{3}-\frac{16}{8} = \frac{8}{3}-2 \\
&= \frac{8}{3}-\frac{6}{3} = \frac{2}{3}.
\end{aligned}$$

$$\boldsymbol{V(W)=\frac{2}{3}}.$$

EXERCISE 20B

Exercise. *Compute the triple integral* $\iiint_W zy\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *in* $W = \{(x,y,z) \in \mathbb{R}^3 \colon 0 \leq z \leq x^2+y^2, 0 \leq y \leq \sqrt{2x-x^2}\}$.

Resolution: $z = x^2+y^2$ is a paraboloid with vertex $(0,0,0)$ and if we expand $y = \sqrt{2x-x^2}$ we get

$$y = \sqrt{2x-x^2} \iff y^2 = 2x-x^2 \iff y^2 = -(x^2-2x) \iff y^2 = -(x-1)^2+1 \iff (x-1)^2+y^2 = 1,$$

so $y = \sqrt{2x-x^2}$ is a cylinder with radius 1 around the vertical line $\{(1,0,z), z \in \mathbb{R}\}$. The graph shows us the figures seen from the y-z axes. We have to calculate the yellow zone which is the one bounded by the cylinder, paraboloid, and the plane $z = 0$. One of the easiest ways to solve the problem is using the Cartesian coordinates. The statement gives us the integration limits on y-axis and z-axis, but we can also calculate the x ones. As we know that the cylinder has radius 1 and center $(1,0)$ for each z, we can deduce that the integration limits on the x-axis will be from 0 to 2. So,

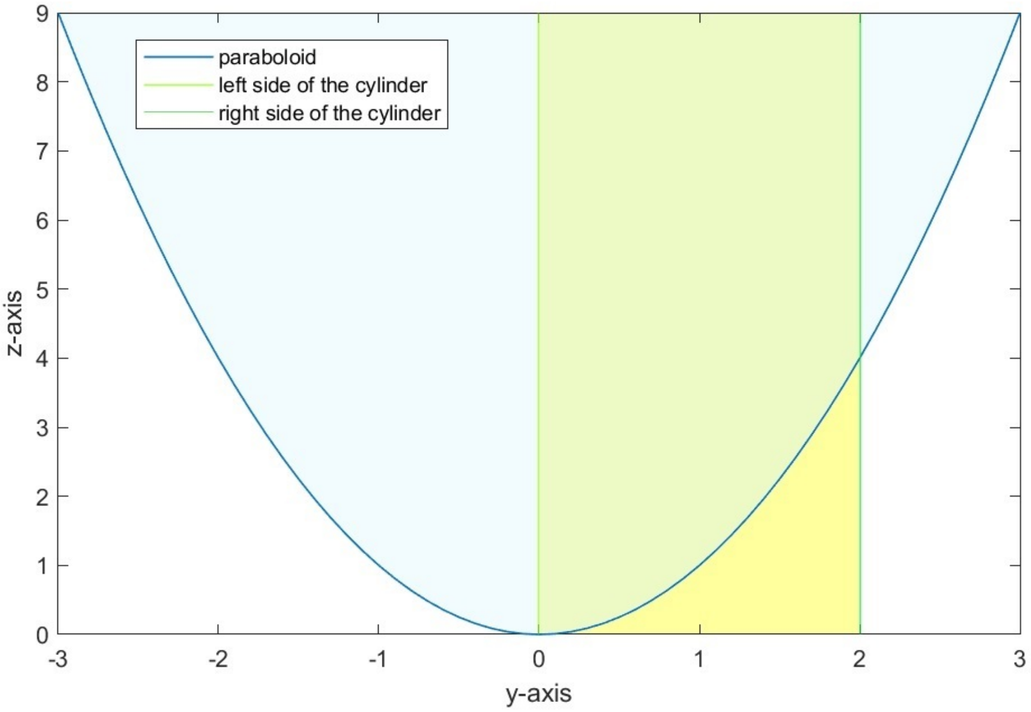


FIGURE 53. W from Exercise 20b: The yellow zone is the volume to be calculated.

$$W = \{(x,y,z) \colon 0 \leq x \leq 2,\ 0 \leq y \leq \sqrt{2x-x^2},\ 0 \leq z \leq x^2+y^2\}$$

Now, as we have the integration limits and the function that must be integrated, we only have to do the following operation:

$$I(W) = V = \int_0^2 \mathrm{d}x \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\mathrm{d}y \int_0^{x^2+y^2} z\,\mathrm{d}z = \int_0^2 \mathrm{d}x \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\left[\frac{z^2}{2}\right]_{z=0}^{z=x^2+y^2} \mathrm{d}y =$$

$$= \frac{1}{2}\int_0^2 \mathrm{d}x \int_0^{\sqrt{2x-x^2}} y(x^2+y^2)^{\frac{5}{2}}\,\mathrm{d}y = \frac{1}{14}\int_0^2 \left[(x^2+y^2)^{\frac{7}{2}}\right]_{y=0}^{y=\sqrt{2x-x^2}} \mathrm{d}x =$$

$$= \frac{1}{14}\int_0^2 \left((x^2+2x-x^2)^{\frac{7}{2}} - x^7\right)\mathrm{d}x == \frac{1}{14}\left[2 \cdot 2^{\frac{7}{2}} \cdot \frac{x^{\frac{9}{2}}}{9} - \frac{x^8}{8}\right]_{x=0}^{x=2} = \frac{16}{9}.$$

The volume bounded by the specified region is $\boldsymbol{\frac{16}{9}}$.

EXERCISE 20C

Exercise. *Compute the triple integral* $\iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *in the region* $W = \{(x,y,z) \in \mathbb{R}^3 \colon 1 \leq x \leq 3,\ 1 \leq y \leq 3,\ 0 \leq z \leq xy\}$.

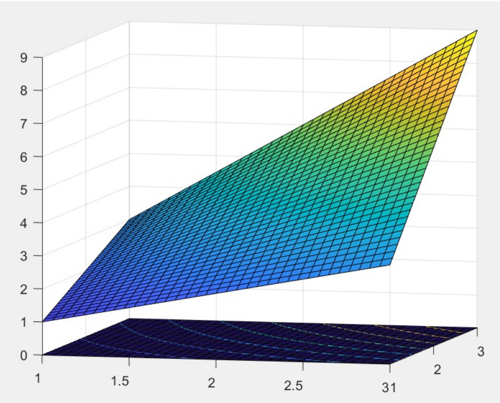


FIGURE 54. $z = xy$ and $z = 0$ over $\{1 \leq x \leq 3,\ 1 \leq y \leq 3\}$ of Exercise 20c.

Resolution: The figure illustrates the surfaces $z = 0$ and $z = xy$, which are the lower face and the upper face of the boundary of W. As we have already been given the limits of integration of the three variables of the triple integral, let us just compute it:

$$\begin{aligned}\iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_1^3 \mathrm{d}y \int_1^3 \mathrm{d}x \int_0^{xy} \mathrm{d}z = \int_1^3 \mathrm{d}y \int_1^3 [z]_{z=0}^{z=xy}\,\mathrm{d}x \\ &= \int_1^3 y\,\mathrm{d}y \int_1^3 x\,\mathrm{d}x = \left[\frac{y^2}{2}\right]_{y=1}^{y=3} \left[\frac{x^2}{2}\right]_{x=1}^{x=3} = 4 \cdot 4 = 16.\end{aligned}$$

EXERCISE 20D

Exercise. *Compute the following triple integral* $\iiint_W xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *in the region* W *bounded by the surfaces* $y = x^2$, $x = y^2$, $z = xy$, $z = 0$.

Resolution: First of all let's rewrite the equations so as to make our lives easier. We can change $x = y^2$ for $y = \sqrt{x}$ which will help us to put the bounds on **y**. If we plot all the surfaces we can see

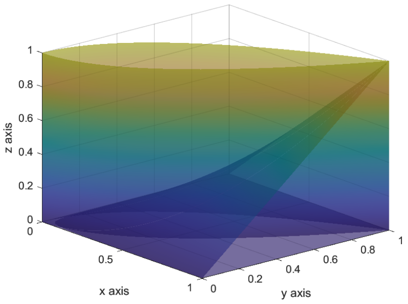


FIGURE 55. Boundary surfaces of W from Exercise 20d

that we can fix **x** from 0 to 1, then we can bound **y** from x^2 to $\sqrt{x}$ and finally **z** from 0 to xy. Once we have the bounds settled we can start integrating:

$$
\begin{aligned}
V(W) &= \iiint_W xyz \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_0^1 x \, \mathrm{d}x \int_{x^2}^{\sqrt{x}} y \, \mathrm{d}y \int_0^{xy} z \, \mathrm{d}z \\
&= \frac{1}{2} \int_0^1 x \, \mathrm{d}x \int_{x^2}^{\sqrt{x}} x^2 y^3 \, \mathrm{d}y = \frac{1}{2} \int_0^1 x^3 \left[\frac{y^4}{4} \right]_{y=x^2}^{y=\sqrt{x}} \mathrm{d}x \\
&= \frac{1}{2} \int_0^1 x^3 \left(\frac{x^2}{4} - \frac{x^8}{4} \right) \mathrm{d}x = \frac{1}{8} \int_0^1 x^5 - x^{11} \, \mathrm{d}x \\
&= \frac{1}{8} \left(\frac{1}{6} - \frac{1}{12} \right) = \frac{1}{96}.
\end{aligned}
$$

Exercise 20e

Exercise. *Compute* $I = \iiint_W x \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$, *where* W *tetrahedron bounded by the planes* $x = 0$, $y = 0$, $z = 0$, $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

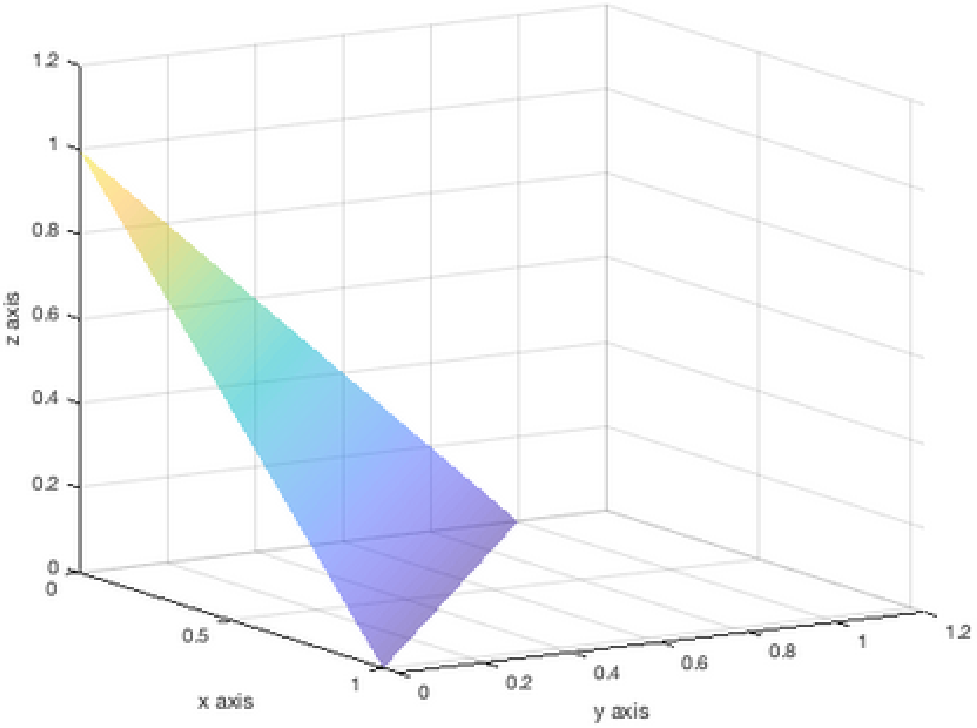


Figure 56. W from Exercise 20e for $a = b = c = 1$: tetrahedron

Resolution: The first step we should always take when solving a double or triple integral is defining W, which is a region in space. As it can be seen in Figure 4, and it is also said in the statement of the problem, the region is a tetrahedron. It is easy to see from Figure 4 that x, y and z will be bigger than 0. As we want the region below the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, we will impose the inequality $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1$. Finally, we can write W as:

$$W = \left\{ (x, y, z) \in \mathbb{R}^3 \colon x \geq 0, \, y \geq 0, \, z \geq 0, \, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1 \right\}$$

We can solve this exercise by fixing one variable (I chose x) and writing the others in terms of the fixed variable. We have to take into account that this procedure will determine the order in which we

will integrate. For instance, x will be the outermost variable in the integral (that is, the last variable with which we will integrate).
<u>If we fix the variable x</u>:

$$x \leq a\left(1 - \frac{y}{b} - \frac{z}{c}\right),\ x \geq 0$$

We can see from the previous inequality that x achieves its maximum value when $y = 0$ and $z = 0$:

$$0 \leq x \leq a$$

Now, if we take a look to variable y:

$$y \leq b(\underbrace{1 - \frac{x}{a}}_{\text{fixed value}} - \frac{z}{c}),\ y \geq 0$$

So, y achieves its maximum value when $z = 0$ because x is already fixed:

$$0 \leq y \leq b\left(1 - \frac{x}{a}\right)$$

Finally:

$$0 \leq z \leq c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

Once we have determined the upper and lower limit of each variable we can rewrite the W as:

$$W = \left\{(x,y,z) \in \mathbb{R}^3 \colon 0 \leq x \leq a, 0 \leq y \leq b\left(1 - \frac{x}{a}\right), 0 \leq z \leq c\left(1 - \frac{x}{a} - \frac{y}{b}\right)\right\}$$

$$\begin{aligned}
I &= \iiint_W x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^a x\,\mathrm{d}x \int_0^{b\left(1-\frac{x}{a}\right)} \mathrm{d}y \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} \mathrm{d}z \\
&= \int_0^a x\,\mathrm{d}x \int_0^{b\left(1-\frac{x}{a}\right)} \mathrm{d}y \Big[z\Big]_{z=0}^{z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)} = c\int_0^a x\,\mathrm{d}x \int_0^{b\left(1-\frac{x}{a}\right)} \left(1 - \frac{x}{a} - \frac{y}{b}\right) \mathrm{d}y \\
&= c\int_0^a x\,\mathrm{d}x \left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_{y=0}^{y=b\left(1-\frac{x}{a}\right)} = bc\int_0^a x\left(\frac{1}{2} + \frac{x^2}{2a^2} - \frac{x}{a}\right) \mathrm{d}x = bc\left[\frac{x^2}{4} + \frac{x^4}{8} - \frac{x^3}{3a}\right]_0^a \\
&= bc\left(\frac{a^2}{4} + \frac{a^2}{8} - \frac{a^2}{3}\right) = \frac{a^2bc}{24}.
\end{aligned}$$

Exercise 21

Exercise. *The formula for a change of variables $(x,y) = T(u,v)$, for (u, v) in a bounded domain D of R^2 is*

$$\iint_{\Omega=T(D)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_D f\left(T(u,v)\right)\left|\det\left(DT(u,v)\right)\right|\,\mathrm{d}u\,\mathrm{d}v, where DT(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}(u,v).$$

Application: Compute the double integral $\iint_\Omega (x^2+y^2)^2\,\mathrm{d}x\,\mathrm{d}y$ on the domain

$$\Omega = \{(x,y) \in \mathbb{R}^2 \colon 1 \leq x^3 - 3xy^2 \leq 4,\ 2 \leq 3x^2y - y^3 \leq 5, x \geq 0, y \geq 0\}.$$

Resolution: We make a change of variable such that $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$, now $\Omega = T(D)$ where

$$D = \{(u,v) \colon 1 \leq u \leq 4, 2 \leq v \leq 5\}$$

After this change of variable, instead of a complicated domain Ω we have a simple square D.

In this example we'll use the inverse jacobian, $\frac{\partial(u,v)}{\partial(x,y)}$ instead of $\frac{\partial(x,y)}{\partial(u,v)}$, so we won't need to put x and y as a function of (u,v), we will need to take into account that $\left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| = \frac{1}{\left|\det\left(\frac{\partial(u,v)}{\partial(x,y)}\right)\right|}$. We

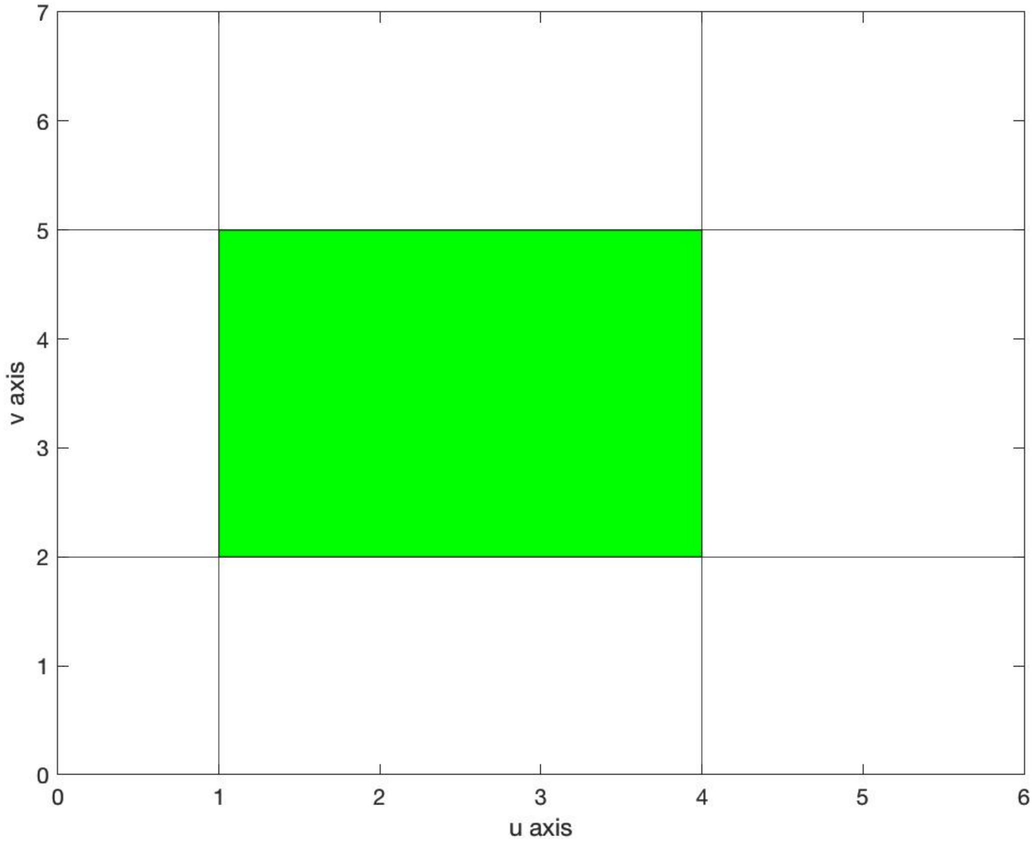


FIGURE 57. Square D that we are left with after the change of variable of Exercise 21.

first compute the Jacobian matrix

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 3x^2-3y^2 & -6xy \\ 6xy & 3x^2-3y^2 \end{pmatrix},$$

and then its determinant

$$\begin{aligned} \det\left(\frac{\partial(u,v)}{\partial(x,y)}\right) &= 9(x^2-y^2)^2+36x^2y^2 = 9(x^4-2x^2y^2+y^4+4x^2y^2) \\ &= 9(x^4+2x^2y^2+y^4) = 9(x^2+y^2)^2. \end{aligned}$$

As $9(x^2+y^2)^2$ will always be positive there is no need for the absolute value, so we plug this into the integral:

$$\begin{aligned} I &= \iint_\Omega (x^2+y^2)^2 \,\mathrm{d}x\,\mathrm{d}y = \iint_R (x^2+y^2)^2 \left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| \mathrm{d}u\,\mathrm{d}v \\ &= \iint_R (x^2+y^2)^2 \frac{1}{9(x^2+y^2)^2}\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{9}\int_2^5\int_1^4 \mathrm{d}u\,\mathrm{d}v = \frac{1}{9}\int_2^5 \mathrm{d}v \int_1^4 \mathrm{d}u \\ &= \frac{1}{9}\int_2^5 3\,\mathrm{d}v = \frac{1}{9}\left[3v\right]_{v=2}^{v=5} = \frac{15-6}{9} = 1. \end{aligned}$$

EXERCISE 22A

Exercise. *The change to* Polar coordinates *is defined by* $(x,y) = T(r,\theta) := (r\cos\theta, r\sin\theta)$. *Check that* $|\det DT(x,y)| = r$ *and use this change to compute the double integral* $\iint_\Omega x^2+y^2\,\mathrm{d}x\,\mathrm{d}y$, *where* $\Omega = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le 4\}$.

Resolution: First of all, we check the absolute value of the determinant of the Jacobian matrix:

$$\det DT(x,y) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r, \text{ so that } |\det DT(x,y)| = |r| = r.$$

Secondly, we note that $\Omega = T(D)$ with $D = \{(\theta, r)\colon 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2\}$. Therefore we apply the change of variables

$$\iint_\Omega x^2 + y^2 \,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi} \mathrm{d}\theta \int_0^2 r^2 \cdot r \,\mathrm{d}r = 2\pi \int_0^2 r^3 \,\mathrm{d}r = 2\pi \left[\frac{r^4}{4}\right]_{r=0}^{r=2} = 8\pi.$$

EXERCISE 22B

Exercise. *The change to* <u>*Polar coordinates*</u> *is defined by* $(x, y) = T(r, \theta) := (r\cos\theta, r\sin\theta)$. *Check that* $|\det DT(r, \theta)| = r$ *and use this change to compute the following double integral* $I = \iint_\Omega \cos(x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y$, *where* $\Omega = \left\{(x, y) \in \mathbb{R}^2\colon x^2 + y^2 \leq \frac{\pi}{2}\right\}$.

Resolution: First of all, let's have all the information ordered by following the given data in the statement. From $x^2 + y^2 \leq \frac{\pi}{2}$ we deduce the limits of integration that we are going to apply: $0 \leq r \leq \sqrt{\frac{\pi}{2}}$, $0 \leq \theta \leq 2\pi$. To compute the double integral by applying the polar coordinates, we cannot forgot about the jacobian. Let's calculate it:

$$\det DT(r, \theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r \implies |\det DT(r, \theta)| = |r| = r.$$

Finally the disk Ω in cartesian coordinates is transformed to a rectangle D in polar coordinates: $\Omega = T(D)$, where $D = \left\{(r, \theta)\colon 0 \leq \theta \leq 2\pi, 0 \leq r \leq \sqrt{\frac{\pi}{2}}\right\}$.

We have all the ingredients to compute our double integral:

$$\begin{aligned} I &= \iint_\Omega \cos(x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y = \iint_D \cos r^2\, r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{\frac{\pi}{2}}} r\cos r^2 \,\mathrm{d}r \\ &= \pi \int_0^{\sqrt{\frac{\pi}{2}}} 2r\cos r^2 \,\mathrm{d}r = \pi \left[\sin r^2\right]_{r=0}^{r=\sqrt{\frac{\pi}{2}}} = \pi \sin\left(\frac{\pi}{2}\right) = \pi. \end{aligned}$$

EXERCISE 22C

Exercise. *The change to Polar coordinates is defined by* $(x, y) = T(r, \theta) = (r\cos\theta, r\sin\theta)$. *Check that* $|\det DT(x, y)| = r$ *and use this change to compute the integral* $\iint_\Omega \frac{(x+y)^2}{x^2 + y^2 + 2}\,\mathrm{d}x\,\mathrm{d}y$, *where* $\Omega = \left\{(x, y) \in \mathbb{R}^2\colon x^2 + y^2 \leq 1\right\}$.

Resolution: We first compute the expression for the matrix $DT(x, y)$

$$DT(x, y) = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix},$$

from which it readily follows that $|\det DT(x, y)| = |r| = r$. Then we have the following formula for the change to polar coordinates:

$$\iint_{\Omega = T(D)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_D f\left(r\cos\theta, r\sin\theta\right) r\,\mathrm{d}r\,\mathrm{d}\theta.$$

In our concrete case $\Omega = T(D)$ with $D = \{(r, \theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\}$, so

$$\begin{aligned} \iint_\Omega \frac{(x+y)^2}{x^2 + y^2 + 2}\,\mathrm{d}x\,\mathrm{d}y &= \iint_D \frac{r^2 + 2r^2\cos\theta\sin\theta}{r^2 + 2} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi} 1 + 2\cos\theta\sin\theta\,\mathrm{d}\theta \int_0^1 \frac{r^3}{r^2 + 2}\,\mathrm{d}r \\ &= \left[\theta - \frac{\cos^2\theta}{2}\right]_{\theta=0}^{\theta=2\pi} \left[\frac{r^2}{2} - \ln\left(r^2 + 2\right)\right]_{r=0}^{r=1} \\ &= \left(2\pi - \left(\tfrac{1}{2} - \tfrac{1}{2}\right)\right)\left(\left(\tfrac{1}{2} - \ln 3\right) - \left(-\ln 2\right)\right) = 2\pi\left(\frac{1}{2} - \ln\frac{2}{3}\right). \end{aligned}$$

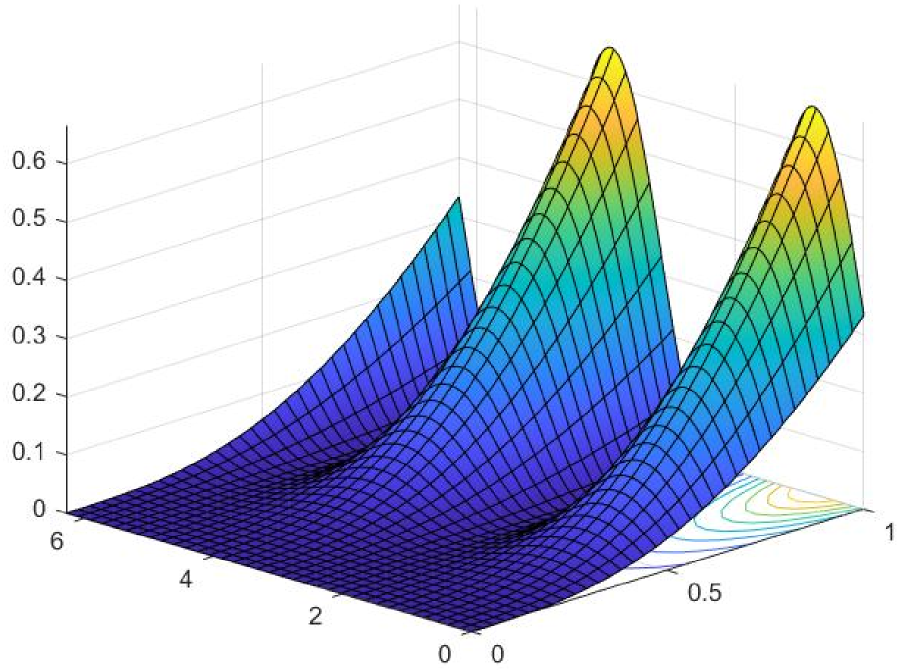


FIGURE 58. Graph of $z = \dfrac{r^2 + 2r^2 \cos\theta \sin\theta}{r^2+2} r$ over D from Exercise 22c.

EXERCISE 22D

Exercise. *The change to polar coordinates is (x, y)* $= T(r,\theta) = (r\cos\theta, r\sin\theta)$. *Use this change to compute the following double integral* $I = \displaystyle\iint_\Omega \frac{1}{(1+x^2+y^2)^2\sqrt{x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y$, *where* $\Omega = \{(x,y) \in \mathbb{R}^2 \colon x^2+y^2 \le R^2\}$

Resolution: In polar coordinates, take D such that $\Omega = T(D)$:

$$D = \{\boldsymbol{r^2} \le R^2\} = \{\boldsymbol{r} \le R\} = \{(r,\theta) \colon 0 \le \theta \le 2\pi, 0 \le r \le R\}.$$

Then

$$
\begin{aligned}
I &= \iint_\Omega \frac{r}{(1+r^2)^2\sqrt{r^2}}\,\mathrm{d}r\,\mathrm{d}\theta = \iint_\Omega \frac{\mathrm{d}r\,\mathrm{d}\theta}{(1+r^2)^2}\\
&= \int_0^{2\pi} \mathrm{d}\theta \int_0^R \frac{1}{(1+r^2)\sqrt{1+r^2}\sqrt{1+r^2}}\,\mathrm{d}r\\
&= \int_0^{2\pi} \mathrm{d}\theta \int_0^R \frac{1}{1+r^2}\cos^2\left(\arctan r\right)\mathrm{d}r\\
&= \int_0^{2\pi} \mathrm{d}\theta \int_0^R \frac{1}{1+r^2}\left(\frac{1}{2}\cos\left(2\arctan r\right)+\frac{1}{2}\right)\mathrm{d}r\\
&= \int_0^{2\pi} \left[\frac{1}{2}\arctan r + \frac{1}{4}\sin\left(2\arctan r\right)\right]_{r=0}^{r=R} \mathrm{d}\theta\\
&= \int_0^{2\pi} \left[\frac{1}{2}\arctan r + \frac{1}{2}(\sin r\cos r)\right]_{r=0}^{r=R} \mathrm{d}\theta\\
&= \int_0^{2\pi} \left[\frac{1}{2}\arctan r + \frac{1}{2}\left(\frac{r}{\sqrt{1+r^2}}\frac{1}{\sqrt{1+r^2}}\right)\right]_{r=0}^{r=R} \mathrm{d}\theta\\
&= \int_0^{2\pi} \frac{\arctan R}{2} + \frac{R}{2+2R^2}\,\mathrm{d}\theta\\
&= \frac{\arctan R}{2} + \frac{R}{2+2R^2}\,[\theta]_{\theta=0}^{\theta=2\pi} = \pi\left(\arctan R + \frac{R}{1+R^2}\right).
\end{aligned}
$$

Or alternatively,

$$
\begin{aligned}
I &= \iint_\Omega \frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{(1+r^2)^2 r} = \int_0^{2\pi}\mathrm{d}\theta\int_0^R \frac{\mathrm{d}r}{(1+r^2)^2}\\
&= 2\pi\left[\frac{\arctan x}{2} + \frac{x}{2(1+x^2)}\right]_{r=0}^{r=R} = \pi\left(\arctan R + \frac{R}{1+R^2}\right),
\end{aligned}
$$

where the primitive $\int_0^R \frac{\mathrm{d}r}{(1+r^2)^2} = \frac{\arctan x}{2} + \frac{x}{2(1+x^2)}$ is obtained through the change $x = \tan u$.

Exercise 22e

Exercise. *Compute the double integral* $I = \iint_D \sqrt{x^2+y^2-9}\,\mathrm{d}x\,\mathrm{d}y$, *where the domain* D *is given by* $D = \left\{(x,y)\colon 9 \le x^2+y^2 \le 25\right\}$.

Resolution: First we change to polar coordinates:

$$(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta).$$

As we know, this change satisfies $|\det DT(r,\theta)| = r$. So, $D = \left\{(x,y)\colon 9 \le x^2+y^2 \le 25\right\}$ in polar coordinates takes the form $B = \{(r,\theta)\colon 0 \le \theta \le 2\pi, 9 \le r^2 \le 25\}$ that is the same as $B = \{(r,\theta)\colon 0 \le$

$\theta \leq 2\pi,\ 3 \leq r \leq 5\}$. Now we have a double integral, with a trivial integral respect θ and a semi-immediate one respect to r. Therefore:

$$\begin{aligned} I &= \iint_D \sqrt{x^2+y^2-9}\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_B r\sqrt{r^2-9}\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int\limits_0^{2\pi} \mathrm{d}\theta \int\limits_3^5 r\sqrt{r^2-9}\,\mathrm{d}r \\ &= 2\pi\frac{1}{2}\int\limits_3^5 2r(r^2-9)^{\frac{1}{2}}\,\mathrm{d}r \\ &= 2\pi\frac{1}{2}\left[\frac{2(r^2-9)^{\frac{3}{2}}}{3}\right]_{r=5}^{r=3} \\ &= 2\pi\left[\frac{(r^2-9)^{\frac{3}{2}}}{3}\right]_{r=5}^{r=3} \\ &= 2\pi(\frac{64}{3}-0) = \boxed{\frac{128\pi}{3}} \end{aligned}$$

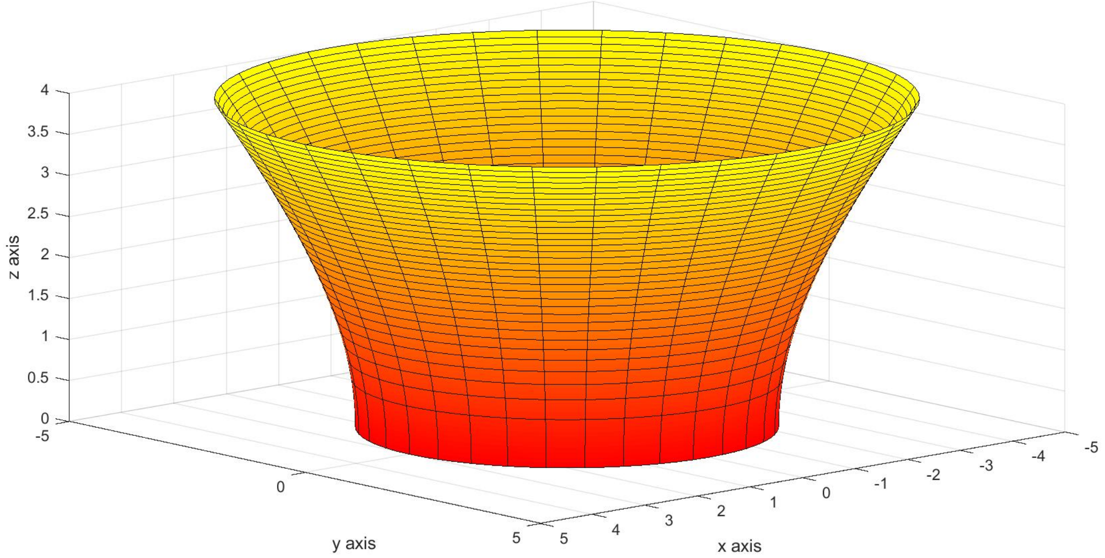


Figure 59. Plot of the integrand of $I = \iint_D \sqrt{x^2+y^2-9}\,\mathrm{d}x\,\mathrm{d}y$ on the domain D. Exercise 22e.

Exercise 22f

Exercise. *Compute the integral* $I = \displaystyle\iint_\Omega xy\,\mathrm{d}x\,\mathrm{d}y$ *on a region* Ω *which is the first quadrant of the annulus with center* $(0,0)$*, interior radius* 1 *and exterior radius* 2*:*

Resolution: From the figure and the data of the exercise we find the domain:

$$\Omega = \left\{x, y) \colon 1 \leq x^2+y^2, 4 \geq x^2+y^2, 0 \leq x, 0 \leq y\right\}.$$

Now we express Ω in polar coordinates following the next steps and checking that $\Omega = T(D)$:

- The change to polar coordinates is $(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta)$.

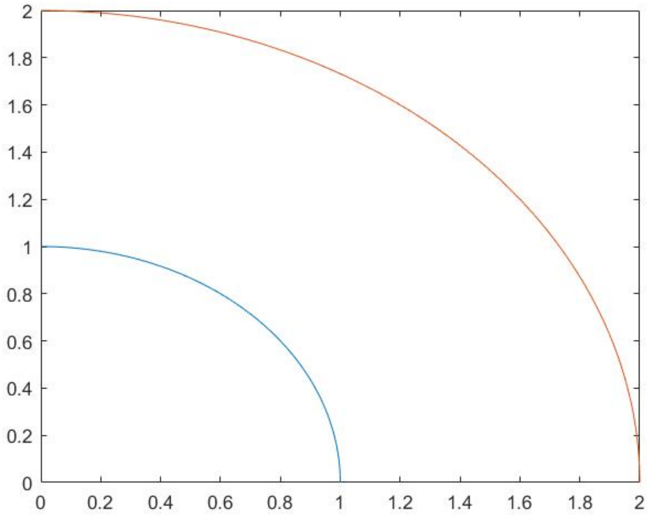


Figure 60. Domain Ω of Exercise 22f.

- We compute the Jacobian and the absolute value of its determinant:

$$DT(r,\theta)=\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix},\qquad |\det(DT(r,\theta))|=\left|r\cos^2\theta+r\sin^2\theta\right|=|r|=r.$$

- We find the new domain D in polar coordinates bearing in mind that $r^2=x^2+y^2$:

$$D=\{(r,\theta)\colon 1\le r\le 2, 0\le\theta\le\pi/2\}.$$

  We can see this domain represented in the figure by the space between both circles.
- Now we use the change and the absolute value of the determinant of the Jacobian to compute the integral in its domain:

$$I=\iint_\Omega xy\,\mathrm{d}x\,\mathrm{d}y=\iint_D r\sin\theta\, r\cos\theta\, r\,\mathrm{d}r\,\mathrm{d}\theta=\int_0^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta\int_1^2 r^3\,\mathrm{d}r$$
$$=\left[\frac{\sin^2\theta}{2}\right]_{\theta=0}^{\theta=\pi/2}\left[\frac{r^4}{4}\right]_{r=1}^{r=2}=\frac{1}{2}\cdot\frac{15}{4}=\boxed{\frac{15}{8}}.$$

Exercise 22g

Exercise. *The change to Polar coordinates is* $(x,y)=T(r,\theta)=(r\cos(\theta),r\sin(\theta))$. *Check that* $|\det(DT(u,v))|=r$ *and use this change to compute the integral* $I=\iint_\Omega x(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$ *where* Ω *is a circular sector of center* $(0,0)$ *and radius* R*, forming angles between* $\pi/3$ *and* $\pi/6$ *with the positive* x *axis.*

Resolution: Firt we compute that $|\det(DT(u,v))|=r$ so we have $x=r\cos(\theta)$ and $y=r\sin(\theta)$ with Jacobian matrix:

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{pmatrix}\cos(\theta) & -r\sin(\theta)\\ \sin(\theta) & r\cos(\theta)\end{pmatrix}\ \text{so}\ \det\frac{\partial(x,y)}{\partial(r,\theta)}=r\cos^2(\theta)+r\sin^2(\theta)=r(\cos^2(\theta)+\sin^2(\theta))=r$$

The domain of integration $\Omega=T(D)$ becomes a rectangle D in polar coordinates:

$$D=\{(r,\theta)\colon 0\le r\le R, \pi/6\le\theta\le\pi/3\}.$$

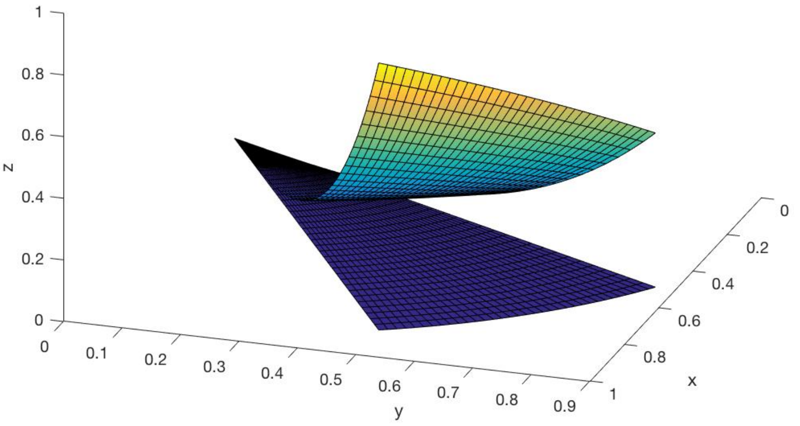


FIGURE 61. Integrand function $f(x,y) = x(x^2+y^2)$ just on Ω of Exercise 22g.

$$\begin{aligned}
I &= \iint_D x(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y = \iint_\Omega r\cos(\theta)(r^2)r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \iint_\Omega r^4\cos(\theta)\,\mathrm{d}r\,\mathrm{d}\theta = \int_{\pi/6}^{\pi/3} \frac{R^5}{5}\cos(\theta)\,\mathrm{d}\theta \\
&= \frac{R^5}{5}\left[\sin(\theta)\right]_{\theta=\pi/6}^{\theta=\pi/3} = \frac{R^5}{5}\left(\frac{\sqrt{3}}{1} - \frac{1}{2}\right) = \boxed{\frac{\sqrt{3}-1}{10}R^5}
\end{aligned}$$

EXERCISE 23A

Exercise. *Compute the areas of the domain* $\Omega \subset \mathbb{R}^2$ *defined in polar coordinates by* $a\cos\theta \le r \le a(1+\cos\theta)$ *(*$a > 0$*). Hint: Notice that the left expression only makes sense when* $\cos\theta \ge 0$*. Drawing graphs of* $a\cos\theta$ *and* $a(1+\cos\theta)$ *can help you find the admissible values of* r*.*

Resolution: First, let us plot the functions $r = a(1+\cos\theta)$ and $r = a\cos\theta$. Please note that in the following graph $a = 1$. However, as the problem states that ($a > 0$), the form of the graph will not vary if we change a, it will only move the cosine functions up or down.

To simplify the integral of the domain, we will compute the area as $A_T = A_1 - A_2$ where A_1 is the area under $r = a(1+\cos\theta)$ and A_2 is the area under $r = a\cos\theta$. Recall that in the polar plane the domain of the functions are restricted by $\theta \in [0, 2\pi]$ and $r \ge 0$.

On the one hand, the limits of the area under $r = a(1+\cos\theta)$ are defined as $\begin{cases} 0 \le \theta \le 2\pi, \\ 0 \le r \le a(1+\cos\theta). \end{cases}$

$$\begin{aligned}
A_1 &= \int_0^{2\pi}\int_0^{a(1+\cos\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_{r=0}^{r=a(1+\cos\theta)}\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi}(a(1+\cos\theta))^2\,\mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi}a^2(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta \\
&= \frac{a^2}{2}\int_0^{2\pi}\frac{3}{2}+\frac{1}{2}\cos 2\theta + 2\cos\theta\,\mathrm{d}\theta = \frac{a^2}{2}\left[\frac{3}{2}\theta + \frac{1}{4}\sin 2\theta + 2\sin\theta\right]_0^{2\pi} \\
&= \frac{3a^2\pi}{2}.
\end{aligned}$$

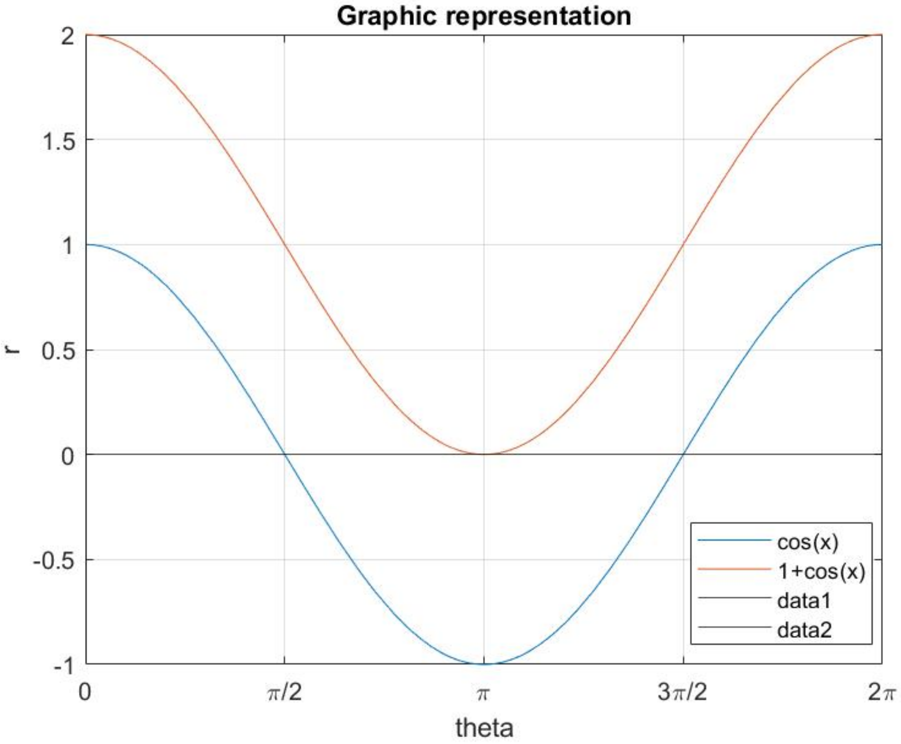


FIGURE 62. Graph of function $r = a(1+\cos\theta)$, in blue, and function $r = a\cos\theta$, in red, both in polar coordinates from Exercise 23a if $a = 1$.

On the other hand, as $a\cos\theta$ is symmetric, the area under this function can be computed as twice the integral defined by $\begin{cases} 0 \le \theta \le \pi/2, \\ 0 \le r \le a\cos\theta. \end{cases}$

$$\begin{aligned} A_2 &= 2\int_0^{\pi/2}\int_0^{a\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta = 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{r=0}^{r=a\cos\theta}\mathrm{d}\theta \\ &= \int_0^{\pi/2} a^2\cos^2\theta\,\mathrm{d}\theta = a^2\int_0^{\pi/2}\frac{1}{2}+\frac{1}{2}\cos 2\theta\,\mathrm{d}\theta \\ &= a^2\left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_{\theta=0}^{\theta=\pi/2} = \frac{a^2\pi}{4}. \end{aligned}$$

Finally,

$$A_T = \frac{3\pi a^2}{2} - \frac{\pi a^2}{4} = \frac{5\pi a^2}{4}.$$

EXERCISE 23C

Exercise. *Compute the area of the following domain* $\Omega \subset \mathbb{R}^2$ *defined in polar coordinates:* $\frac{1}{2} \le r \le |\sin(2\theta)|$. Hint: $|\sin(2\theta)| \ge \frac{1}{2}$ *is needed so as to the expression makes sense.*

Resolution:

$$\sin 2\theta = \frac{1}{2} \implies \theta_1 = \frac{\pi}{12} \text{ and } \theta_2 = \frac{5\pi}{12}.$$

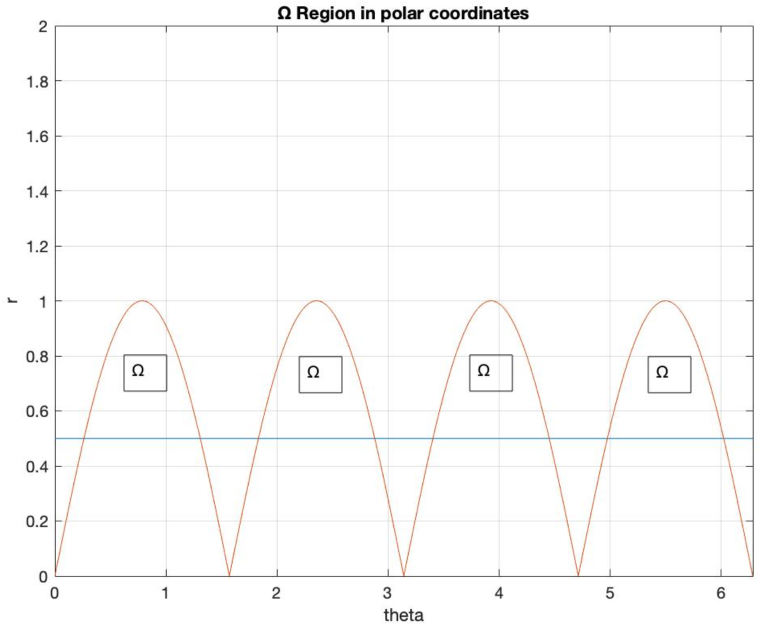


Figure 63. Ω region in polar coordinates from Exercise 23c.

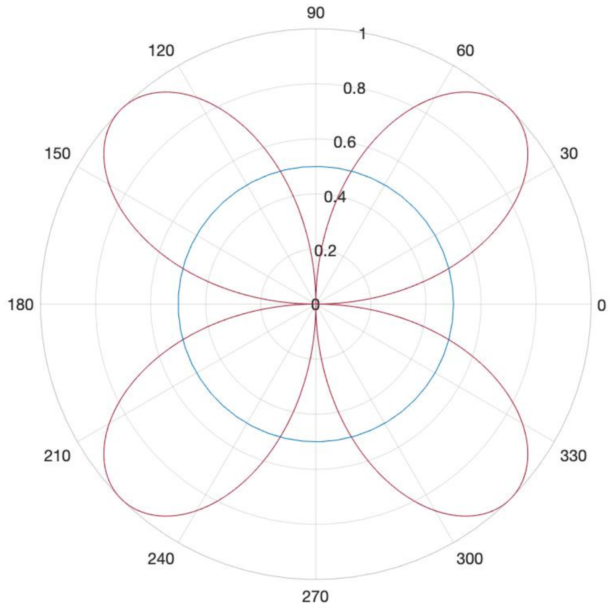


Figure 64. Ω region in cartesian coordinates from Exercise 23c.

$$A(\Omega) = 4\int_{\theta_1=\frac{\pi}{12}}^{\theta_2=\frac{5\pi}{12}} \int_{r=\frac{1}{2}}^{r=\sin(2\theta)} r \, \mathrm{d}r \, \mathrm{d}\theta = 4\int_{\theta_1=\frac{\pi}{12}}^{\theta_2=\frac{5\pi}{12}} \left[\frac{r^2}{2}\right]_{r=\frac{1}{2}}^{r=\sin(2\theta)} \mathrm{d}\theta =$$

$$= 4\int_{\theta_1=\frac{\pi}{12}}^{\theta_2=\frac{5\pi}{12}} \frac{\sin^2(2\theta)-\frac{1}{4}}{2} \, \mathrm{d}\theta = 4\int_{\theta_1=\frac{\pi}{12}}^{\theta_2=\frac{5\pi}{12}} \frac{\frac{1}{2}-\frac{1}{2}\cos(4\theta)-\frac{1}{4}}{2} \, \mathrm{d}\theta =$$

$$= 4\cdot\frac{1}{2}\left[\frac{1}{4}\left[\theta\right]_{\theta_1=\frac{\pi}{12}}^{\theta_2=\frac{5\pi}{12}} - \frac{1}{8}\int_{\theta_1=\frac{\pi}{12}}^{\theta_2=\frac{5\pi}{12}} 4\cdot\cos 4\theta \, \mathrm{d}\theta\right] =$$

$$= 2\left[\frac{\pi}{12} - \frac{1}{8}\left[\sin 4\theta\right]_{\theta_1=\frac{\pi}{12}}^{\theta_2=\frac{5\pi}{12}}\right] = 2\left[\frac{\pi}{12} - \frac{1}{8}\cdot(-\sqrt{3})\right] =$$

The total area of the domain Ω is equal to $\frac{\pi}{6}+\frac{\sqrt{3}}{4}$.

Exercise 23d

Exercise. *Compute the double integral* $\iint_\Omega \arcsin\left(x^2+y^2\right)\mathrm{d}x\,\mathrm{d}y$*, where* Ω *is the region bounded by the curve* $r=\sqrt{\sin(\theta)}$ *for* $0\leq\theta\leq\frac{\pi}{2}$.

Resolution: First, note that r is always positive, so we have $0\leq r\leq\sqrt{\sin(\theta)}$. Also note that there is no problem with the square root, since $\sin(\theta)$ is always positive for $0\leq\theta\leq\frac{\pi}{2}$.

Therefore, we have $\Omega=\left\{(r,\theta):0\leq r\leq\sqrt{\sin(\theta)},0\leq\theta\leq\frac{\pi}{2}\right\}$.

To compute the integral, we do the change of variables to polar coordinates: $x=r\cos(\theta)$ and $y=r\sin(\theta)$. Then, we find that $x^2+y^2=r^2$ and we have

$$I=\iint_\Omega \arcsin(r^2)r\,\mathrm{d}r\,\mathrm{d}\theta$$

Introducing the limits of integration, we can now compute the integral.

$$I=\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{\sqrt{\sin(\theta)}}\arcsin(r^2)r\,\mathrm{d}r$$

We first compute the second part of the integral by parts, with $u=\arcsin(r^2)$ and $\mathrm{d}v=r\,\mathrm{d}r$, so then $\mathrm{d}u=\frac{2r}{\sqrt{1-r^4}}\mathrm{d}r$ and $v=\frac{r^2}{2}$.

$$\int_0^{\sqrt{\sin(\theta)}}\arcsin(r^2)r\,\mathrm{d}r=\left[\arcsin(r^2)\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{\sin(\theta)}}-\int_0^{\sqrt{\sin(\theta)}}\frac{r^2}{2}\frac{2r}{\sqrt{1-r^4}}\mathrm{d}r=$$

$$=\left[\arcsin(r^2)\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{\sin(\theta)}}-\int_0^{\sqrt{\sin(\theta)}}\frac{r^3}{\sqrt{1-r^4}}\mathrm{d}r=\left[\arcsin(r^2)\frac{r^2}{2}+\frac{\sqrt{1-r^4}}{2}\right]_{r=0}^{r=\sqrt{\sin(\theta)}}=$$

$$=\left(\arcsin(\sin(\theta))\frac{\sin(\theta)}{2}+\frac{\sqrt{1-\sin^2(\theta)}}{2}\right)-\left(\cancel{\arcsin(0)\frac{0}{2}}+\frac{\sqrt{1-0}}{2}\right)=$$

$$=\frac{\arcsin(\sin(\theta))\sin(\theta)+\sqrt{1-\sin^2(\theta)}-1}{2}$$

We can simplify this expression so that we have

$$I=\int_0^{\frac{\pi}{2}}\frac{\theta\sin(\theta)+\cos(\theta)-1}{2}\mathrm{d}\theta=\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\theta\sin(\theta)\,\mathrm{d}\theta+\int_0^{\frac{\pi}{2}}\cos(\theta)\,\mathrm{d}\theta-\int_0^{\frac{\pi}{2}}1\,\mathrm{d}\theta\right)$$

We can solve the first integral by parts, while the second and third are trivial. To compute the first we have $u=\theta$ and $\mathrm{d}v=\sin(\theta)\,\mathrm{d}\theta$, and then $\mathrm{d}u=\mathrm{d}\theta$ and $v=-\cos(\theta)$. Therefore,

$$\int_0^{\frac{\pi}{2}}\theta\sin(\theta)\,\mathrm{d}\theta=\left[-\theta\cos(\theta)\right]_{\theta=0}^{\theta=\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos(\theta)\,\mathrm{d}\theta=\left[-\theta\cos(\theta)+\sin(\theta)\right]_{\theta=0}^{\theta=\frac{\pi}{2}}=$$

$$=\left(\sin\frac{\pi}{2}-\cancelto{0}{\frac{\pi}{2}\cos\frac{\pi}{2}}\right)-\left(\cancelto{0}{\sin 0-0\cos 0}\right)=1$$

Finally, introducing what we have just computed and computing the rest of the integrals, we have

$$I=\frac{1}{2}\left(1+\left[\sin\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}}-\left[\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}}\right)=\frac{1}{2}\left(1+1-\frac{\pi}{2}\right)=1-\frac{\pi}{4}$$

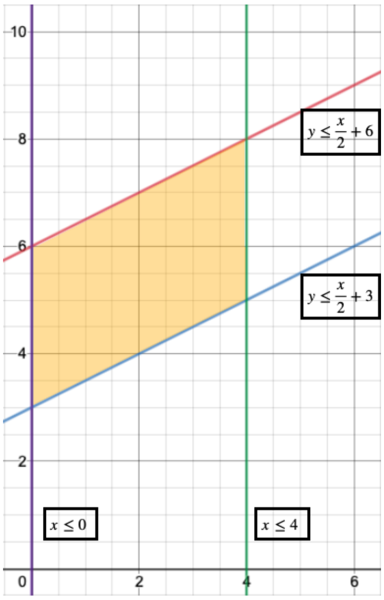


FIGURE 65. Initial representation

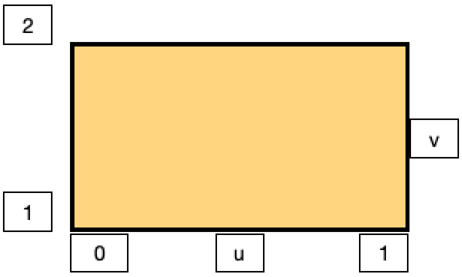


FIGURE 66. After change representation

EXERCISE 24A

Exercise. *Compute the double integral* $\iint_\Omega x\ y\ \,\mathrm{d}x\,\mathrm{d}y$*,* $\Omega = \{(x, y) \in \mathbb{R}^2 \colon 6 \le 2y - x \le 12, 0 \le x \le 4\}$*, using the change of variables* $x = 4u,\ y = 2u + 3v$*.*

Resolution:
The change suggested by the statement $(x, y) = T(u, v) = (4u, 2u + 3v)$ will be very useful because $\Omega = T(R)$, where R is the rectangle

$$R = \{(u, v) \colon 0 \le u \le 1, 1 \le v \le 2\}.$$

We now compute the absolute value of the determinant of the matrix $DT(u, v)$ in order to place this value inside the integral once the change has been done:

$$DT(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix} \implies \left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| = 12.$$

We can now proceed to apply the formula of change of variables:

$$\begin{aligned} \iint_\Omega xy\,\mathrm{d}x\,\mathrm{d}y &= \iint_D 4u(2u+3v)\cdot 12\,\mathrm{d}u\,\mathrm{d}v = 12\int_0^1 4u\,\mathrm{d}u \int_1^2 (2u+3v)\,\mathrm{d}v \\ &= 12\left(8\int_0^1 u^2\,\mathrm{d}u\int_1^2 \mathrm{d}v + 12\int_0^1 u\,\mathrm{d}u\int_1^2 v\,\mathrm{d}v\right) = 12\left(8\cdot\frac{1}{3}\cdot 1 + 12\cdot\frac{1}{2}\cdot\frac{3}{2}\right) \\ &= 12\left(\frac{8}{3}+9\right) = 12\frac{108+32}{12} = \cancel{12}\cdot\frac{140}{\cancel{12}} = \mathbf{140}. \end{aligned}$$

EXERCISE 24B

Exercise. *Compute the double integral* $I = \iint_\Omega \frac{\mathrm{d}x\,\mathrm{d}y}{(1+x+y)^5}$*, where* $\Omega = \{(x, y) \in \mathbb{R}^2 \colon 0 \le x, 0 \le y, x + y \le 1\}$*, using the change of variables* $u = x + y, v = y$*.*

Resolution: The change of variables $(u, v) = (x + y, y)$ can be easily inverted to $(x, y) = T(u, v) = (u-v, v)$. The limiting conditions $0 \le x, 0 \le y, x+y \le 1$ for Ω are transformed to $0 \le u-v, 0 \le v, v \le 1$ in the variables u, v, so $\Omega = T(D)$ with

$$D = \{(u, v) \colon 0 \le u \le 1, 0 \le v \le u\}.$$

Notice that both Ω and D are right rectangles. Moreover

$$DT(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \implies \left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| = 1.$$

Finally, before to start computing our integral, we also need to apply the change of variables in the function we are going to integrate:

$$\frac{1}{(1+x+y)^5} = \frac{1}{(1+u)^5}.$$

Now that we have everything ready, we proceed to compute the integral:

$$I = \int_0^1 \frac{\mathrm{d}u}{(1+u)^5} \int_0^u \mathrm{d}v = \int_0^1 \frac{u\,\mathrm{d}u}{(1+u)^5}$$

we integrate by parts: $a = u$, $\mathrm{d}b = \dfrac{\mathrm{d}u}{(1+u)^5} \implies \mathrm{d}a = \mathrm{d}u, b = \dfrac{-1}{4(1+u)^4}$

$$= \left[\frac{-u}{4(1+u)^4}\right]_{u=0}^{u=1} + \int_0^1 \frac{\mathrm{d}u}{4(1+u)^4} = -\frac{1}{64} - \left[\frac{1}{12(1+u)^3}\right]_{u=0}^{u=1} = -\frac{1}{64} - \frac{1}{96} + \frac{1}{12} = \frac{11}{192}.$$

Exercise 24c

Exercise. *Compute the following double integral* $I = \displaystyle\iint_\Omega \frac{\mathrm{d}x\,\mathrm{d}y}{(x+y)^{n+1}}$, *where* $\Omega = \{(x,y) \in \mathbb{R}^2 \colon 1 \leq x+y \leq 2,\ x \geq 0,\ y \geq 0\}$, *taking* $u = x+y$ *and* $v = y$.

Resolution: We are told to make a change of variable, so we first look for the Jacobian matrix to apply it.

$$DT(u,v) = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix},$$

so that

$$\mathrm{d}x\,\mathrm{d}y = \left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v = \mathrm{d}u\,\mathrm{d}v.$$

Under this change $\Omega = T(D)$ and $D = \{(u,v) \in \mathbb{R}^2 \colon 1 \leq u \leq 2,\ v \geq 0,\ u-v \geq 0\}$ and we can also write the limits of integration as

$$D = \{(u,v) \in \mathbb{R}^2 \colon 1 \leq u \leq 2,\ 0 \leq v \leq u\}.$$

Putting all this together,

$$\begin{aligned} I &= \iint_\Omega \frac{\mathrm{d}x\,\mathrm{d}y}{(x+y)^{n+1}} = \iint_D \frac{\mathrm{d}u\,\mathrm{d}v}{u^{n+1}} \\ &= \int_1^2 \frac{\mathrm{d}u}{u^{n+1}} \int_0^u \mathrm{d}v = \int_1^2 \frac{\mathrm{d}u}{u^{n+1}} [v]_{v=0}^{v=u} = \int_1^2 \frac{u\,\mathrm{d}u}{u^{n+1}} = \int_1^2 \frac{\cancel{u}\mathrm{d}u}{u^n \cancel{u}} = \int_1^2 \frac{\mathrm{d}u}{u^n} \\ &= \left[-\frac{1}{(n-1)u^{n-1}}\right]_{u=1}^{u=2} = -\frac{1}{n-1}\left(\frac{1}{2^{n-1}} - \frac{1}{1^{n-1}}\right) = \frac{1}{n-1}\left(1 + \frac{1}{2^{n-1}}\right). \end{aligned}$$

Exercise 24d

Exercise. *Compute the following double integral using the change of variables indicated:* $I = \displaystyle\iint_\Omega \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{3}{2}} \mathrm{d}x\,\mathrm{d}y$, *where* $\Omega = \{(x,y) \in \mathbb{R}^2 \colon \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1$, *taking* $x = ar\cos\theta$ *and* $y = br\sin\theta$.

Resolution: Since the condition given, $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 \leq 1$, is the interior of an ellipse, we use modified polar coordinates method, which consists on changing the variables $(x/a, y/b)$ to (r,θ). To do this, we need to do the change $x = ar\cos\theta$ and $y = br\sin\theta$, with Jacobian matrix:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{pmatrix} \text{ and } \det\frac{\partial(x,y)}{\partial(r,\theta)} = abr\cos^2\theta + abr\sin^2\theta = abr(\cos^2\theta + \sin^2\theta) = abr.$$

With this information, we just need the integration limits to compute the integral, which in these modified polar coordinates are the following: Since $r^2 = 1$, $r = \pm 1$, but we only need the positive radius, so $r = 1$:

$$0 \leq \theta \leq 2\pi$$
$$0 \leq r \leq 1$$

Putting all this together, we can now compute the integral:

$$\begin{aligned} I &= \iint_\Omega \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{3}{2}} \mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^1 abr(1-r^2)^{\frac{3}{2}}\,\mathrm{d}r\,\mathrm{d}\theta \\ &= ab\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r(1-r^2)^{\frac{3}{2}}\,\mathrm{d}r = 2\pi ab\left(-\frac{1}{2}\right)\int_0^1(-2r)(1-r^2)^{\frac{3}{2}}\,\mathrm{d}r \\ &= -\frac{2\pi ab}{2}\left[\frac{(1-r^2)^{\frac{5}{2}}}{\frac{5}{2}}\right]_{r=0}^{r=1} = -\pi ab\left[\frac{2}{5}0 - \frac{2}{5}(1)^{\frac{5}{2}}\right] \\ &= \frac{2}{5}\pi ab. \end{aligned}$$

That is, the final result of the integral is I=$\frac{2}{5}\pi ab$.

Exercise 24e

Exercise. *Compute the following double integrals using the change of variables indicated in each case.* $I = \iint_\Omega \arctan\left(x^2 + \frac{y^2}{2}\right)\mathrm{d}x\,\mathrm{d}y$, *where* $\Omega = \{(x,y) \in \mathbb{R}^2 \colon x^2 + \frac{y^2}{2} \leq 1,\ x \geq 0,\ y \geq 0\}$, *taking* $x = r\cos\theta$ *and* $y = \sqrt{2}r\sin\theta$.

Resolution: We try the change $x = r\cos\theta$, $y = \sqrt{2}r\sin\theta$ with Jacobian matrix

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{pmatrix}\cos\theta & -\sqrt{2}r\sin\theta \\ \sin\theta & \sqrt{2}r\cos\theta\end{pmatrix} \text{ and } \det\frac{\partial(r,\theta)}{\partial(x,y)} = \sqrt{2}r\cos^2\theta + \sqrt{2}r\sin^2\theta = \sqrt{2}r,$$

Under this change Ω is transformed into the circumference

$$D = \{(r,\theta) \colon 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1\}.$$

We also realize that

$$x^2 + \frac{y^2}{2} = r^2$$

Putting all this together

$$\begin{aligned} I = \iint_\Omega \arctan\left(x^2 + \frac{y^2}{2}\right)\mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi}\int_0^1 \arctan(r^2)\sqrt{2}r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \sqrt{2}\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^1 r\ \ \arctan(r^2)\,\mathrm{d}r \end{aligned}$$

We integrate by parts doing the change $u = \arctan(r^2)$, $\mathrm{d}v = r\,\mathrm{d}r$.

$$\left[\frac{r^2}{2}\arctan(r^2) - \int\frac{r^3}{1+r^4}\,\mathrm{d}r\right]_{r=0}^{r=1}$$

$$= \left[\frac{r^2}{2}\arctan(r^2) - \frac{1}{4}\ln 1 + r^4\right]_{r=0}^{r=1} = \frac{\pi}{8} - \frac{\ln 2}{4}$$

and so we have

$$\frac{\sqrt{2}\pi}{2}\left(\frac{\pi}{8} - \frac{\ln 2}{4}\right) = \frac{\sqrt{2}\pi}{8}\left(\frac{\pi}{2} - \ln 2\right)$$

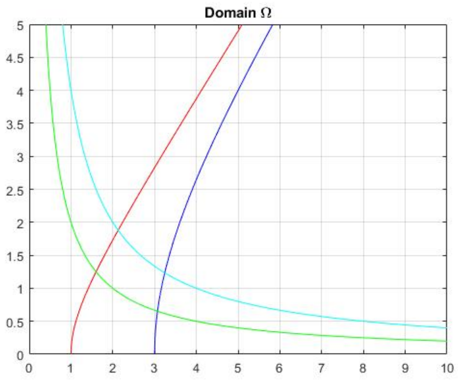


Figure 67. Domain Ω enclosed by the curves $x^2-y^2=1,9$ and $xy=2,4$ of Exercise 24f.

Exercise 24f

Exercise. *Compute the double integral* $I=\iint_\Omega (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$, *where* $\Omega=\{(x,y)\in\mathbb{R}^2\colon 1\le x^2-y^2\le 9,\ 2\le xy\le 4,\ x\ge 0,\ y\ge 0\}$.

Resolution: By the form of the inequalities bounding Ω and from the figure, we see that our domain is like a deformed rectangle, so we try the change of variables

$$(u,v)=T(x,y)=(x^2-y^2,2xy),\ \text{with Jacobian } J(u,v)=DT(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{pmatrix}2x & -2y\\ 2y & 2x\end{pmatrix},$$

and $\det\dfrac{\partial(x,y)}{\partial(u,v)}=4(x^2+y^2)$, so that

$$\mathrm{d}u\,\mathrm{d}v=\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}x\,\mathrm{d}y=4(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y,\ \text{or}\ \mathrm{d}x\,\mathrm{d}y=\frac{1}{4(x^2+y^2)}\,\mathrm{d}u\,\mathrm{d}v,$$

where we have used that $\left|\det\dfrac{\partial(u,v)}{\partial(x,y)}\right|=\dfrac{1}{4(x^2+y^2)}$ for the inverse change $(u,v)\to(x,y)=T^{-1}(u,v)$. Under this change of variables, the domain Ω is transformed to a rectangle R with area 32:

$$R=T(\Omega)=\left\{(u,v)\colon 1\le u\le 9, 2\le\frac{v}{2}\le 4\right\}=\{(u,v)\colon 1\le u\le 9, 4\le\ v\le 8\}.$$

Applying the formula of change of variables we get:

$$I=\iint_\Omega (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y=\iint_R (x^2+y^2)\frac{1}{4(x^2+y^2)}\,\mathrm{d}u\,\mathrm{d}v=\iint_R \mathrm{d}u\,\mathrm{d}v=\frac{1}{4}A(R)=\boxed{8}.$$

Exercise 24g

Exercise. *Compute the following double integral* $I=\iint_\Omega \dfrac{x+2xy}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$, *where* $\Omega=\{(x,y)\in\mathbb{R}^2\colon x^2\le y\le x^2+1,\ 1\le x^2+y^2\le\ \mathrm{e}^2,\ x\ge 0\}$.

Resolution: Since $\Omega=\{(x,y)\in\mathbb{R}^2\colon 0\le y-x^2\le 1,\ 1\le x^2+y^2\le\ \mathrm{e}^2,\ x\ge 0\}$, we try the change $u=x^2+y^2, v=y-x^2$ with Jacobian matrix

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{pmatrix}2x & 2y\\ -2x & 1\end{pmatrix}\ \text{and}\ \det\frac{\partial(u,v)}{\partial(x,y)}=2x+4xy=2(x+2xy),$$

which is positive on Ω, so that

$$\mathrm{d}u\,\mathrm{d}v=\left|\det\frac{\partial(u,v)}{\partial(x,y)}\right|\mathrm{d}x\,\mathrm{d}y=2(x+2xy)\,\mathrm{d}x\,\mathrm{d}y$$

or

$$(x+2xy)\,\mathrm{d}x\,\mathrm{d}y=\frac{\mathrm{d}u\,\mathrm{d}v}{2}.$$

Under this change Ω is transformed into the rectangle

$$D=\{(u,v)\colon 1\le u\le\ \mathrm{e}^2, 0\le v\le 1\}.$$

Putting all this together

$$I = \iint_{\Omega} \frac{x+2xy}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\iint_{D} \frac{\mathrm{d}u\,\mathrm{d}v}{u}$$
$$= \frac{1}{2}\int_1^{\mathrm{e}^2} \frac{\mathrm{d}u}{u}\int_0^1 \mathrm{d}v = \frac{1}{2}\left[\ln u\right]_1^{\mathrm{e}^2}\left[v\right]_0^1 = 1.$$

EXERCISE 25A

Exercise. *The change to Cylindrical coordinates is given by* $(x,y,z) = T(r,\theta,z)\colon = (r\cos\theta, r\sin\theta, z)$*. Check that* $|\det(DT(x,y,z))| = r$ *and use this change to compute the volume of a solid bounded by the cone* $z^2 = x^2+y^2$ *and the paraboloid* $z = x^2+y^2$*, for* $z > 0$*.*

Resolution: First we have to check that $|\det(DT(x,y))| = r$. As the change to cylindrical coordinates is $(x,y,z) = T(r,\theta,z)\colon = (r\cos\theta, r\sin\theta, z)$,

$$|\det(DT(x,y))| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\cdot r\cos\theta - (-r\sin\theta)\cdot\sin\theta = r\cos^2\theta + r\sin^2\theta = \boldsymbol{r}.$$

Now we are asked to use this change to compute the volume of the solid $W = T(D)$:

$$V(W) = \iiint\limits_{W} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_{D} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$

Once we have done the change, we have to find the limits of integration D. To do so, we can look at the graph or calculate them. We can find the limits for z by getting the value z in which the cone and paraboloid intersect: $z^2 = x^2+y^2 \cap z = x^2+y^2$, that is $z = 1$. Then, as the minimum value for z is determined by the plane $z = 0$, we have $0 \le z \le 1$.

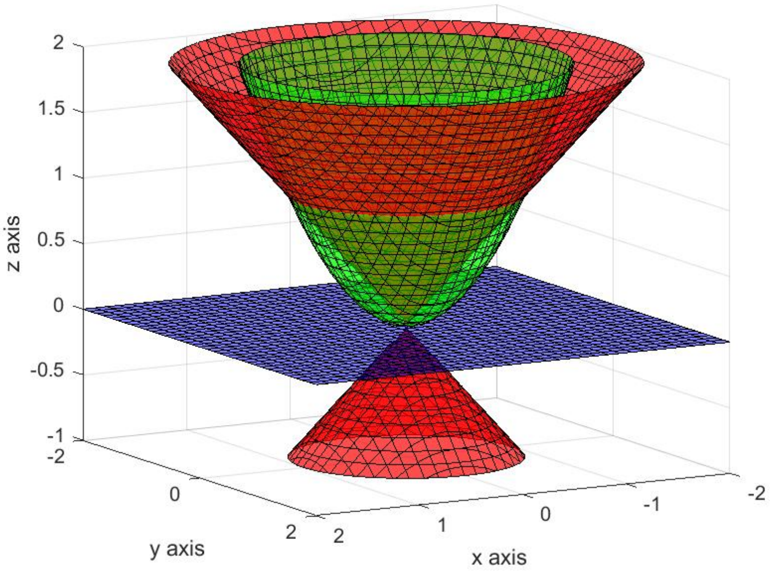


FIGURE 68. W from Exercise 25a: a cone, a paraboloid and a plane.

To find the limits for r, doing the change we get that $z^2 = r^2$ and $z = r^2$. This way, $z \le r \le \sqrt{z}$. So,

$$D = \{(r,\theta,z)\colon 0 \le \theta \le 2\pi, 0 \le z \le 1, z \le r \le \sqrt{z}\}.$$

Finally, we have to compute the integral:

$$\begin{aligned}
V(W) &= \iiint\limits_D r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}z \int_z^{\sqrt{z}} r\,\mathrm{d}r = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \left[\frac{r^2}{2}\right]_{r=z}^{r=\sqrt{z}} \mathrm{d}z \\
&= \frac{1}{2}\int_0^{2\pi} \mathrm{d}\theta \int_0^1 z - z^2\,\mathrm{d}z = \frac{1}{2}\int_0^{2\pi} \left[\frac{z^2}{2} - \frac{z^3}{3}\right]_{z=0}^{z=1} \mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi} \frac{1}{2} - \frac{1}{3}\,\mathrm{d}\theta \\
&= \frac{1}{2}\left[\frac{\theta}{2} - \frac{\theta}{3}\right]_{\theta=0}^{\theta=2\pi} = \frac{1}{2}\left(\frac{2\pi}{2} - \frac{2\pi}{3}\right) = \frac{1}{2}\cdot\frac{2\pi}{6} = \boldsymbol{\frac{\pi}{6}}.
\end{aligned}$$

Exercise 25b

Exercise. *Compute the volume of the solid bounded by the sphere* $x^2 + y^2 + z^2 = a^2$ *and the cylinder* $x^2 + y^2 = b^2$ $(a > b > 0)$.

Resolution: In cylindrical coordinates, take B such that $T(B) = W$:

$$B = \{(r, \theta, z) \colon 0 \le \theta \le 2\pi, b \le r \le a, -\sqrt{a^2 - r^2} \le z \le \sqrt{a^2 - r^2}\},$$

$$T(B) = W = \{(x, y, z) \colon \boldsymbol{x^2 + y^2} + z^2 \le a^2, \boldsymbol{x^2 + y^2} \ge b^2\}.$$

Then

$$\begin{aligned}
V(W) &= \iiint\limits_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_B r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \int\limits_0^{2\pi} \mathrm{d}\theta \int\limits_b^a r\,\mathrm{d}r \int\limits_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} \mathrm{d}z \\
&= 4\pi \int\limits_b^a \sqrt{a^2 - r^2}\, r\,\mathrm{d}r = 4\pi \left[-\frac{(a^2 - r^2)^{3/2}}{3}\right]_{r=b}^{r=a} = \frac{4\pi}{3}(a^2 - b^2)^{3/2}.
\end{aligned}$$

Exercise 26a

Exercise. *Compute the integral* $I = \iiint_W \sqrt{x^2 + y^2 + z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *on the domain* $W = \{(x, y, z) \in \mathbb{R}^3 \colon \sqrt{x^2 + y^2} \le z \le 4\}$.

Resolution: Looking at the domain, we can deduce that z goes from the upper surface part of a cone to z=4. We should use the cylindrical coordinates.

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z \\
\text{Jacobian} &= r
\end{aligned}$$

Now computing the following integral is much easier:

$$I = \int\limits_0^{2\pi} \mathrm{d}\theta \int\limits_0^4 \mathrm{d}z \int\limits_0^z r\sqrt{r^2 + z^2}\,\mathrm{d}r.$$

If we take a look we'll see that we can obtain the derivative of the function inside the square root with respect to r, multiplying by 2. We also can apply the Fubini theorem to evaluate the integral with respect to θ.

$$
\begin{aligned}
I &= \pi \int_0^4 \left[\frac{2}{3}(r^2+z^2)^{\frac{2}{3}}\right]_{r=0}^{r=z} \mathrm{d}z \\
&= \pi \int_0^4 \frac{4\sqrt{2}}{3}z^3 - \frac{2}{3}z^3 \,\mathrm{d}z \\
&= \pi \left[\frac{\sqrt{2}}{3}z^4 - \frac{1}{6}z^4\right]_{z=0}^{z=4} \\
&= \frac{128\pi}{3}(2\sqrt{2}-1).
\end{aligned}
$$

Exercise 26b

Exercise. *Compute* $I = \iiint_W ze^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *on the domain* $W = \{(x,y,z) \in \mathbb{R}^3 \colon z^2 - 1 \leq x^2+y^2 \leq \frac{z^2}{2}, z \leq 0\}$.

Resolution: The boundary surfaces of W are

$$z^2-1 = x^2+y^2 \iff x^2+y^2-z^2 = -1 \iff z^2-x^2-y^2 = 1,$$

which is an hyperboloid around the z-axis, and

$$x^2+y^2 = \frac{z^2}{2} \iff x^2+y^2-\frac{z^2}{2} = 0,$$

which is a cone.

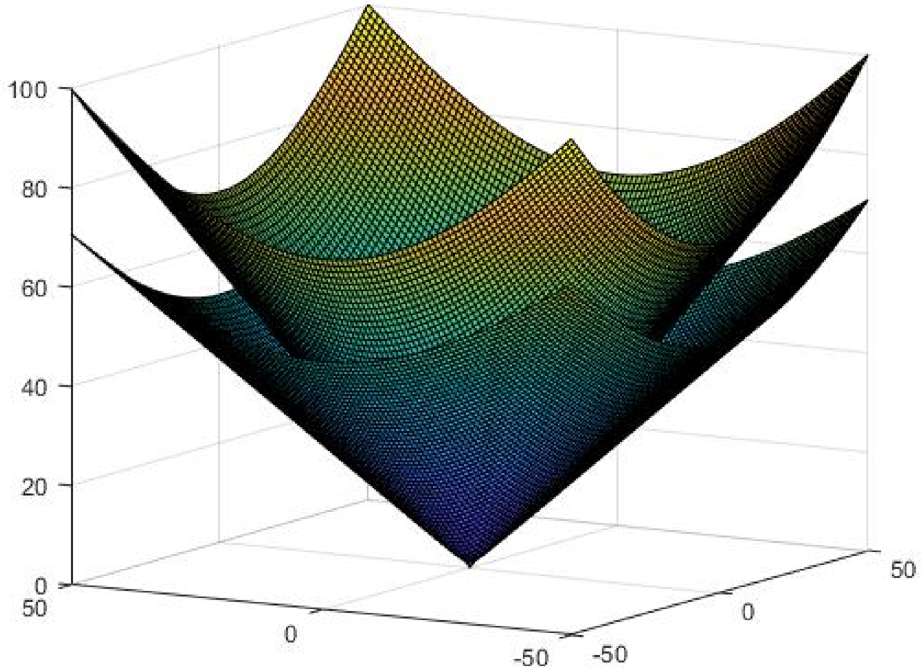


FIGURE 69. domain W from exercise 26b

These boundary surfaces intersect at

$$1+x^2+y^2 = 2x^2+2y^2 \iff x^2+y^2 = 1,$$

so we are going to use cylindrical coordinates $(x,y,z) = T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$, whose Jacobian satisfies $|\det(DT(r,\theta,z))| = r$, where $W = T(B)$ with

$$B = \{(r,\theta,z)\colon 0 \le \theta \le 2\pi, 0 \le r \le 1, \sqrt{2}r \le z \le \sqrt{1+r^2}\}.$$

Then

$$\begin{aligned} I &= \iiint_W z\,\mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_B z\,\mathrm{e}^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta \int_0^1 \mathrm{e}^{-r^2} r\,\mathrm{d}r \int_{\sqrt{2}r}^{\sqrt{1+r^2}} \mathrm{d}z \\ &= 2\pi \int_0^1 re^{-r^2}\left(\frac{1+r^2}{2} - r^2\right)\mathrm{d}r = 2\pi \int_0^1 re^{-r^2}\left(\frac{1-r^2}{2}\right)\mathrm{d}r = 2\pi \int_0^1 \left(\frac{re^{-r^2} - r^3e^{-r^2}}{2}\right)\mathrm{d}r \\ &= 2\pi\frac{1}{4e} = \frac{\pi}{2e}. \end{aligned}$$

EXERCISE 26C

Exercise. *Compute the integral* $I = \iiint_W (x+y-2z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *using cylindrical coordinates, where* $W = \{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2 \le z^2, 0 \le z \le 3\}$.

Resolution: In cylindrical coordinates, take B such that $T(B) = W$:

$$B = \{(r,\theta,z)\colon 0 \le \theta \le 2\pi, 0 \le r \le 3, r \le z \le 3\},$$

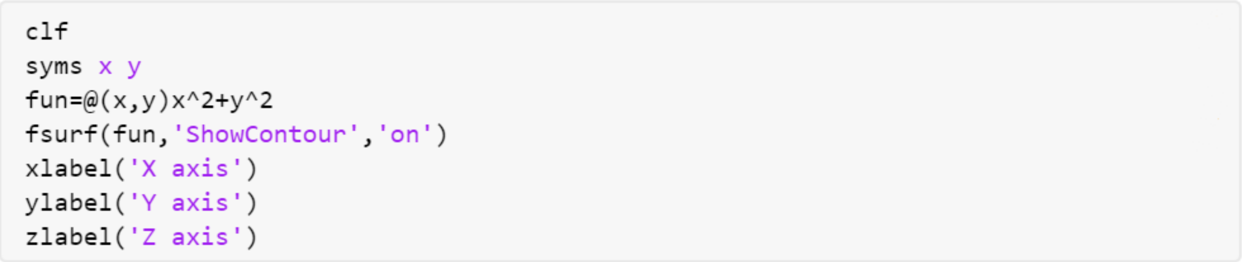
Applying the cylindrical change we obtain:

$$\begin{aligned} V(W) &= \iiint\limits_B (r\cos(\theta) + r\sin(\theta) - 2z)r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = \int\limits_0^{2\pi}\mathrm{d}\theta \int\limits_0^3 \mathrm{d}r \int\limits_r^3 (r^2\cos(\theta) + r^2\sin(\theta) - 2zr)\,\mathrm{d}z \\ &= \int\limits_0^{2\pi}\mathrm{d}\theta \int\limits_0^3 \left[r^2\cos(\theta)z + r^2\sin(\theta)z - z^2r\right]_{z=r}^{z=3}\mathrm{d}r \\ &= \int\limits_0^{2\pi}\mathrm{d}\theta \int\limits_0^3 (3r^2\cos(\theta) + 3r^2\sin(\theta) - 9r - r^3\cos(\theta) - r^3\sin(\theta) + r^3)\,\mathrm{d}r \\ &= \int\limits_0^{2\pi}\left[r^3\cos(\theta) + r^3\sin(\theta) - \frac{9r^2}{2} - \frac{r^4\cos(\theta)}{4} - \frac{r^4\sin(\theta)}{4} + \frac{r^4}{4}\right]_{r=0}^{r=3}\mathrm{d}\theta \\ &= \int\limits_0^{2\pi} 27\cos(\theta) + 27\sin(\theta) - \frac{81}{2} - \frac{81\cos(\theta)}{4} - \frac{81\sin(\theta)}{4} + \frac{81}{4}\,\mathrm{d}\theta \\ &= \left[27\sin(\theta) - 27\cos(\theta) - \frac{81}{2}\theta - \frac{81\sin(\theta)}{4} + \frac{81\cos(\theta)}{4} + \frac{81}{4}\theta\right]_0^{2\pi} \\ &= -\frac{81}{2}2\pi + \frac{81}{4}2\pi = -\frac{81\pi}{2}. \end{aligned}$$

EXERCISE 26D

Exercise. *Compute the following triple integral* $I = \iiint_W \left(x^2+y^2\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$*, where* $W = \{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2 \le 3z \le 9\}$*, using cylindrical coordinates.*

Resolution: We write it in cylindrical coordinates, such that $B = T(W)$ and $x^2+y^2 = r^2$. As we can move between all quadrants, $B = \{(r,\theta,z) : r^2 \le 3z \le 9, 0 \le \theta \le 2\pi\}$. Now we write both z and r so that they are inside a determined interval. In this case, we write r in terms of z. $B = \{(r,\theta,z) : 0 \le z \le 3, 0 \le \theta \le 2\pi, 0 \le r \le \sqrt{3z}\}$. Also, recall that $\iiint_{W=T(B)} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z =$



```
clf
syms x y
fun=@(x,y)x^2+y^2
fsurf(fun,'ShowContour','on')
xlabel('X axis')
ylabel('Y axis')
zlabel('Z axis')
```

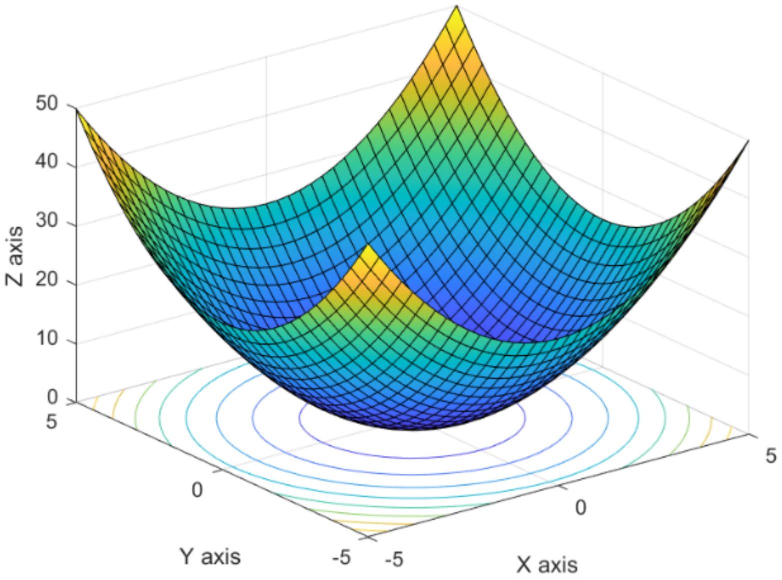


FIGURE 70. Integrand of Exercise 26d plotted with Matlab.

$\iiint_B f(r\cos\theta, r\sin\theta, z)\cdot r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$. Therefore

$$
\begin{aligned}
I &= \iiint_W (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_B r^2\cdot r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta\int_0^3\mathrm{d}z\int_0^{\sqrt{3z}} r^3\,\mathrm{d}r \\
&= 2\pi\int_0^3 \mathrm{d}z\left[\frac{r^4}{4}\right]_{r=0}^{r=\sqrt{3z}} = 2\pi\int_0^3 \frac{\left(\sqrt{3z}\right)^4}{4}-\frac{0}{4}\,\mathrm{d}z = 2\pi\int_0^3\frac{9z^2}{4}\,\mathrm{d}z \\
&= \frac{9\cdot 2\pi}{4}\left[\frac{z^3}{3}\right]_{z=0}^{z=3} = \frac{9\pi}{2}\cdot\frac{3^3}{3} = \frac{9\cdot 9\pi}{2} = \frac{81\pi}{2}.
\end{aligned}
$$

EXERCISE 26E

Exercise. *Compute the triple integral* $I = \iiint_W z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *using cylindrical coordinates, where*

$$W = \left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le 6,\ x^2+y^2\le z,\ z\ge 0\right\}.$$

Resolution: The two surfaces that restrict our domain are $x^2+y^2+z^2=6$ which is a sphere of radius R= $\sqrt{6}$ and center =(0,0,0), and $x^2+y^2\le z$, which is a paraboloid with center = (0,0,0). Moreover, there is a third restriction by the plane $z=0$ which is a consequence of the two first ones. It is worth saying that the domain W specified is the inner part of both the sphere and the paraboloid, see the figure.

From this visual illustration, we can write now W in an equivalent way

$$W = \left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le z\le\sqrt{6-x^2-y^2}\right\}.$$

As the exercise indicates us, we may transform to cylindrical coordinates $(x,y,z) = T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$, which satisfy $|\mathrm{det}DT(x,y)| = r$.

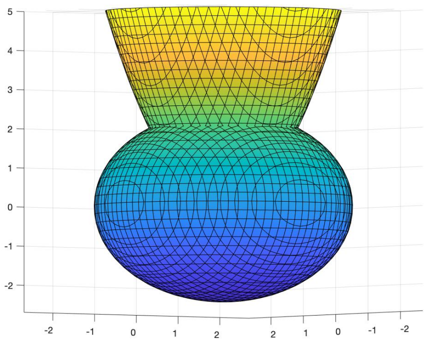


FIGURE 71. Domain W in 2D

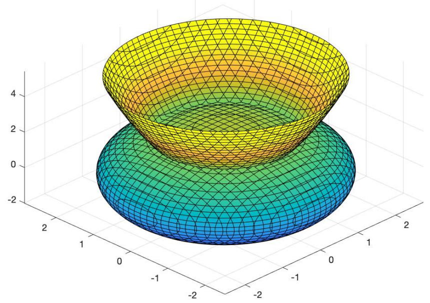


FIGURE 72. Domain W in 3D

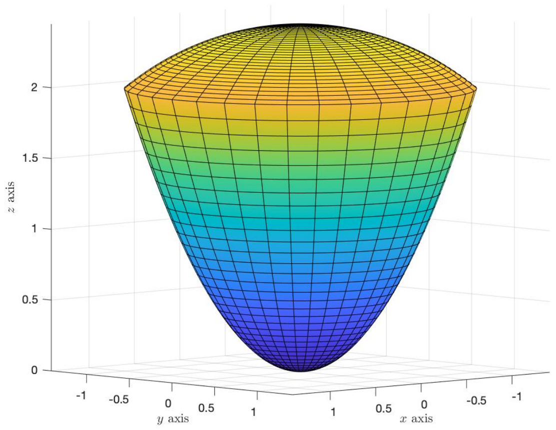


FIGURE 73. Domain W from Exercise 26e.

To proceed with the exercise, we need to fix the limits of integration of r, θ and z. The last one will be a function of the others. If we focus on θ, we see that with respect to the z axis this asked volume is symmetric and there is no type of restriction in that sense. Then, referring to r, we see that r goes from 0 to a point where the two functions intersect:

$$\left.\begin{aligned} r^2 &= z \\ r^2 + z^2 &= 6 \end{aligned}\right\} \implies z + z^2 = 6 \implies z = 2 \text{ (for } z \geq 0) \implies r^2 + 4 = 6 \implies r^2 = 2 \implies r = \sqrt{2}.$$

We see that they intersect at $z = 2$ on a circle $C = \{(x, y, z) : x^2 + y^2 = 2, z = 2\}$ of radius $\sqrt{2}$. In W, inside this circle r runs from 0 to $\sqrt{2}$.

Therefore, we can state that $W = T(B)$, with

$$B = \left\{(r, \theta, z) : 0 \leq \theta \leq 2\pi, 0 \leq r \leq \sqrt{2}, r^2 \leq z \leq \sqrt{6 - r^2}\right\},$$

We are now ready to compute the integral I over W:

$$I = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{\sqrt{6-r^2}} zr \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\mathrm{d}\theta \int_0^{\sqrt{2}} r\,\mathrm{d}r \int_{r^2}^{\sqrt{6-r^2}} z\,\mathrm{d}z = 2\pi \int_0^{\sqrt{2}} r\left[\frac{z^2}{2}\right]_{z=r^2}^{z=\sqrt{6-r^2}} \mathrm{d}r$$

$$= 2\pi \int_0^{\sqrt{2}} \frac{1}{2} r\left(6-r^2-r^4\right)\mathrm{d}r = 2\pi\int_0^{\sqrt{2}} 3r - \frac{1}{2}r^3 - \frac{1}{2}r^5\,\mathrm{d}r = 2\pi\left[\frac{3}{2}r^2 - \frac{1}{8}r^4 - \frac{1}{12}r^6\right]_{r=0}^{r=\sqrt{2}}$$

$$= 2\pi\left(\frac{3\cdot 2}{2} - \frac{2^2}{8} - \frac{2^3}{12}\right) = 2\pi\left(3 - \frac{1}{2} - \frac{2^3}{12}\right) = 2\pi\left(\frac{36-6-8}{12}\right) = 2\pi\left(\frac{22}{12}\right)$$

$$= 2\pi\frac{11\cdot 2}{3\cdot 2\cdot 2} = \frac{11\pi}{3}.$$

Exercise. Exercise 27a

The change to Spherical coordinates is $(x,y,z) = T(r,\theta,\varphi) := (r\cos\varphi\cos\theta, r\cos\varphi\sin\theta, r\sin\varphi)$. *Check that* $|\det(DT(x,y))| = r^2\cos\varphi$ *and use this change to the volume of the following solids. The ball* $B_r(0)$ *in* $\mathbb{R}^3$ *of radius* $R>0$ *and center at the origin.*

Resolution: First of all, check the determinant of the Jacobian matrix:

$$\begin{vmatrix} \cos\varphi\cos\theta & -r\cos\varphi\sin\theta & -r\sin\varphi\cos\theta \\ \cos\varphi\sin\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi & 0 & r\cos\varphi \end{vmatrix} = (r^2\cos\varphi\sin\varphi\sin^2\theta + r^2\cos\varphi\sin\varphi\cos^2\theta)\sin\varphi$$

$$= (r\sin^2\varphi\cos^2\theta + r\cos^2\varphi\sin^2\theta)r\cos\varphi$$
$$= r^2\cos\varphi(\sin^2\varphi+\cos^2\varphi)$$
$$= r^2\cos\varphi.$$

Then, in spherical coordinates, take B such that $T(B) = W$:

$$W = \{(r,\theta,\varphi)\colon 0\le\theta\le 2\pi, -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}, 0\le r\le R\}.$$

Thus, the volume of the figure is defined by:

$$V(W) = \iiint_{Br}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta\int_{-\pi/2}^{\pi/2}\cos\varphi\,\mathrm{d}\varphi\int_0^R r^2\,\mathrm{d}r.$$

We can calculate each integral separated because the result of each integral is a constant. Then we just multiply the integrals.

$$\int_0^{2\pi}\mathrm{d}\theta = 2\pi, \qquad \int_{-\pi/2}^{\pi/2}\cos\varphi\,\mathrm{d}\varphi = 2, \qquad \int_0^R r^2\,\mathrm{d}r = \frac{R^3}{3},$$

$$V(W) = \int_0^{2\pi}\mathrm{d}\theta\int_{-\pi/2}^{\pi/2}\cos\varphi\,\mathrm{d}\varphi\int_0^R r^2\,\mathrm{d}r = \frac{4\pi R^3}{3}.$$

Exercise 27b

Exercise. *Spherical coordinates are given by* $(x,y,z) = T(r,\theta,\phi)\colon = (r\cos\phi\cos\theta, r\cos\phi\sin\theta, r\sin\phi)$. *Check that* $|\det(DT(r,\theta,\phi)| = r^2\cos\phi$ *and use this change of variables to compute the volume of the solid* W *in the ball* $r\le a$ *cut by the cone* $\alpha\le\phi\le\pi/2$ *(*$a>0$*,* $0<\alpha<\pi/2$*).*

Resolution: Firstly we compute the determinant of the Jacobian:

$$\begin{aligned}\det(DT(r,\theta,\phi)) = \det\left(\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right) &= \begin{vmatrix}\cos\phi\cos\theta & -r\cos\phi\sin\theta & -r\sin\phi\cos\theta\\ \cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta\\ \sin\phi & 0 & r\cos\phi\end{vmatrix}\\ &= r^2\cos^3\phi\cos^2\theta + r^2\cos\phi\sin^2\theta\sin^2\phi +^2\sin^2\phi\cos^2\theta\cos\phi + r^2\cos^3\phi\sin^2\theta\\ &= r^2[\cos^3\phi(\sin^2\theta+\cos^2\theta) + \cos\phi\sin^2\phi(\sin^2\theta+\cos^2\theta)] = r^2\cos\phi(\cos^2\phi+\sin^2\phi)\\ &= r^2\cos\phi > 0 \text{ as } -\pi/2 \le \phi \le \pi/2,\end{aligned}$$

so that $|\det(DT(r,\theta,\phi)| = \left|r^2\cos\phi\right| = r^2\cos\phi$.

Imposing the inequalities defining W, from $r < a$ and $a > 0$, we get $0 \le r \le a$, the angle ϕ goes from α to $\pi/2$, so $\alpha \le \phi \le \pi/2$. As there are no more restrictions, the angle θ goes from 0 to 2π, so $0 \le \theta \le 2\pi$. The solid in spherical coordinates is a parallelepiped, so we can use Fubini theorem in any order to compute the integral:

$$\begin{aligned}V(W) = \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^{2\pi}\int_\alpha^{\pi/2}\int_0^a r^2\cos\phi\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta\\ &= \int_0^{2\pi}\mathrm{d}\theta\int_\alpha^{\pi/2}\cos\phi\,\mathrm{d}\phi\int_0^a r^2\,\mathrm{d}r\\ &= 2\pi\left[\sin\phi\right]_{\phi=\alpha}^{\phi=\pi/2}\left[\frac{r^3}{3}\right]_0^a = \frac{2\pi a^3}{3}(1-\sin\alpha).\end{aligned}$$

Exercise 28a

Exercise. *Compute the following triple integral using spherical coordinates:*

$$I = \iiint_W x^4y^2z^3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \text{ where } W = \left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^{^2}+z^2 \le a^2\right\}$$

Resolution: As we see that the structure $x^2+y^2+z^2$ is familiar to us and also because the statement says it, we will solve the integral using spherical coordinates. For this reason : $r^2 = x^2+y^2+z^2$. The first step that we will make is redefine the domain:

$$B = \left\{(\rho,\theta,\varphi):\ 0\le\rho\le a\ ;\ 0\le\theta\le 2\pi\ ;\ \frac{-\pi}{2}\le\varphi\le\frac{\pi}{2}\right\}$$

Then, we look for the equivalence between the old variables and those of now:

$$(x,y,z) = (\rho\cos\varphi\cos\theta,\ \rho\cos\varphi\sin\theta,\ \rho\sin\varphi)$$

and also we must change the differentials:

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2 cos\varphi\ \ \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

Now we can proceed to the computation of the integral:

$$\begin{aligned}I &= \iiint_B \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi\ \rho^2\cos\varphi\,(\rho\cos\varphi\cos\theta)^4\,(\rho\cos\varphi\sin\theta)^2\,(\rho\sin\varphi)^3 =\\ &= \iiint_B \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi\ \left(\rho^2\cos\varphi\rho^4\cos^4\varphi\cos^4\theta\rho^2\cos^2\varphi\sin^2\theta\rho^3\sin^3\varphi\right) =\\ &= \iiint_B \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi\ \left(\rho^{11}\cos^7\varphi\cos^4\theta\sin^2\theta\sin^3\varphi\right) =\\ &= \int_0^{2\pi}\mathrm{d}\theta\left(\cos^4\theta\sin\theta\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\varphi\left(\cos^7\varphi\sin^3\varphi\right)\int_0^a\mathrm{d}\rho\left(\rho^{11}\right) =\\ &= \left[\frac{\cos^5\theta}{5}\right]_{\theta=0}^{\theta=2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\varphi\left(\cos^7\varphi\sin^3\varphi\right)\left[\frac{\rho^{12}}{12}\right]_{x=0}^{x=a} =\\ &= \left(\frac{1}{5}-\frac{1}{5}\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\varphi\left(\cos^7\varphi\sin^3\varphi\right)\left[\frac{\rho^{12}}{12}\right]_{x=0}^{x=a} =\\ &= (0)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\varphi\left(\cos^7\varphi\sin^3\varphi\right)\left[\frac{\rho^{12}}{12}\right]_{x=0}^{x=a} = 0.\end{aligned}$$

Exercise 28b

Exercise. *Compute the following triple integral using spherical coordinates:* $\iiint_W z(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, *with* $W=\{(x,y,z)\in\mathbb{R}^3\colon x^2+y^2+z^2\le a^2,\ z\ge 0\}$.

Resolution:

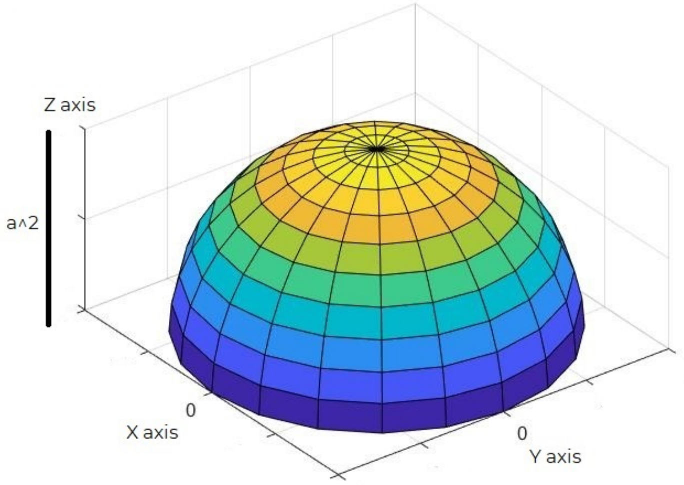


FIGURE 74. from Exercise 28b): domain in x, y, z coordinates

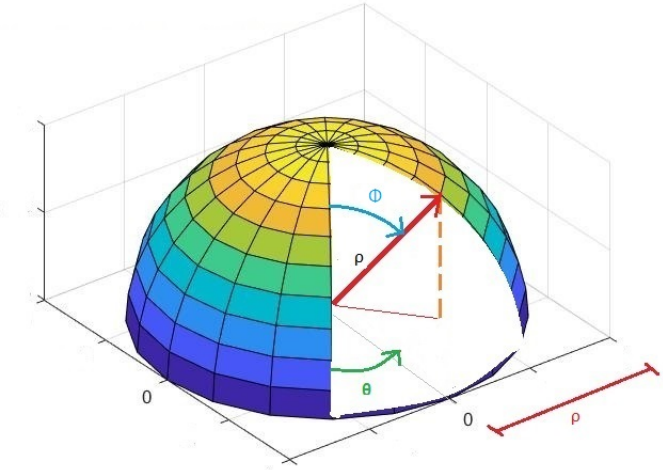


FIGURE 75. from Exercise 28b): domain in $\rho\ \theta\ \phi$ coordinates

In spherical coordinates, take B such that $T(B)=W$:

$$B=\{(\rho,\theta,\phi)\colon 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{2}, 0\le\rho\le a\}$$

Then:

$$I=\iiint_B \rho\sin\phi\left(\rho^2\cos^2\phi\cos^2\theta+\rho^2\cos^2\phi\sin^2\theta\right)\rho^2\cos\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/2}\sin\phi\cos^3\phi\,\mathrm{d}\phi\int_0^{a}\rho^5\,\mathrm{d}\rho$$

$$=\frac{1}{12}\pi a^6$$

where $\int_0^{\frac{\pi}{2}}\sin\phi\cos^3\phi\,\mathrm{d}\phi$ is integrated by parts using: $u=\cos^3\phi$; $\mathrm{d}u=-\sin\phi\,\mathrm{d}\phi$

Exercise 28c

Exercise. *Compute the following triple integral using spherical coordinates:*
$$I = \iiint_W \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(x^2+y^2+z^2)^{3/2}}, \text{ where } W = \{(x,y,z) \in \mathbb{R}^3 \colon a^2 \le x^2+y^2+z^2 \le b^2\} .$$

Resolution: In spherical coordinates, take B such that $T(B) = W$:

$$B = \{(\rho,\theta,\phi) \colon a \le \rho \le b, 0 \le \theta \le 2\pi, -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}\}.$$

Then, since B is the parallelepiped $[a,b] \times [0,2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$I = \int\limits_0^{2\pi} \mathrm{d}\theta \int\limits_{-\pi/2}^{\pi/2} \cos\phi \,\mathrm{d}\phi \int\limits_a^b \frac{1}{\rho}\,\mathrm{d}\rho = 2\pi\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{-\pi}{2}\right)\right]\int_a^b \frac{1}{\rho}\,\mathrm{d}\rho = 4\pi \ln\frac{b}{a}.$$

Exercise 28d

Exercise. *Compute* $I = \iiint_W \sqrt{x^2+y^2+z^2}\,\mathrm{e}^{-(x^2+y^2+z^2)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, *where* $W = \{(x,y,z) \in \mathbb{R}^3 \colon a^2 \le x^2+y^2+z^2 \le b^2\}$.

Resolution: In spherical coordinates, take B such that $T(B) = W$:

$$B = \{(\rho,\theta,\phi) \colon 0 \le \theta \le 2, -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}, a \le \rho \le b\}.$$

Then

$$\begin{aligned} I &= \iiint\limits_B \rho\,\mathrm{e}^{-\rho^2}\rho^2 \cos\phi \,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta = \int\limits_0^{2\pi} \mathrm{d}\theta \int\limits_{-\pi/2}^{\pi/2} \cos\phi\,\mathrm{d}\phi \int\limits_a^b \rho^3\,\mathrm{e}^{-\rho^2}\,\mathrm{d}\rho \\ &= 2\pi(\,\mathrm{e}^{-a^2}(a^2+1) - \,\mathrm{e}^{-b^2}(b^2+1)) \end{aligned}$$

where $\int_a^b \rho^3\,\mathrm{e}^{-\rho^2}\,\mathrm{d}\rho = \left(\mathrm{e}^{-a^2}(a^2+1) - \,\mathrm{e}^{-b^2}(b^2+1)\right)/2$ is integrated by parts $u = -\rho^2/2$, $\mathrm{d}v = -2\rho\,\mathrm{e}^{-\rho^2}\,\mathrm{d}\rho$.

Exercise 29a

Exercise. *Compute the volume of the following domain* $W \subset \mathbb{R}^3$ *defined in spherical coordinates as the sphere* $r \le a$ *cut by the cone* $\alpha \le \phi \le \frac{\pi}{2}$, $(a \ge 0,\ 0 \le \alpha \le \frac{\pi}{2})$.

Resolution. By the statement we know the following:

$$W = \left\{(x,y,z) \in \mathbb{R}^3 \colon x^2+y^2+z^2 \le a^2, \alpha \le \phi \le \frac{\pi}{2}\right\}.$$

As we have to solve the exercise in spherical coordinates we will take B such that $T(B) = W$:

$$B = \left\{(\rho,\theta,\phi) \colon 0 \le \rho \le a, \alpha \le \phi \le \frac{\pi}{2}, 0 \le \theta \le 2\pi\right\}.$$

Then, we have to compute the following triple integral to find the volume (taking into account that the Jacobian for a transformation of volume elements between Cartesian and spherical coordinates is $\rho^2\cos\phi$):

$$\begin{aligned} V(W) &= \iiint\limits_B \rho^2\cos\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta = \int\limits_0^{2\pi}\mathrm{d}\theta \int\limits_\alpha^{\pi/2} \cos\phi\,\mathrm{d}\phi \int\limits_0^a \rho^2\,\mathrm{d}\rho = 2\pi \int\limits_\alpha^{\pi/2} \cos\phi\,\mathrm{d}\phi \int\limits_0^a \rho^2\,\mathrm{d}\rho \\ &= 2\pi\left[\frac{\sin\phi}{1}\right]_{\phi=\alpha}^{\phi=\frac{\pi}{2}} \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=a} = 2\pi(1-\sin\alpha)\frac{a^3}{3} = \frac{2\pi a^3}{3}(1-\sin\alpha). \end{aligned}$$

Exercise 29b

Exercise. *Compute the volume of the domain enclosed by the deformed sphere which is defined by* $r = 1 + 0.2\sin(8\theta)\sin\varphi$ *in spherical coordinates. (This type of solids are used as models of tumours, unfortunately.)*

Resolution: First of all, we have to define the limits of integration. Since our domain is defined in spherical coordinates we know that $0 \le \theta \le 2\pi$ and that $\frac{-\pi}{2} \le \varphi \le \frac{\pi}{2}$. We also know that the variable r is limited by $r = 1 + 0.2\sin(8\theta)\sin\varphi$ and that this expression is always greater than 0 because $0.2\sin(8\theta)\sin\varphi$ cannot be less than -1.

$$0 \le r \le 1 + 0.2\sin(8\theta)\sin\varphi, \quad \frac{-\pi}{2} \le \varphi \le \frac{\pi}{2}, \quad 0 \le \theta \le 2\pi.$$

Once we have defined the limits of integration, note that as the domain is defined in spherical coordinates we have to integrate the jacobian for spherical coordinates, which is $r^2\cos\varphi$.

Taking all this into account, in order to obtain the volume of the given domain, we have to compute the following integral:

$$\begin{aligned}
V &= \int_0^{2\pi} \mathrm{d}\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \,\mathrm{d}\varphi \int_0^{1+0.2\sin(8\theta)\sin\varphi} r^2\,\mathrm{d}r \\
&= \int_0^{2\pi} \mathrm{d}\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^{1+0.2\sin(8\theta)\sin\varphi} \cos\varphi\,\mathrm{d}\varphi \\
&= \frac{1}{3}\int_0^{2\pi} \mathrm{d}\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi(1 + 0.2\sin(8\theta)\sin\varphi)^3\,\mathrm{d}\varphi \\
&= \frac{1}{3}\int_0^{2\pi} \mathrm{d}\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\cdot 1^3 + \cos\varphi\cdot 3\cdot 1^2\cdot 0.2\sin(8\theta)\sin\varphi \\
&\qquad + \cos\varphi\cdot 3\cdot 1\cdot 0.2^2\sin^2(8\theta)\sin^2\varphi + \cos\varphi\cdot 0.2^3\sin^3(8\theta)\sin^3\varphi\,\mathrm{d}\varphi \\
&= \frac{1}{3}\int_0^{2\pi} \mathrm{d}\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi + 0.6\sin(8\theta)\sin\varphi\cos\varphi + 0.12\sin^2(8\theta)\sin^2\varphi\cos\varphi \\
&\qquad + 0.008\sin^3(8\theta)\sin^3\varphi\cos\varphi\,\mathrm{d}\varphi \\
&= \frac{1}{3}\int_0^{2\pi} \left[\sin\varphi + 0.6\sin(8\theta)\frac{\sin^2\varphi}{2} + 0.12\sin^2(8\theta)\frac{\sin^3\varphi}{3} + 0.008\sin^3(8\theta)\frac{\sin^4\varphi}{4}\right]_{\varphi=-\frac{\pi}{2}}^{\varphi=\frac{\pi}{2}} \mathrm{d}\theta \\
&= \frac{1}{3}\int_0^{2\pi} 1 + 0.3\sin(8\theta)\cdot 1^2 + 0.04\sin^2(8\theta)\cdot 1^3 + 0.002\sin^3(8\theta)\cdot 1^4 \\
&\qquad - (-1) - 0.3\sin(8\theta)\cdot(-1)^2 - 0.04\sin^2(8\theta)\cdot(-1)^3 - 0.002\sin^3(8\theta)\cdot(-1)^4\,\mathrm{d}\theta \\
&= \frac{1}{3}\int_0^{2\pi} 2 + 2\cdot 0.04\sin^2(8\theta)\,\mathrm{d}\theta \overset{(*)}{=} \frac{1}{3}\left[2\theta + 0.08\left(\frac{1}{2}\theta - \frac{1}{32}\sin(16\theta)\right)\right]_0^{2\pi} \\
&= \frac{1}{3}\left(2\cdot 2\pi + 0.08\left(\frac{1}{2}\cdot 2\pi\right)\right) = \frac{1}{3}\left(2\cdot 2\pi + 0.08\left(\frac{1}{2}\cdot 2\pi\right)\right) = \frac{4.08}{3}\pi = 1.36\pi.
\end{aligned}$$

$(*)$ To compute $\int \sin^2(8\theta)\,\mathrm{d}\theta$ we have applied the change of variables $t = 8\theta$, $\mathrm{d}t = 8\,\mathrm{d}\theta$ to get

$$\begin{aligned}
\int \sin^2(8\theta)\,\mathrm{d}\theta &= \int \sin^2 t\frac{1}{8}\,dt = \frac{1}{8}\int \frac{1}{2}(1 - \cos(2t))\,dt = \frac{1}{16}\left(t - \frac{\sin(2t)}{2}\right) \\
&= \frac{1}{16}\left(8\theta - \frac{\sin(16\theta)}{2}\right) = \frac{1}{2}\theta - \frac{1}{32}\sin(16\theta).
\end{aligned}$$

EXERCISE 29C

Exercise. *Compute the triple integral* $\iiint_W \frac{1}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$*, where* W *is the region in the first octant of* $\mathbb{R}^3$ *bounded by the cones* $\varphi = \frac{\pi}{4}$ *and* $\varphi = \arctan(2)$ *and the sphere* $r = \sqrt{6}$*. (Recall:* $\sin(\arctan(a)) = \frac{a}{\sqrt{1+a^2}}$*)).*

Resolution: In spherical coordinates, take B such that $T(B) = W$:

$$B = \{(\rho,\theta,\phi)\colon 0 \le \theta \le \frac{\pi}{2}, \frac{\pi}{4} \le \phi \le \arctan(2), 0 \le \rho \le \sqrt{6}\}.$$

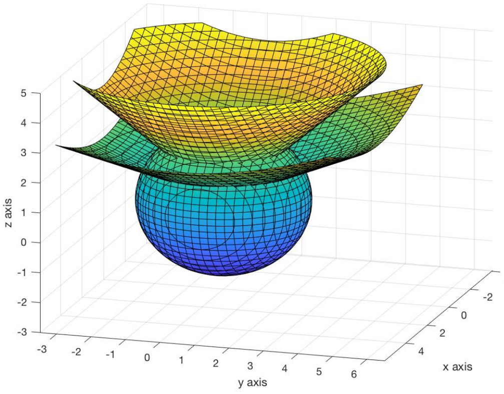


FIGURE 76. The figure formed by the two cones and the sphere of Exercise 29c

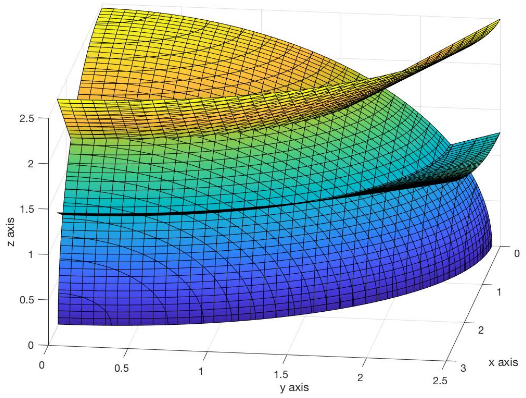


FIGURE 77. W in the first octant of $\mathbb{R}^3$ from Exercise 29c

Then

$$\begin{aligned} I &= \iiint\limits_B \frac{1}{\rho}\rho^2 \cos\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\pi/4}^{\arctan(2)} \cos\phi\,\mathrm{d}\phi \int_0^{\sqrt{6}} \rho\,\mathrm{d}\rho \\ &= [\theta]_0^{\frac{\pi}{2}} \cdot [\sin\phi]_{\frac{\pi}{4}}^{\arctan 2} \cdot \left[\frac{\rho^2}{2}\right]_0^{\sqrt{6}} = \frac{\pi}{2} \cdot \left(\sin(\arctan(2)) - \sin\frac{\pi}{4}\right) \cdot 3 \\ &= \mathbf{\frac{3\pi}{2}\left(\frac{2}{\sqrt{5}} - \frac{\sqrt{2}}{2}\right)} \end{aligned}$$

EXERCISE 30A

Exercise. *Adapt the spherical coordinates in order to compute the following triple integral:* $I = \iiint_W 16z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$*, where* $W = \{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2+\frac{z^2}{4} \le 1, 0 \le z \le 1, 0 \le y \le x\}$*.*

Resolution: $W = \{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2+\frac{z^2}{4} \le 1, 0 \le z \le 1, 0 \le y \le x\}$ in cylindrical coordinates takes the form $B = \left\{(r,\theta,z)\colon 0 \le \theta \le \frac{\pi}{4}, 0 \le z \le 1, 0 \le r \le \sqrt{1-\frac{z^2}{4}}\right\}$, so:

$$\begin{aligned} V(W) &= \int\limits_0^{\pi/4}\int\limits_0^1\int\limits_0^{\sqrt{1-\frac{z^2}{4}}} 16zr\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta = 16\int\limits_0^{\pi/4}\mathrm{d}\theta\int\limits_0^1 z\cdot\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{1-\frac{z^2}{4}}}\mathrm{d}z \\ &= 8\cdot\frac{\pi}{4}\int\limits_0^1 z - \frac{z^3}{4}\,\mathrm{d}z = 2\pi\cdot\left(\left[\frac{z^2}{2}-\frac{z^4}{16}\right]_{z=0}^{z=1}\right) = 2\pi\cdot\left(\frac{1}{2}-\frac{1}{16}\right) = \frac{7\pi}{8}. \end{aligned}$$

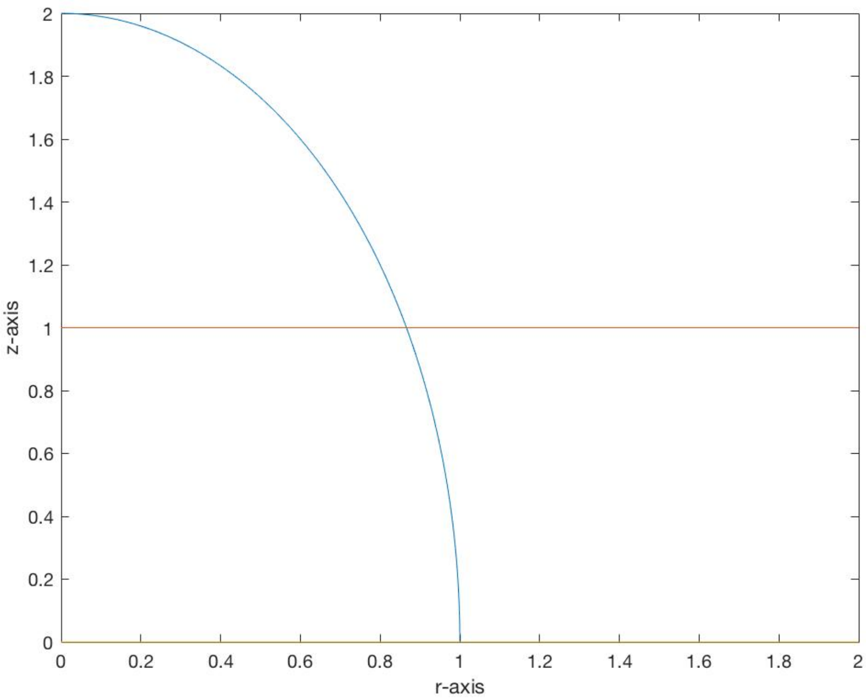


FIGURE 78. Solid W of Exercise 30a in the (r, z) plane

EXERCISE 30B

Exercise. *Adapt the spherical coordinates in order to compute* $I = \iiint_W \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, *where* $W = \left\{(x,y,z) \in \mathbb{R}^3 \colon \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1\right\}$.

Resolution: The coordinates must be changed from cartesian to adapted spherical

$$(x,y,z) = T(r,\theta,\phi) = (ar\cos\theta\cos\phi, br\sin\theta\cos\phi, cr\sin\phi),$$

for which we know (see Exercise 19b) that $|\det DT(x,y)| = abcr^2\cos\phi$.

As the ellipsoid is a sphere-like object, the angles θ and ϕ have no restriction, so their limits are between $0 and 2\pi$ and between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, respectively. As $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = r^2$, the value of r in W goes from 0 to 1. Thus, we have $W = T(B)$, where

$$B = \left\{(r,\theta,\phi)\colon 0 \le \theta \le 2\pi, -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}, 0 \le r \le 1\right\}.$$

We can now compute the triple integral:

$$\begin{aligned} I &= \iiint_W \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_B r^2 \cdot abcr^2\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= abc\int_0^{2\pi} \mathrm{d}\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi\,\mathrm{d}\phi \int_0^1 r^4\,\mathrm{d}r = \frac{4\pi}{5}abc. \end{aligned}$$

EXERCISE 31A

Exercise. *Compute the volume of the domain* W *of* $\mathbb{R}^3$ *bounded by the surfaces* $x^2+z^2 = 1$, $x^2+y^2 = 1$.

Resolution: $W = \{x,y,z) : x^2+z^2 \le 1, x^2+y^2 \le 1\} = \{(x,y,z)\colon\ -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, -\sqrt{1-x^2} \le z \le \sqrt{1-x^2}\}$, therefore

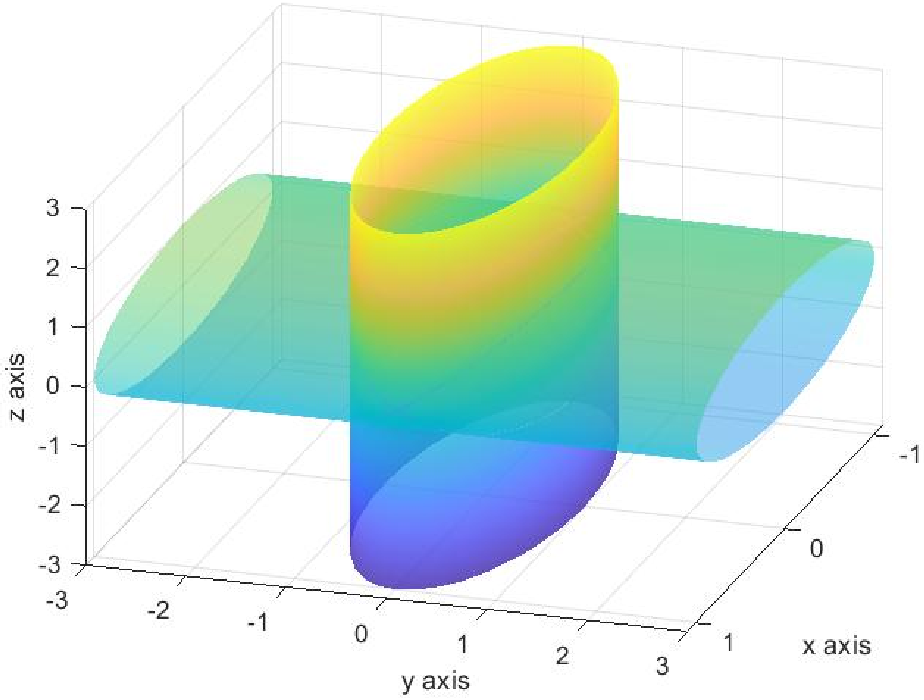


FIGURE 79. W from Exercise 28d: a pipe junction

$$\begin{aligned} V(W) &= \iiint\limits_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int\limits_{-1}^{1} \mathrm{d}x \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \mathrm{d}y \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \mathrm{d}z \\ &=4\int_{-1}^{1}(1-x^2)\,\mathrm{d}x = 16/3. \end{aligned}$$

Second Resolution: Looking at the figure from the x-axis, we notice that we can apply Cavalieri's principle. Slicing

$$W = \{x, y, z) : x^2 + z^2 \leq 1, x^2 + y^2 \leq 1\}$$

along the x-axis, the cross-section is:

$$W_x = \{(y, z) : y^2 \leq 1 - x^2, z^2 \leq 1 - x^2\},$$

that is, just a **square** with $A(x) = 4(1 - x^2)$.

Therefore

$$V(W) = 4\int_{-1}^{1}(1-x^2)\,\mathrm{d}x = 16/3.$$

EXERCISE 31B

Exercise. *Compute the volume of the domain* W *of* $\mathbb{R}^3$ *bounded by the surfaces* $z^2-x^2=1$, $z^2-y^2=1$, $z=\sqrt{2}$.

Resolution: $z^2 = 1 + x^2 \Rightarrow |z| \geq 1$, so

$$\begin{aligned} W &= \{(x, y, z)\colon z^2 - x^2 \geq 1, z^2 - y^2 \geq 1, z \leq \sqrt{2}\} \\ &= \{(x, y, z)\colon 1 \leq z \leq \sqrt{2}, -\sqrt{z^2-1} \leq x, y \leq \sqrt{z^2-1}\}, \end{aligned}$$

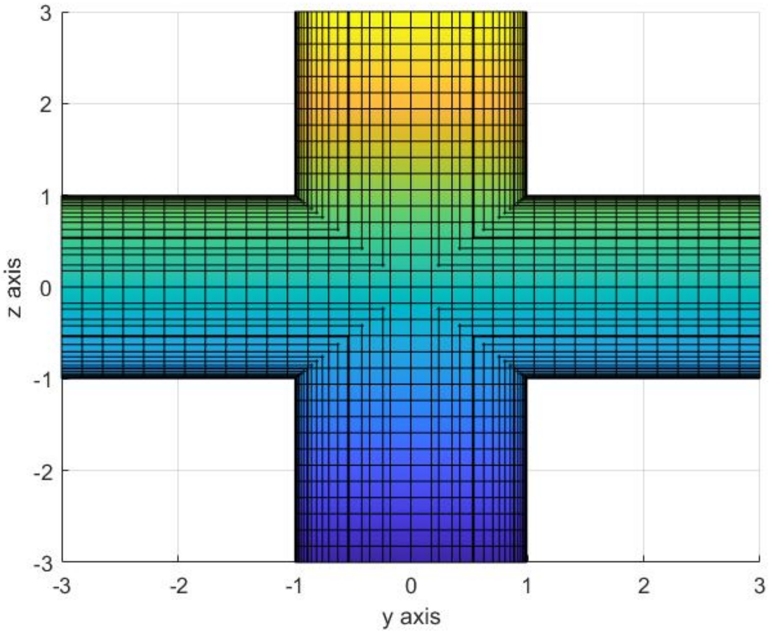


FIGURE 80. W from Exercise 28d viewed from the x-axis

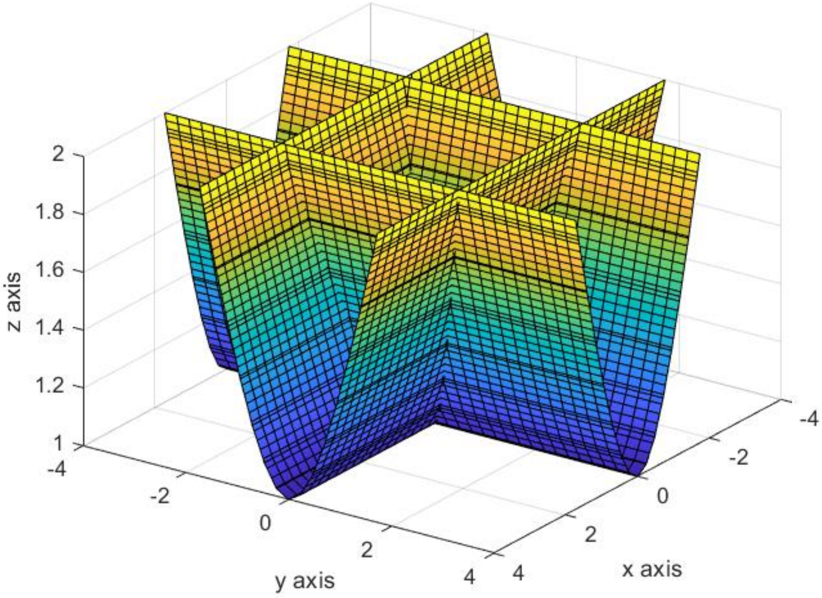


FIGURE 81. W of Exercise 31b

and

$$\begin{aligned} V(W) &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int\limits_1^{\sqrt{2}} \mathrm{d}z \int\limits_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} \mathrm{d}x \int\limits_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} \mathrm{d}y \\ &=4\int_1^{\sqrt{2}} (z^2-1)\,\mathrm{d}z = \frac{4}{3}(2-\sqrt{2}). \end{aligned}$$

EXERCISE 31C

Exercise. *Compute the volume of the domain W of $\mathbb{R}^3$ bounded by the surfaces $z^2 = x^2 + y^2$, $z = x^2 + y^2$, $z \geq 0$.*

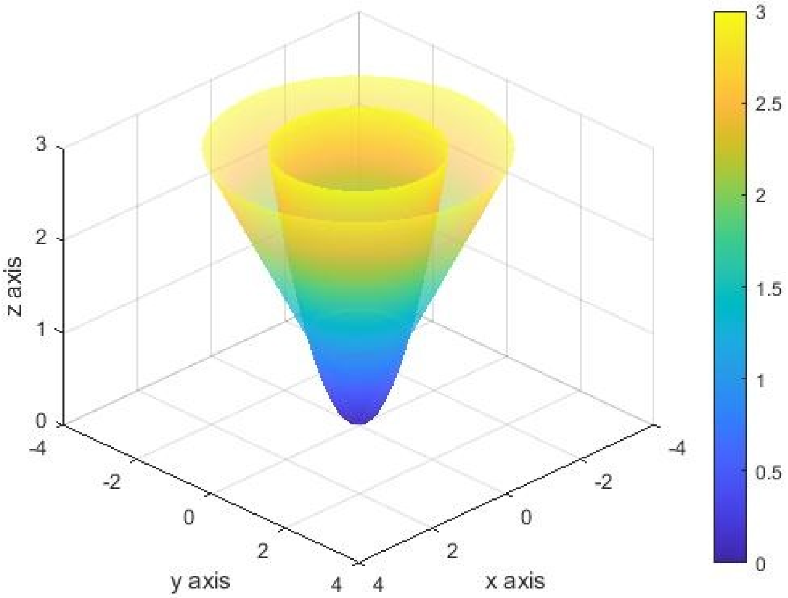


FIGURE 82. $V(W)$ from Exercise 31c

Resolution: $W = \left\{(x, y, z) \in \mathbb{R}^3 \colon z^2 \leq x^2 + y^2 \leq z, z \geq 0\right\}$, which in cylindrical coordinates takes the form

$$B = \{(r, \theta, z) \colon 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, r^2 \leq z \leq r\}.$$

Therefore,

$$V(W) = \iiint\limits_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_B r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$
$$= \int_0^{2\pi} \mathrm{d}\theta \int_0^1 r\,\mathrm{d}r \int_{r^2}^{r} \mathrm{d}z = 2\pi \int_0^1 (r^2 - r^3)\,\mathrm{d}r = \frac{\pi}{6}.$$

EXERCISE 31D

Exercise. *Use cartesian, cylindrical or spherical coordinates (or Cavalieri's principle if needed) so as to compute the volume of the domains of $\mathbb{R}^3$ bounded by the surfaces indicated below. (d) A part of the sphere $x^2 + y^2 + z^2 = a^2$ that is external to the cylinder $x^2 + y^2 = b^2$ $(a > b > 0)$.*

Resolution: In cylindrical coordinates, take B such that $T(B) = W$:

$$B = \{(r, \theta, z) \colon 0 \leq \theta \leq 2\pi, b \leq r \leq a, -\sqrt{a^2 - r^2} \leq z \leq \sqrt{a^2 - r^2}\},$$
$$T(B) = W = \{(x, y, z) \colon x^2 + y^2 + z^2 \leq a^2, x^2 + y^2 \geq b^2\}.$$

Then

$$V(W) = \iiint\limits_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_B r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \int_0^{2\pi} \mathrm{d}\theta \int_b^a r\,\mathrm{d}r \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} \mathrm{d}z$$
$$= 4\pi \int_b^a \sqrt{a^2 - r^2} r\,\mathrm{d}r = 4\pi \left[-\frac{(a^2 - r^2)^{3/2}}{3}\right]_{r=b}^{r=a} = \frac{4\pi}{3}(a^2 - b^2)^{3/2}.$$

Alternatively, we can compute the cylinder's volume and substract it from the sphere's volume

$$B^* = \{(r,\theta,z)\colon 0 \le \theta \le 2\pi, 0 \le r \le b, -\sqrt{a^2-r^2} \le z \le \sqrt{a^2-r^2}\},$$

$$V(W) = \iiint\limits_{W} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_{B^*} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \int\limits_0^{2\pi} \mathrm{d}\theta \int\limits_0^{b} r\,\mathrm{d}r \int\limits_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} \mathrm{d}z$$

$$= 4\pi \int\limits_0^{b} r\sqrt{a^2-r^2}\,\mathrm{d}r = 4\pi \left[-\frac{(a^2-r^2)^{3/2}}{3}\right]_{r=0}^{r=b} = \frac{-4\pi}{3}(a^2-b^2)^{3/2} + \frac{4\pi}{3}a^3.$$

Sphere's Volume:

$$\frac{4\pi}{3}a^3,$$

Total Volume:

$$\frac{4\pi}{3}a^3 - (\frac{-4\pi}{3}(a^2-b^2)^{3/2} + \frac{4\pi}{3}a^3) = \frac{4\pi}{3}(a^2-b^2)^{3/2}.$$

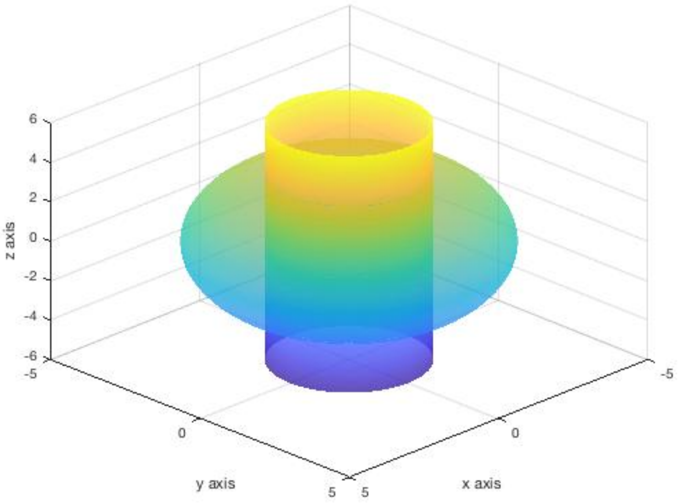


FIGURE 83. 3D perspective of the volume in Exercise 31d.

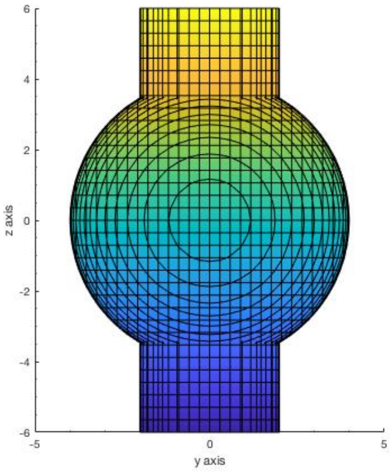


FIGURE 84. W from Exercise 31d viewed from the y,z plane.

EXERCISE 31E

Exercise. *Use cartesian, cylindrical or spherical coordinates (or Cavalieri's principle if needed) so as to compute the volume of the domain W of $\mathbb{R}^3$ bounded by the surfaces $z = x^2 - 4x + 1$, $1 - z = x^2 + y^2$.*

Resolution: The first thing we need to do is to arrange the first equation in an easier way. If we solve the quadratic equation we get to $z + 3 = (x - 2)^2$, so our equations now are:

- $z = -3 + (x - 2)^2$: an increasing parabola in the z direction with vertex $v = (2, -3)$,
- $z = 1 - (x^2 + y^2)$: a decreasing paraboloid in the z direction with vertex $v = (0, 0, 1)$.

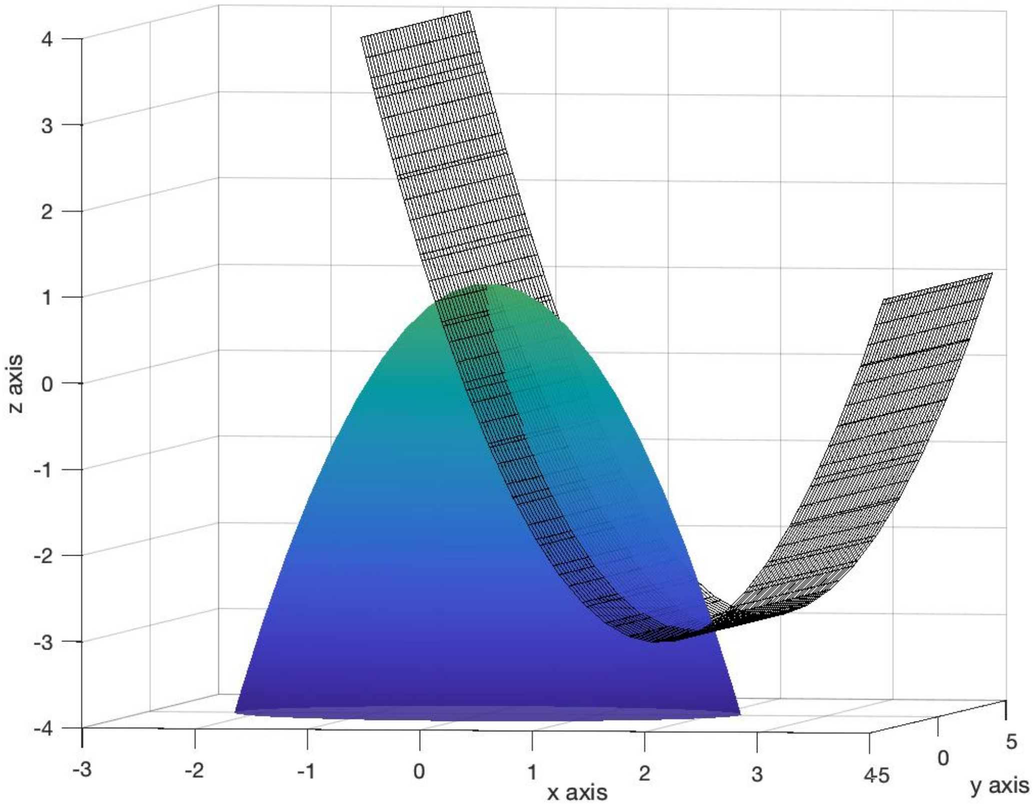


FIGURE 85. Graph for exercise 31 E

Therefore, the points of the domain W are between the increasing parabola and the decreasing paraboloid so they satisfy $-3 + (x - 2)^2 \leq 1 - (x^2 + y^2$ or equivalently $x^2 - 4x + 1 \leq z \leq 1 - x^2 - y^2$. If we calculate the intersection of the equations of the boundary of W we get

$$2x^2 - 4x + y^2 = 0 \iff 2\left(x^2 - 2x + 1 + \frac{y^2}{2} - 1\right) = 0 \iff 2\left((x - 1)^2 + \frac{y^2}{2} - 1\right) = 0,$$

or equivalently

$$(x - 1)^2 + \frac{y^2}{2} = 1,$$

which is an elliptic cylinder with base an ellipse with center $c = (1, 0)$ and semi-axes $a = 1$, $b = \sqrt{2}$. As W is inside this cylinder,

$$W = \{(x, y, z)\colon x^2 - 4x + 1 \leq z \leq 1 - x^2 - y^2, (x - 1)^2 + \frac{y^2}{2} \leq 1\}.$$

As in the variables (x, y) we have an ellipse, we are going to use adapted cylindrical coordinates:

$$(x, y, z) = T(r, \theta, z) = (1 + r\cos\theta, \sqrt{2}r\sin\theta, z), \text{ with } |\det DT(r, \theta, z)| = \sqrt{2}r.$$

With this change $W = T(B)$ and the domain W in these coordinates takes the form

$$B = \{(r, \theta, z)\colon x^2 - 4x + 1 \leq z \leq 1 - x^2 - y^2, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\},$$

where we could write explicitly $x = 1 + r\cos\theta$, $y = \sqrt{2}r\sin\theta$, but it will not be necessary, as we will only use that $-2x^2 + 4x - y^2 = -2\left((x-1)^2 + \frac{y^2}{2} - 1\right) = 2(1-r^2)$:

$$\begin{aligned} V(W) &= \int_0^{2\pi}\int_0^1\int_{x^2-4x+1}^{-x^2-y^1+1} \sqrt{2}r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta == \sqrt{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1\mathrm{d}r\int_{x^2-4x+1}^{-x^2-y^2+1} r\,\mathrm{d}z \\ &= \sqrt{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r(-2x^2+4x-y^2)\,\mathrm{d}r = \sqrt{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 2r(1-r^2)\,\mathrm{d}r \\ &= \sqrt{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 2r - 2r^3\,\mathrm{d}r = 2\pi\sqrt{2}\left(\left[r^2\right]_{r=0}^{r=1} - \left[\frac{r^4}{2}\right]_{r=0}^{r=1}\right) = 2\pi\sqrt{2}\frac{1}{2} = \pi\sqrt{2}. \end{aligned}$$

EXERCISE 31F

Exercise. *Compute the volume of the domain W of $\mathbb{R}^3$ bounded by the surfaces $x^2 = z$, $y^2 = x$, $z^2 = y$, $x^2 = az$, $y^2 = ax$, $z^2 = ay$ ($a > 1$).*

Resolution: We only need to consider the positive octant $x \geq 0$, $y \geq 0$, $z \geq 0$ and inside it $y \leq ay$, $z \leq az$, $x \leq ax$. Therefore

$$\begin{aligned} W &= \{(x,y,z)\colon y \leq z^2 \leq ay, z \leq x^2 \leq az, x \leq y^2 \leq ax\} \\ &= \{(x,y,z)\colon 1 \leq \frac{z^2}{y} \leq a, 1 \leq \frac{x^2}{z} \leq a, 1 \leq \frac{y^2}{x} \leq a\}, \end{aligned}$$

so we try the change $(u,v,w) = \left(\frac{z^2}{y}, \frac{x^2}{z}, \frac{y^2}{x}\right)$ with Jacobian

$$\frac{\partial(u,v,w)}{(x,y,z)} = \begin{pmatrix} 0 & -z^2/y^2 & 2z/y \\ 2x/z & 0 & -x^2/z^2 \\ -y^2/x^2 & 2y/x & 0 \end{pmatrix}$$

and $\det\frac{\partial(u,v,w)}{(x,y,z)} = 7$ so that

$$\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w = \left|\det\frac{\partial(u,v,w)}{(x,y,z)}\right|\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 7\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

or $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{7}\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$. Under this change W is transformed into the parallelepiped

$$B = \{(u,v,w)\colon 1 \leq u \leq a, 1 \leq v \leq a, 1 \leq w \leq a\}$$

with volume $V(B) = (a-1)^3$. Therefore

$$\begin{aligned} V(W) &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{7}\iiint_B \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \\ &= \frac{1}{7}V(B) = \frac{(a-1)^3}{7}. \end{aligned}$$

EXERCISE 31G

Exercise. *Compute the volume of the domain W of $\mathbb{R}^3$ bounded by the surfaces $z^2 = y$, $x^2 = 1 - y$.*

Resolution:

$$\begin{aligned} W &= \{(x,y,z)\colon z^2 \leq y, x^2 \leq 1-y\} \\ &= \{(x,y,z)\colon z^2 \leq y \leq 1 - x^2\} \\ &= \{(x,y,z)\colon \boldsymbol{x^2 + z^2} \leq y + x^2 \leq 1\}, \end{aligned}$$

so we introduce a change similar to cylindrical coordinates:

$$(x,y,z) = T(r,\theta,y) = (r\cos\theta, y, r\sin\theta)$$

The change $(x,y,z) = T(r,\theta,y) = (r\cos\theta, y, r\sin\theta)$ satisfies

$$|\det DT(r,\theta,y)| = r.$$

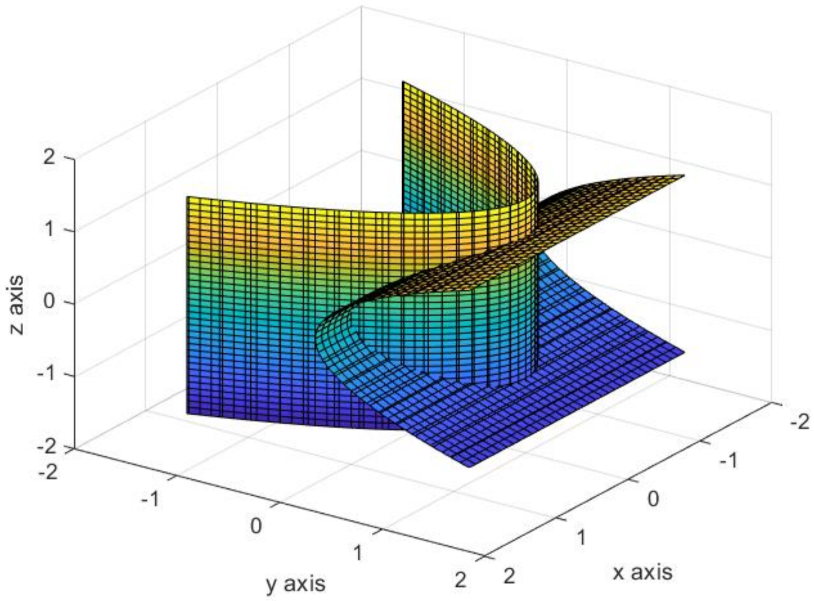


Figure 86. W of Exercise 31g between $z^2 = y$, $x^2 = 1 - y$

In these coordinates the solid $W = \{(x, y, z) \colon \boldsymbol{x^2 + z^2} \leq y + x^2 \leq 1\}$ takes the form

$$\begin{aligned} B &= \{(r, \theta, y) \colon r^2 \leq y + r^2 \cos^2\theta \leq 1\} \\ &= \{(r, \theta, y) \colon 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, r^2 \sin^2\theta \leq y \leq 1 - r^2 \cos\theta\} \end{aligned}$$

so that

$$\begin{aligned} V(W) &= \iiint\limits_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_B r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}y = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 r\,\mathrm{d}r \int_{r^2\sin^2\theta}^{1-r^2\cos^2\theta} \mathrm{d}y \\ &= 2\pi \int_0^1 r(1 - r^2)\,\mathrm{d}r = 2\pi \int_0^1 r - r^3\,\mathrm{d}r = 2\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{2}. \end{aligned}$$

Exercise 31h

Exercise. *Compute the volume of the domain W of $\mathbb{R}^3$ bounded by the surfaces $x^2+y^2 = 1$, $x^2+y^2 = 2$, $z(x^2 + y^2) = 1$, $z = 0$.*

Resolution: $W = \left\{(x, y, z) \colon 1 \leq \boldsymbol{x^2 + y^2} \leq 2, 0 \leq z \leq \dfrac{1}{\boldsymbol{x^2 + y^2}}\right\}$, which in cylindrical coordinates takes the form

$$B = \{(r, \theta, z) \colon 0 \leq \theta \leq 2\pi, 1 \leq r \leq \sqrt{2}, 0 \leq z \leq 1/r^2\}.$$

Therefore

$$\begin{aligned} V(W) &= \iiint\limits_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_B r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}y \\ &= \int_0^{2\pi} \mathrm{d}\theta \int_1^{\sqrt{2}} r\,\mathrm{d}r \int_0^{1/r^2} \mathrm{d}z = 2\pi \int_1^{\sqrt{2}} \frac{\mathrm{d}r}{r} = \pi \ln 2. \end{aligned}$$

Exercise 31i

Exercise. *Compute the volume of the domain W of $\mathbb{R}^3$ bounded by the surfaces $x^2 + y^2 = 2z^2$, $x^2 + y^2 = z^2 + 1$ ($x \geq 0$, $y \geq 0$, $z \geq 0$).*

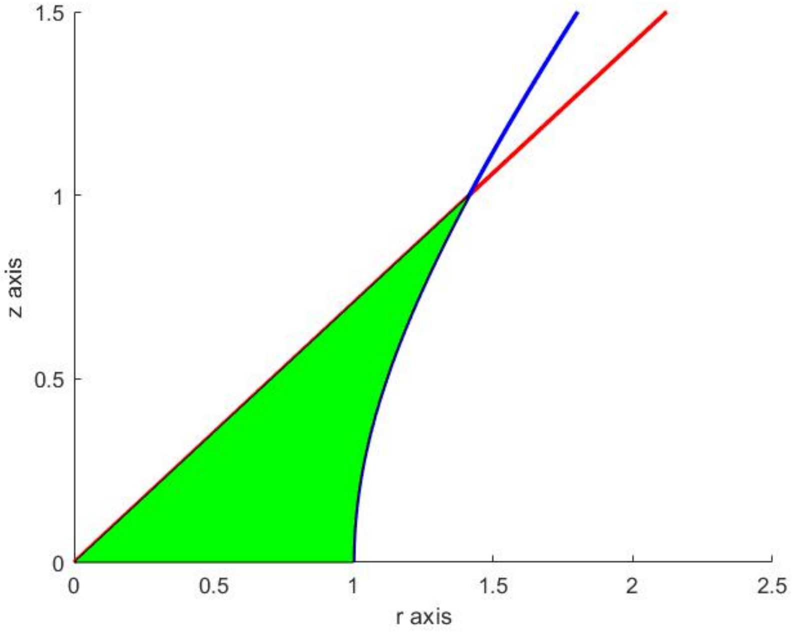


FIGURE 87. W of Exercise 31i between $r^2 = 2z^2$ and $r^2 = z^2 + 1$ in (r, z) plane

Resolution: $W = \{(x, y, z)\colon 2z^2 \le \boldsymbol{x^2 + y^2} \le z^2 + 1, x \ge 0, y \ge 0\}$ in cylindrical coordinates takes the form $B = \{(r, \theta, z)\colon 0 \le \theta \le 2\pi,\ 0 \le z \le 1,\ \sqrt{2}z \le r \le \sqrt{z^2 + 1}\}$,

$$V(W) = \int\limits_0^{\pi/2} \mathrm{d}\theta \int\limits_0^1 \mathrm{d}z \int\limits_{\sqrt{2}z}^{\sqrt{z^2+1}} r\,\mathrm{d}r$$

$$= \frac{\pi}{4} \int\limits_0^1 1 - z^2\,\mathrm{d}z = \frac{\pi}{6}.$$

EXERCISE 31J

Exercise. *Compute the volume of the domain W of $\mathbb{R}^3$ bounded by the surfaces $x^2 + y^2 + z^2 = 2a^2$, $z = \dfrac{x^2 + y^2}{a}$ $(z \ge 0,\ a > 0)$.*

Resolution: $W = \left\{(x, y, z)\colon \boldsymbol{x^2 + y^2} + z^2 \le 2a^2, z \ge \dfrac{\boldsymbol{x^2 + y^2}}{a}\right\}$ in cylindrical coordinates takes the form $B = \{(r, \theta, z)\colon 0 \le \theta \le 2\pi, 0 \le r \le a, r^2/a \le z \le \sqrt{2a^2 - r^2}\}$, so

$$V(W) = \int_0^{2\pi} \mathrm{d}\theta \int_0^a r\,\mathrm{d}r \int_{r^2/a}^{\sqrt{2a^2-r^2}} \mathrm{d}z$$

$$= 2\pi \int_0^a r\sqrt{2a^2 - r^2} - r^3/a\,\mathrm{d}r$$

$$= 2\pi a^3 \left(\frac{2^{3/2}}{3} - \frac{7}{12}\right).$$

EXERCISE 31K

Exercise. *Compute the volume of the solid domain W in $\mathbb{R}^3$ bounded by the surfaces: $x^2+y^2 = 4$,$z = x + y$, $x \ge 0, y \ge 0, z \ge 0$.*

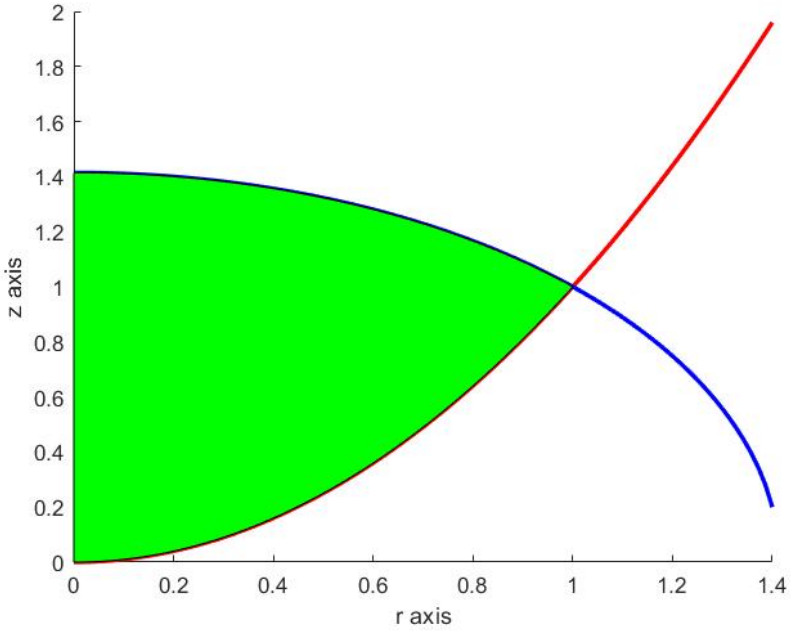


FIGURE 88. W of Exercise 31j between $r^2+z^2=2a^2$ and $z=\dfrac{r^2}{a}$ for $a=1$

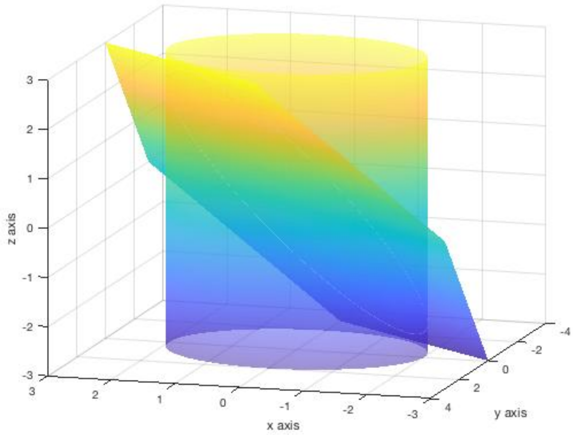


FIGURE 89. Domain W of Exercise 31k.

Resolution: To compute $I=\iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, first we need to decide on a change of variable according to the maximum range:

$$x^2+y^2=4.$$

As it is a centered circle we apply the cylindrical change of variables

$$(x,y,z)=(r\cos\theta, r\sin\theta, z)=T(r,\theta,z), \text{ with } |\det DT(r,\theta,z)|=r.$$

Let us now proceed to fix θ and r:

$$0\le\theta\le\pi/2,$$

since the θ angle only goes from 0 to $\pi/2$ as we need to consider $x\ge 0, y\ge 0, z\ge 0$, and

$$0\le r\le 2,$$

since if we apply the cylindrical change of variables we observe that the maximum range of r in W is $\sqrt{4}=2$. In order to find the last integral limits we must rise in the z axis starting from 0 and towards the equation $z=x+y$ or $z=r\cos\theta+r\sin\theta$ with the change of variables applied:

$$0\le z\le r\cos\theta+r\sin\theta.$$

With this we have that $W = T(B)$ with

$$B = \{(r,\theta,z)\colon 0 \le r \le 2, 0 \le \theta \le \pi/2, 0 \le z \le r\cos\theta + r\sin\theta\}.$$

Let's now lay out and proceed to solve the triple integral:

$$\begin{aligned} I &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_B r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \int_0^{\pi/2} \mathrm{d}\theta \int_0^2 r\,\mathrm{d}r \int_0^{r\cos\theta + r\sin\theta} \mathrm{d}\theta \\ &= \int_0^{\pi/2} \cos\theta + \sin\theta\,\mathrm{d}\theta \int_0^2 r^2\,\mathrm{d}r = [\sin\theta - \cos\theta]_{\theta=0}^{\theta=\pi/2} \cdot \left[r^3/3\right]_{r=0}^{r=2} = 16/3. \end{aligned}$$

EXERCISE 31L

Exercise. *Compute the volume of the ice cream cone W defined by $x^2 + y^2 \le z^2/5$, $0 \le z \le 5 + \sqrt{5 - x^2 - y^2}$.*

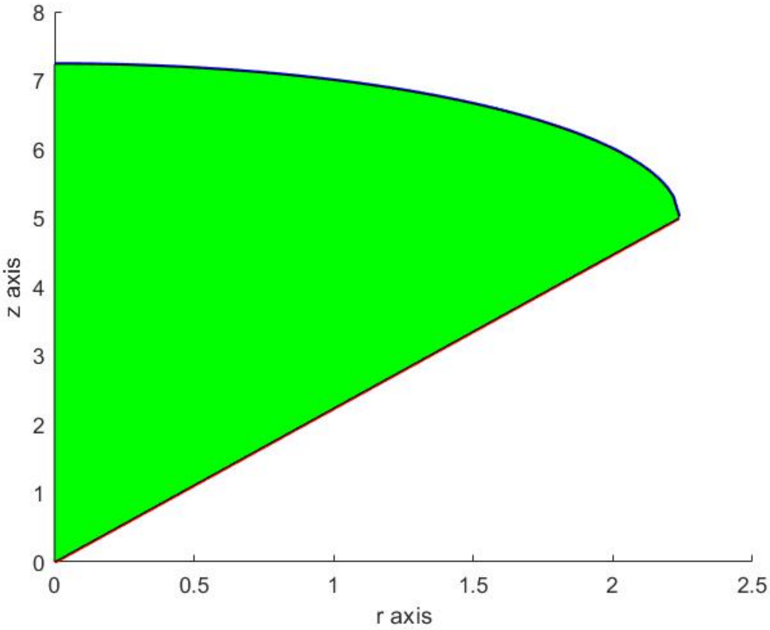


FIGURE 90. Ice cream of Exercise 31l in (r,z) plane

Resolution: The ice cream W in cylindrical coordinates takes the form $B = \{(r,\theta,z)\colon r^2 \le z^2/5, 0 \le z \le 5 + \sqrt{5 - r^2}\} = B_1 \cup B_2$ with $B_1 = \{(r,\theta,z)\colon (z-5)^2 + r^2 \le 5, z \ge 5\}$, hemisphere of radius $\sqrt{5}$, and $B_2 = \{(r,\theta,z)\colon r \le z/\sqrt{5}, z \le 5\}$, cone of base a circle of radius $\sqrt{5}$ and height 5.

Therefore

$$\begin{aligned} V(W) &= V(B_1) + V(B_2) \\ &= \frac{1}{2}\frac{4\pi\sqrt{5}^3}{3} + \frac{\pi\sqrt{5}^2 \cdot 5}{3} \\ &= \pi(10\sqrt{5}/3 + 25/3). \end{aligned}$$

Exercise. *Find the center of mass of the solid W homogeneous mass distribution*

$$W = \{(x,y,z)\colon x^2 + y^2 + z^2 \le R^2, x \ge 0, y \ge 0, z \ge 0\}.$$

Resolution: The restrictions tell us that our solid is the first octant of a sphere of radius R.

Let's calculate its mass. We assume its density is 1:

$$M(W) = \frac{4\pi}{3}R^3\frac{1}{8} = \frac{\pi R^3}{6}.$$

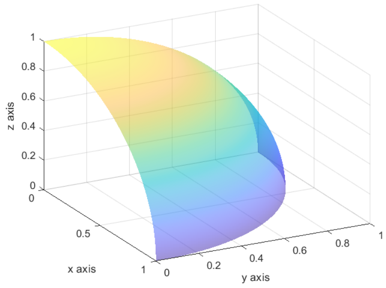


FIGURE 91. The solid W of Exercise 18a is the inner volume of this octant of the sphere.

The center of mass is $x_{c_m} = \dfrac{\iiint\limits_W x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{M(W)}$. Let's set the integral in spherical coordinates

$$B = \{(r,\theta,\phi)\colon 0 \le r \le R, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}\}.$$

$$\iiint\limits_W x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_B (r^2\cos\phi)(r\cos\theta\cos\phi)\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta = \int\limits_0^{\frac{\pi}{2}} \cos\theta\,\mathrm{d}\theta \int\limits_0^{\frac{\pi}{2}} \cos^2\phi\,\mathrm{d}\phi \int\limits_0^{R} r^3\,\mathrm{d}r =$$

$$= \int\limits_0^{\frac{\pi}{2}} \cos\theta\,\mathrm{d}\theta \int\limits_0^{\frac{\pi}{2}} \frac{R^4\cos^2\phi}{4}\,\mathrm{d}\phi = \int\limits_0^{\frac{\pi}{2}} \cos\theta \left[\frac{\phi}{2} + \frac{\sin 2\phi}{4}\right]_{\phi=0}^{\phi=\frac{\pi}{2}} \mathrm{d}\theta = \left[\frac{R^4\pi\sin\theta}{16}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} =$$

$$= \frac{R^4\pi}{16}.$$

Therefore $x_{c_m} = \dfrac{\frac{R^4\pi}{16}}{\frac{R^3\pi}{6}} = \dfrac{3R}{8}$. By symmetry, the center of mass is $\left(\dfrac{3R}{8}, \dfrac{3R}{8}, \dfrac{3R}{8}\right)$.

EXERCISE 32

Exercise. *$B = B_R(0)$ is the ball in $\mathbb{R}^3$ of radius $R > 0$ and center at the origin. We consider $T(x,y,z)$ as the temperature in the point (x,y,z) and we suppose that it is proportional to the distance between the point and the origin. In what points of B is the temperature equal to the average one?*

Resolution: First of all we must build the temperature function $T(x,y,z) = k\sqrt{x^2+y^2+z^2}$ Considering k as a constant. The temperature T has this equation because it depends on the distance which in $\mathbb{R}^3$ is given by this expression.

To proceed we must know the Average formula of a function which is:

$$\overline{f} = \frac{\iiint_\Omega f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\iiint_\Omega \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}.$$

If we want to know the average of our function T:

$$\overline{T} = \frac{\iiint_\Omega T(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\iiint_\Omega \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}.$$

We must first of all change to spherical coordinates as we are dealing with spheres:

$$(x,y,z) = T(\rho,\theta,\phi) = (\rho\cos\theta\cos\phi, \rho\sin\theta\cos\phi, \rho\sin\phi), \text{ with } |\det DT(\rho,\theta,\phi)| = \rho^2 cos\phi.$$

Knowing it is a sphere, the limits of integration will also change to:

$$0 \le \theta \le 2\pi, \frac{-\pi}{2} \le \Psi \le \frac{\pi}{2}, 0 \le \rho \le R$$

Substituting the new changes and limits and changing $\sqrt{x^2+y^2+z^2}$ for ρ:

$$\overline{T} = k\frac{\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^R \rho^3\cos(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta}{\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^R \rho^2\cos(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta}$$

If we take a closer look we can see that the denominator is actually the volume of the sphere, therefore equal to $4/3\pi R^3$:

$$\overline{T} = k\frac{\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^R \rho^3 cos(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta}{\frac{4}{3}\pi R^3}.$$

Applying Fubini theorem on the numerator:

$$\overline{T} = k\frac{2\pi\frac{R^4}{4}\left(\sin\frac{\pi}{2}-\sin(-\frac{\pi}{2})\right)}{\frac{4}{3}\pi R^3} = \frac{kR^4}{\frac{4}{3}R^3} = \frac{3Rk}{4}.$$

Once we have found the average temperature $\overline{T}$, we must see where it is equal to the temperature function:

$$\frac{3Rk}{4} = k\sqrt{x^2+y^2+z^2}$$

We can now reach this expression:

$$\left(\frac{3R}{4}\right)^2 = x^2+y^2+z^2$$

We can conclude that the average temperature will be located on the surface of a sphere of radius $\frac{3}{4}R$ centered at the origin.

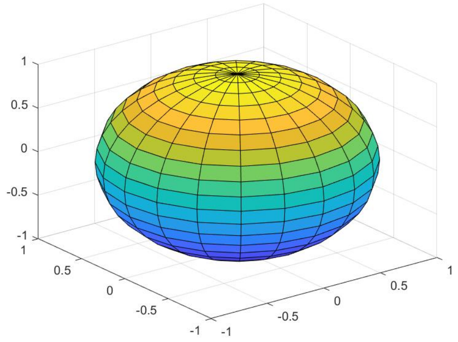


FIGURE 92. Sphere plot of Exercise 32.

EXERCISE 33A

Exercise. *Find the center of mass of a circular sector defined by an annulus with internal radius a, external radius A and central angle 2α, that is symmetric with respect to the positive x axis, assuming constant density $\rho(x,y)=1$.*

Resolution:

In order to solve the exercise, we consider the center of mass equations:

$$\overline{x} = \frac{\iint_D \rho\, x\,\mathrm{d}x\,\mathrm{d}y}{\iint_D \mathrm{d}x\,\mathrm{d}y}, \qquad \overline{y} = \frac{\iint_D \rho\, y\,\mathrm{d}x\,\mathrm{d}y}{\iint_D \mathrm{d}x\,\mathrm{d}y}.$$

We start finding the area of the annulus (which corresponds to the denominator of both equations) using Fubini's theorem:

$$\iint_D \mathrm{d}x\,\mathrm{d}y = \int_{-\alpha}^{\alpha}\int_a^A r\,\mathrm{d}r\,\mathrm{d}\theta = \int_{-\alpha}^{\alpha}\mathrm{d}\theta\int_a^A r\,\mathrm{d}r\left[\frac{r^2}{2}\right]_{r=a}^{r=A}[\theta]_{\theta=-\alpha}^{\theta=\alpha} = \frac{A^2-a^2}{2}2\alpha = \alpha(A^2-a^2).$$

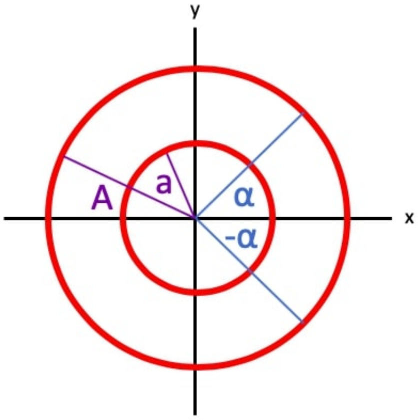


Figure 93. Plot of the annulus of Exercise 33a.

Then, we compute the numerator of both equations. We will need to make the change to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ with $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$. For the numerator of $\overline{x}$:

$$\begin{aligned}\iint_D \rho\, x\, \mathrm{d}x\,\mathrm{d}y &= \int_{-\alpha}^{\alpha} \cos\theta\,\mathrm{d}\theta \int_a^A r^2\,\mathrm{d}r = \left[\frac{r^3}{3}\right]_{r=a}^{r=A} [\sin\theta]_{\theta=-\alpha}^{\theta=\alpha} \\ &= \frac{A^3-a^3}{3}(\sin\alpha - \sin(-\alpha)) = \frac{2}{3}\sin\alpha(A^3-a^3),\end{aligned}$$

so that

$$\overline{x} = \frac{\frac{2}{3}\sin\alpha(A^3-a^3)}{\alpha(A^2-a^2)} = \frac{2}{3}\frac{\sin\alpha}{\alpha}\cdot\frac{A^3-a^3}{A^2-a^2}.$$

For the numerator of $\overline{y}$:

$$\begin{aligned}\iint_D \rho\, y\, \mathrm{d}x\,\mathrm{d}y &= \int_{-\alpha}^{\alpha}\int_a^A r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta = \int_{-\alpha}^{\alpha}\mathrm{d}\theta\int_a^A r^2\,\mathrm{d}r = \left[\frac{r^3}{3}\right]_{r=a}^{r=A}[-\cos\theta]_{\theta=-\alpha}^{\theta=\alpha} \\ &= \frac{A^3-a^3}{3}(-\cos\alpha) + \cos(-\alpha)) = 0,\end{aligned}$$

so that $\overline{y} = 0$. Notice that this is clear from the symmetry of the circular sector with respect to the x-axis.

Finally, the center of mass of this circular sector is:

$$(\overline{x}, \overline{y}) = \left(\frac{2}{3}\frac{\sin\alpha}{\alpha}\cdot\frac{A^3-a^3}{A^2-a^2}, 0\right).$$

Exercise 33c

Exercise. *Find the* center of mass *of the planar region between* $y = 0$ *and* $y = x^2$ $\left(0 \le x \le \frac{1}{2}\right)$ *with density* $\rho(x, y) = 1$.

Resolution: The domain of the planar region is:

$$\Omega = \left\{(x,y):\ 0 \le x \le \frac{1}{2},\ 0 \le y \le x^2\right\}$$

Its mass is:

$$m = \iint_\Omega \rho\,(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{1}{2}} 1\,\mathrm{d}x \int_0^{x^2} 1\,\mathrm{d}y = \int_0^{\frac{1}{2}} [y]_{y=0}^{y=x^2}\,\mathrm{d}x = \int_0^{\frac{1}{2}} x^2\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_{x=0}^{x=\frac{1}{2}} = \frac{1}{24}$$

The x-component of the center of mass is:

$$\begin{aligned}\bar{x} &= \frac{1}{m}\iint_\Omega x\rho\,(x,y)\,\mathrm{d}x\,\mathrm{d}y = 24\int_0^{\frac{1}{2}} x\,\mathrm{d}x\int_0^{x^2} 1\,\mathrm{d}y = 24\int_0^{\frac{1}{2}} x\,[y]_{y=0}^{y=x^2}\,\mathrm{d}x = \\ &= 24\int_0^{\frac{1}{2}} x^3\,\mathrm{d}x = 24\left[\frac{x^4}{4}\right]_{x=0}^{x=\frac{1}{2}} = \frac{3}{8}.\end{aligned}$$

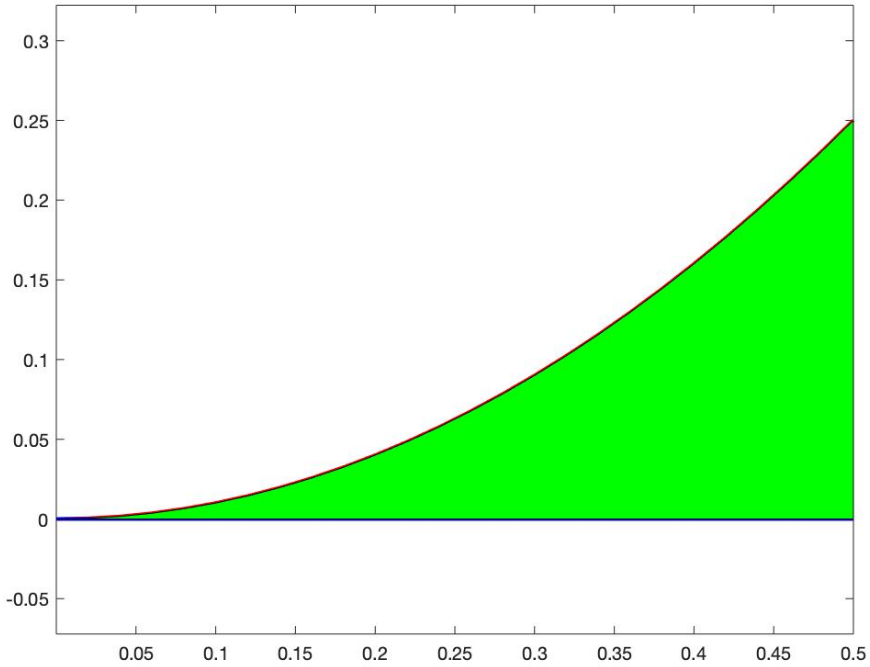


FIGURE 94. Domain of the planar region from Exercise 33c.

The y-component of the center of mass is:

$$\bar{y} = \frac{1}{m}\iint_{\Omega} y\rho\,(x,y)\,\mathrm{d}x\,\mathrm{d}y = 24\int_0^{\frac{1}{2}} 1\,\mathrm{d}x \int_0^{x^2} y\,\mathrm{d}y = 24\int_0^{\frac{1}{2}} \left[\frac{y^2}{2}\right]_{y=0}^{y=x^2} \mathrm{d}x =$$
$$= 24\int_0^{\frac{1}{2}} \frac{x^4}{2}\,\mathrm{d}x = 24\left[\frac{x^5}{10}\right]_{x=0}^{x=\frac{1}{2}} = \frac{3}{40}.$$

Therefore, the center of mass of the planar region is: $\vec{r}_{cm} = \left(\frac{3}{8}, \frac{3}{40}\right)$.

EXERCISE 34B

Exercise. *Find the center of mass of the solid W considering homogeneous mass distribution:*

$$W = \left\{(x,y,z) \in \mathbb{R}^3 \colon 0 \le x \le 2, 0 \le y \le 6, 0 \le z \le 4 - x^2\right\}.$$

Resolution: As long as we are told that the solid has a homogeneous mass distribution, we know that the density will be equal to one. The formula for the center of mass is:

$$\bar{\mathbf{x}} = \frac{\iiint_W \rho x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{V(W)}, \quad \bar{\mathbf{y}} = \frac{\iiint_W \rho y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{V(W)}, \quad \bar{\mathbf{z}} = \frac{\iiint_W \rho z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{V(W)}.$$

First, we compute the volume:

$$V(W) = \int_0^6 \mathrm{d}y \int_0^2 \mathrm{d}x \int_0^{4-x^2} \mathrm{d}z = \int_0^6 \mathrm{d}y \int_0^2 (4-x^2)\,\mathrm{d}x$$
$$= \int_0^6 \left[4x - \frac{x^3}{3}\right]_0^2 \mathrm{d}y = \int_0^6 \left(8 - \frac{8}{3}\right) \mathrm{d}y = 32.$$

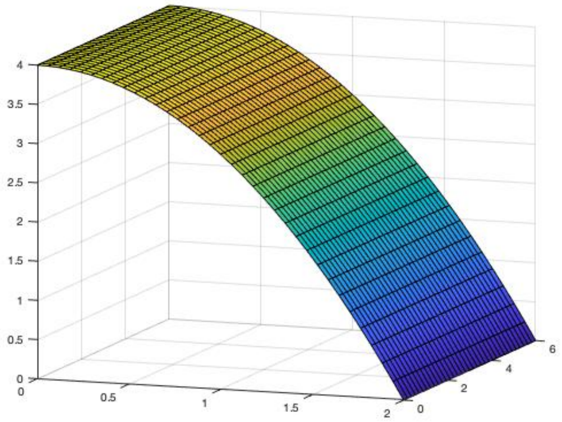


FIGURE 95. plot of $z = 4 - x^2$ on $0 \le x \le 2, 0 \le y \le 6$ from Exercise 34b.

Now, we can compute the center of mass of the solid:

$$\begin{aligned}\bar{\mathbf{x}} &= \frac{1}{32}\iiint_W \rho x \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{32}\int_0^6 \mathrm{d}y \int_0^2 \mathrm{d}x \int_0^{4-x^2} x\,\mathrm{d}z = \frac{1}{32}\int_0^6 \mathrm{d}y \int_0^2 x\,[z]_{z=0}^{z=4-x^2}\,\mathrm{d}x \\ &= \frac{1}{32}\int_0^6 \mathrm{d}y \int_0^2 x(4-x^2)\,\mathrm{d}x = \frac{1}{32}\int_0^6 \mathrm{d}y \int_0^2 4x - x^3\,\mathrm{d}x = \frac{1}{32}\int_0^6 \left[\frac{4x^2}{2} - \frac{x^4}{4}\right]_{x=0}^{x=2} \mathrm{d}y \\ &= \frac{1}{32}\int_0^6 4\,\mathrm{d}y = \frac{1}{8}\,[y]_{y=0}^{y=6} = \frac{3}{4}.\end{aligned}$$

$$\begin{aligned}\bar{\mathbf{y}} &= \frac{1}{32}\iiint_W \rho y \,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{32}\int_0^6 \mathrm{d}y \int_0^2 \mathrm{d}x \int_0^{4-x^2} y\,\mathrm{d}z = \frac{1}{32}\int_0^6 \int_0^2 y\,[z]_{z=0}^{z=4-x^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{32}\int_0^6 \mathrm{d}y \int_0^2 4y - yx^2\,\mathrm{d}x = \frac{1}{32}\int_0^6 y\left[4x - \frac{x^3}{3}\right]_{x=0}^{x=2} \mathrm{d}y = \frac{1}{32}\int_0^6 y\left(8 - \frac{8}{3}\right)\mathrm{d}y \\ &= \frac{1}{32}\frac{16}{3}\left[\frac{y^2}{2}\right]_0^6 = \frac{1}{6}\cdot\frac{36}{2} = 3.\end{aligned}$$

$$\begin{aligned}\bar{\mathbf{z}} &= \frac{1}{32}\iiint_W \rho z \,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{32}\int_0^6 \mathrm{d}y \int_0^2 \mathrm{d}x \int_0^{4-x^2} z\,\mathrm{d}z = \frac{1}{32}\int_0^6 \mathrm{d}y \int_0^2 \left[\frac{z^2}{2}\right]_{z=0}^{z=4-x^2} \mathrm{d}x \\ &= \frac{1}{32}\frac{1}{2}\int_0^6 \mathrm{d}y \int_0^2 (4-x^2)^2\,\mathrm{d}x = \frac{1}{64}\int_0^6 \mathrm{d}y \int_0^2 16 - 8x^2 + x^4\,\mathrm{d}x = \frac{1}{64}\int_0^6 \left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{x=0}^{x=2} \mathrm{d}y \\ &= \frac{1}{64}\int_0^6 \frac{256}{15}\,\mathrm{d}y = \frac{1}{64}\cdot\frac{256}{15}\cdot 6 = \frac{8}{5}.\end{aligned}$$

To conclude, the center of mass of the solid W is $\mathbf{c} = \left(\frac{3}{4}, 3, \frac{8}{5}\right)$.

EXERCISE 34C

Exercise. *Find the center of mass of the solid W considering homogeneous mass distribution, where*

$$W = \left\{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 \le 4a^2, (x-a)^2 + y^2 + z^2 \ge a^2\right\}.$$

Resolution: The solid W can be written as the region between two spheres $W = W_1 \setminus W_2$, see Figure 96, where

$W_1 = \left\{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 \le 4a^2\right\}$ is a sphere with center $(0,0,0)$ and radius $\sqrt{4a^2} = 2a$,

$W_2 = \left\{(x,y,z) \in \mathbb{R}^3 \colon (x-a)^2 + y^2 + z^2 \ge a^2\right\}$ is a sphere with center $(a,0,0)$ and radius $\sqrt{a^2} = a$.

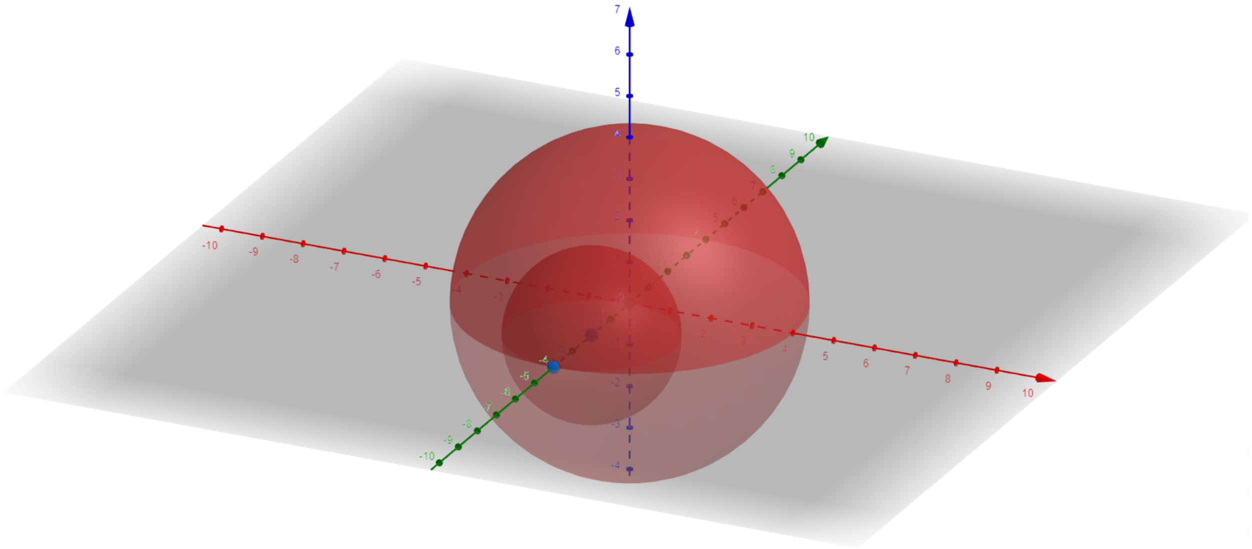


Figure 96. Solid W between two spheres from Exercise 34c.

As W is homogeneously distributed $\rho = 1$, and thus $M(W) = V(W)$:

$$M(W) = \iiint_W \rho \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \rho \iiint_W \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = 1 \iiint_W \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iiint_W \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = V(W).$$

Firstly, to find the center of mass we will check if there are some symmetries. A good way to identify them clearly is to plot the projection of both figures in the three planes. As it can clearly be seen in Figures 97 and 98, there is a symmetry with respect to the x-axis. Also, in the projection on the plane $x = 0$ in Figure 99 we can identify that the center of mass in the z and y axes will be $(0, 0)$, that is, the center of mass of W will be of the form $\overline{\boldsymbol{c}} = (\overline{x}, 0, 0)$.

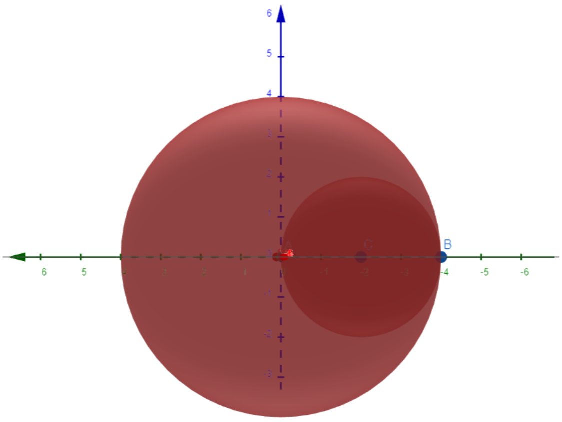


Figure 97. Projection of W on the the plane $(x, y) : z = 0$ from Exercise 34c.

To obtain $\overline{x}$, we will use the following formula:

$$\overline{x} = \frac{\iiint_W x \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z}{\iiint_W \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z} = \frac{\iiint_{W_2} x \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z - \iiint_{W_1} x \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z}{\iiint_{W_2} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z - \iiint_{W_1} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z} = \frac{M_2 \overline{x}_2 - M_2 \overline{x}_1}{M_2 - M_1},$$

where x_1, x_2 are the centers of mass of W_1, W_2, and M_1, M_2 their masses: $M_i = M(W_i) = V(W_i)$, $i = 1, 2$. Let us calculate x_1, x_2, M_1, M_2:

$\overline{x}_1 = 0$ (Looking at plane (x, y) or (z, x)), $M_1 = V(W_1) = \dfrac{4}{3}\pi(2a)^3 = \dfrac{32}{3}\pi a^3$ (sphere of radius $2a$),

$\overline{x}_2 = a$ (Looking at plane (x, y) or (z, x)), $M_2 = V(W_2) = \dfrac{4}{3}\pi a^3$ (sphere of radius a).

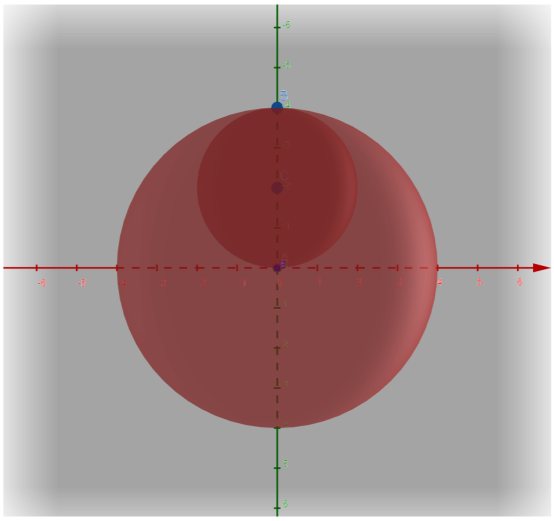


Figure 98. Projection of W on the the plane $(z,x) : y = 0$ from Exercise 34c.

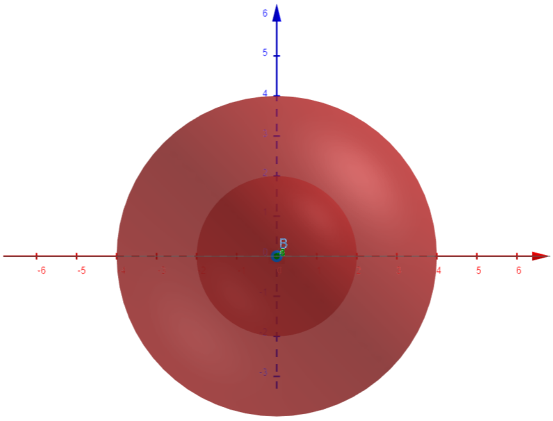


Figure 99. Projection of W on the the plane $(z,y) : x = 0$ from Exercise 34c.

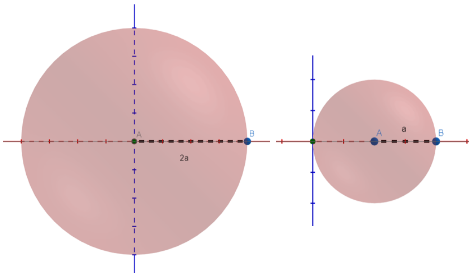


Figure 100. Projections of W_1 and W_2 on the plane (x,y) from Exercise 34c.

We now substitute the obtained values in the formula for $\overline{x}$:

$$\overline{x} = \frac{M_2\overline{x}_2 - M_2\overline{x}_1}{M_2 - M_1} = \frac{\frac{4}{3}\pi a^3 \cdot a - \frac{4}{3}\pi(2a)^3 \cdot 0}{\frac{4}{3}\pi a^3 - \frac{32}{3}\pi a^3} = -\frac{a}{7}.$$

Finally, the center of mass of the domain W is $\overline{\boldsymbol{c}} = \left(-\frac{a}{7}, 0, 0\right)$.

We end this exercise by noticing that we can solve our exercise in an easier way by looking at our previous calculations. Indeed, we can see easily that $M_2 = 8M_1$ and $\overline{x}_2 = 0$, $\overline{x}_1 = a$, so $\overline{x} = -a/7$ follows easily. The same formula above for $\overline{x}$ is also true replacing x by y or z, but since we can clearly deduce by looking at the pictures that $\overline{y}_1 = \overline{y}_2 = \overline{z}_1 = \overline{z}_2 = 0$, then $\overline{y} = \overline{z} = 0$.

EXERCISE 35

Exercise. *Find the total mass of the cylinder* $W = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 2, 0 \leq z \leq 3\}$ *if its density is* $\rho(x,y,z) = z\mathrm{e}^{-z^2}(x^2+y^2)$.

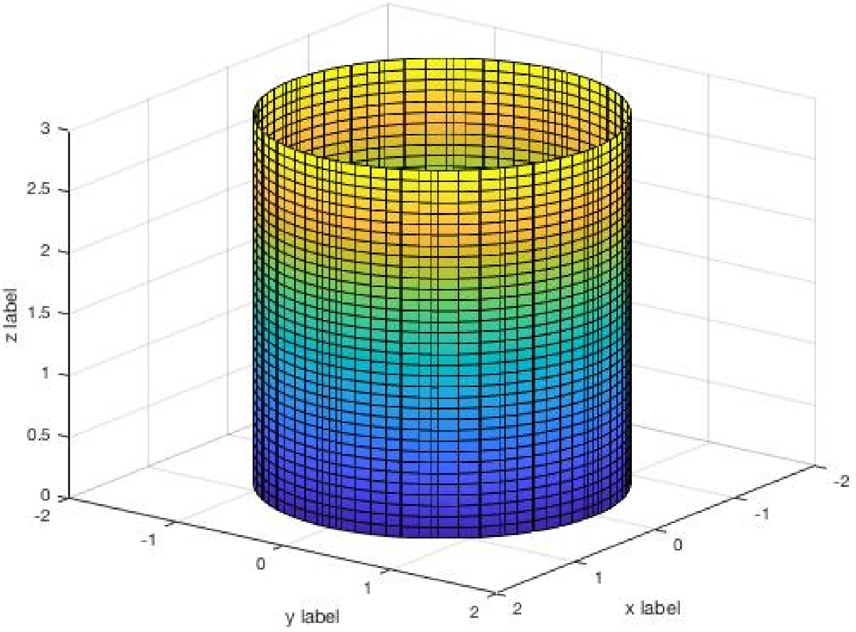


FIGURE 101. The cylinder W from Exercise 35.

Resolution: The first step is to acknowledge that this exercise can be solved using cylindrical coordinates, taking B such that $W = T(B)$ and $(x,y,z) = T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$ with $|\det(DT(x,y,z))| = r$. Now we can rewrite our density function as $\rho\left(T(r,\theta,z)\right) = r^2 z\mathrm{e}^{-z^2}$.

We have to compute the following integral:

$$M(W) = \iiint_W \rho(x,y,x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_B \rho(z,r,\theta)\,r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

In cylindrical coordinates W takes the form of a parallelepiped:

$$B = \left\{(r,\theta,z)\colon 0 \leq z \leq 3, 0 \leq \theta \leq 2\pi, 0 \leq r \leq \sqrt{2}\right\}.$$

We finally substitute the limits of our domain in cylindrical coordinates:

$$M(W) = \int\limits_0^{2\pi} \mathrm{d}\theta \int\limits_0^{\sqrt{2}} r^3\,\mathrm{d}r \int\limits_0^{3} z\mathrm{e}^{-z^2}\,\mathrm{d}z = [\theta]_{\theta=0}^{\theta=2\pi}\left[r^4/4\right]_{r=0}^{r=\sqrt{2}}\left[\frac{\mathrm{e}^{-z^2}}{-2}\right]_0^3 = 2\pi\cdot 1\cdot\left(\frac{-\mathrm{e}^{-9}}{2}+\frac{1}{2}\right).$$

So the total mass is $\boxed{M = \pi(1-e^{-9})}$.

EXERCISE 37

Exercise. $W \subset \mathbb{R}^3$ *is a solid object with density* $\rho(x,y,z)$. *If we divide* W *into two subsolids,* $W = W_1 \cup W_2$, $V(W_1 \cap W_2) = 0$ *and we consider* $(\overline{x}_1, \overline{y}_1, \overline{z}_1)$ *and* $(\overline{x}_2, \overline{y}_2, \overline{z}_2)$ *as the centers of mass of* W_1 *and* W_2, *respectively, prove that the center of mass* $(\overline{x}, \overline{y}, \overline{z})$ *of* W *is the same as if we assume that all the mass in* W_1 *is concentrated in* $(\overline{x}_1, \overline{y}_1, \overline{z}_1)$ *and all the mass of* W_2 *is in* $(\overline{x}_2, \overline{y}_2, \overline{z}_2)$

Plan: The idea is to find intuitively which would be the center of mass in the case we had point particles, which in theory will be easier. Afterwards, we will prove the equality by developing the formula for the union of centers of mass.

Resolution: Let us assume that all three components (x, y, z) work in the same way. In that case, we will prove the equality for just one component instead of three. The result for the other two will be exactly equal. Let us work now with the horizontal component x.

For two point particles at (x_1, y_1, z_1), (x_2, y_2, z_2) with masses m_1, m_2, the first component of the center of mass $(\overline{x}, \overline{y}, \overline{z})$ is

$$\overline{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2},$$

Using this formula for our case, we get the following expression:

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}, \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2} \right).$$

Now, let us move to the second part of the exercise:

For $W = W_1 \cup W_2$, $V(W_1 \cap W_2) = 0$,

$$M = \iiint_W \rho \, \mathrm{d}V = \iiint_{W_1} \rho \, \mathrm{d}V + \iiint_{W_2} \rho \, \mathrm{d}V = M_1 + M_2,$$

where we have used the short forms $M = M(W)$, $M_1 = M(W_1)$, $M_2 = M(W_2)$ and $\mathrm{d}V = \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$. Therefore

$$\overline{x} = \frac{\iiint_W x\rho \, \mathrm{d}V}{M} = \frac{\iiint_{W_1} x\rho \, \mathrm{d}V + \iiint_{W_2} x\rho \, \mathrm{d}V}{M_1 + M_2} = \frac{M_1 \overline{x}_1 + M_2 \overline{x}_2}{M_1 + M_2},$$

where $\overline{x}$, $\overline{x}_1$ and $\overline{x}_2$ are the centers of mass of W, W_1, W_2.

Finally, as all three components work following the same procedure, we get that for $W = W_1 \cup W_2$, $V(W_1 \cap W_2) = 0$, the whole center of mass is:

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{M_1 \overline{x}_1 + M_2 \overline{x_2}}{M_1 + M_2}, \frac{M_1 \overline{y}_1 + M_2 \overline{y_2}}{M_1 + M_2}, \frac{M_1 \overline{z}_1 + M_2 \overline{z_2}}{M_1 + M_2} \right),$$

which is indeed the same result obtained for the first case.

Exercise 38

Exercise. *Ω is a planar domain that is contained in the half-plane $\{y = 0, \, x > 0\}$ of $\mathbb{R}^3$. If we consider $(\bar{x}, 0, \bar{z})$ as the geometric center of Ω (equal to its center of mass if density is constantly equal to* 1*), prove that the volume of the solid of revolution W obtained by rotating Ω around the z axis is $V(W) = 2\pi\bar{x}A(\Omega)$, where $2\pi\bar{x}$ is the the length of the circle obtained by rotating $(\bar{x}, 0, \bar{z})$.* Hint: *Use cylindrical coordinates and check that $(\theta, r, z) \in \Omega^* = [0, 2\pi] \times \Omega$.*

Resolution: The solid of revolution W generated by rotating the planar domain Ω around the z axis is given by

$$W = \{(x, y, z) = (r\cos\theta, r\sin\theta, z), \, 0 \leq \theta \leq 2\pi, \, (r, 0, z) \in \Omega\},$$

where we have used the coordinates $(r, 0, z)$ in the planar domain Ω. If we change to cylindrical coordinates

$$(x, y, z) = T(\theta, r, z) = (r\cos\theta, r\sin\theta, z), \qquad |\det DT((\theta, r, z)| = r,$$

it turns out that $W = T(\Omega^*)$ with

$$\Omega^* = \{(\theta, r, z) \colon 0 \leq \theta \leq 2\pi, \, (r, 0, z) \in \Omega\} = [0, 2\pi] \times \Omega.$$

Therefore

$$\begin{aligned} V(W) &= \iiint_W \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iiint_{\Omega^*} r \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}z = \int_0^{2\pi} \int_\Omega r \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}z \\ &= \int_0^{2\pi} \mathrm{d}\theta \int_\Omega r \, \mathrm{d}r \, \mathrm{d}z = 2\pi \int_\Omega r \, \mathrm{d}r \, \mathrm{d}z, \end{aligned}$$

where we have used the coordinates $(r, 0, z)$ for the points in Ω. Indeed, $x = r$ for $\theta = 0$, or equivalently for $(x, y, z) = (r, 0, z) \in \Omega$, so we can write the above expression in terms of the variable x instead of the variable r and get the desired formula:

$$V(W) = 2\pi \int_\Omega x \, \mathrm{d}x \, \mathrm{d}z = 2\pi \frac{\int_\Omega x \, \mathrm{d}x \, \mathrm{d}z}{\int_\Omega \mathrm{d}x \, \mathrm{d}z} \int_\Omega \mathrm{d}x \, \mathrm{d}z = 2\pi\bar{x} \, A(\Omega).$$

Exercise 39

Exercise. *Compute the volume of* W *if* $\Omega = \left\{(x,y) \in \mathbb{R}^2 : z \leq x^2, x+z \leq 2, x \geq 0, z \geq 0\right\}$ *rotates around the* z *axis.*

Resolution: To calculate this volume we will use the formula of the previous exercise: $V(W) = 2\pi\overline{x}A(\Omega)$. The domain of Ω is:

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : z \leq x^2, x+z \leq 2, x \geq 0, z \geq 0\right\}$$

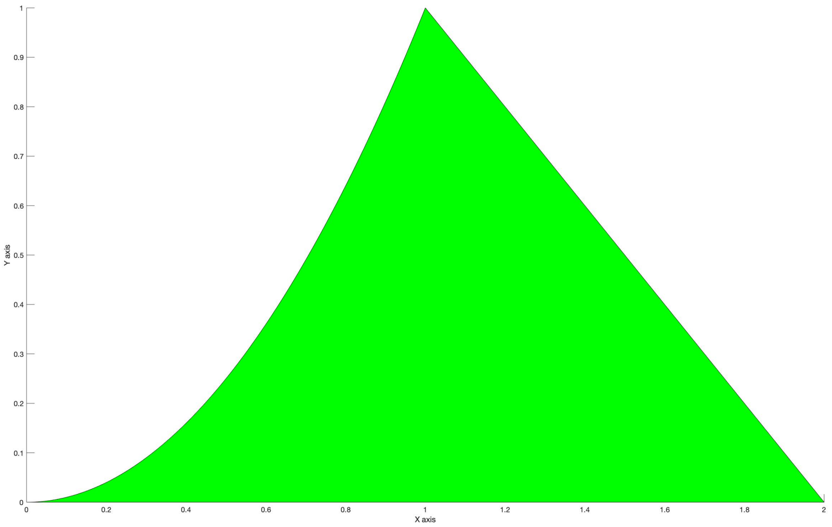


Figure 102. Domain Ω from exercise 39.

The volume is the following:

$$V(W) = 2\pi\overline{x}A(\Omega) = 2\pi \iint_{\Omega} x \,\mathrm{d}x \,\mathrm{d}z.$$

Next, we will define the limits of integration:

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : 0 \leq z \leq 1, \sqrt{z} \leq x \leq 2-z\right\}.$$

The double integral would be the following:

$$V(W) = 2\pi \iint_{\Omega} x \,\mathrm{d}x \,\mathrm{d}z = 2\pi \int_0^1 \int_{\sqrt{z}}^{2-z} x \,\mathrm{d}x \,\mathrm{d}z.$$

Finally, we can calculate the volume of W:

$$\begin{aligned}
V(W) &= 2\pi \int_0^1 \int_{\sqrt{z}}^{2-z} x \,\mathrm{d}x \,\mathrm{d}z \\
&= 2\pi \frac{1}{2} \int_0^1 \left[x^2\right]_{\sqrt{z}}^{2-z} \,\mathrm{d}z \\
&= \pi \int_0^1 \Big((2-z)^2 - z\Big) \,\mathrm{d}z \\
&= \pi \int_0^1 \left(4 - 4z + z^2 - z\right) \,\mathrm{d}z \\
&= \pi \int_0^1 \left(4 - 5z + z^2\right) \,\mathrm{d}z \\
&= \pi \left(\int_0^1 4 \,\mathrm{d}z - \int_0^1 5z \,\mathrm{d}z + \int_0^1 z^2 \,\mathrm{d}z \right) \\
&= \pi \left(4 \left[z\right]_0^1 - \frac{5}{2}\left[z^2\right]_0^1 + \frac{1}{3}\left[z^3\right]_0^1 \right) \\
&= \pi \left(4 - \frac{5}{2} + \frac{1}{3} \right) = \pi \frac{11}{6}.
\end{aligned}$$

$$\boldsymbol{V(W) = \frac{11\pi}{6}.}$$