

## TEMA 2. Integració de funcions de varies variables.

3) Apliquen el principi de Cavalieri per calcular els següents volums a partir de l'àrea de seccions amb plans paral·lels als plans coordenats (triades de forma adequada)

(a) Volum envoltat per l'elipsoide:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solució. Fixant  $0 \leq x \leq a$ :  $\frac{y^2}{b^2(1-\frac{x^2}{a^2})} + \frac{z^2}{c^2(1-\frac{x^2}{a^2})} = 1$ , d'on  $S(x) = \pi bc \left(1 - \frac{x^2}{a^2}\right)$

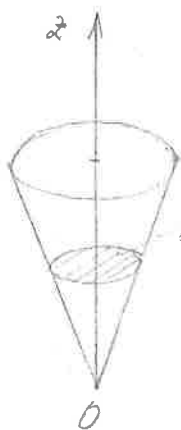
Nota: recordem que l'àrea d'una el·lipse ve donada per  $\pi AB$  on  $A$  i  $B$  són la longitud dels seus semieixos.

Lavors, aplicant el principi de Cavalieri: ...

$$V = 2 \int_0^a S(x) dx = 2\pi bc \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi bc \left(x - \frac{x^3}{3a^2}\right) \Big|_0^a = 2\pi bc \left(a - \frac{a}{3}\right)$$

$$= \boxed{\frac{4}{3} \pi abc}$$

(b) Volum envoltat pel con de base el·líptica  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2$ , amb  $0 \leq z \leq h$



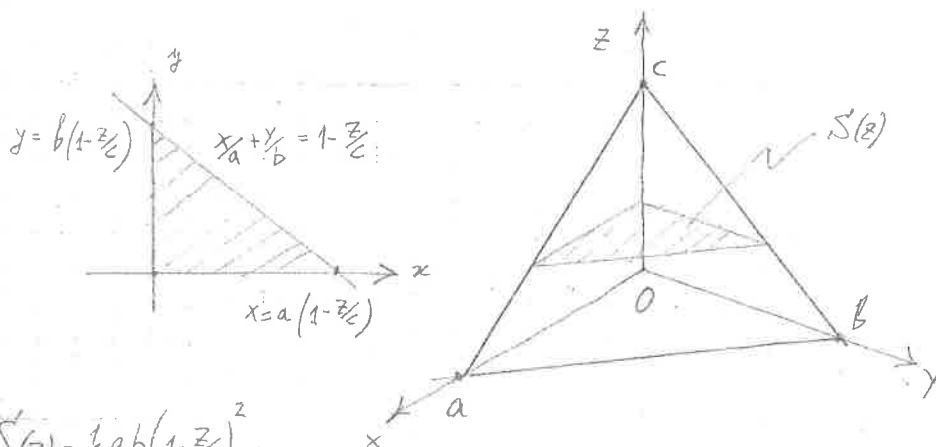
Solució. Fixant  $0 \leq z \leq h$ :  $\frac{x^2}{(az)^2} + \frac{y^2}{(bz)^2} = 1$ , d'on  $S(z) = \pi ab z^2$

Aleshores, aplicant el principi de Cavalieri:

$$V = \int_0^h S(z) dz = \int_0^h \pi ab z^2 dz = \pi ab \frac{z^3}{3} \Big|_0^h = \boxed{\frac{\pi}{3} ab h^3}$$

Nota:  $V = \frac{1}{3} \underbrace{\pi (ah) \cdot (bh)}_{\text{Àrea de la base}} \cdot h$

c) Volum de tetraedre limitat pels plans  $x=0, y=0, z=0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$  ( $a, b, c > 0$ )

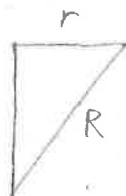
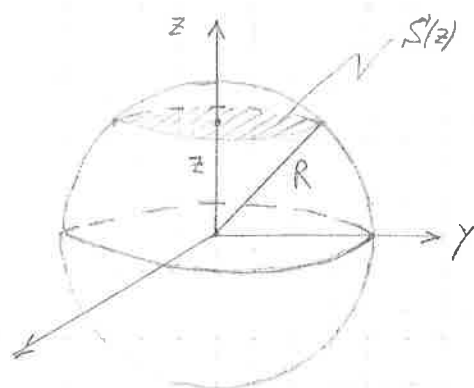


Solució. Fixant  $z$ :  $S(z) = \frac{1}{2} ab \left(1 - \frac{z}{c}\right)^2$ ,

d'om, aplicant el principi de Cavalieri:

$$V = \int_0^c S(z) dz = \frac{1}{2} ab \int_0^c \left(1 - \frac{z}{c}\right)^2 dz = -\frac{1}{6} abc \left(1 - \frac{z}{c}\right)^3 \Big|_0^c = \boxed{\frac{abc}{6}}$$

d) Volum envoltat pel casquet esfèric determinat per l'esfera  $x^2 + y^2 + z^2 = R^2$  i la condició  $R-h \leq z \leq R$ .



Solució.

$$S(z) = \pi r^2(z) = \pi (R^2 - z^2)$$

D'om, aplicant el principi de Cavalieri:

$$V = \int_{R-h}^R S(z) dz = \pi \int_{R-h}^R (R^2 - z^2) dz$$

$$= \pi \left( R^2 z - \frac{z^3}{3} \right) \Big|_{R-h}^R = \pi \left( R^3 - \frac{R^3}{3} - R^3 + R^3 h + \frac{R^3}{3} - R^2 h + R h^2 - \frac{h^3}{3} \right) = \boxed{\frac{\pi h^2}{3} (3R-h)}$$

4) Generalitzen el principi de Cavalieri al càlcul de volums en  $\mathbb{R}^4$  i calculeu el volum de la bola 4-dimensional  $B = \{ (x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + t^2 \leq R^2 \}$

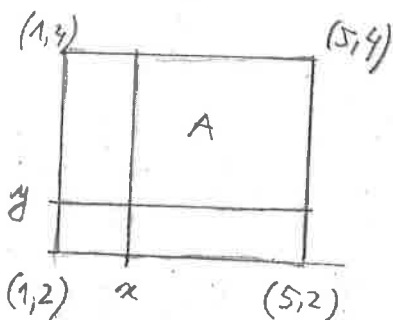
Solució: Fixem  $t$ :  $0 \leq x^2 + y^2 + z^2 \leq R^2 - t^2$ , amb  $-R \leq t \leq R$ . Aleshores  $V(t) = \frac{4}{3} \pi (R^2 - t^2)^{3/2}$ . D'om, aplicant el principi de Cavalieri:  $M = \int_{-R}^R V(t) dt = \frac{4}{3} \pi \int_{-R}^R (R^2 - t^2)^{3/2} dt = \frac{8}{3} \pi R^3 \int_0^R (1 - \frac{t^2}{R^2})^{3/2} dt = \left\{ \text{c.v. } \frac{t}{R} = \sin \theta \right\} =$   
 $= \frac{8}{3} \pi R^4 \int_0^{\pi/2} \cos^4 \theta d\theta = \frac{8}{3} \pi R^4 \int_0^{\pi/2} \left( \frac{1 + \cos(2\theta)}{2} - \frac{1 - \cos(4\theta)}{8} \right) d\theta = \frac{8}{3} \pi R^4 \left( \frac{\pi}{4} - \frac{\pi}{6} \right) = \boxed{\frac{\pi^2 R^4}{12}}$

(\*) Nota:  $\cos^4 \theta = \cos^2 \theta (1 - \sin^2 \theta) = \cos^2 \theta - \frac{1}{4} (2 \sin \theta \cos \theta)^2 = \cos^2 \theta - \frac{1}{4} \sin^2(2\theta) = \frac{1 + \cos(2\theta)}{2} - \frac{1 - \cos(4\theta)}{8}$  fent servir les fórmules de l'anada doble.

8) Per a les regions  $A \subset \mathbb{R}^2$  indicades, escriu la integral doble  $\iint_A f(x,y) dx dy$  en termes d'integrals iterades preses en diferents ordres  $\int \left( \int f dx \right) dy$  i  $\int \left( \int f dy \right) dx$ , donats quins són els extrems d'integració per a  $x$  i  $y$  en cada cas.

a) A rectangle de vèrtexs  $(1,2)$ ,  $(5,2)$ ,  $(5,4)$  i  $(1,4)$ .

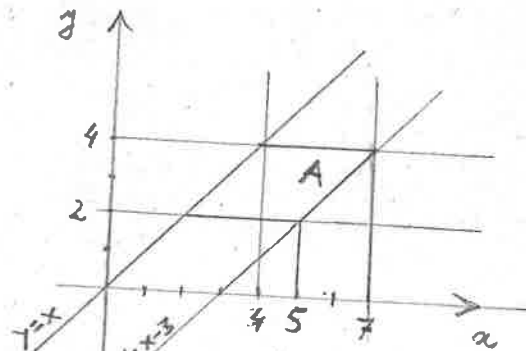
Solució.



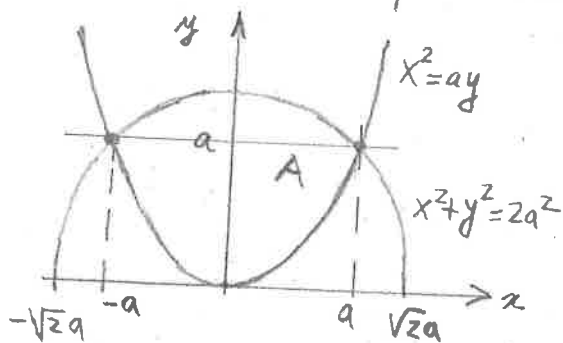
$$\iint_A f(x,y) dx dy = \int_1^5 dx \int_2^4 f(x,y) dy = \int_2^4 dy \int_1^5 f(x,y) dx$$

b) A paral·lelogram limitat per les rectes  $y=x$ ,  $y=x-3$ ,  $y=2$ ,  $y=4$ .

$$I = \iint_A f(x,y) dx dy = \int_2^4 dx \int_x^{x+3} f(x,y) dy + \int_4^5 dx \int_x^{x+3} f(x,y) dy + \int_5^7 dx \int_{x-3}^x f(x,y) dy = \int_2^4 dy \int_y^{y+3} f(x,y) dx$$



c) A regió limitada per les corbes  $x^2+y^2=2a^2$ ,  $x^2=ay$  ( $y \geq 0$ ,  $a > 0$ ).



$$I = \iint_A f(x,y) dx dy = \int_{-a}^a dx \int_{\frac{x^2}{a}}^{\sqrt{2a^2-x^2}} f(x,y) dy = \int_0^a dy \int_{-\sqrt{ay}}^{\sqrt{ay}} f(x,y) dx + \int_a^{2a} dy \int_{-\sqrt{2a^2-y^2}}^{\sqrt{2a^2-y^2}} f(x,y) dx$$

$$ay + y^2 = 2a^2 \Leftrightarrow y^2 + ay - 2a^2 = 0$$

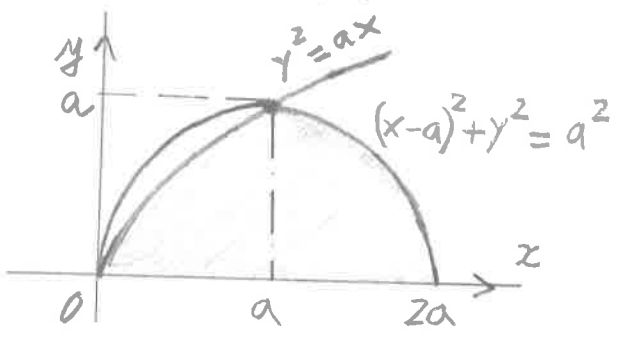
$$\text{donc: } y = \frac{-a \pm \sqrt{a^2 + 8a^2}}{2}$$

$$= \frac{-a \pm 3a}{2} = \begin{cases} a \\ -2a \text{ (No)} \end{cases}$$

d) A regió limitada per les corbes  $y^2 = ax$ ,  $x^2 + y^2 = 2ax$  ( $y \geq 0, a > 0$ )

$$I = \iint_A f(x,y) dx dy = \int_0^a \left( \int_0^{\sqrt{ax}} f(x,y) dy \right) dx + \int_a^{2a} \left( \int_0^{\sqrt{2ax-x^2}} f(x,y) dy \right) dx$$

$$= \int_0^a \left( \int_{y^2/a}^{a+\sqrt{a^2-y^2}} f(x,y) dx \right) dy$$

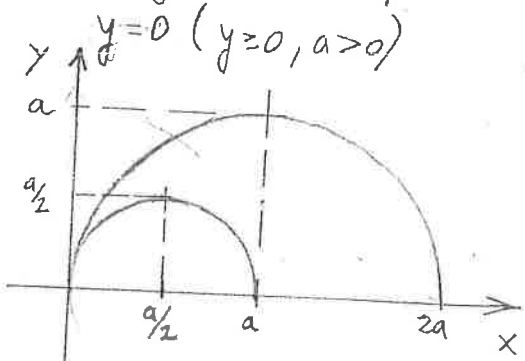


Nota:

$$2ax - x^2 - ax = ax - x^2 = x(a-x) \geq 0$$

$\forall 0 \leq x \leq a$ .

e) A regió limitada per les corbes  $x^2 + y^2 = ax$ ,  $x^2 + y^2 = 2ax$ ,  $y = 0$  ( $y \geq 0, a > 0$ )



$$I = \iint_A f(x,y) dx dy$$

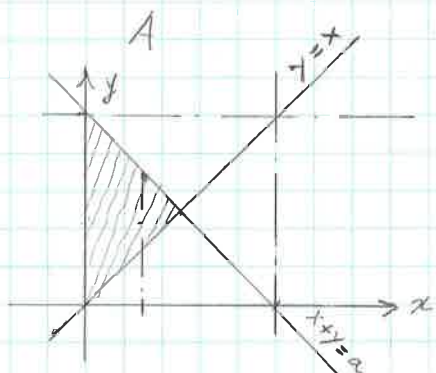
$$= \int_0^a dx \int_{\sqrt{ax-x^2}}^{\sqrt{2ax-x^2}} f(x,y) dy + \int_a^{2a} dx \int_0^{\sqrt{2ax-x^2}} f(x,y) dy$$

$$= \int_0^{a/2} dy \int_{a-\sqrt{a^2-y^2}}^{a/2-\sqrt{a^2/4-y^2}} f(x,y) dx + \int_{a/2}^a dy \int_{a/2+\sqrt{a^2/4-y^2}}^a f(x,y) dx$$

$$+ \int_0^{a/2} dy \int_a^{a+\sqrt{a^2-y^2}} f(x,y) dx + \int_{a/2}^a dy \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y) dx. \square$$

12. Calculen los siguientes integrales dobles en els dominis de  $\mathbb{R}^2$  que s'indiquen

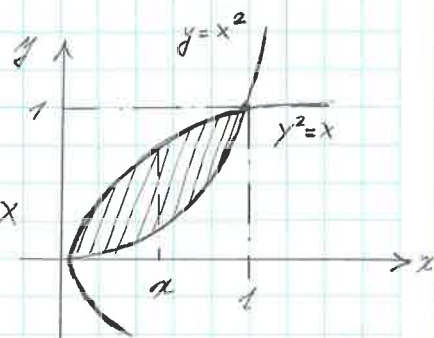
(a)  $I = \iint_A (x^2 + y^2) dx dy$  A limitat per les rectes  $y = x$ ,  $x + y = 2a$ ,  $x = 0$  ( $a > 0$ )



$$\begin{aligned}
 I &= \iint_A (x^2 + y^2) dx dy = \int_0^a dx \int_x^{2a-x} (x^2 + y^2) dy \\
 &= \int_0^a dx \left( x^2 y + \frac{y^3}{3} \right) \Big|_{y=x}^{y=2a-x} = \int_0^a \left( x^2(2a-x) + \frac{1}{3}(2a-x)^3 - x^3 - \frac{x^3}{3} \right) dx \\
 &\stackrel{(*)}{=} \int_0^a \left( \frac{8a^2}{3} - 4a^2 x + 4ax^2 - \frac{8}{3}x^3 \right) dx = \left[ \frac{8}{3}a^2 x - 2a^2 x^2 + \frac{4}{3}ax^3 - \frac{2}{3}x^4 \right]_0^a \\
 &= \boxed{\frac{4a^4}{3}} \quad \square
 \end{aligned}$$

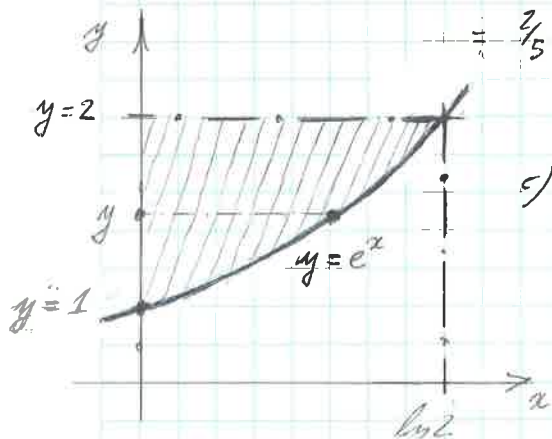
$$\begin{aligned}
 (*) \quad & 2ax^2 - x^3 + \frac{8}{3}a^3 - 4a^2x + 2ax^2 - \frac{x^3}{3} - x^3 - \frac{x^3}{3} \\
 &= \frac{8}{3}a^3 - 4a^2x + 4ax^2 - \frac{8}{3}x^3
 \end{aligned}$$

(b)  $I = \iint_A (x + 2y) dx dy$ , A limitat per les corbes  $y = x^2$ ,  $y^2 = x$



Solució

$$\begin{aligned}
 I &= \iint_A (x + 2y) dx dy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (x + 2y) dy = \int_0^1 dx \left( xy + y^2 \right) \Big|_{y=x^2}^{y=\sqrt{x}} = \int_0^1 \left( x^{3/2} + x - x^3 - x^4 \right) dx \\
 &= \frac{2}{5} x^{5/2} + \frac{x^2}{2} - \frac{x^4}{4} - \frac{x^5}{5} \Big|_0^1 = \frac{8+10-5-4}{20} = \boxed{\frac{9}{20}} \quad \square
 \end{aligned}$$



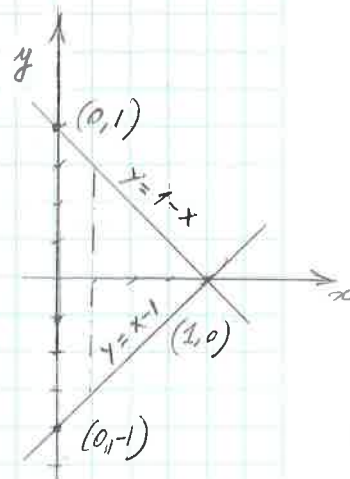
c)  $I = \iint_A e^{x+y} dx dy$ . A limitat per les corbes  $y = e^x$ ,  $x = 0$ ,  $y = 2$

$$\begin{aligned}
 I &= \int_1^2 e^y dy \int_0^{\ln y} e^x dx = \int_1^2 e^y dy \left[ e^x \right]_0^{\ln y} = \int_1^2 e^y (y-1) dy \\
 &= \left[ ye^y - 2e^y \right]_1^2 = 2e^2 - 2e - e + 2e = \boxed{e} \quad \square
 \end{aligned}$$

Alternativament:

$$I = \int_0^{\ln 2} e^z dx \int_{e^x}^2 e^y dy = \int_0^{\ln 2} e^x \left[ e^y \right]_{e^x}^2 dx = \int_0^{\ln 2} e^x (e^2 - e^{e^x}) dx$$

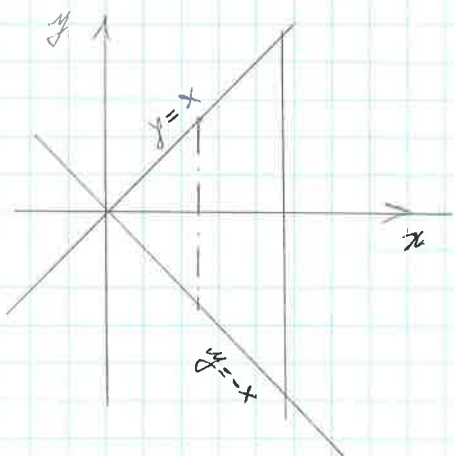
$$= \left( e^{2+x} - e^{e^x} \right) \Big|_0^{\ln 2} = 2e^2 - e^2 - e^2 + e = \boxed{e} \quad \square$$



d)  $I = \iint_A e^y dx dy$ , A triangle de vèrtexs:  $(1,0), (0,1), (0,-1)$

$$I = \iint_A e^y dx dy = \int_0^1 dx \int_{x-1}^{1-x} e^y dy = \int_0^1 dx (e^{1-x} - e^{x-1})$$

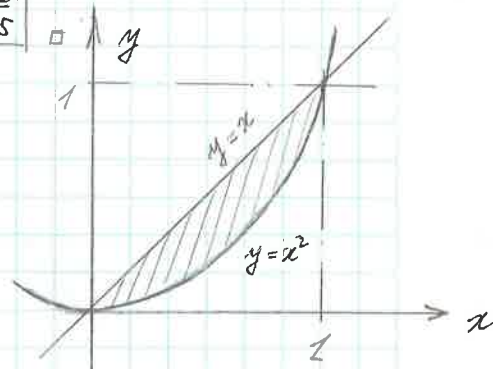
$$= -e^{1-x} - e^{x-1} \Big|_0^1 = \boxed{e + \frac{1}{e} - 2} \quad \square$$



e)  $I = \iint_A xy^2 dx dy$ , A limitat per les rectes  $x=1, x=y, x+y=0$ .

$$I = \iint_A xy^2 dx dy = \int_0^1 x dx \int_{-x}^x y^2 dy = \int_0^1 x \left( \frac{y^3}{3} \right)_{y=-x}^{y=x} dx$$

$$= \frac{2}{3} \int_0^1 x^4 dx = \frac{2}{15} \left( x^5 \right) \Big|_0^1 = \boxed{\frac{2}{15}} \quad \square$$



f)  $I = \iint_A xy dx dy$ . A limitat per les corbes  $x=y, y=x^2$

$$I = \iint_A xy dx dy = \int_0^1 x dx \int_{x^2}^x y dy = \int_0^1 x \left( \frac{y^2}{2} \right)_{x^2}^x dx$$

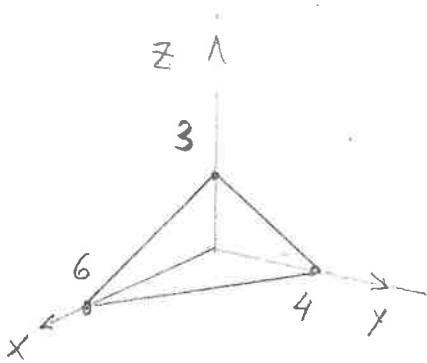
$$= \frac{1}{2} \int_0^1 (x^3 - x^5) dx = \frac{1}{2} \left( \frac{1}{4} - \frac{1}{6} \right) = \boxed{\frac{1}{24}} \quad \square$$

$$x=y, y=x^2$$

$$x=x^2 \Leftrightarrow x(x-1)=0: \begin{matrix} x=0 \\ x=1 \end{matrix}$$

14) Per les regions de  $\mathbb{R}^3$  indicades escriu la integral triple  $\iiint_A f(x,y,z) dx dy dz =: I$  en termes d'integrals iterades preses en diferents ordres.

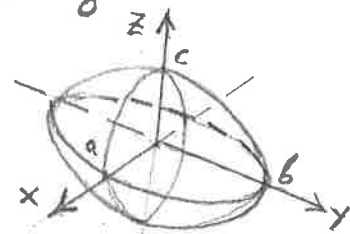
(a) A tetraedre limitat pels plans  $x=0, y=0, z=0, 2x+3y+4z=12$ .



$$\begin{aligned}
 I &= \int_0^6 dx \int_0^{4-\frac{2}{3}x} dy \int_0^{3-\frac{1}{2}x-\frac{3}{4}y} f(x,y,z) dz = \int_0^4 dy \int_0^{6-\frac{3}{2}y} dx \int_0^{3-\frac{1}{2}x-\frac{3}{4}y} f(x,y,z) dz \\
 &= \int_0^6 dx \int_0^{3-\frac{1}{2}x} dz \int_0^{4-\frac{2}{3}x-\frac{4}{3}z} f(x,y,z) dy = \int_0^3 dz \int_0^{6-2z} dx \int_0^{4-\frac{2}{3}x-\frac{4}{3}z} f(x,y,z) dy \\
 &= \int_0^4 dy \int_0^{3-\frac{3}{4}y} dz \int_0^{6-\frac{3}{2}y-2z} f(x,y,z) dx = \int_0^3 dz \int_0^{4-\frac{4}{3}z} dy \int_0^{6-\frac{3}{2}y-2z} f(x,y,z) dx. \square
 \end{aligned}$$

- $4z = 12 - 2x - 3y \rightarrow z = 3 - \frac{1}{2}x - \frac{3}{4}y$
- $z=0, 2x+3y=12 \rightarrow y = 4 - \frac{2}{3}x$
- $x = 6 - \frac{3}{2}y$

(b) A interior del el·lipsoide



- $3y = 12 - 2x - 4z \rightarrow y = 4 - \frac{2}{3}x - \frac{4}{3}z$
- $y=0, 2x+4z=12 \rightarrow z = 3 - \frac{1}{2}x$
- $x = 6 - 2z$

$$I = \int_{-a}^a dx \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} dy \int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} f(x,y,z) dz$$

- $2x = 12 - 3y - 4z \rightarrow x = 6 - \frac{3}{2}y - 2z$
- $x=0: 3y+4z=12 \rightarrow z = 3 - \frac{3}{4}y$
- $y = 4 - \frac{4}{3}z$

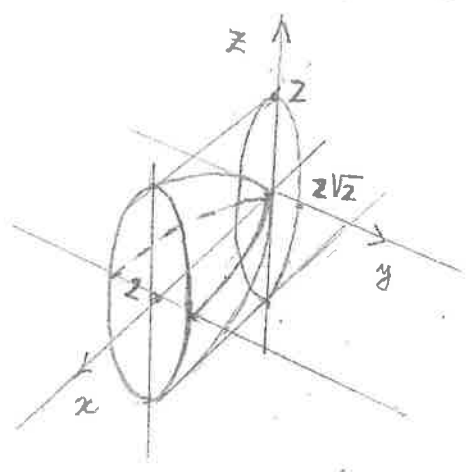
$$= \int_{-b}^b dy \int_{-a\sqrt{1-y^2/b^2}}^{a\sqrt{1-y^2/b^2}} dx \int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} f(x,y,z) dz$$

$$= \int_{-a}^a dx \int_{-c\sqrt{1-x^2/a^2}}^{c\sqrt{1-x^2/a^2}} dz \int_{-b\sqrt{1-x^2/a^2-z^2/c^2}}^{b\sqrt{1-x^2/a^2-z^2/c^2}} f(x,y,z) dy = \int_{-c}^c dz \int_{-a\sqrt{1-z^2/c^2}}^{a\sqrt{1-z^2/c^2}} dx \int_{-b\sqrt{1-x^2/a^2-z^2/c^2}}^{b\sqrt{1-x^2/a^2-z^2/c^2}} f(x,y,z) dy$$

$$= \int_{-b}^b dy \int_{-c\sqrt{1-y^2/b^2}}^{c\sqrt{1-y^2/b^2}} dz \int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z) dx = \int_{-c}^c dz \int_{-b\sqrt{1-z^2/c^2}}^{b\sqrt{1-z^2/c^2}} dy \int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z) dx. \square$$

$$\frac{y^2}{4} + \frac{z^2}{2} = x$$

(c) A cos limitat per les superfícies  $y^2 + 2z^2 = 4x$ ,  $x = 2$ .



$$I = \int_{-2}^2 dz \int_{-\sqrt{8-2z^2}}^{\sqrt{8-2z^2}} dy \int_{\frac{y^2}{4} + \frac{z^2}{2}}^2 f(x,y,z) dx = \int_{-2\sqrt{2}}^{2\sqrt{2}} dy \int_{-\sqrt{4-y^2/2}}^{\sqrt{4-y^2/2}} dz \int_{\frac{y^2}{4} + \frac{z^2}{2}}^2 f(x,y,z) dx$$

$$= \int_0^2 dx \int_{-\sqrt{2x}}^{\sqrt{2x}} dz \int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}} f(x,y,z) dy = \int_{-2}^2 dz \int_{z^2/2}^2 dx \int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}} f(x,y,z) dy$$

$$= \int_{-2\sqrt{2}}^{2\sqrt{2}} dy \int_{\frac{1}{4}y^2}^2 dx \int_{-\sqrt{2x-\frac{1}{2}y^2}}^{\sqrt{2x-\frac{1}{2}y^2}} f(x,y,z) dz = \int_0^2 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} dy \int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}} f(x,y,z) dz. \square$$

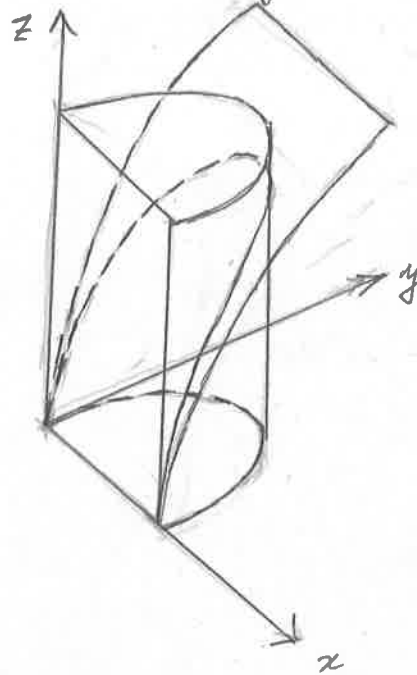
15) Calculeu les integrals triples següents en les regions de  $\mathbb{R}^3$  que s'indiquen

a)  $I = \iiint_A xz \, dx \, dy \, dz$ , A limitat pel cilindre de base circular  $x^2 + y^2 - 2x = 0$  i

la superfície:  $z^2 = 2y$  ( $y, z \geq 0$ )

$$\begin{aligned} I &= \int_0^2 x \, dx \int_0^{\sqrt{2x-x^2}} dy \int_0^{\sqrt{2y}} z \, dz \\ &= \int_0^2 x \, dx \int_0^{\sqrt{2x-x^2}} y \, dy \\ &= \int_0^2 x \left( \frac{y^2}{2} \right) \Big|_0^{\sqrt{2x-x^2}} dx = \int_0^2 \left( x^2 - \frac{x^3}{3} \right) dx \end{aligned}$$

$$= \left[ \frac{x^3}{3} - \frac{x^4}{8} \right]_0^2 = \frac{8}{3} - \frac{16}{8} = \frac{8}{3} - 2 = \frac{2}{3} \quad \square$$



b)  $I = \iiint_A zy \sqrt{x^2 + y^2} \, dz \, dy \, dx$ ,  $A = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq x^2 + y^2, 0 \leq y \leq \sqrt{2x - x^2}\}$

$$\begin{aligned} I &= \int_0^2 dx \int_0^{\sqrt{2x-x^2}} y \sqrt{x^2 + y^2} \, dy \int_0^{x^2 + y^2} z \, dz = \int_0^2 dx \int_0^{\sqrt{2x-x^2}} y \sqrt{x^2 + y^2} \left[ \frac{z^2}{2} \right]_0^{x^2 + y^2} dy \\ &= \frac{1}{2} \int_0^2 dx \int_0^{\sqrt{2x-x^2}} y (x^2 + y^2)^{3/2} dy = \frac{1}{14} \int_0^2 dx \left[ (x^2 + y^2)^{7/2} \right]_0^{\sqrt{2x-x^2}} = \frac{1}{14} \int_0^2 dx \left[ (x^2 + 2x - x^2)^{7/2} - x^7 \right] \\ &= \frac{1}{14} \left[ 2 \cdot 2^{7/2} \frac{x^{9/2}}{9} - \frac{x^8}{8} \right]_0^2 = \frac{1}{14} \cdot \frac{1}{72} \cdot (16 \cdot 2^8 - 9 \cdot 2^8) = \frac{1}{14} \cdot \frac{1}{72} \cdot 7 \cdot 2^8 = \frac{16}{9} \quad \square \end{aligned}$$

c)  $I = \iiint_A dx \, dy \, dz$ ,  $A = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x \leq 3, 1 \leq y \leq 3, 0 \leq z \leq xy\}$ .

$$I = \int_1^3 dx \int_1^3 dy \int_0^{xy} dz = \int_1^3 x \, dx \int_1^3 y \, dy = \left( \int_1^3 x \, dx \right)^2 = \left( \left[ \frac{x^2}{2} \right]_1^3 \right)^2 = \left( \frac{9}{2} - \frac{1}{2} \right)^2 = 16 \quad \square$$

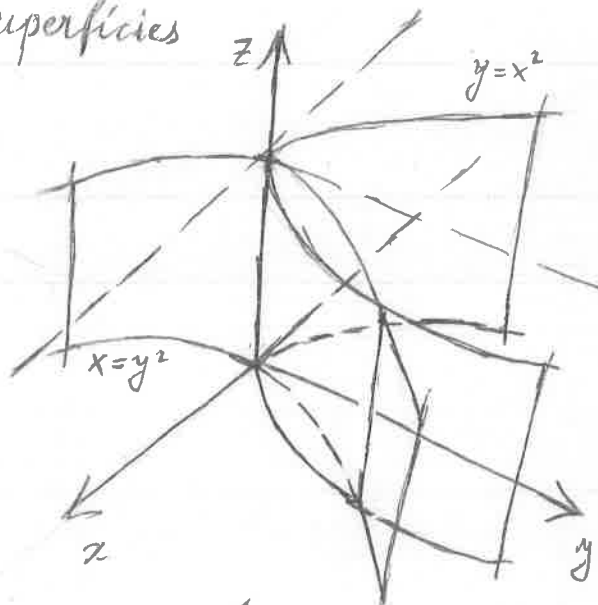
(d)  $I = \iiint_A xyz \, dx \, dy \, dz$ , A limitat per les superfícies

$$y = x^2, x = y^2, z = xy, z = 0$$

$$I = \int_0^1 x \, dx \int_{x^2}^{\sqrt{x}} y \, dy \int_0^{xy} z \, dz$$

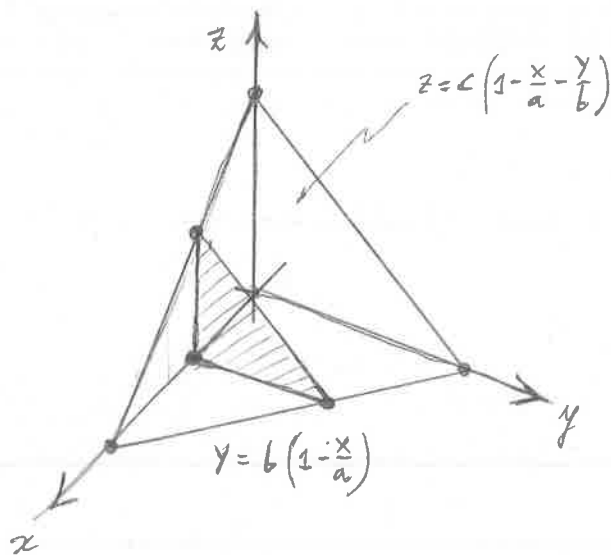
$$= \frac{1}{2} \int_0^1 x^3 \, dx \int_{x^2}^{\sqrt{x}} y^3 \, dy$$

$$= \frac{1}{8} \int_0^1 dx \, x^3 (x^2 - x^8) = \frac{1}{8} \int_0^1 (x^5 - x^{11}) \, dx = \frac{1}{8} \left( \frac{x^6}{6} - \frac{x^{12}}{12} \right) \Big|_0^1 = \frac{1}{8} \left( \frac{1}{6} - \frac{1}{12} \right) = \boxed{\frac{1}{96}} \quad \square$$



(e)  $I = \iiint_A x \, dx \, dy \, dz$ , A tetraedre format pels plans  $x=0, y=0, z=0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ,

amb  $a, b, c > 0$



$$I = \int_a^b x \, dx \int_0^{b(1-x/a)} dy \int_0^{c(1-x/a-y/b)} dz$$

$$= c \int_0^a x \, dx \int_0^{b(1-x/a)} dy (1 - x/a - y/b)$$

$$= c \int_0^a dx \, x \left( y - \frac{xy}{a} - \frac{y^2}{2b} \right) \Big|_0^{b(1-x/a)}$$

$$= c \int_0^a x \left[ b(1-x/a) - \frac{bx}{a}(1-x/a) - \frac{b}{2}(1-x/a)^2 \right] dx$$

$$(*) = \frac{cb}{2} \int_0^a x(1-x/a)^2 dx = \frac{cb}{2} \int_0^a x \left( 1 - \frac{2x}{a} + \frac{x^2}{a^2} \right) dx = \frac{cb}{2} \left( \frac{x^2}{2} - \frac{2x^3}{3a} + \frac{x^4}{4a^2} \right) \Big|_0^a$$

$$= \frac{cb}{2} \left( \frac{a^2}{2} - \frac{2a^2}{3} + \frac{a^2}{4} \right) = \boxed{\frac{cba^2}{24}}$$

$$(*) \left( 1 - \frac{x}{a} \right) \left( 1 - \frac{x}{a} - \frac{1}{2} + \frac{x}{2a} \right) = \frac{b}{2} \left( 1 - \frac{x}{a} \right) \left( 1 - \frac{x}{a} \right) = \frac{b}{2} \left( 1 - \frac{x}{a} \right)^2$$

16) Useu coordenades polars per calcular les següents integrals dobles.

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r.$$

$$\iint_{T(D)} f(x,y) \, dx \, dy = \iint_D f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.$$

a)  $I = \iint_A (x^2 + y^2) \, dx \, dy, \quad A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$

$$I = \int_0^{2\pi} d\theta \int_0^2 r^3 \, dr = 2\pi \left[ \frac{r^4}{4} \right]_0^2 = \boxed{8\pi}.$$

b)  $I = \iint_A \cos(x^2 + y^2) \, dx \, dy, \quad A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq \frac{\pi}{2}\}$

$$I = \int_0^{2\pi} d\theta \int_0^{\sqrt{\pi/2}} r \cos(r^2) \, dr = 2\pi \frac{1}{2} \left[ \sin(r^2) \right]_0^{\sqrt{\pi/2}} = \boxed{\pi}.$$

c)  $I = \iint_A \frac{(x+y)^2}{x^2 + y^2 + 2} \, dx \, dy, \quad A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

$$I = \int_0^{2\pi} d\theta \int_0^1 \frac{r^3}{r^2 + 2} \, dr = 2\pi \int_0^1 \left( 1 - \frac{r}{r^2 + 2} \right) \, dr = 2\pi \left[ \frac{r^2}{2} - \ln(r^2 + 2) \right]_0^1$$

$$= 2\pi \left[ \frac{1}{2} - \ln 3 + \ln 2 \right] = \boxed{2\pi \left[ \frac{1}{2} + \ln \left( \frac{2}{3} \right) \right]} \quad \square$$

d)  $I = \iint_A \frac{dx \, dy}{(1+x^2+y^2)^2 \sqrt{x^2+y^2}}, \quad A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq R^2\}$ . (Indicació: usen propietats elementals del sin i cos per veure que  $\sin(\arctan(R)) = \frac{R}{\sqrt{1+R^2}}$  i  $\cos(\arctan(R)) = \frac{1}{\sqrt{1+R^2}}$ ).

$$I = \int_0^{2\pi} d\theta \int_0^R \frac{r dr}{(1+r^2)^2 \sqrt{r^2}} = 2\pi \int_0^R \frac{dr}{(1+r^2)^2} = \begin{cases} r = \operatorname{tg} t \Rightarrow dr = \frac{dt}{\cos^2 t} \\ r=0 \Rightarrow t=0 \\ r=R \Rightarrow t = \arctan(R) \end{cases}$$

$$= 2\pi \int_0^{\arctan(R)} \frac{\cos^4 t}{\cos^2 t} dt = 2\pi \int_0^{\arctan(R)} \frac{1+\cos(2t)}{2} dt = 2\pi \left( \frac{t}{2} + \frac{\sin(2t)}{4} \right) \Big|_0^{\arctan(R)}$$

$$\stackrel{(*)}{=} \boxed{\pi \left( \arctan(R) + \frac{R}{1+R^2} \right)} \quad \square \quad (*) \text{ Notem que:}$$

$$\sin(2t) = 2 \sin t \cos t = \frac{2 \sin t \cos^2 t}{\cos t}$$

$$e) I = \iint_A \sqrt{x^2+y^2-9} dx dy, A = \{(x,y) \in \mathbb{R}^2 : 9 \leq x^2+y^2 \leq 25\}, = 2 \operatorname{tg} t \cos^2 t = \frac{2 \operatorname{tg} t}{1+\operatorname{tg}^2 t}$$

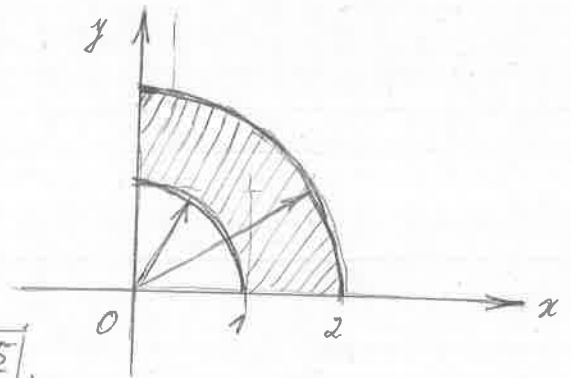
$$I = \int_0^{2\pi} d\theta \int_3^5 \sqrt{r^2-9} r dr = \frac{2\pi}{3} (r^2-9)^{3/2} \Big|_3^5 = \frac{2\pi}{3} 4^3 = \boxed{\frac{128\pi}{3}} \quad \square$$

f)  $I = \iint_A xy dx dy$ , A intersecció amb el 1<sup>er</sup> quadrant de la corona circular de centre (0,0), radi interior 1 i radi exterior 2

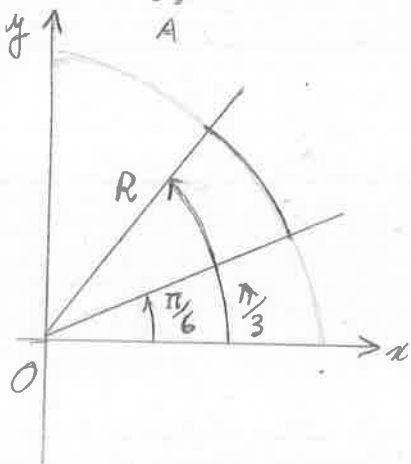
$$I = \int_0^{\pi/2} d\theta \int_1^2 r^3 \cos\theta \sin\theta dr$$

$$= \frac{1}{2} \int_0^{\pi/2} \sin(2\theta) d\theta \int_1^2 r^3 dr$$

$$= \left[ -\frac{1}{2} \cos(2\theta) \right]_0^{\pi/2} \cdot \left[ \frac{r^4}{4} \right]_1^2 = \frac{1}{2} \left( 4 - \frac{1}{4} \right) = \boxed{\frac{15}{8}} \quad \square$$



g)  $I = \iint_A x(x^2+y^2) dx dy$ , A sector circular de centre (0,0) i radi R format per angles entre  $\frac{\pi}{3}$  i  $\frac{\pi}{6}$  amb l'eix x positiu.



$$I = \int_{\pi/6}^{\pi/3} d\theta \int_0^R r^4 \cos\theta dr = \int_{\pi/6}^{\pi/3} \cos\theta d\theta \int_0^R r^4 dr$$

$$= \left[ \sin\theta \right]_{\pi/6}^{\pi/3} \cdot \left[ \frac{r^5}{5} \right]_0^R = \boxed{\frac{R^5}{10} (\sqrt{3}-1)}$$

20.) Useu coordenades esfèriques per calcular les següents integrals triples

(a)  $I = \iiint_B x^4 y^2 z^3 dx dy dz, B = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2\} = B_a(0,0,0) \quad (a > 0)$

Solució:  $I = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \cos^4 \theta \sin^2 \theta d\theta \int_{-\pi/2}^{\pi/2} \cos^2 \varphi \sin^3 \varphi d\varphi \int_0^a r^4 dr = 0. \square$

(b)  $I = \iiint_B z(x^2 + y^2) dx dy dz, B = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2, z \geq 0\}$

Solució:  $I = \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos^3 \varphi \sin \varphi \int_0^a r^5 dr = 2\pi \left[ -\frac{\cos^4 \varphi}{4} \right]_0^{\pi/2} \cdot \left[ \frac{r^6}{6} \right]_0^a = \boxed{\frac{\pi a^6}{12}} \square$

(c)  $I = \iiint_B \frac{dx dy dz}{(x^2 + y^2 + z^2)^{3/2}}, B = \{(x,y,z) \in \mathbb{R}^3 : a^2 \leq x^2 + y^2 + z^2 \leq b^2\}$

Solució:  $I = \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos \varphi d\varphi \int_a^b r^2 \frac{dr}{r^3} = 4\pi \ln\left(\frac{b}{a}\right)$

(d)  $I = \iiint_B \sqrt{x^2 + y^2 + z^2} e^{-(x^2 + y^2 + z^2)}, B$  el domini de l'apartat anterior.

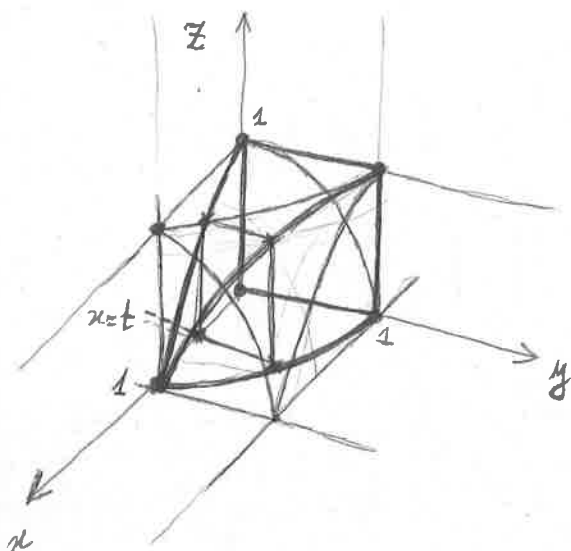
Solució:  $I = \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos \varphi d\varphi \int_a^b r^3 e^{-r^2} dr = 4\pi \int_a^b r^3 e^{-r^2} dr = 2\pi \left( e^{-a^2} (a^2 + 1) - e^{-b^2} (b^2 + 1) \right)$

(\*) Parts:  $\int_a^b r^3 e^{-r^2} dr = \frac{1}{2} \int_a^b r^2 d(e^{-r^2}) = \left[ -\frac{r^2 e^{-r^2}}{2} \right]_a^b + \int_a^b r e^{-r^2} dr =$   
 $= \frac{a^2 e^{-a^2}}{2} - \frac{b^2 e^{-b^2}}{2} + \frac{e^{-a^2}}{2} - \frac{e^{-b^2}}{2} = \frac{1}{2} (a^2 + 1) e^{-a^2} - \frac{1}{2} (b^2 + 1) e^{-b^2}$



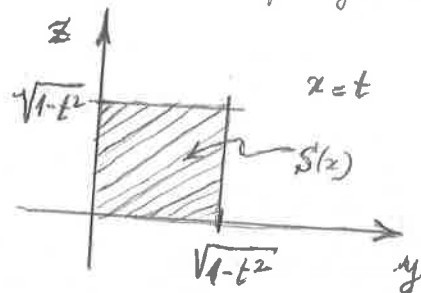
23) Usen coordenades cartesianes, cilíndriques o esfèriques (o bé el principi de Cavalieri per calcular el volum dels dominis de  $\mathbb{R}^3$  limitats per les superfícies que s'indiquen.

a)  $x^2+z^2=1, x^2+y^2=1$



Parametrització de la intersecció per  $y \geq 0, z \geq 0$ .

$x=t$   
 $y=\sqrt{1-t^2}$   
 $z=\sqrt{1-t^2}$



Per tant:  $S(x) = 1-x^2$

$$V = 8 \int_0^1 S(x) dx = 8 \int_0^1 (1-x^2) dx$$
$$= 8 \left[ x - \frac{x^3}{3} \right]_0^1 = 8 \left( 1 - \frac{1}{3} \right) = \boxed{\frac{16}{3}}$$

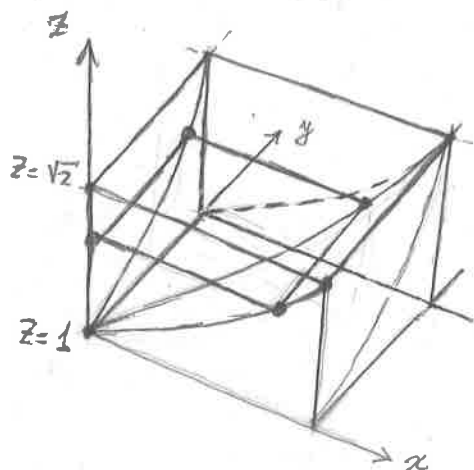
I ho podem calcular com una integral triple. En efecte, en el 1<sup>er</sup> octant (veure figura):  $0 \leq z \leq \sqrt{1-x^2}, 0 \leq y \leq \sqrt{1-x^2}, 0 \leq x \leq 1$ . Aleshores

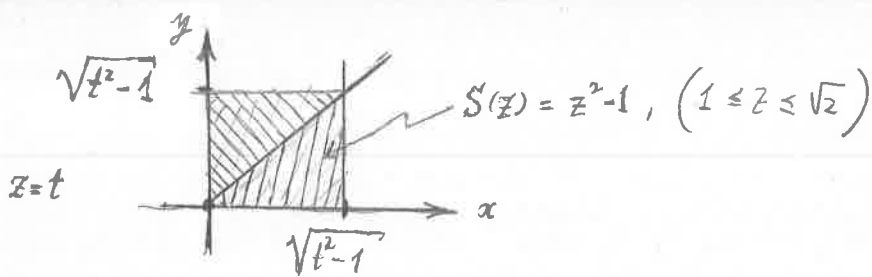
$$V = 8 \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2}} dz$$
$$= 8 \int_0^1 (1-x^2) dx$$
$$= 8 \left( x - \frac{x^3}{3} \right) \Big|_{x=0}^{x=1} = \boxed{\frac{16}{3}}$$

b)  $z^2-x^2=1$   
 $z^2-y^2=1$   
 $z=\sqrt{2}$

Parametrització de la intersecció per  $x \geq 0, y \geq 0, z \geq 0$ .

$x=\sqrt{t^2-1}, y=\sqrt{t^2-1}, z=t, (t \geq 1)$ .





aplicant el principi de Cavalieri:

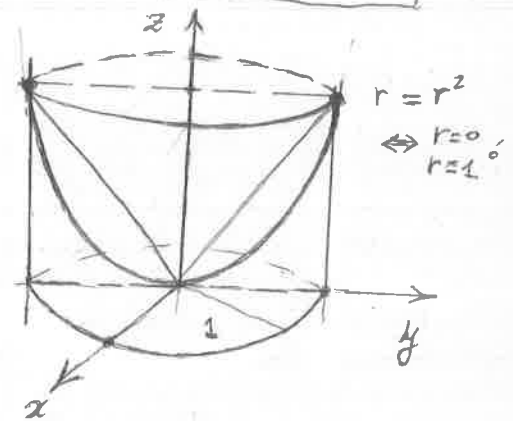
$$V = 4 \cdot \int_1^{\sqrt{2}} S(z) dz = 4 \cdot \int_1^{\sqrt{2}} (z^2 - 1) dz = 4 \left( \frac{z^3}{3} - z \right) \Big|_1^{\sqrt{2}} = 4 \left( \frac{2\sqrt{2}}{3} - \sqrt{2} + \frac{2}{3} \right) = \boxed{\frac{4}{3} (2 - \sqrt{2})} \quad \square$$

Alternativament:

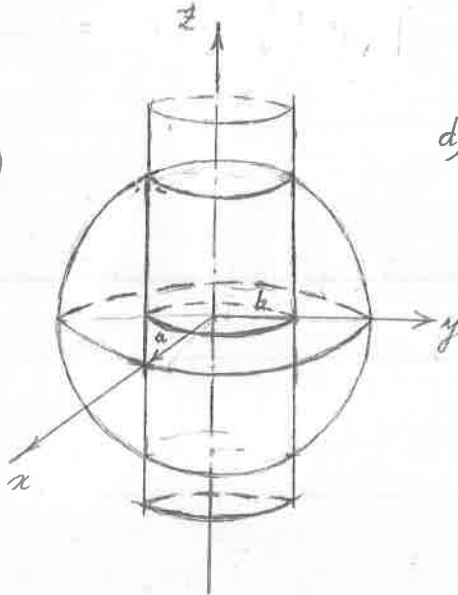
$$\begin{aligned} \frac{V}{8} &= \int_0^1 dx \int_0^x dy \int_{\sqrt{1+x^2}}^{\sqrt{2}} dz = \int_0^1 dx \int_0^x (\sqrt{2} - \sqrt{1+x^2}) dy = \int_0^1 (\sqrt{2}x - x\sqrt{1+x^2}) dx \\ &= \left[ -\frac{1}{3}(1+x^2)^{3/2} + \frac{\sqrt{2}}{2}x^2 \right]_0^1 = -\frac{1}{3}2\sqrt{2} + \frac{\sqrt{2}}{2} + \frac{1}{3} = \frac{1}{6}(2 - \sqrt{2}) \Rightarrow \boxed{V = \frac{4}{3}(2 - \sqrt{2})} \end{aligned}$$

c)  $z^2 = x^2 + y^2, z = x^2 + y^2, (z \geq 0)$

$$\begin{aligned} V &= \int_0^{2\pi} d\theta \int_0^1 r dr \int_{r^2}^r dz = 2\pi \int_0^1 (r^2 - r^3) dr = 2\pi \left( \frac{1}{3} - \frac{1}{4} \right) \\ &= \boxed{\frac{\pi}{6}} \quad \square \end{aligned}$$



d)



d) Part de l'esfera  $x^2 + y^2 + z^2 = a^2$  que és exterior al cilindre  $x^2 + y^2 = b^2$  ( $a > b > 0$ ).

$$\begin{aligned} \frac{V}{2} &= \int_0^{2\pi} d\theta \int_b^a r dr \int_0^{\sqrt{a^2 - r^2}} dz = 2\pi \int_b^a r (a^2 - r^2)^{1/2} dr \\ &= -\frac{2}{3} (a^2 - r^2)^{3/2} \pi \Big|_b^a = \frac{2\pi}{3} (a^2 - b^2)^{3/2} \Rightarrow \boxed{V = \frac{4\pi}{3} (a^2 - b^2)^{3/2}} \end{aligned}$$

e)  $z = x^2 - 4x + 1, \quad 1 - z = x^2 + y^2$

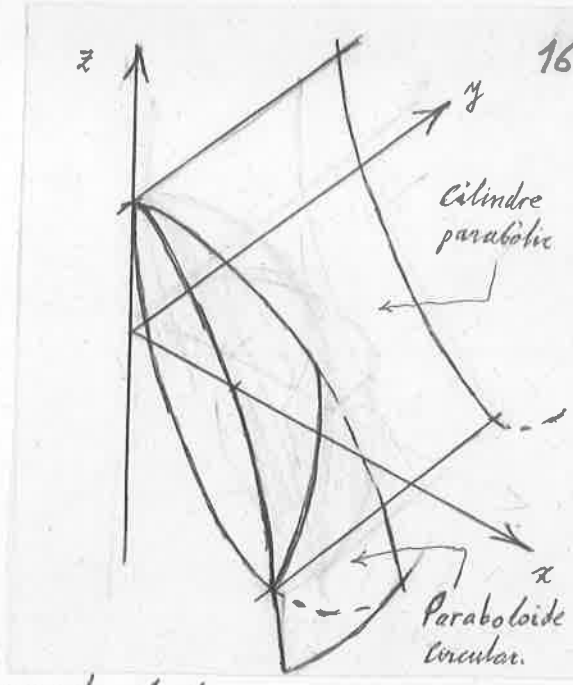
Intersecció de les superfícies per  $y \geq 0$

$1 - x^2 - y^2 = x^2 - 4x + 1 \Rightarrow y^2 = 4x - 2x^2$

quan,  $y \geq 0$ , llavors  $y = \sqrt{4x - 2x^2}$ , i tenim la següent parametrització de la intersecció de les superfícies (per  $y \geq 0$ )

$$\left. \begin{aligned} x &= t \\ y &= \sqrt{4t - 2t^2} \\ z &= t^2 - 4t + 1 \end{aligned} \right\} \text{ amb } 0 \leq t \leq 2.$$

La projecció d'aquesta parametrització sobre el pla  $z=0$  ve donada per:  $y = \sqrt{4x - 2x^2}$ . Amb la qual cosa, per calcular el volum, fem:



Aleshores:

$$\begin{aligned} V/2 &= \int_0^2 dx \int_0^{\sqrt{4x-x^2}} dy \int_{x^2-4x+1}^{1-x^2-y^2} dz = \int_0^2 dx \int_0^{\sqrt{4x-2x^2}} (1-z^2-y^2-x^2+4x-1) dy = \int_0^2 dx \int_0^{\sqrt{4x-2x^2}} (4x-2x^2-y^2) dy \\ &= \int_0^2 dx \left( 4xy - 2x^2y - \frac{y^3}{3} \right) \Big|_0^{\sqrt{4x-2x^2}} = \int_0^2 dx \left( 4x\sqrt{4x-2x^2} - 2x^2\sqrt{4x-2x^2} - \frac{1}{3}(4x-2x^2)\sqrt{4x-2x^2} \right) \\ &= \int_0^2 \sqrt{4x-2x^2} \left( 4x - 2x^2 - \frac{4}{3}x + \frac{2}{3}x^2 \right) dx = \frac{2}{3} \int_0^2 (4x-2x^2)^{3/2} dx = \frac{2}{3} \int_0^2 x^{3/2} \left( 1 - \frac{x}{2} \right)^{3/2} dx \\ &= \left\{ \begin{aligned} x &= 2 \sin^2 t \\ dx &= 4 \sin t \cos t dt \end{aligned} \right\} = \frac{64}{3} 2^{3/2} \int_0^{\pi/2} \sin^4 t \cos^4 t dt = \frac{2 \cdot 2^6}{3 \cdot 2^4} \int_0^{\pi/2} \sin^4(2t) dt \\ &\stackrel{(*)}{=} \frac{8\sqrt{3}}{3} \cdot \frac{3\pi}{16} = \frac{\pi\sqrt{3}}{2} \Rightarrow \boxed{V = \pi\sqrt{3}} \end{aligned}$$

(\*)  $\int_0^{\pi/2} \sin^4(2t) dt = \int_0^{\pi/2} \sin^2(2t) (1 - \cos^2(2t)) dt = \int_0^{\pi/2} \left[ \sin^2(2t) - \frac{1}{4} \sin^2(4t) \right] dt$

$$= \int_0^{\pi/2} \left[ \frac{1 - \cos(4t)}{2} - \frac{1 - \cos(8t)}{8} \right] dt = \frac{\pi}{4} - \frac{\pi}{16} = \frac{3\pi}{16}$$

$$f) \quad x^2 = z, \quad y^2 = x, \quad z^2 = y$$

$$x^2 = az, \quad y^2 = ax, \quad z^2 = ay \quad (a > 1)$$

$$D: \left. \begin{array}{l} \frac{y^2}{a} \leq x \leq y^2 \\ \frac{z^2}{a} \leq y \leq z^2 \\ \frac{x^2}{a} \leq z \leq x^2 \end{array} \right\} \text{ Així suggereix el canvi següent:}$$

$$u = \frac{x}{y^2}, \quad v = \frac{y}{z^2}, \quad w = \frac{z}{x^2}$$

$$\text{ llavors } D': \frac{1}{a} \leq u \leq 1, \frac{1}{a} \leq v \leq 1, \frac{1}{a} \leq w \leq 1 \text{ és el domini en}$$

Càlcul del Jacobini de la transformació:

$$(x, y, z) \mapsto (u, v, w) = f(x, y, z) = \left( \frac{x}{y^2}, \frac{y}{z^2}, \frac{z}{x^2} \right), \text{ donc podem calcular } Df(x, y, z),$$

però de fet, necessitem:  $(u, v, w) \mapsto (x, y, z) = f^{-1}(u, v, w)$  i el determinant de la corresponent matriu Jacobiana, i.e.  $\det Df^{-1}(u, v, w)$ .

$$f \circ f^{-1}(u, v, w) = (u, v, w) \Rightarrow Df \circ f^{-1}(u, v, w) Df^{-1}(u, v, w) = Df(x, y, z) \cdot Df^{-1}(u, v, w) = Id$$

$$\Rightarrow Df^{-1}(u, v, w) = Df(x, y, z)^{-1} \Rightarrow \det Df^{-1}(u, v, w) = \frac{1}{\det Df(x, y, z)}$$

(i.e., apliquem la regla de la cadena a la identitat:  $f \circ f^{-1} = Id$ ). Fent els càlculs:

$$\det Df(x, y, z) = \begin{vmatrix} \frac{1}{y^2} & -\frac{2x}{y^3} & 0 \\ 0 & \frac{1}{z^2} & -\frac{2y}{z^3} \\ -\frac{2z}{x^3} & 0 & \frac{1}{x^2} \end{vmatrix} = \frac{1}{x^3 y^3 z^3} \begin{vmatrix} y & -2x & 0 \\ 0 & z & -2y \\ -2z & 0 & x \end{vmatrix} = \frac{1}{x^3 y^3 z^3} (xy^2z - 8x^2yz) =$$

$$= -\frac{7}{x^2 y^2 z^2} \Rightarrow \left| \det Df^{-1}(u, v, w) \right| = \frac{x^2 y^2 z^2}{7} = \frac{1}{7} \frac{1}{u^2 v^2 w^2}$$

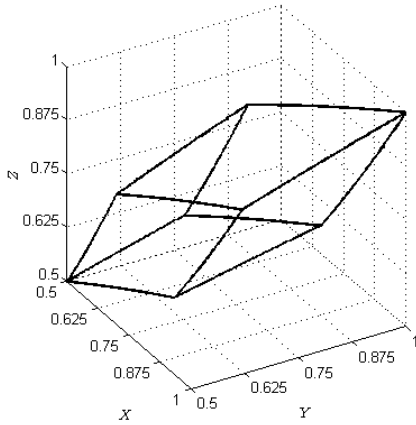
Aleshores:

$$V = \iiint_D dx dy dz = \iiint_{D'} \left| \det Df^{-1}(u, v, w) \right| du dv dw = \frac{1}{7} \iiint_{D'} \frac{du dv dw}{u^2 v^2 w^2}$$

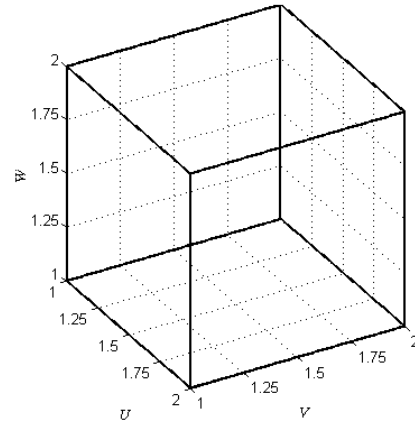
$$D' = f(D) \quad D' = \left[ \frac{1}{a}, 1 \right] \times \left[ \frac{1}{a}, 1 \right] \times \left[ \frac{1}{a}, 1 \right]$$

$$= \frac{1}{7} \int_{\frac{1}{a}}^1 \frac{du}{u^2} \int_{\frac{1}{a}}^1 \frac{dv}{v^2} \int_{\frac{1}{a}}^1 \frac{dw}{w^2} = \frac{1}{7} \left( \int_{\frac{1}{a}}^1 \frac{du}{u^2} \right)^3 = \frac{1}{7} \left( \left[ -\frac{1}{u} \right]_{\frac{1}{a}}^1 \right)^3 = \frac{(a-1)^3}{7} \quad \square$$

Nota: amb el canvi:  $1 \leq u = \frac{y^2}{x} \leq a$ ,  $1 \leq v = \frac{x^2}{y} \leq a$ ,  $1 \leq w = \frac{z^2}{z} \leq a$  el domini  $D$  es transforma en  $D' = [1, a] \times [1, a] \times [1, a]$ . Llavors el determinant del Jacobini corresponent resulta  $\det \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{1}{\det \frac{\partial(u, v, w)}{\partial(x, y, z)}} = -\frac{1}{7}$ .



(a)  $\mathcal{D}$ : Domini definit per les superfícies,  
 $x^2 = z, \quad y^2 = x, \quad z^2 = y,$   
 $x^2 = az, \quad y^2 = ax, \quad z^2 = ay.$



(b)  $\mathcal{D}' : 1 \leq u \leq a, 1 \leq v \leq a, 1 \leq w \leq a.$

FIGURA 1. Transformació del domini  $\mathcal{D}$  en  $\mathcal{D}'$  pel canvi (1).

#### NOTA

Alternativament, es comprova d'immediat que el canvi

$$u = \frac{y^2}{x}, \quad v = \frac{z^2}{y}, \quad w = \frac{x^2}{z}, \quad (1)$$

transforma el domini original  $\mathcal{D}$ , en  $\mathcal{D}' = [1, a] \times [1, a] \times [1, a]$ . És a dir, en un cub d'aresta  $a - 1$  (veure Figura 1). Aleshores el Jacobià corresponent surt més senzill. En efecte:

$$\det \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} -\frac{y^2}{x^2} & 2\frac{y}{x} & 0 \\ 0 & -\frac{z^2}{y^2} & 2\frac{z}{y} \\ 2\frac{x}{z} & 0 & -\frac{x^2}{z^2} \end{vmatrix} = -7,$$

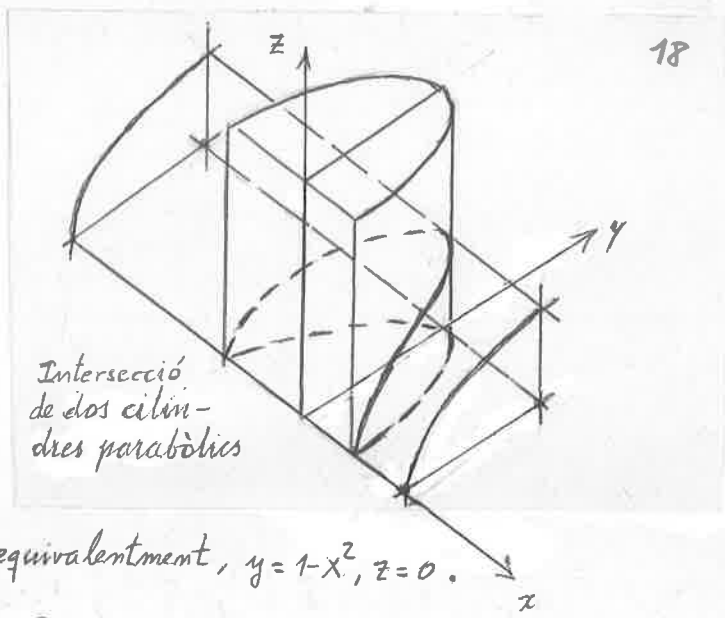
d'on:

$$\left| \det \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \frac{1}{\left| \det \frac{\partial(u, v, w)}{\partial(x, y, z)} \right|} = \frac{1}{7}.$$

Lavors el càlcul del volum es simplifica encara més:

$$\begin{aligned} V &= \iiint_{\mathcal{D}} dx \, dy \, dz = \iiint_{\mathcal{D}'} \left| \det \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du \, dv \, dw \\ &= \frac{1}{7} \int_1^a du \int_1^a dv \int_1^a dw = \frac{1}{7} \left( \int_1^a du \right)^3 = \frac{(a-1)^3}{7}. \end{aligned}$$

g)  $z^2 = y, x^2 = 1 - y$



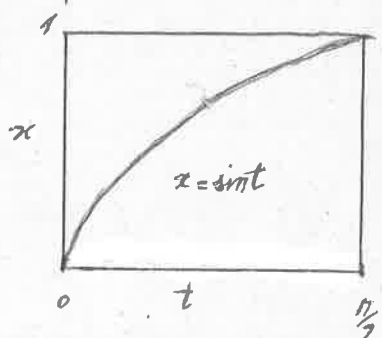
parametrització de la intersecció per a  $x \geq 0, z \geq 0$

$$\left. \begin{aligned} x &= \sqrt{1-t} \\ y &= t \\ z &= \sqrt{t} \end{aligned} \right\} 0 \leq t \leq 1$$

Projecció sobre el pla  $z=0$ :  
 $x = \sqrt{1-t}, y = t, z = 0$ ; o, equivalentment,  $y = 1 - x^2, z = 0$ .

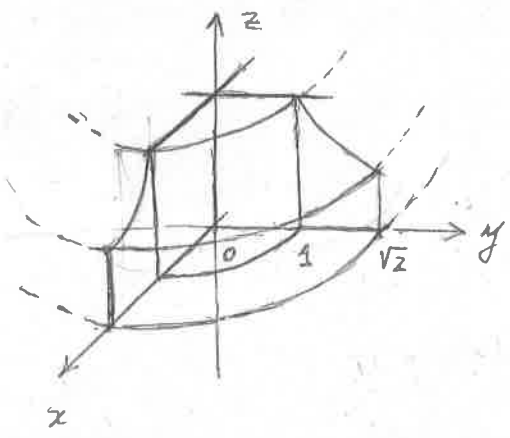
Aleshores:

$$\begin{aligned} V_{1/4} &= \int_0^1 dx \int_0^{1-x^2} dy \int_0^{\sqrt{y}} dz = \int_0^1 dx \int_0^{1-x^2} dy \sqrt{y} = \frac{2}{3} \int_0^1 dx \left[ y^{3/2} \right]_0^{1-x^2} \\ &= \frac{2}{3} \int_0^1 (1-x^2)^{3/2} dx = \left\{ \begin{array}{l} \text{c.v. (x)} \\ x = \sin t \\ dx = \cos t dt \end{array} \right\} = \frac{2}{3} \int_0^{\pi/2} \cos^4 t dt = \frac{\pi}{8} \Rightarrow \boxed{V = \frac{\pi}{2}} \end{aligned}$$



h)  $x^2 + y^2 = 1, x^2 + y^2 = 2, z(x^2 + y^2) = 1, z = 0$ .

En coordenades cilíndriques:  $1 \leq r \leq \sqrt{2}, 0 \leq z \leq \frac{1}{r^2}$

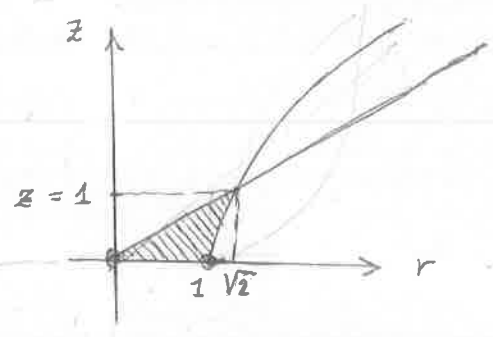


$$\begin{aligned} V &= \int_0^{2\pi} d\theta \int_1^{\sqrt{2}} r dr \int_0^{1/r^2} dz = 2\pi \int_1^{\sqrt{2}} \frac{dr}{r} \\ &= 2\pi \left[ \ln r \right]_1^{\sqrt{2}} = \boxed{\pi \ln 2} \end{aligned}$$

(\*) Canvi de variables a la integral

i)  $x^2 + y^2 = z^2, x^2 + y^2 = z^2 + 1 (x \geq 0, y \geq 0, z \geq 0)$

Apliquem coordenades cilíndriques (pàg. següent)



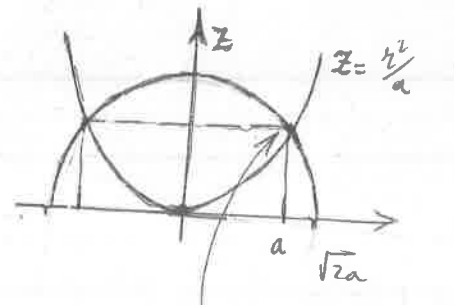
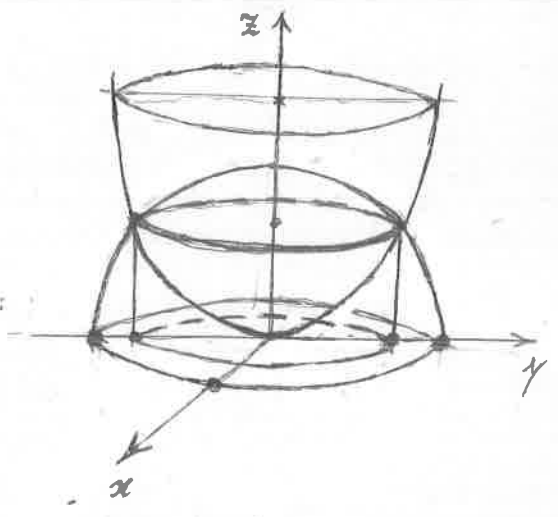
$$\begin{aligned}
 V &= \int_0^{\pi/2} d\theta \int_0^1 r dr \int_0^{r/\sqrt{2}} dz \\
 &+ \int_0^{\pi/2} d\theta \int_1^{\sqrt{2}} r dr \int_{\sqrt{r^2-1}}^{r/\sqrt{2}} dz \\
 &= \frac{\pi}{2} \int_0^1 \frac{r^2}{\sqrt{2}} dr + \frac{\pi}{2} \int_1^{\sqrt{2}} \left( \frac{r^2}{\sqrt{2}} - r\sqrt{r^2-1} \right) dr \\
 &= \frac{\pi}{4} \sqrt{2} \left[ \frac{r^3}{3} \right]_0^1 + \frac{\pi}{2} \left[ \frac{\sqrt{2}}{2} \cdot \frac{r^3}{3} - \frac{1}{3} (r^2-1)^{3/2} \right]_1^{\sqrt{2}} = \frac{\sqrt{2}\pi}{12} + \frac{\pi}{2} \left[ \frac{2}{3} - \frac{1}{3} - \frac{\sqrt{2}}{6} \right] = \boxed{\frac{\pi}{6}}
 \end{aligned}$$

Alternativement (més 'facil!'):

$$\begin{aligned}
 V &= \int_0^{\pi/2} d\theta \int_0^1 dz \int_{\sqrt{z^2+1}}^1 r dr = \frac{\pi}{4} \int_0^1 (z^2+1 - z z^2) dz = \frac{\pi}{4} \int_0^1 (1-z^2) dz \\
 &= \frac{\pi}{4} \left[ z - \frac{z^3}{3} \right]_0^1 = \frac{\pi}{4} \left( 1 - \frac{1}{3} \right) = \boxed{\frac{\pi}{6}}
 \end{aligned}$$

(j)  $x^2+y^2+z^2 \leq 2a^2$ ,  $z \geq \frac{x^2+y^2}{a}$  ( $a > 0$ ). En cylindriques:

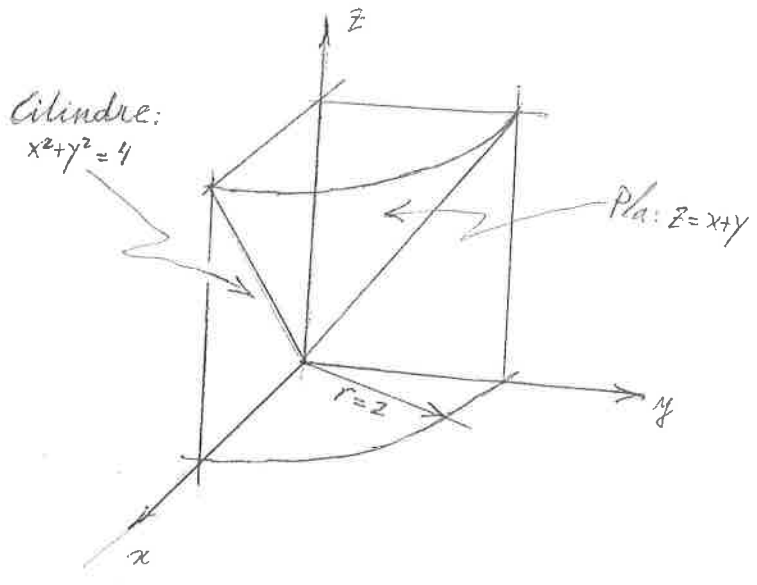
$$\begin{aligned}
 V &= \int_0^{2\pi} d\theta \int_0^a r dr \int_{\frac{r^2}{a}}^{\sqrt{2a^2-r^2}} dz = 2\pi \int_0^a \left( r\sqrt{2a^2-r^2} - \frac{r^3}{a} \right) dr \\
 &= 2\pi \left[ -\frac{1}{3} (2a^2-r^2)^{3/2} - \frac{r^4}{4a} \right]_0^a \\
 &= 2\pi \left[ \frac{1}{3} (2\sqrt{2}-1) - \frac{1}{4} \right] a^3 = \boxed{2\pi a^3 \left( \frac{2^{3/2}}{3} - \frac{7}{12} \right)}
 \end{aligned}$$



$$\begin{aligned}
 &az + z^2 = 2a^2 \\
 &\Leftrightarrow z^2 + az - 2a^2 = 0 \\
 &\text{d'où:} \\
 &z = \frac{-a \pm \sqrt{a^2 + 8a^2}}{2} = \frac{-a \pm 3a}{2} \\
 &= \begin{cases} a \rightarrow r=a \\ -2a \text{ (No)} \end{cases}
 \end{aligned}$$

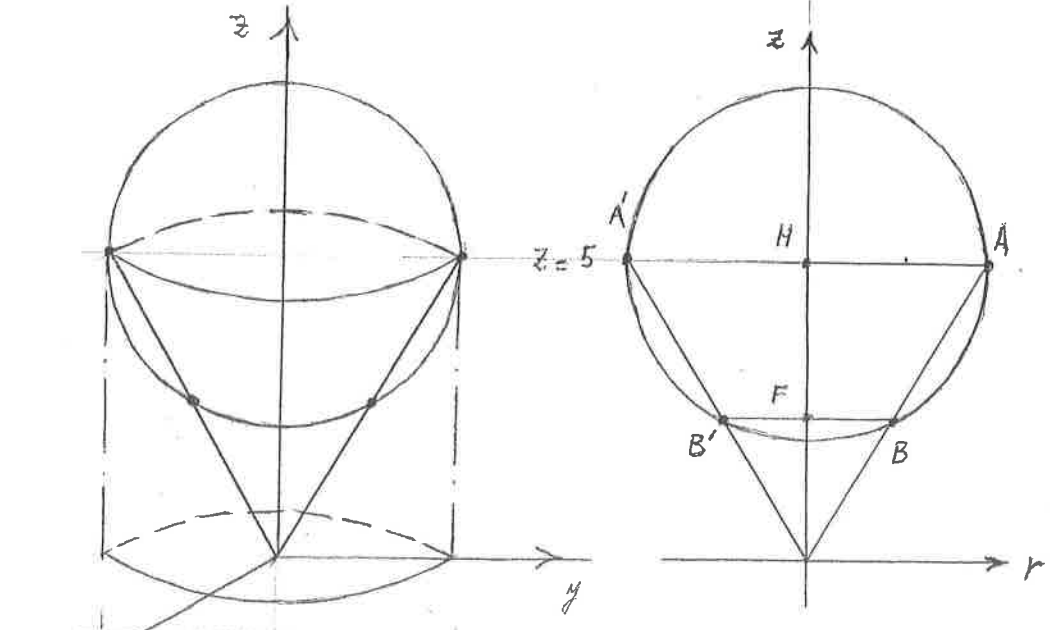
(K)  $x^2 + y^2 = 4, z = x + y$  ( $x \geq 0, y \geq 0, z \geq 0$ )

$$\begin{aligned}
 V &= \int_0^{\pi/2} d\theta \int_0^z r dr \int_0^z r \cos\theta + r \sin\theta dz \\
 &= \int_0^{\pi/2} d\theta \int_0^z r^2 (\cos\theta + \sin\theta) dr \\
 &= \int_0^{\pi/2} (\cos\theta + \sin\theta) d\theta \int_0^z r^2 dr \\
 &= \left[ \sin\theta - \cos\theta \right]_0^{\pi/2} \cdot \left[ \frac{r^3}{3} \right]_0^z = \boxed{\frac{16}{3}} \square
 \end{aligned}$$



Exercici: substituir  $z = x + y \rightarrow z = x + y - 1$  i repetir el càlcul...

(L) Con de gelat definit per:  $x^2 + y^2 \leq \frac{z^2}{5}, 0 \leq z \leq 5 + \sqrt{5 - x^2 - y^2}$



Alçada dels punts d'intersecció A, B, A', B'

$$\begin{aligned}
 (z-5)^2 &= 5 - \frac{z^2}{5} \\
 \Leftrightarrow 6z^2 - 50z + 100 &= 0.
 \end{aligned}$$

d'on,  $z = 5$  : alçada de A, A'  
 $z = \frac{10}{3}$  : " " B, B'

i els 'radis' respectius són :

$$\begin{aligned}
 r &= \sqrt{5} : \text{distància } \overline{HA} = \overline{A'H} \\
 r &= \frac{2\sqrt{5}}{3} : \text{" } \overline{FB} = \overline{B'F}
 \end{aligned}$$

Aleshores el volum del cos ve donat per

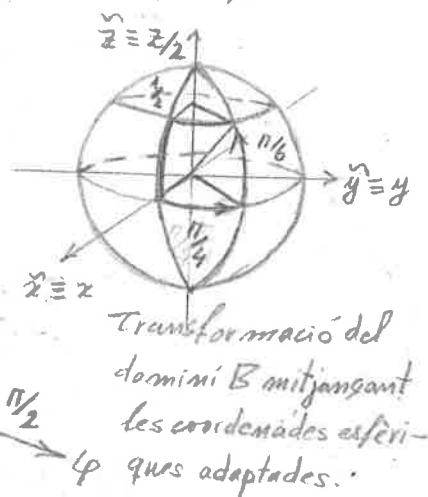
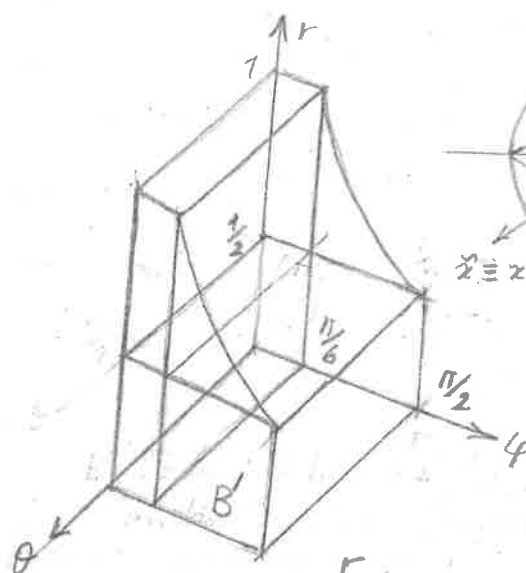
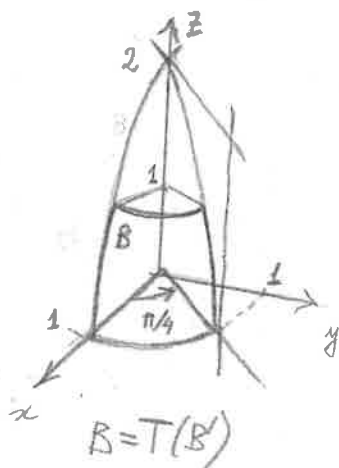
$$V = \frac{2}{3} \pi \sqrt{5}^3 + \frac{1}{3} \pi \sqrt{5}^2 \cdot 5 = \boxed{\left( \frac{10\sqrt{5}}{3} + \frac{25}{3} \right) \pi}$$

Volum de la semiesfera      Volum del con.

22) Adapten les coordenades esfèriques per calcular les següents integrals triples.

(a)  $I = \iiint_B 16z \, dx \, dy \, dz$ ,  $B = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, 0 \leq z \leq 1, 0 \leq y \leq x\}$

$B$   
Coordenades esfèriques adaptades:  $T(r, \theta, \varphi) = (r \cos \theta \cos \varphi, r \sin \theta \cos \varphi, r \sin \varphi)$ .



$$x^2 + y^2 + z^2 \leq 1 \iff 0 \leq r \leq 1$$

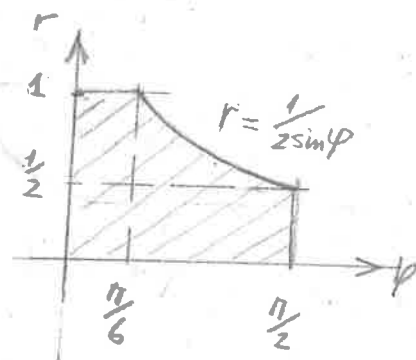
$$0 \leq z \leq 1 \iff 0 \leq r \leq \frac{1}{2 \sin \varphi}, \text{ amb } 0 \leq \varphi \leq \frac{\pi}{2}$$

$$0 \leq y \leq x \iff 0 \leq r \cos \theta \cos \varphi \leq r \sin \theta \cos \varphi$$

$$\text{amb } r > 0, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\iff 0 \leq \sin \theta \leq \cos \theta \text{ amb } 0 \leq \theta \leq 2\pi$$

$$\iff 0 \leq \theta \leq \frac{\pi}{4}$$



Aleshores  $I = I_1 + I_2$ , amb:

$$I_1 = 64 \int_0^{\pi/4} d\theta \int_0^{\pi/6} \sin \varphi \cos \varphi d\varphi \int_0^1 r^3 dr = 64 \frac{\pi}{4} \left[ \frac{\sin^2 \varphi}{2} \right]_0^{\pi/6} \left[ \frac{r^4}{4} \right]_0^1 = 16\pi \frac{1}{8} \frac{1}{4} = \frac{\pi}{2}$$

$$I_2 = 64 \int_0^{\pi/4} d\theta \int_{\pi/6}^{\pi/2} \sin \varphi \cos \varphi d\varphi \int_0^{\frac{1}{2 \sin \varphi}} r^3 dr = \frac{64 \cdot \pi}{16 \cdot 16} \int_{\pi/6}^{\pi/2} \frac{\cos \varphi}{\sin^3 \varphi} d\varphi = \frac{\pi}{8} \left[ -\frac{1}{\sin^2 \varphi} \right]_{\pi/6}^{\pi/2}$$

$$= \frac{\pi}{8} (4-1) = \frac{3\pi}{8}$$

Llavors:  $I = I_1 + I_2 = \frac{\pi}{2} + \frac{3\pi}{8} = \frac{7\pi}{8}$ .

(\*)  $\det DT(r, \theta, \varphi) = 2r^2 \cos \varphi$ ,

$z = 2r \sin \varphi$ . Aleshores  $16z \, dx \, dy \, dz = 16 \cdot 2r \sin \varphi \cdot 2r^2 \cos \varphi \, d\theta \, d\varphi \, dr = 64r^3 \sin \varphi \cos \varphi \, d\theta \, d\varphi \, dr$ .

Alternativament:

$$I = I'_1 + I'_2$$

amb:  $I'_1 = 64 \int_0^{\pi/4} d\theta \int_{1/2}^1 r^3 dr \int_0^{\arcsin \frac{1}{2r}} \cos\varphi \sin\varphi d\varphi = 64 \int_0^{\pi/4} d\theta \int_{1/2}^1 r^3 \frac{1}{8r^2} dr = 64 \cdot \frac{3}{64} \cdot \frac{\pi}{4} = \frac{3\pi}{4}$

$$\frac{1}{8} \int_{1/2}^1 r dr = \frac{1}{16} \left(1 - \frac{1}{4}\right) = \frac{3}{64}$$

$$I'_2 = 64 \int_0^{\pi/4} d\theta \int_0^{1/2} r^3 dr \int_0^{\pi/2} \cos\varphi \sin\varphi d\varphi = 64 \cdot \frac{\pi}{4} \cdot \frac{1}{64} \cdot \frac{1}{2} = \frac{\pi}{8}$$

Aleshores:

$$I = I'_1 + I'_2 = \frac{3\pi}{4} + \frac{\pi}{8} = \frac{7\pi}{8}$$

Exercici: intentem aquest mateix apartat, fent servir coordenades cilíndriques 😊

(b)  $I = \iiint_B \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dx dy dz$ ,  $B = \{ (x,y,z) \in \mathbb{R}^3 : x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1 \}$ .

Solució: adaptem les coordenades esfèriques:

$x = ar \cos\theta \cos\varphi$   
 $y = br \sin\theta \cos\varphi$ ,  $T: (r, \theta, \varphi) \rightarrow (x, y, z) = T(r, \theta, \varphi)$   
 $z = cr \sin\varphi$

Aleshores:  $B' = \{ (r, \theta, \varphi) \in [0, +\infty) \times [0, 2\pi] \times [-\pi/2, \pi/2] : 0 \leq r \leq 1 \} = B'_1(0, 90)$ , és el domini transformat i llavors la integral I es pot calcular com:

$$I = \iiint_{B=T(B')} \underbrace{\left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)}_{r^2} dx dy dz = \iiint_{B'} r^2 \underbrace{\left| \det \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} \right|}_{abc r^2 \cos\varphi} d\theta d\varphi dr$$

Per què?

$$= abc \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi d\varphi \int_0^1 r^4 dr = abc \cdot 2\pi \left[ \sin\varphi \right]_{-\pi/2}^{\pi/2} \left[ \frac{r^5}{5} \right]_0^1 = \frac{4\pi}{5} abc$$

24) Signi B la bola de centre  $(0,0,0)$  i radi  $R$ . Denotem per  $T(x,y,z)$  la temperatura en el punt  $(x,y,z)$  i suposem que és proporcional a la distància del punt a l'origen. En quins punts de B la temperatura coincideix amb la temperatura mitjana de la bola?

Solució

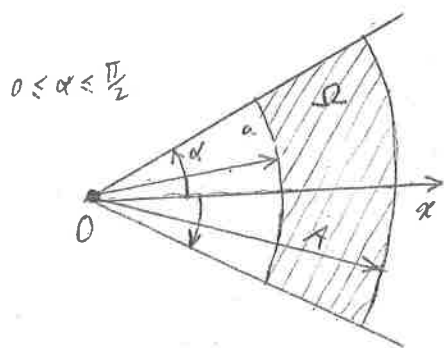
$$\langle T \rangle = \frac{\alpha}{\frac{4}{3}\pi R^3} \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi d\varphi \int_0^R r^3 dr = \alpha \frac{4\pi \frac{R^4}{4}}{\frac{4}{3}\pi R^3} = \alpha \frac{3}{4} R$$

$$T(x,y,z) = \alpha \sqrt{x^2+y^2+z^2} \\ = \alpha r$$

$$T(x,y,z) = \alpha \frac{3}{4} R \iff \boxed{x^2+y^2+z^2 = \frac{9}{16} R^2}$$

25) Troben el centre de masses de les regions planes amb les densitats que s'indiquen

(a) Sector pla definit per una corona de radi interior  $a$  i radi exterior  $A$ , un angle d'obertura  $2\alpha$  i que és simètrica respecte l'eix  $x$  positiva, suposant densitat constant  $\rho(x,y) = 1$ .



Solució:

$$M = \iint_{\Omega} \rho(x,y) dx dy = \int_{-\alpha}^{\alpha} d\theta \int_a^A r dr = \frac{2\alpha}{2} (A^2 - a^2) = \alpha (A^2 - a^2)$$

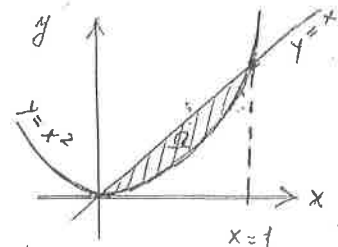
$$X = \frac{1}{M} \iint_{\Omega} x \rho(x,y) dx dy = \frac{1}{M} \int_{-\alpha}^{\alpha} \cos\theta d\theta \int_a^A r^2 dr = \frac{1}{3M} (\sin\alpha) \cdot (A^3 - a^3) \\ = \frac{2}{3} \frac{\sin\alpha}{\alpha} \frac{A^3 - a^3}{A^2 - a^2}$$

$$Y = \frac{1}{M} \iint_{\Omega} y \rho(x,y) dx dy = \frac{1}{M} \int_{-\alpha}^{\alpha} \sin\theta d\theta \int_a^A r^2 dr = 0$$

Posició del Centre de Masses:

$$(X, Y) = \left( \frac{2}{3} \frac{\sin\alpha}{\alpha} \frac{A^3 - a^3}{A^2 - a^2}, 0 \right) \square$$

b) Regió entre  $y=x^2$  i  $y=x$  amb  $p(x,y)=x+y$



Punt de tall:  $x^2=x \Leftrightarrow x(x-1)=0$   
 $\Leftrightarrow x=0, x=1.$

$$M = \iint_{\Omega} p(x,y) dx dy = \int_0^1 dx \int_{x^2}^x (x+y) dy$$

$$= \int_0^1 dx \left( xy + \frac{y^2}{2} \right) \Big|_{x^2}^x = \int_0^1 dx \left( x^2 + \frac{x^2}{2} - x^3 - \frac{x^4}{2} \right)$$

$$= \int_0^1 \left( \frac{3}{2}x^2 - x^3 - \frac{x^4}{2} \right) dx = \left[ \frac{x^3}{2} - \frac{x^4}{4} - \frac{x^5}{10} \right]_0^1 = \frac{1}{2} - \frac{1}{4} - \frac{1}{10} = \frac{10-5-2}{20} = \frac{3}{20}$$

$$X = \frac{1}{M} \int_0^1 dx \int_{x^2}^x x(x+y) dy = \frac{1}{M} \int_0^1 dx \left[ x^2 y + x \frac{y^2}{2} \right]_{x^2}^x = \int_0^1 \left( \frac{3x^3}{2} - x^4 - \frac{x^5}{2} \right) dx$$

$$= \frac{1}{M} \left( \frac{3x^4}{8} - \frac{x^5}{5} - \frac{x^6}{12} \right) \Big|_0^1 = \frac{1}{M} \left( \frac{3}{8} - \frac{1}{5} - \frac{1}{12} \right) = \frac{45-24-10}{120M} = \frac{11}{120M} = \frac{11}{120} \cdot \frac{20}{3} = \frac{11}{18}$$

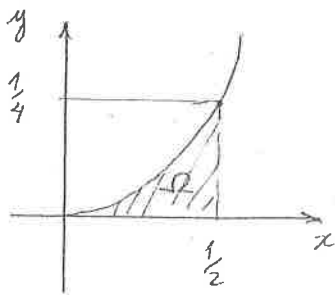
$$Y = \frac{1}{M} \int_0^1 dx \int_{x^2}^x y(x+y) dy = \frac{1}{M} \int_0^1 dx \left[ \frac{y^2}{2}x + \frac{y^3}{3} \right]_{x^2}^x = \frac{1}{M} \int_0^1 \left( \frac{5x^3}{2} - \frac{x^5}{2} - \frac{x^6}{3} \right) dx$$

$$= \frac{1}{M} \left( \frac{5x^4}{24} - \frac{x^6}{12} - \frac{x^7}{21} \right) \Big|_0^1 = \frac{1}{M} \left( \frac{1}{24} - \frac{1}{12} - \frac{1}{21} \right) = \frac{1}{M} \cdot \frac{35-14-8}{168} = \frac{13}{168} \cdot \frac{20}{3} = \frac{65}{126}$$

$\therefore$  Posició del CDM:

$$(X, Y) = \left( \frac{11}{18}, \frac{65}{126} \right) \quad \square$$

c) Regió entre  $y=0$  i  $y=x^2$  ( $0 \leq x \leq \frac{1}{2}$ ), amb  $p(x,y)=1$ .



$$M = \iint_{\Omega} p(x,y) dx dy = \int_0^{\frac{1}{2}} dx \int_0^{x^2} dy = \int_0^{\frac{1}{2}} x^2 dx = \left[ \frac{x^3}{3} \right]_0^{\frac{1}{2}} = \frac{1}{24}$$

$$X = \frac{1}{M} \iint_{\Omega} x p(x,y) dx dy = \frac{1}{M} \int_0^{\frac{1}{2}} x dx \int_0^{x^2} dy = \frac{1}{M} \int_0^{\frac{1}{2}} x^3 dx = \frac{1}{M} \left[ \frac{x^4}{4} \right]_0^{\frac{1}{2}} = \frac{3}{8}$$

$$Y = \frac{1}{M} \iint_{\Omega} y p(x,y) dx dy = \frac{1}{M} \int_0^{\frac{1}{2}} dx \int_0^{x^2} y dy = \frac{1}{M} \int_0^{\frac{1}{2}} \frac{y^2}{2} dx = \frac{1}{M} \int_0^{\frac{1}{2}} \frac{x^6}{2} dx = \frac{1}{M} \left[ \frac{x^7}{14} \right]_0^{\frac{1}{2}} = \frac{3}{40}$$

$\therefore$  Posició del CDM:

$$(X, Y) = \left( \frac{3}{8}, \frac{3}{40} \right) \quad \square$$

26) Em cadascun dels casos següents, troben el centre de masses del sòlid A suposant distribució de masses homogènica.

(a)  $A = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq R^2, x \geq 0, y \geq 0, z \geq 0 \right\}$ .

Solució.  $\rho = \rho_0 = \text{const}$

$$m(A) = \frac{4}{3} \pi R^3 \rho_0 \cdot \frac{1}{8} = \frac{\rho_0}{6} \pi R^3$$

$$X = \frac{1}{M} \iiint_A \underbrace{x}_{\rho_0} \rho(x, y, z) dx dy dz = \frac{\rho_0}{M} \int_0^{\pi/2} \cos \theta d\theta \int_0^{\pi/2} \cos^2 \varphi d\varphi \int_0^R r^3 dr$$

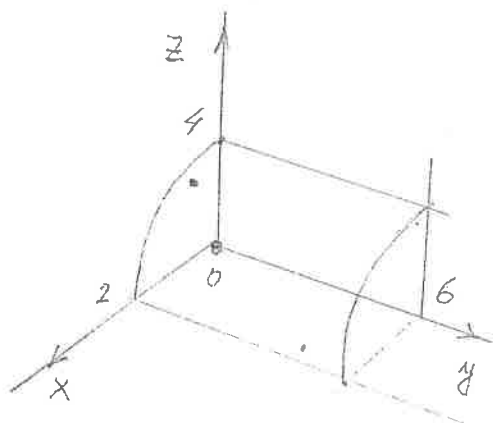
$$= \frac{\rho_0}{M} \cdot \frac{\pi}{4} \cdot \frac{R^4}{4} = \frac{\rho_0/16 \pi R^4}{\rho_0/6 \pi R^3} = \boxed{\frac{3}{8} R}$$

$$Y = \frac{1}{M} \iiint_A y \rho(x, y, z) dx dy dz = \frac{\rho_0}{M} \int_0^{\pi/2} \sin \theta d\theta \int_0^{\pi/2} \cos^2 \varphi d\varphi \int_0^R r^3 dr = \dots = \boxed{\frac{3}{8} R}$$

$$Z = \frac{1}{M} \iiint_A z \rho(x, y, z) dx dy dz = \frac{\rho_0}{M} \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin \varphi \cos \varphi d\varphi \int_0^R r^3 dr = \frac{\rho_0 \frac{\pi}{16} R^4}{\rho_0 \pi R^3} = \boxed{\frac{3}{8} R}$$

$\therefore$  Posició del CDM:  $(X, Y, Z) = \left( \frac{3}{8} R, \frac{3}{8} R, \frac{3}{8} R \right)$

(b)  $A = \left\{ (x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq z, 0 \leq y \leq 6, 0 \leq z \leq 4 - x^2 \right\}, \rho(x, y, z) = \rho_0 = \text{const}$ .



$$M = \iiint_A \underbrace{\rho(x, y, z)}_{\rho_0} dx dy dz = \rho_0 \int_0^2 dx \int_0^6 dy \int_0^{4-x^2} dz = 6\rho_0 \int_0^2 (4-x^2) dx = 6\rho_0 \left( 4x - \frac{x^3}{3} \right) \Big|_0^2 = 6\rho_0 \left( 8 - \frac{8}{3} \right) = 32\rho_0$$

$$X = \frac{1}{M} \iiint_A \underbrace{x}_{\rho_0} \rho(x, y, z) dx dy dz = \frac{1}{M} \int_0^2 x dx \int_0^6 dy \int_0^{4-x^2} dz$$

$$= \frac{6\rho_0}{M} \int_0^2 x(4-x^2) dx = \frac{6\rho_0}{M} \left( 2x^2 - \frac{x^4}{4} \right) \Big|_0^2 = \frac{6\rho_0}{M} (8-4) = \frac{24\rho_0}{32\rho_0} = \boxed{\frac{3}{4}}$$

$$Y = \frac{1}{M} \iiint_{\Omega} y \rho(x,y,z) dx dy dz = \frac{\rho_0}{M} \int_0^2 dx \int_0^6 y dy \int_0^{4-x^2} dz = \frac{\rho_0}{M} \left[ \frac{y^2}{2} \right]_0^6 \left[ 4x - \frac{x^3}{3} \right]_0^2$$

$$= \frac{\rho_0}{M} 18 \cdot \frac{16}{3} = \frac{6 \cdot 16 \cdot \rho_0}{32 \cdot \rho_0} = \frac{3}{5}$$

$$Z = \frac{1}{M} \iiint_{\Omega} z \rho(x,y,z) dx dy dz = \frac{\rho_0}{M} \int_0^2 dx \int_0^6 dy \int_0^{4-x^2} z dz = \frac{6\rho_0}{M} \int_0^2 \frac{(4-x^2)^2}{2} dx$$

$$= \frac{3\rho_0}{M} \int_0^2 (16 - 8x^2 + x^4) dx = \frac{3\rho_0}{M} \left( 16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right) \Big|_0^2 = \frac{3\rho_0}{M} \left( 32 - \frac{64}{3} + \frac{32}{5} \right) = \frac{3\rho_0}{M} \frac{320 - 320 + 96}{15}$$

$$= \frac{3\rho_0}{32\rho_0} \cdot \frac{256}{15} = \frac{8}{5}$$

Posició del CDM:  $(X, Y, Z) = \left( \frac{3}{4}, \frac{3}{5}, \frac{8}{5} \right)$ .  $\square$

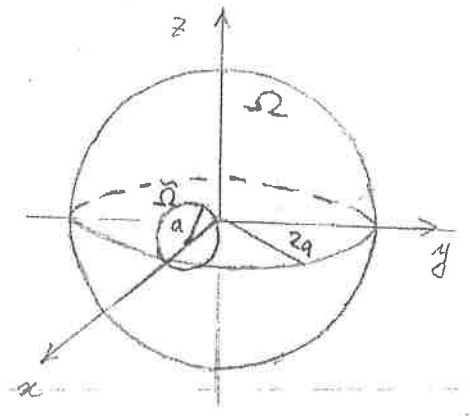
(c)  $A = \left\{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4a^2, (x-a)^2 + y^2 + z^2 \geq a^2 \right\}$ ,  $\rho(x,y,z) = \rho_0 = \text{const.}$

Solució. Tenim:  $\tilde{\Omega} = B_a(a,0,0)$ ,  $\Omega = B_{2a}(0,0,0) \setminus \tilde{\Omega}$ , amb:

$$m(\tilde{\Omega}) = \frac{4}{3} \pi a^3 \rho_0$$

$$m(\Omega) = m(B_{2a}(0,0,0)) - m(\tilde{\Omega}) = \frac{4}{3} \pi 8a^3 \rho_0 - \frac{4}{3} \pi a^3 \rho_0$$

$$= \frac{4}{3} \pi 7a^3 \rho_0$$



$$m(\Omega) X + m(\tilde{\Omega}) \tilde{X} = \frac{4}{3} \pi \rho_0 7a^3 X + \frac{4}{3} \pi \rho_0 a^4 = 0 \Rightarrow X = - \frac{\frac{4}{3} \pi \rho_0 a^4}{\frac{4}{3} \pi \rho_0 7a^3} = - \frac{a}{7}$$

$$m(\Omega) Y + m(\tilde{\Omega}) \tilde{Y} = 0 \Rightarrow \tilde{Y} = 0$$

$$m(\Omega) Z + m(\tilde{\Omega}) \tilde{Z} = 0 \Rightarrow \tilde{Z} = 0$$

Posició del CDM:  $(X, Y, Z) = \left( -\frac{a}{7}, 0, 0 \right)$ .  $\square$

28) Troben el centre de masses de la semiesfera definida per  $x^2 + y^2 + z^2 \leq R^2$  i  $z \geq 0$ , si la densitat en cada punt és proporcional a la distància d'aquest punt al centre.

Solució.

$$x^2 + y^2 + z^2 \leq R^2$$

$$z \geq 0$$

$$\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2} =: r$$

$$M = \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos\varphi d\varphi \int_0^R r^3 dr = 2\pi \frac{R^4}{4} = \boxed{\frac{\pi R^4}{2}}$$

$$X = \frac{1}{M} \int_0^{2\pi} \underbrace{\cos\theta d\theta}_0 \int_0^{\pi/2} \cos\varphi d\varphi \int_0^R r^4 dr = 0$$

$$Y = \frac{1}{M} \int_0^{2\pi} \underbrace{\sin\theta d\theta}_0 \int_0^{\pi/2} \cos\varphi d\varphi \int_0^R r^4 dr = 0$$

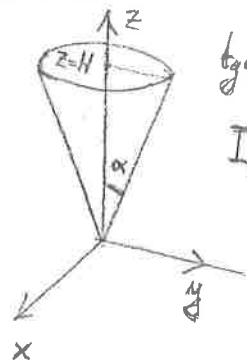
$$Z = \frac{1}{M} \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos\varphi \sin\varphi d\varphi \int_0^R r^4 dr = \frac{\pi R^5/5}{\pi R^4/2} = \boxed{\frac{2}{5}R}$$

Posició del CDM:  $(X, Y, Z) = (0, 0, \frac{2R}{5})$ .  $\square$

34) Pels sòlids següents, calculeu els moments d'inèrcia que es demanen en cada cas tot suposant densitat homogènia igual a 1.

(a) Calculeu  $I_z$  pel sòlid limitat pel con de revolució d'altura  $H$  i radi de base  $R$  donat per  $x^2 + y^2 \leq R^2 \frac{z^2}{H^2}$  ( $0 \leq z \leq H$ ).

Solució.



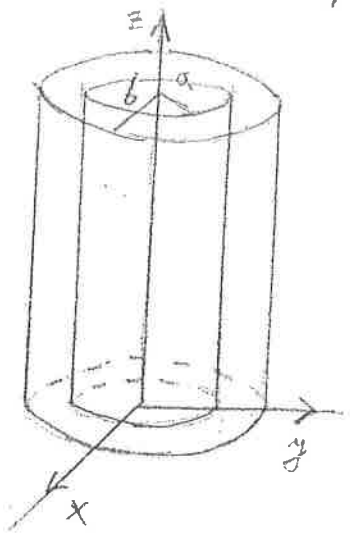
$$\tan\alpha = \frac{R}{H}, \quad \rho = 1 \text{ const.}$$

$$I_z = \int_0^{2\pi} d\theta \int_0^R r^3 dr \int_{H/R}^H \rho dz = 2\pi \int_0^R \left( Hr^3 - \frac{H}{R} r^4 \right) dr = 2\pi H R^4 \left( \frac{1}{4} - \frac{1}{5} \right) = \boxed{\frac{\pi H R^4}{10}} \quad \square$$

(b) Calculeu  $I_z$  pel sòlid limitat per dos cilindres d'altura  $h$ ,  $a^2 \leq x^2 + y^2 \leq b^2$ ,  $0 \leq z \leq h$

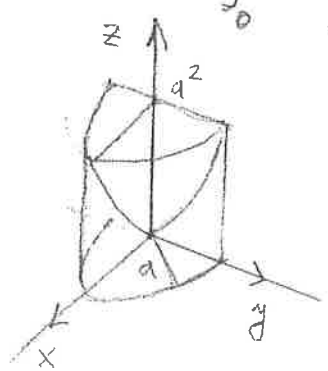
Solució.

$$I_z = \int_0^{2\pi} d\theta \int_a^b r^3 dr \int_0^h dz = 2\pi h \left[ \frac{r^4}{4} \right]_a^b = \frac{\pi h}{4} (b^4 - a^4)$$



(c) Calculeu  $I_z$  pel sòlid limitat pel paraboloida  $z = x^2 + y^2$ , el cilindre  $x^2 + y^2 = a^2$  ( $z \geq 0$ )

$$I_z = \int_0^{2\pi} d\theta \int_0^a r^3 dr \int_0^{r^2} dz = 2\pi \int_0^a r^5 dr = 2\pi \left[ \frac{r^6}{6} \right]_0^a = \frac{\pi a^6}{3}$$



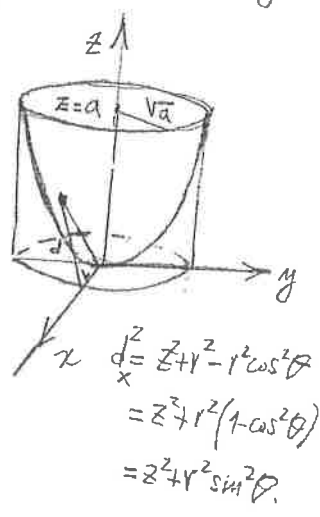
(d) calculeu  $I_x, I_y, I_z$  pel sòlid tancat pel paraboloida  $z = x^2 + y^2$  i el pla  $z = a$  ( $a > 0$ )

$$I_x = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r dr \int_{r^2}^a \rho d_x^2 dz$$

$d_x$ : distància a l'eix x.

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r dr \int_{r^2}^a (z^2 + r^2 \sin^2 \theta) dz$$

$$= \underbrace{\int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r dr \int_{r^2}^a z^2 dz}_{=: I_1} + \underbrace{\int_0^{2\pi} \sin^2 \theta d\theta \int_0^{\sqrt{a}} r^3 dr \int_{r^2}^a dz}_{=: I_2}$$



$= I_1 + I_2$

$$I_1 = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r dr \int_{r^2}^a z^2 dz = \frac{2\pi}{3} \int_0^{\sqrt{a}} r (a^3 - r^6) dr = \frac{2\pi}{3} \left( \frac{a^4}{2} - \frac{a^4}{8} \right) = \frac{\pi a^4}{4}$$

$$I_2 = \int_0^{2\pi} \sin^2 \theta d\theta \int_0^{\sqrt{a}} r^3 dr \int_{r^2}^a dz = \pi \int_0^{\sqrt{a}} r^3 (a - r^2) dr = \pi a^3 \left( \frac{1}{4} - \frac{1}{6} \right) = \frac{\pi a^3}{12}$$

$\therefore I_x = I_1 + I_2 = \frac{\pi a^3}{12} (1 + 3a)$ .  $\square$

$$I_y = I_x = \frac{\pi a^3}{12} (1+3a) \quad (\text{per simetria}). \quad \square$$

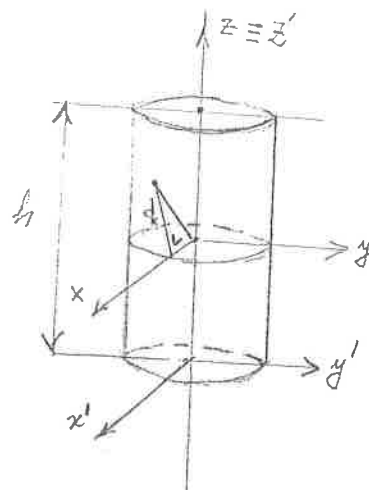
$$I_z = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r^3 dr \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho dz = 2\pi \int_0^{\sqrt{a}} r^3 (a-r^2) dr = 2\pi \left( a \frac{r^4}{4} - \frac{r^6}{6} \right) \Big|_0^{\sqrt{a}} = \frac{\pi a^3}{6} \quad \square$$

(e) Calculeu  $I_x, I_y, I_z$  pel cilindre  $x^2+y^2 \leq R^2, -\frac{h}{2} \leq z \leq \frac{h}{2}$

$$I_x = \int_0^{2\pi} d\theta \int_0^R r dr \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho d_x^2 dz$$

$d_x$ : distància a l'eix x:

$$\begin{aligned} d_x &= z^2 + r^2 - r^2 \cos^2 \theta \\ &= z^2 + r^2 (1 - \cos^2 \theta) \\ &= z^2 + r^2 \sin^2 \theta. \end{aligned}$$



$$= \int_0^{2\pi} d\theta \int_0^R r dr \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho (z^2 + r^2 \sin^2 \theta) dz$$

$$= \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_0^R r dr}_{\frac{R^2}{2}} \underbrace{\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 dz}_{\frac{h^3}{12}} + \underbrace{\int_0^{2\pi} \sin^2 \theta d\theta}_{\pi} \underbrace{\int_0^R r^3 dr}_{\frac{R^4}{4}} \underbrace{\int_{-\frac{h}{2}}^{\frac{h}{2}} dz}_{h} = 2\pi \frac{R^2}{2} \frac{h^3}{12} + \pi \frac{R^4}{4} h = \frac{\pi R^2 h}{12} (h^2 + 3R^2)$$

$$I_y = I_x = \frac{\pi R^2 h}{12} (h^2 + 3R^2) \quad (\text{per simetria})$$

Nota:  $I_{x'} = I_{y'} = \frac{\pi R^2 h}{12} (h^2 + 3R^2) + \pi R^2 h \cdot \frac{h^2}{4} = \frac{\pi R^2 h}{12} \left( \frac{h^2}{12} + \frac{h^2}{4} + 3R^2 \right) = \frac{\pi R^2 h}{12} (4h^2 + 3R^2)$

Teorema de Steiner  $I_r = I_c + m(N) d^2(CM, r)$

Moment d'inèrcia respecte de l'eix  $x$ , que passa pel CDM

Massa del cos ( $\rho=1$ )

$d^2(x, x')$

$$I_z = \int_0^{2\pi} d\theta \int_0^R r^3 dr \int_{-\frac{h}{2}}^{\frac{h}{2}} dz = 2\pi \frac{R^4}{4} h = \frac{\pi R^2 h}{2} \rho^2 \quad \square$$

Nota: posant  $M := \pi R^2 h$  (la "massa" del cos):

$$I_x = \frac{M}{12} (h^2 + 3R^2) = I_y, \quad I_{x'} = \frac{M}{12} (4h^2 + 3R^2) = I_{y'}, \quad I_z = \frac{M}{2} h^2 \quad \square$$

## TEMA 2

Problemes: 1, 2, 5, 6, 7, 9, 10, 11, 13, 17, 18, 19, 21, 27, 29, 30, 31, 32, 33, 35.

1. Usant el teorema del valor mig per integrals, proveu les següents desigualtats

$$(a) 4e^5 \iint_A e^{x^2+y^2} dz dy \leq 4e^{25}, \text{ on } A = [1,3] \times [2,4]$$

S. Recordem el Teorema del Valor Mig (veure apunts de teoria, Tema 2):

△ Signi  $f: \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\Omega$  acotat i arc-connex, i  $f \in C^0(\bar{\Omega})$ , llavors,  $\exists c \in \bar{\Omega}$  tal que:

$$\int_{\Omega} f(x) = f(c) m(\Omega),$$

i on  $m(\Omega)$  és la mesura del conjunt  $\Omega$  (longitud en dimensió 1, àrea en dimensió 2, volumen en dimensió 3, ...). llavors tenim les acotacions següents:

$$m(\Omega) \inf_{x \in \Omega} f(x) \leq \int_{\Omega} f(x) \leq m(\Omega) \sup_{x \in \Omega} f(x). \quad (1)$$

(a)  $f(x,y) = e^{x^2+y^2}$ , d'on:  $f'_x(x,y) = 2xe^{x^2+y^2} = 0 \Leftrightarrow x=0$ ,  $f'_y(x,y) = 2ye^{x^2+y^2} = 0$ . llavors l'únic punt crític de la funció és  $(x,y) = (0,0) \notin A = [1,3] \times [2,4]$

•  $x=1$ :  $f(1,y) = e^{1+y^2}$ :  $f'(1,y) = 2y e^{1+y^2} = 0 \Leftrightarrow y=0$ :  $(1,0) \notin A$ .

•  $x=3$ :  $f(3,y) = e^{9+y^2}$ :  $f'(3,y) = 2y e^{9+y^2} = 0 \Leftrightarrow y=0$ :  $(3,0) \notin A$ .

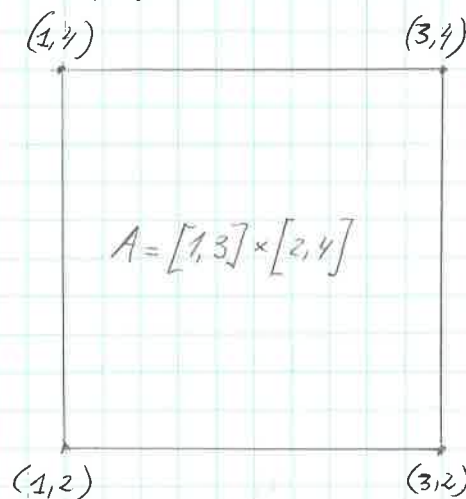
•  $y=2$ :  $f(x,2) = e^{x^2+4}$ :  $f'(x,2) = 2x e^{x^2+4} = 0 \Leftrightarrow x=0$ :  $(0,2) \notin A$ .

•  $y=4$ :  $f(x,4) = e^{x^2+16}$ :  $f'(x,4) = 2x e^{x^2+16} = 0$  (1,4) (3,4)

$\Leftrightarrow x=0$ ,  $(0,4) \notin A$ .

Veiem que el punt crític de  $f$  no pertany a  $A$  i que les restriccions de  $f$  sobre els costats d' $A$  no tenen punts crítics. Avaluem  $f$  als vèrtexs d' $A$ :

$$f(1,2) = e^5, \quad f(3,2) = e^{13}, \quad f(1,4) = e^{17}, \quad f(3,4) = e^{25}$$



D'on:  $\max_{(x,y) \in A} e^{x^2+y^2} = e^{25}$  i  $\min_{(x,y) \in A} e^{x^2+y^2} = e^5$ ; s'atanyen als punts

$(3,4)$  i  $(1,2)$  respectivament. D'altra banda:  $m(A) = \iint dx dy = (3-1) \cdot (4-2) = 4$ .

Així doncs, aplicant el Teorema del Valor Mig (fórmula (1)),  $A$

$$4e^5 \leq \iint e^{x^2+y^2} dx dy \leq 4e^{25} \quad \square$$

$$A = [1,3] \times [2,4]$$

$$(b) \frac{1}{e} \leq \frac{1}{4\pi^2} \iint_A e^{\sin(x+y)} dx dy \leq e, \quad A = [-\pi, \pi] \times [-\pi, \pi]$$

Solució: Clarament:  $\max_{(x,y) \in A} e^{\sin(x+y)} = e$ ,  $\min_{(x,y) \in A} e^{\sin(x+y)} = \frac{1}{e}$ , mentre que

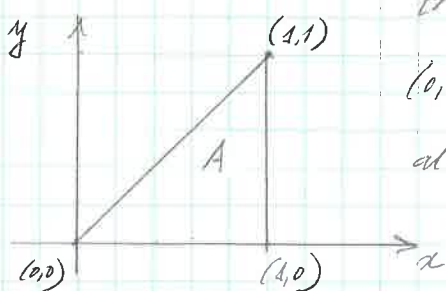
$m(A) = 4\pi^2$ . Alleshores, aplicant (1) resulta:

$$\frac{4\pi^2}{e} \leq \iint e^{\sin(x+y)} dx dy \leq 4\pi^2 e, \quad A = [-\pi, \pi] \times [-\pi, \pi]$$

i dividint les dues desigualtats per  $4\pi^2$  s'obtenen les acotacions buscades.  $\square$

$$(c) \frac{1}{6} \leq \iint_A \frac{dx dy}{y-x+3} \leq \frac{1}{4}, \text{ on } A \text{ és el triangle de vèrtexs } (0,0), (1,1) \text{ i } (1,0)$$

Solució.



El màxim de  $y-x+3$  sobre el triangle  $A$  s'atany als vèrtexs  $(0,0)$  i  $(1,1)$ ; val 3, mentre que el mínim té lloc al vèrtex  $(1,0)$  i val 2. Per tant:

$$\frac{1}{3} \leq \frac{1}{y-x+3} \leq \frac{1}{2}$$

per tot punt del triangle donat; mentre que l'àrea corresponent és  $m(A) = \frac{1}{2}$ .

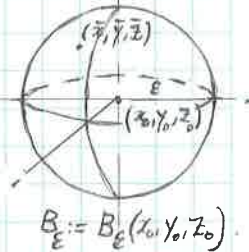
Finalment doncs, aplicant la fórmula (1) del Teorema del Valor Mitjà, obtenim:

$$\frac{1}{6} \leq \iint_A \frac{dx dy}{y-x+3} \leq \frac{1}{4}, \text{ que són les desigualtats buscades. } \square$$

2) Sigui  $f(x, y, z)$  una funció contínua i  $B_\varepsilon$  la bola de centre  $(x_0, y_0, z_0)$  i radi  $\varepsilon$ .  
 Proven que se satisfà que  $f(x_0, y_0, z_0) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\text{volum}(B_\varepsilon)} \iiint_{B_\varepsilon} f(x, y, z) dx dy dz$

Solució: Pel Teorema del Valor Mig, existeix  $(\bar{x}, \bar{y}, \bar{z}) \in B_\varepsilon = B_\varepsilon(x_0, y_0, z_0)$  t.q.:

$$\iiint_{B_\varepsilon(x_0, y_0, z_0)} f(x, y, z) dx dy dz = f(\bar{x}, \bar{y}, \bar{z}) \cdot \text{Volum } B_\varepsilon(x_0, y_0, z_0)$$



D'altra banda, com que  $f$  és contínua i  $(\bar{x}, \bar{y}, \bar{z}) \rightarrow (x_0, y_0, z_0)$  quan  $\varepsilon \rightarrow 0$ , prement límits a l'expressió de dalt hem obté:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\text{Volum}(B_\varepsilon(x_0, y_0, z_0))} \iiint_{B_\varepsilon(x_0, y_0, z_0)} f(x, y, z) dx dy dz = \lim_{(\bar{x}, \bar{y}, \bar{z}) \rightarrow (x_0, y_0, z_0)} f(\bar{x}, \bar{y}, \bar{z}) = f(x_0, y_0, z_0)$$

que és el resultat que es buscava.  $\square$

5) Troben les següents integrals dobles en els rectangles que s'indiquen

(a)  $I = \iint_R x^2 y dx dy$ ,  $R = [0, 1] \times [0, 1]$ .

Solució:  $I = \iint_R x^2 y dx dy = \left( \int_0^1 x^2 dx \right) \left( \int_0^1 y dy \right) = \left[ \frac{x^3}{3} \right]_0^1 \cdot \left[ \frac{y^2}{2} \right]_0^1 = \frac{1}{6}$

(b)  $I = \iint_R \frac{x^2}{1+y^2} dx dy$ ,  $R = [0, 1] \times [0, 1]$

Solució:  $I = \iint_{R=[0,1] \times [0,1]} \frac{x^2}{1+y^2} dx dy = \int_0^1 x^2 dx \int_0^1 \frac{dy}{1+y^2} = \left[ \frac{x^3}{3} \right]_0^1 \cdot \left[ \arctan y \right]_0^1 = \frac{\pi}{12}$

$$(c) I = \iint_R y \ln x \, dx dy, \quad R = [1, e] \times [1, e].$$

$$\text{Solució. } I = \iint_{R=[1,e] \times [1,e]} y \ln x \, dx dy = \left( \int_1^e \ln x \, dx \right) \cdot \left( \int_1^e y \, dy \right)^{(*)} = \boxed{\frac{e^2 - 1}{2}}$$

$$(*) \int_1^e \ln x \, dx = \left[ x \ln x - x \right]_1^e = e - e + 1 = 1 \quad (\text{primitivització per parts + Barrow}).$$

$$\int_1^e y \, dy = \left[ \frac{y^2}{2} \right]_1^e = \frac{1}{2}(e^2 - 1)$$

$$(d) \iint_R (x^2 + y) \, dx dy, \quad R = [0, 1] \times [0, 2]$$

$$\text{Solució. } I = \iint_{R=[0,1] \times [0,2]} (x^2 + y) \, dx dy = \int_0^1 dx \int_0^2 (x^2 + y) \, dy = \int_0^1 dx \left[ x^2 y + \frac{y^2}{2} \right]_0^2 =$$

$$= \int_0^1 (2x^2 + 2) \, dx = \left[ \frac{2x^3}{3} + 2x \right]_0^1 = \boxed{\frac{8}{3}}$$

$$(e) I = \iint_R \frac{1}{(x+2y)^2} \, dx dy, \quad R = [2, 5] \times [1, 3]$$

$$\text{Solució. } I = \iint_R \frac{dx dy}{(x+2y)^2} = \int_2^5 dx \int_1^3 \frac{dy}{(x+2y)^2} = \int_2^5 \left[ \frac{-1/2}{x+2y} \right]_{y=1}^{y=3} dx = \frac{1}{2} \int_2^5 \left( \frac{1}{x+2} - \frac{1}{x+6} \right) dx$$

$$= \frac{1}{2} \cdot \ln \left( \frac{x+2}{x+6} \right) \Big|_2^5 = \frac{1}{2} \left( \ln \frac{7}{11} - \ln \frac{4}{8} \right) = \boxed{\frac{1}{2} \ln \frac{14}{11}}$$

$$(f) I = \iint_R e^y \sin \left( \frac{x}{y} \right) \, dx dy, \quad R = \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times [1, 2]$$

$$\text{Solució. } I = \iint_{R=[-\pi/2, \pi/2] \times [1,2]} e^y \sin \left( \frac{x}{y} \right) \, dx dy = \int_1^2 e^y dy \int_{-\pi/2}^{\pi/2} \sin \left( \frac{x}{y} \right) \, dx = - \int_1^2 e^y y \left[ \cos \left( \frac{x}{y} \right) \right]_{x=-\pi/2}^{x=\pi/2} dx \stackrel{(*)}{=} \boxed{0}$$

$$(*) \left[ \dots \right]_{x=-\pi/2}^{x=\pi/2} = \cos \left( \frac{\pi}{2y} \right) - \cos \left( -\frac{\pi}{2y} \right) = 0.$$

$$(g) I = \iint_R (x+y)^{27} dx dy, R = [-1,1] \times [-1,1].$$

Solució.  $I = \iint_{R=[-1,1] \times [-1,1]} (x+y)^{27} dx dy = \int_{-1}^1 dx \int_{-1}^1 (x+y)^{27} dy = \int_{-1}^1 \left[ \frac{(x+y)^{28}}{28} \right]_{y=-1}^{y=1} dx$

$$= \int_{-1}^1 \left[ \frac{(x+1)^{28}}{28} - \frac{(x-1)^{28}}{28} \right] dx = \left[ \frac{(x+1)^{29}}{29 \cdot 28} - \frac{(x-1)^{29}}{29 \cdot 28} \right]_{-1}^1 = \frac{1}{29 \cdot 28} \left( 2^{29} + (-2)^{29} \right) = \boxed{0}.$$

6) Calculeu  $I = \iint_R x^y dx dy$  on  $R = [0,1] \times [a,b]$ , essent  $0 < a < b$ , i dedueu el valor de la integral  $\int_0^b \frac{x^b - x^a}{\ln x} dx$ .

Solució.  $I = \iint_{R=[0,1] \times [a,b]} x^y dx dy = \int_a^b dy \int_0^1 x^y dx = \int_a^b \left[ \frac{x^{y+1}}{y+1} \right]_{x=0}^{x=1} dy = \int_a^b \frac{dy}{y+1} = \ln \frac{b+1}{a+1}$ .

D'altra banda, invertint l'ordre d'integració (Fubini):

$$I = \iint_{R=[0,1] \times [a,b]} x^y dx dy = \int_0^1 dx \int_a^b e^{y \ln x} dy = \int_0^1 \frac{1}{\ln x} \left[ x^y \right]_a^b dx = \int_0^1 \frac{x^b - x^a}{\ln x} dx,$$

d'on es segueix que:

$$\int_0^1 \frac{x^b - x^a}{\ln x} dx = I = \ln \frac{b+1}{a+1}$$

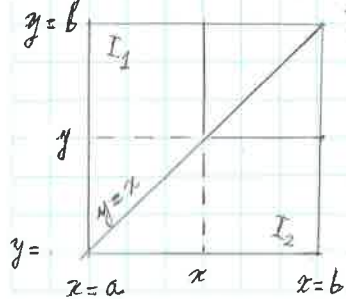
7) Proveu que  $2 \int_a^b \int_x^b f(x) f(y) dy dx = \left( \int_a^b f(x) dx \right)^2$ . (Indicació  $\left( \int_a^b f(x) dx \right)^2 =$

$$= \iint_{[a,b] \times [a,b]} f(x) f(y) dx dy.$$

canviant les variables  $x \leftrightarrow y$ :

Solució.  $2 \int_a^b \int_x^b f(x) f(y) dx dy = \int_a^b f(x) dx \int_x^b f(y) dy + \int_a^b f(y) dy \int_y^b f(x) dx$

$$\stackrel{(*)}{=} \iint_{I_1} f(x) f(y) dx dy + \iint_{I_2} f(x) f(y) dx dy = \iint_{I_1 \cup I_2} f(x) f(y) dx dy = \text{pàgina següent}$$



(\*) On  $I_1 = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq y \leq b\}$ ,  $I_2 = \{(x,y) \in \mathbb{R}^2 : a \leq y \leq x \leq b\}$ , i notem

que  $m(I_1 \cap I_2) = 0$ , per tant  $\iint_{I_1} + \iint_{I_2} = \iint_{I_1 \cup I_2}$ .

$$= \iint_{I \cup J} f(x)f(y) dx dy = \left( \int_a^b f(x) dx \right) \left( \int_a^b f(y) dy \right) = \left( \int_a^b f(x) dx \right)^2$$

$I \cup J = [a,b] \times [a,b]$

g) Calculeu les següents integrals dobles en els dominis de  $\mathbb{R}^2$  que s'indiquen

(a)  $I = \iint_A y^3 dx dy, A = \left\{ (x,y) \in \mathbb{R}^2 : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, 0 \leq y \leq 2 \cos x \right\}$

Solució.  $I = \int_{-\pi/2}^{\pi/2} dx \int_0^{2 \cos x} y^3 dy = 4 \int_{-\pi/2}^{\pi/2} \cos^4 x dx = 8 \int_0^{\pi/2} \cos^4 x dx = 8 \int_0^{\pi/2} \cos^2 x (1 - \sin^2 x) dx$

$$= 8 \int_0^{\pi/2} \left( \cos^2 x - \frac{1}{4} 4 \sin^2 x \cos^2 x \right) dx = 8 \int_0^{\pi/2} \left( \cos^2 x - \frac{1}{4} \sin^2(2x) \right) dx$$

$$= 8 \int_0^{\pi/2} \left( \frac{1 + \cos(2x)}{2} - \frac{1 - \cos(4x)}{8} \right) dx = 8 \left( \frac{\pi}{4} - \frac{\pi}{16} \right) = 8 \frac{3\pi}{16} = \boxed{\frac{3\pi}{2}}$$

(b)  $I = \iint_A x dx dy, A = \left\{ (x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq e^x \right\}$

Solució.  $I = \iint_A x dx dy = \int_0^1 x dx \int_0^{e^x} dy = \int_0^1 x e^x dx = \left( x e^x - e^x \right) \Big|_0^1 = \boxed{1}$

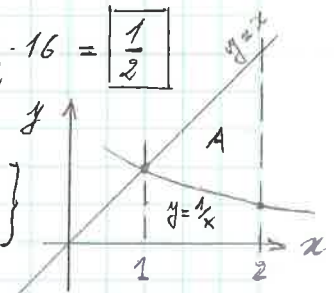
*integració per parts.*

(c)  $I = \iint_A xy dx dy, A = \left\{ (x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2, 0 \leq y \leq \frac{x}{2} \right\}$

Solució.  $I = \iint_A xy dx dy = \int_0^2 x dx \int_0^{x/2} y dy = \frac{1}{8} \int_0^2 x^3 dx = \frac{1}{32} \cdot 16 = \boxed{\frac{1}{2}}$

(d)  $I = \iint_A \frac{x^2}{y^2} dx dy, A = \left\{ (x,y) \in \mathbb{R}^2 : 1 \leq x \leq 2, \frac{1}{x} \leq y \leq x \right\}$

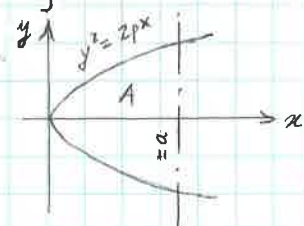
Solució.  $I = \iint_A \frac{x^2}{y^2} dx dy = \int_1^2 x^2 dx \int_{1/x}^x \frac{dy}{y^2} = \int_1^2 x^2 \left( x - \frac{1}{x} \right) dx = \left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_1^2 = 4 - 2 - \frac{1}{4} + \frac{1}{2} = \boxed{\frac{9}{4}}$



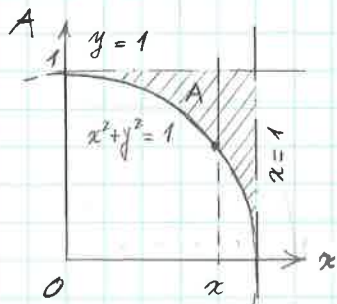
$$e) I = \iint_A (x+zy) dx dy, \quad A = \{(x,y) \in \mathbb{R}^2; -3 \leq y \leq 3, y^2-4 \leq x \leq 5\}$$

$$\begin{aligned} \text{Solució. } I &= \int_{-3}^3 dy \int_{y^2-4}^5 (x+zy) dx = \int_{-3}^3 dy \left( \frac{x^2}{2} + zy^2x \right) \Big|_{y^2-4}^5 \\ &= 2 \cdot \int_0^3 \left( \frac{9}{2} + 4y^2 - \frac{1}{2}y^4 \right) dy = 2 \cdot \left( \frac{27}{2} + \frac{4}{3} \cdot 27 - \frac{9}{10} \cdot 27 \right) = \frac{54}{30} (15 + 40 - 27) \\ &= \frac{54}{30} \cdot 28 = \boxed{\frac{252}{5}} \end{aligned}$$

$$f) I = \iint_A y^3 dx dy, \quad A = \{(x,y) \in \mathbb{R}^2; 0 \leq x \leq a, y^2 \leq 2px\}, \quad a > 0, p > 0$$

$$\text{Solució. } I = \int_0^a dx \int_{-\sqrt{2px}}^{\sqrt{2px}} y^3 dy = \int_0^a \left[ \frac{y^4}{4} \right]_{y=-\sqrt{2px}}^{y=\sqrt{2px}} dy = \boxed{0}$$


$$g) I = \iint_A \frac{y}{1+x^3} dx dy, \quad A = \{(x,y) \in \mathbb{R}^2; 0 \leq x \leq 1, 0 \leq y \leq 1, x^2+y^2 \geq 1\}$$



Solució:

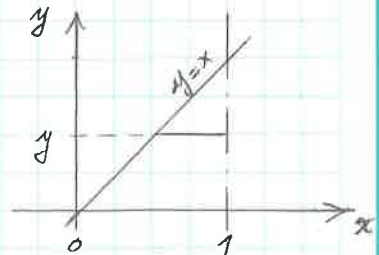
$$\begin{aligned} I &= \iint_A \frac{y}{1+x^3} dx dy = \int_0^1 \frac{dx}{1+x^3} \int_{\sqrt{1-x^2}}^1 y dy \\ &= \frac{1}{2} \int_0^1 \frac{x^2}{1+x^3} dx = \frac{1}{6} \ln(1+x^3) \Big|_0^1 = \frac{\ln 2}{6} \end{aligned}$$

$$h) I = \iint_A x^2 \sin(xy) dx dy, \quad A = \{(x,y) \in \mathbb{R}^2; 0 \leq y \leq 1, y \leq x \leq 1\}$$

$$\text{Solució. } I = \iint_A x^2 \sin(xy) dx dy = \int_0^1 x^2 dx \int_0^x \sin(xy) dy$$

$$= - \int_0^1 x^2 \cdot \left[ \frac{\cos(xy)}{x} \right]_{y=0}^{y=x} dx = - \int_0^1 x (\cos(x^2) - 1) dx$$

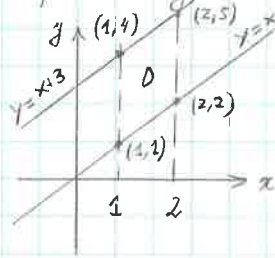
$$= \left[ -\frac{\sin(x^2)}{2} + \frac{x^2}{2} \right]_0^1 = \boxed{\frac{1 - \sin(1)}{2}}$$



10) Per a les integrals iterades següents escriu les equacions de les corbes que limiten les regions d'integració i dibuixen aquestes regions.

(a)  $\int_1^2 \left( \int_x^{x+3} f(x,y) dy \right) dx$

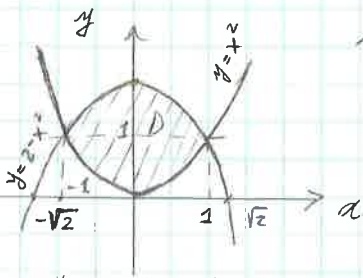
Solució:  $y=x, y=x+3, x=1, x=2$



(b)  $\int_{-1}^1 \left( \int_{x^2}^{2-x^2} f(x,y) dy \right) dx$

Solució:

$y=x^2, y=2-x^2, x=1, x=-1$  (veure figura).

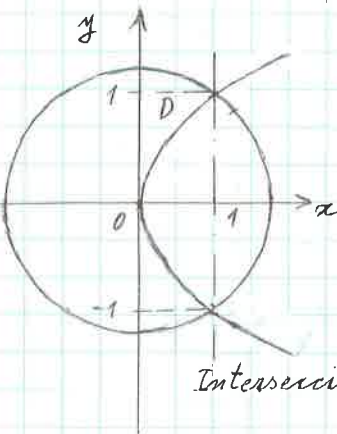
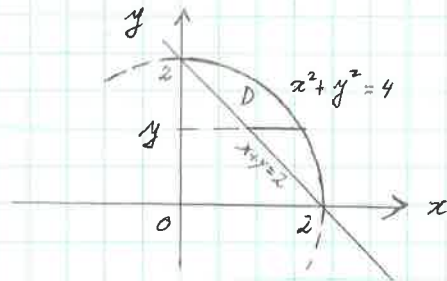


Intersecció:  $2-x^2=x^2 \Leftrightarrow x=\pm 1, y=1$

(c)  $\int_0^2 \left( \int_{2-y}^{\sqrt{4-y^2}} f(x,y) dx \right) dy$

Solució:

$x+y=2, x^2+y^2=4, y=0, y=2$  (veure figura).



(d)  $\int_0^1 \left( \int_{\sqrt{x}}^{\sqrt{2-x^2}} f(x,y) dy \right) dx$

Solució:

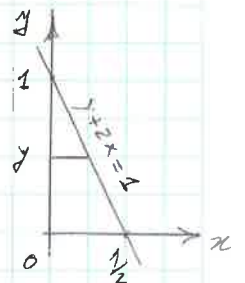
$y^2=x, x^2+y^2=2, x=0, x=1$ . (veure figura).

Intersecció:  $\begin{cases} x^2+y^2=2 \\ -x+y^2=0 \end{cases}$  d'on:  $x^2+x-2=0$ , llavors  $x = \frac{-1 \pm \sqrt{1+8}}{2}$ ;  $-2$  (No)  $1$  d'on  $y = \pm 1$

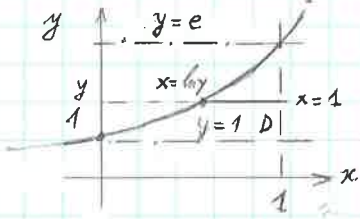
11) Invertiu l'ordre d'integració en les integrals iterades següents

(a)  $I = \int_0^{1/2} \left( \int_0^{1-2x} f(x,y) dy \right) dx$

Solució  $I = \int_0^{1/2} \left( \int_0^{1-2x} f(x,y) dy \right) dx = \int_0^1 \left( \int_0^{1-y} f(x,y) dx \right) dy$  □

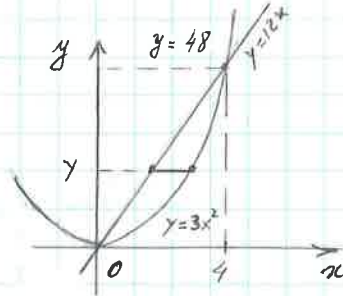


(b)  $I = \int_0^1 \left( \int_1^e f(x,y) dy \right) dx$



Solució.  $I = \int_0^1 \left( \int_1^e f(x,y) dy \right) dx = \int_1^e \left( \int_0^1 f(x,y) dx \right) dy. \square$

c)  $I = \int_0^4 \left( \int_{3x^2}^{12x} f(x,y) dy \right) dx$

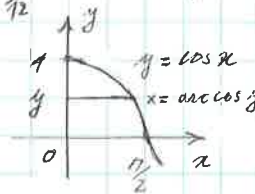


Intersecció:  $3x^2 = 12x$   
 $\Leftrightarrow x=0, x=4$   
 $y=0, y=48$

Solució.

$I = \int_0^4 \left( \int_{3x^2}^{12x} f(x,y) dy \right) dx = \int_0^{48} \left( \int_{y/12}^{\sqrt{y/3}} f(x,y) dx \right) dy. \square$

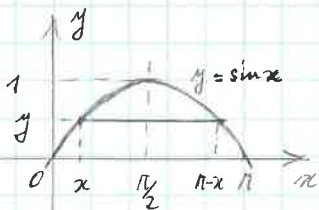
d)  $I = \int_0^{\pi/2} \left( \int_0^{\cos x} f(x,y) dy \right) dx$



Solució.

$I = \int_0^{\pi/2} \left( \int_0^{\cos x} f(x,y) dy \right) dx = \int_0^1 \left( \int_0^{\arccos y} f(x,y) dx \right) dy. \square$

e)  $\int_0^{\pi} \left( \int_0^{\sin x} f(x,y) dy \right) dx$



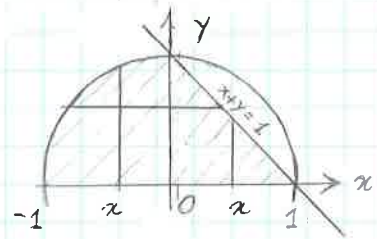
Solució.  $I = \int_0^{\pi} \left( \int_0^{\sin x} f(x,y) dy \right) dx = \int_0^1 \left( \int_{\arcsin y}^{\pi - \arcsin y} f(x,y) dx \right) dy. \square$

(\*) Definim:  $\arcsin: [-1,1] \rightarrow [-\pi/2, \pi/2]$  de manera que  $v = \arcsin u$ , amb  $-1 \leq u \leq 1$ , si i només si  $u = \sin v$ , amb  $-\pi/2 \leq v \leq \pi/2$ .



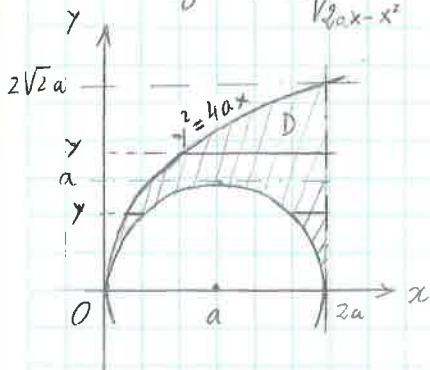
funció arcsin

f)  $I = \int_0^1 \left( \int_{-\sqrt{1-y^2}}^{1-y} f(x,y) dx \right) dy$



Solució.  $I = \int_0^1 \left( \int_{-\sqrt{1-y^2}}^{1-y} f(x,y) dx \right) dy = \int_{-1}^0 \left( \int_0^{\sqrt{1-x^2}} f(x,y) dy \right) dx + \int_0^1 \left( \int_0^{1-x} f(x,y) dy \right) dx. \square$

$$g) I = \int_0^{2a} \left( \int_{\sqrt{2ax-x^2}}^{\sqrt{4ax}} f(x,y) dy \right) dx, \quad a > 0.$$



Solució.

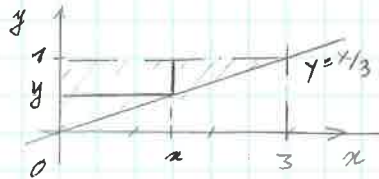
$$I = \int_0^{2a} \left( \int_{\sqrt{2ax-x^2}}^{\sqrt{4ax}} f(x,y) dy \right) dx$$

$$= \int_0^a \left( \int_{y^2/4a}^{a-\sqrt{a^2-y}} f(x,y) dx \right) dy + \int_0^a \left( \int_{a+\sqrt{a^2-y}}^{2a} f(x,y) dx \right) dy + \int_a^{2\sqrt{2}a} \left( \int_{y^2/4a}^{2a} f(x,y) dx \right) dy. \quad \square$$

Intersecció:

$$(x-a)^2 + y^2 = a^2 \quad \text{d'on: } x^2 - 2ax + a^2 + 4ax = a^2 \Leftrightarrow x(x+2a) = 0 \Leftrightarrow \begin{matrix} x=0 \\ x=-2a < 0 \text{ (No)} \end{matrix}$$

$$y^2 = 4ax$$

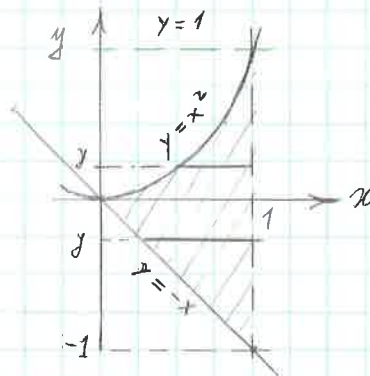


$$h) I = \int_0^3 \left( \int_{y/3}^1 f(x,y) dy \right) dx$$

Solució.

$$I = \int_0^3 \left( \int_{y/3}^1 f(x,y) dy \right) dx = \int_0^1 \left( \int_0^{3y} f(x,y) dx \right) dy \quad \square$$

$$i) I = \int_0^1 \left( \int_{-x}^{x^2} f(x,y) dy \right) dx$$



Solució.

$$I = \int_0^1 \left( \int_{-x}^{x^2} f(x,y) dy \right) dx$$

$$= \int_{-1}^0 \left( \int_{-y}^1 f(x,y) dx \right) dy + \int_0^1 \left( \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y) dx \right) dy \quad \square$$

13) Calculen les següents integrals iterades.

$$(a) I = \int_{-1}^2 \int_0^1 \int_0^{\pi/2} x^2 y^3 \sin z \, dz \, dy \, dx$$

Solució.

$$\begin{aligned} I &= \int_{-1}^2 \int_0^1 \int_0^{\pi/2} x^2 y^3 \sin z \, dz \, dy \, dx = \int_{-1}^2 x^2 dx \int_0^1 y^3 dy \int_0^{\pi/2} \sin z \, dz \\ &= \left[ \frac{x^3}{3} \right]_{-1}^2 \cdot \left[ \frac{y^4}{4} \right]_0^1 \cdot (-\cos z) \Big|_0^{\pi/2} = \left( \frac{8}{3} - \frac{1}{3} \right) \cdot \frac{1}{4} = \boxed{\frac{7}{12}} \end{aligned}$$

$$(b) I = \int_0^1 \int_0^x \int_0^{\sqrt{x^2+y^2}} z \, dz \, dy \, dx$$

Solució.

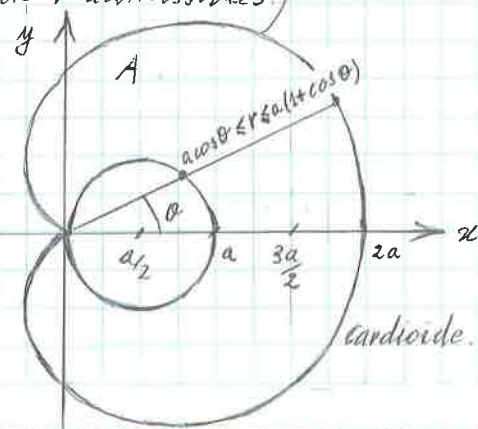
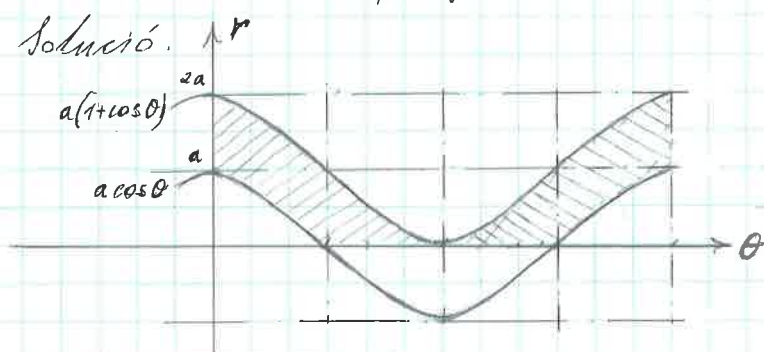
$$\begin{aligned} I &= \int_0^1 \int_0^x \int_0^{\sqrt{x^2+y^2}} z \, dz \, dy \, dx = \frac{1}{2} \int_0^1 dx \int_0^x (z^2+y^2) dy = \frac{1}{2} \int_0^1 \left[ x^2 y + \frac{y^3}{3} \right]_{y=0}^{y=x} dx \\ &= \frac{1}{2} \cdot \frac{4}{3} \int_0^1 x^3 dx = \boxed{\frac{1}{6}} \end{aligned}$$

$$\begin{aligned} (c) I &= \int_0^3 \int_0^{2x} \int_0^{\sqrt{xy}} z \, dz \, dy \, dx = \int_0^3 dx \int_0^{2x} dy \int_0^{\sqrt{xy}} z \, dz = \frac{1}{2} \int_0^3 x \, dx \int_0^{2x} y \, dy \\ &= \int_0^3 x^3 \, dx = \boxed{\frac{81}{4}} \end{aligned}$$

17) Calculen les àrees dels dominis  $A \subset \mathbb{R}^2$  definites en coordenades polars,  $x = r \cos \theta$ ,  $y = r \sin \theta$ , que s'indiquen tot seguit.

(a) A figura definida per  $a \cos \theta \leq r \leq a(1 + \cos \theta)$  ( $a > 0$ ). (Indicació: Observem que l'expressió té sentit quan  $\cos \theta \geq 0$ . Dibuixar les gràfiques de  $a \cos \theta$  i  $a(1 + \cos \theta)$  pot ajudar a veure els valors de  $r$  admissibles.)

Solució.



Dibuix amb Matlab/Octave

$\Rightarrow a=2$ ; % per exemple

$\Rightarrow \theta = \text{linspace}(0, 2\pi, 201)$ ;

$\Rightarrow \rho_1 = a * \cos(\theta)$ ;

$\Rightarrow \rho_2 = a * (1 + \cos(\theta))$ ;

$\Rightarrow \text{polar}(\theta, \rho_1, 'b-')$ ;

$\Rightarrow \text{axis equal}$

$\Rightarrow \text{hold on}$

$\Rightarrow \text{polar}(\theta, \rho_2, 'r-')$

$\Rightarrow \text{hold off}$

$$I_1 = \int_0^{\frac{\pi}{2}} d\theta \int_{a \cos \theta}^{a(1+\cos \theta)} r dr + \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{a \cos \theta}^{a(1+\cos \theta)} r dr$$

$$= 2 \int_0^{\frac{\pi}{2}} d\theta \int_{a \cos \theta}^{a(1+\cos \theta)} r dr = a^2 \int_0^{\frac{\pi}{2}} (1 + 2 \cos \theta) d\theta$$

$$= a^2 \left( 2 + \frac{\pi}{2} \right)$$

$$I_2 = 2 \int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^{a(1+\cos \theta)} r dr = a^2 \int_{\frac{\pi}{2}}^{\pi} (1 + \cos \theta)^2 d\theta$$

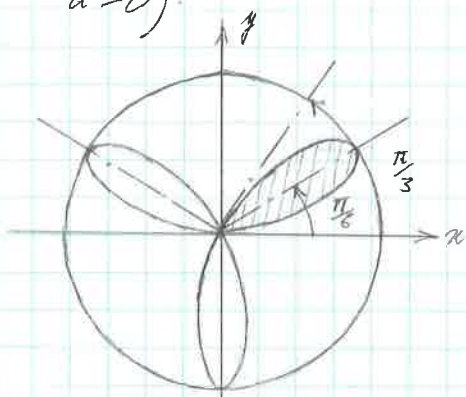
$$= a^2 \int_{\frac{\pi}{2}}^{\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta = a^2 \left( \frac{\pi}{2} - 2 + \frac{\pi}{2} - \frac{\pi}{4} \right)$$

$$= a^2 \left( \frac{3\pi}{4} - 2 \right)$$

$$\therefore I = \iint_A dx dy = I_1 + I_2 = a^2 \left( 2 + \frac{\pi}{2} + \frac{3\pi}{4} - 2 \right) = \boxed{a^2 \frac{5\pi}{4}}$$

(b) A regió limitada per un pètal de la rosa definit per  $r = a \sin 3\theta$  ( $0 \leq \theta \leq \frac{\pi}{3}$ ,

$a > 0$ ).

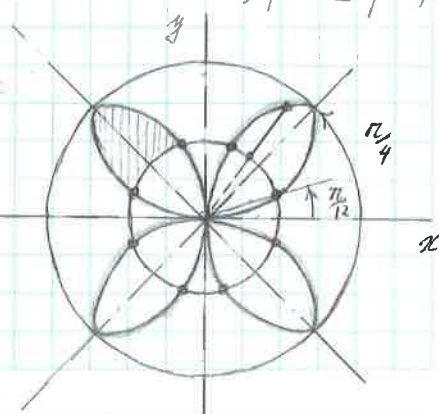
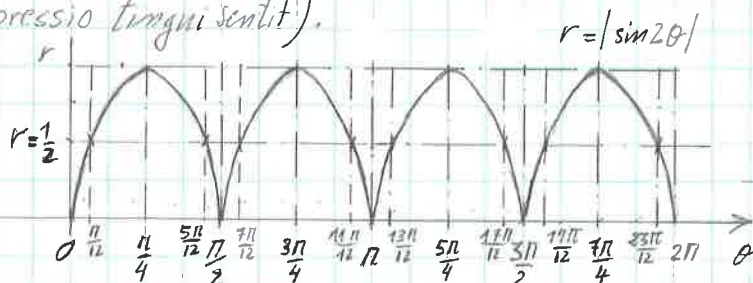


Solució.

$$A = \int_0^{\frac{\pi}{3}} d\theta \int_0^{a \sin(3\theta)} r dr = \frac{a^2}{2} \int_0^{\frac{\pi}{3}} \frac{1 - \cos(6\theta)}{2} d\theta$$

$$= \boxed{\frac{a^2 \pi}{12}}$$

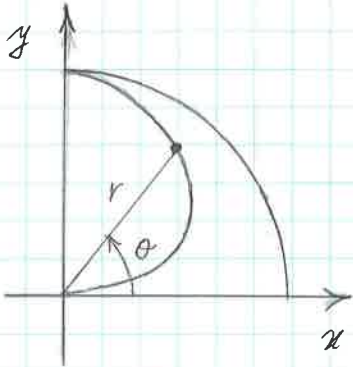
(c) A regió definida per  $\frac{1}{2} \leq r \leq |\sin(2\theta)|$ . (Indicació: Cal  $|\sin(2\theta)| \geq \frac{1}{2}$  perquè l'expressió tingui sentit).



$$\begin{aligned}
 A &= 4 \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} d\theta \int_{\frac{1}{2}}^{|\sin 2\theta|} r dr = 2 \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \left( \sin^2 2\theta - \frac{1}{4} \right) d\theta = 2 \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \left( \frac{1 - \cos 4\theta}{2} - \frac{1}{4} \right) d\theta \\
 &= 2 \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{1 - 2\cos 4\theta}{4} d\theta = 2 \left[ \frac{\theta}{4} - \frac{\sin 4\theta}{8} \right]_{\frac{\pi}{12}}^{\frac{5\pi}{12}} = 2 \left( \frac{5\pi}{48} - \frac{1}{8} \sin \left( \frac{5\pi}{3} \right) - \frac{\pi}{48} + \frac{1}{8} \sin \frac{\pi}{3} \right) \\
 &= 2 \left( \frac{\pi}{12} + \frac{\sqrt{3}}{8} \right) = \frac{2\pi + 3\sqrt{3}}{12} = \boxed{\frac{\pi}{6} + \frac{\sqrt{3}}{4}}
 \end{aligned}$$

(d) Anàlogament, calculeu la integral doble  $\iint_A \arcsin(x^2 + y^2) dx dy$ , on  $A$  és la regió limitada per la corba  $r = \sqrt{\sin \theta}$  ( $0 \leq \theta \leq \frac{\pi}{2}$ ).

Solució.



$$\begin{aligned}
 A &= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\sqrt{\sin \theta}} r \arcsin(r^2) dr = \begin{cases} \text{Integració per parts:} \\ f = \arcsin r^2 \Rightarrow f' = \frac{2r}{\sqrt{1-r^4}} \\ g' = r \Rightarrow g = \frac{r^2}{2} \end{cases} \\
 &= \int_0^{\frac{\pi}{2}} d\theta \left\{ \left[ \frac{r^2}{2} \arcsin(r^2) \right]_0^{\sqrt{\sin \theta}} - \int_0^{\sqrt{\sin \theta}} \frac{r^3}{\sqrt{1-r^4}} dr \right\} \\
 &\stackrel{(*)}{=} \int_0^{\frac{\pi}{2}} \frac{1}{2} (\theta \sin \theta + \cos \theta - 1) d\theta = \begin{cases} \text{parts:} \\ f = \theta \Rightarrow f' = 1 \\ g' = \sin \theta \Rightarrow g = -\cos \theta \end{cases}
 \end{aligned}$$

$$= \frac{1}{2} [-\theta \cos \theta]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \left( \cos \theta - \frac{1}{2} \right) d\theta = \left[ \sin \theta - \frac{\theta}{2} \right]_0^{\frac{\pi}{2}} = \boxed{1 - \frac{\pi}{4}}$$

(\*) Nota:

$$\int_0^{\sqrt{\sin \theta}} \frac{r^3}{\sqrt{1-r^4}} dr = -\frac{1}{2} \sqrt{1-r^4} \Big|_0^{\sqrt{\sin \theta}} = \frac{1}{2} - \frac{1}{2} |\cos \theta| = \frac{1}{2} - \frac{1}{2} \cos \theta$$

18) Calculeu les integrals dobles, fent servir el canvi de variable adequat en cada cas.

(a)  $\iint_D xy dx dy$ ,  $D = \{ (x,y) \in \mathbb{R}^2 : 6 \leq 2y-x \leq 12, 0 \leq x \leq 4 \}$ .

Solució. Introduïm el canvi:  $x = 4u$ ,  $6v = 2y-x \Leftrightarrow y = 3v + 2u$ ,

és a dir:  $(u,v) \in \mathbb{R}^2 \xrightarrow{T} (x,y) = T(u,v) = (4u, 3v+2u)$ , d'on  $\det DT(u,v) = \begin{vmatrix} 4 & 0 \\ 2 & 3 \end{vmatrix} = 12$ .

D'altra banda:  $6 \leq 6v = 2y-x \leq 12$ ,  $0 \leq 4u \leq 4 \Leftrightarrow 0 \leq u \leq 1$ ,  $1 \leq v \leq 2$ , és a dir el domini transformat és,  $D' = T^{-1}(D) = [0,1] \times [1,2]$ . Aleshores:

$$\begin{aligned}
 I &= \iint_D xy \, dx \, dy = \iint_{D'=T^{-1}(D)} x(u,v) \cdot y(u,v) \cdot |\det DT(u,v)| \, du \, dv = 12 \iint_{D'=[0,1] \times [1,2]} 4u(3v+2u) \, du \, dv \\
 &= 48 \int_1^2 dv \int_0^1 (3uv+2u^2) \, du = 48 \int_1^2 \left[ \frac{3}{2}u^2v + \frac{2}{3}u^3 \right]_{u=0}^{u=1} dv = 48 \int_1^2 \left( \frac{3}{2}v + \frac{2}{3} \right) dv \\
 &= 48 \left[ \frac{3}{4}v^2 + \frac{2}{3}v \right]_1^2 = 48 \left( 3 + \frac{4}{3} - \frac{3}{4} - \frac{2}{3} \right) = 48 \frac{36+8-9}{12} = 4 \cdot 35 = \boxed{140}
 \end{aligned}$$

(b)  $I = \iint_D \frac{1}{(1+x+y)^5} \, dx \, dy$ ,  $D = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x+y \leq 1\}$

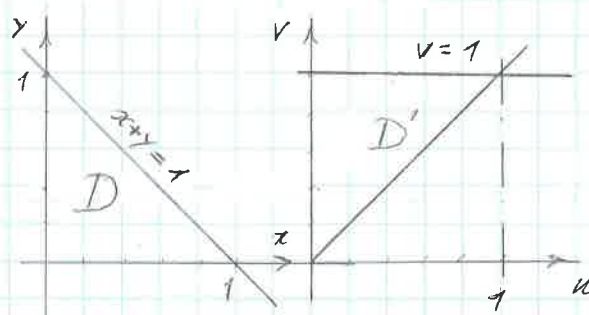
Solució. Fem el canvi:  $\begin{cases} u=x \\ v=x+y \end{cases} \Leftrightarrow \begin{cases} x=u \\ y=v-u \end{cases}$ , i.e.:  $(u,v) \in \mathbb{R}^2 \xrightarrow{T} (x,y) = T(u,v) = (u, v-u)$

$$DT(u,v) = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1,$$

d'altra banda, el domini  $D$  es

transforma (veure figura) en:

$$D' = T^{-1}(D) = \{(u,v) \in \mathbb{R}^2 : 0 \leq u \leq 1, u \leq v \leq 1\}.$$



I podem calcular la integral,

$$\begin{aligned}
 I &= \iint_D \frac{1}{(1+x+y)^5} \, dx \, dy = \iint_{D'=T^{-1}(D)} \frac{1}{(1+x(u,v)+y(u,v))^5} |\det DT(u,v)| \, du \, dv \\
 &= \iint_{D'} \frac{1}{(1+v)^5} \, du \, dv = \int_0^1 du \int_u^1 \frac{dv}{(1+v)^5} = \int_0^1 \left[ \frac{(1+v)^{-4}}{-4} \right]_u^1 du
 \end{aligned}$$

$$= \frac{1}{4} \int_0^1 \left[ \frac{1}{(1+u)^4} - \frac{1}{16} \right] du = \frac{1}{4} \left[ \frac{(1+u)^{-3}}{-3} - \frac{u}{16} \right]_0^1 = \frac{1}{4} \left( \frac{1}{3} - \frac{1}{24} - \frac{1}{16} \right) = \boxed{\frac{11}{192}}$$

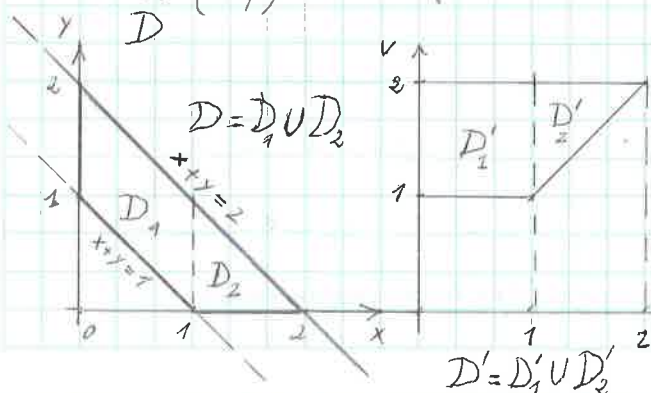
(c)  $I = \iint_D \frac{dx \, dy}{(x+y)^{n+1}}$ ,  $D = \{(x,y) \in \mathbb{R}^2 : 1 \leq x+y \leq 2, x \geq 0, y \geq 0\}$ .

Solució.

Considerem el canvi:  $\begin{cases} u=x \\ v=x+y \end{cases} \Leftrightarrow \begin{cases} x=u \\ y=v-u \end{cases}$ ,

i.e.:  $(u,v) \in \mathbb{R}^2 \xrightarrow{T} (x,y) = T(u,v) = (u, v-u)$ .

$\det DT(u,v) = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$ , mentre que



els dominis es transformen com:  $D' = T^{-1}(D) = \{(u,v) \in \mathbb{R}^2 : u \leq v, 1 \leq v \leq 2\}$ , i podem calcular  $I$  fent:

$$\begin{aligned} I &= \iint_D \frac{dx dy}{(x+y)^{m+1}} = \iint_{D'=T^{-1}(D)} \frac{1}{(x(u,v)+y(u,v))^{m+1}} |\det DT(u,v)| du dv \\ &= \int_1^2 \frac{dv}{v^{m+1}} \int_0^v du = \int_1^2 \frac{dv}{v^m} = \left[ \frac{v^{-m+1}}{-m+1} \right]_1^2 = \boxed{\frac{1}{m-1} \left( 1 - \frac{1}{2^{m-1}} \right)} \end{aligned}$$

Alternativament, podem fer

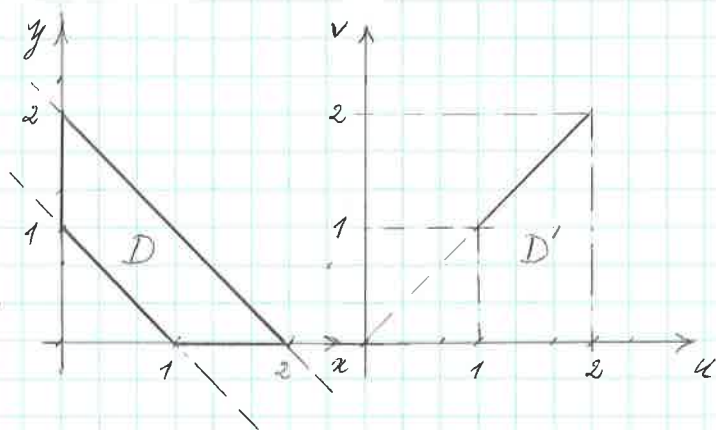
el canvi:

$$(u,v) \in \mathbb{R}^2 \xrightarrow{T} (x,y) = T(u,v) = (v, u-v);$$

el qual transforma el domini  $D$  en

$$D' = T^{-1}(D) = \{(u,v) \in \mathbb{R}^2 : v \leq u, 1 \leq u \leq 2\},$$

(veure figura al costat).



També mateix:

$$\det DT(u,v) = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1.$$

i llavors el càlcul de la integral resulta:

$$\begin{aligned} I &= \iint_D \frac{dx dy}{(x+y)^{m+1}} = \iint_{D'=T^{-1}(D)} \frac{1}{(x(u,v)+y(u,v))^{m+1}} |\det DT(u,v)| du dv \\ &= \int_1^2 \frac{du}{u^{m+1}} \int_0^u dv = \int_1^2 \frac{du}{u^m} = \left[ \frac{u^{-m+1}}{-m+1} \right]_1^2 = \frac{1}{m-1} - \frac{2^{-m+1}}{m-1} = \boxed{\frac{1}{m-1} \left( 1 - \frac{1}{2^{m-1}} \right)} \end{aligned}$$

$$(d) I = \iint_D \left( 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^{3/2} dx dy, D = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}.$$

Solució. Fem servir coordenades polars adaptades:  $x = |a| r \cos \theta$ ,  $y = |b| r \sin \theta$ , que transformem el domini  $D$  en  $D'$ :  $0 \leq r \leq 1$ ,  $0 \leq \theta \leq 2\pi$ , mentre que  $\det \frac{\partial(x,y)}{\partial(r,\theta)} = |ab| r$ . Aleshores:

$$\begin{aligned} I &= \iint_D \left( 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^{3/2} dx dy = \iint_{D'} \left( 1 - x^2(r,\theta)/a^2 - y^2(r,\theta)/b^2 \right)^{3/2} \left| \det \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr d\theta \\ &= |ab| \int_0^{2\pi} d\theta \int_0^1 r (1-r^2)^{3/2} dr = 2\pi |ab| \left[ \frac{(-1/5)(1-r^2)^{5/2}}{5/2} \right]_0^1 = \boxed{\frac{2\pi |ab|}{5}} \end{aligned}$$

(e)  $I = \iint_D \arctan\left(x^2 + \frac{y^2}{2}\right) dx dy, D = \left\{ (x,y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{2} \leq 1, x \geq 0, y \geq 0 \right\}.$

Solució. Apliquem, de nou, coordenades polars adaptades, aquest cop de la forma  $(r, \theta) \in [0, \infty) \times [0, \frac{\pi}{2}] \xrightarrow{T} (x,y) = T(r, \theta) = (r \cos \theta, \sqrt{2} r \sin \theta)$ , amb la qual cosa el domini es transforma en:  $D' = T^{-1}(D) = [0, 1] \times [0, \frac{\pi}{2}]$ , i podem calcular I tot fent:

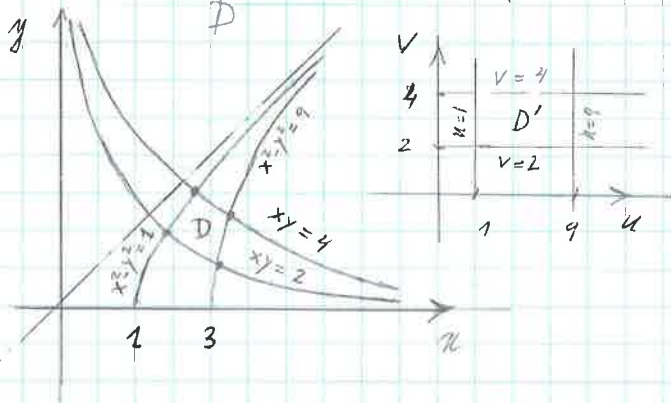
$$I = \iint_D \arctan\left(x^2 + \frac{y^2}{2}\right) dx dy = \iint_{D'=T^{-1}(D)} \arctan\left(x^2(r, \theta) + \frac{y^2(r, \theta)}{2}\right) \underbrace{\left| \det \frac{\partial(x,y)}{\partial(r, \theta)} \right|}_{\sqrt{2}r} dr d\theta =$$

$$\sqrt{2} \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r \arctan(r^2) dr = \frac{\pi \sqrt{2}}{2} \int_0^1 r \arctan(r^2) dr = \begin{cases} f = \arctan(r^2) \Rightarrow f' = \frac{2r}{1+r^4} \\ g' = r \Rightarrow g = r^2/2 \end{cases}$$

$$= \frac{\sqrt{2} \pi}{4} \left[ r^2 \arctan(r^2) \right]_0^1 - \frac{\sqrt{2} \pi}{2} \int_0^1 \frac{r^3}{1+r^4} dr = \frac{\sqrt{2} \pi^2}{16} - \frac{\pi \sqrt{2}}{8} \ln(1+r^4) \Big|_0^1 =$$

$$= \frac{\sqrt{2} \pi}{2} - \frac{\sqrt{2} \pi}{8} \ln 2 = \boxed{\frac{\pi \sqrt{2}}{8} \left( \frac{\pi}{2} - \ln 2 \right)}.$$

(f)  $I = \iint_D (x^2 + y^2) dx dy, D = \left\{ (x,y) \in \mathbb{R}^2 : 1 \leq x^2 - y^2 \leq 9, 2 \leq xy \leq 4, x \geq 0, y \geq 0 \right\}$



Solució. Fem el canvi T, on T<sup>-1</sup> ve donat per:  $\begin{cases} u = x^2 - y^2 \\ v = xy \end{cases}$ , d'on:  $\begin{cases} 1 \leq u = x^2 - y^2 \leq 9 \\ 2 \leq v = xy \leq 4 \end{cases}$ .

Lavors tenim, pel domini transformat:

$$D' = T^{-1}(D) = [1, 9] \times [2, 4],$$

(veure figura).

D'altra banda, pel determinant del Jacobia, tenim:

$$\det \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2 + y^2), \text{ d'on: } \det \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\det \frac{\partial(u,v)}{\partial(x,y)}} = \frac{1/2}{x^2 + y^2}$$

i podem calcular la integral fent,

$$I = \iint_D (x^2 + y^2) dx dy = \iint_{D'=T^{-1}(D)} (x^2(u,v) + y^2(u,v)) \left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| du dv = \frac{1}{2} \iint_{D'=[1,9] \times [2,4]} \frac{x^2(u,v) + y^2(u,v)}{x^2(u,v) + y^2(u,v)} du dv$$

$$= \frac{1}{2} \left( \int_1^9 du \right) \cdot \left( \int_2^4 dv \right) = \frac{1}{2} (9-1) \cdot (4-2) = \boxed{8}.$$

$$\iint_D \frac{x+2xy}{x^2+y^2} dx dy, D = \{(x,y) \in \mathbb{R}^2 : x^2 \leq y \leq x^2+1, 1 \leq x^2+y^2 \leq e^2, x \geq 0\}$$

Introduïm el canvi:

$$(x,y) \in \mathbb{R}^2 \xrightarrow{T^{-1}} (u,v) = T^{-1}(x,y) = (y-x^2, x^2+y^2)$$

Lavors:

$0 \leq u = y - x^2 \leq 1$   
 $1 \leq v = x^2 + y^2 \leq e^2$  d'on, clarament:  $D' = T^{-1}(D) = [0,1] \times [1,e^2]$ , mentre que el determinant del Jacobí de  $T$  es pot calcular a partir del de  $T^{-1}$ . En efecte:

$$\det DT^{-1}(x,y) = \det \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -2x & 1 \\ 2x & 2y \end{vmatrix} = -4xy - 2x = -2(x+2xy),$$

d'on:

$$\det DT(u,v) = \det \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\det \frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\det DT^{-1}(x,y)} = \frac{-1/2}{x+2xy},$$

el càlcul de la integral  $I$  es redueix a:

$$I = \iint_D \frac{x+2xy}{x^2+y^2} dx dy = \iint_{D'=T^{-1}(D)} \frac{x(u,v)+2x(u,v)y(u,v)}{x^2(u,v)+y^2(u,v)} \left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

$$= \iint_{D'=T^{-1}(D)} \frac{x(u,v)+2x(u,v)y(u,v)}{x^2(u,v)+y^2(u,v)} \cdot \frac{1/2}{|x(u,v)+2x(u,v)y(u,v)|} du dv$$

$$D'=T^{-1}(D) = [0,1] \times [1,e^2]$$

$$\stackrel{(*)}{=} \iint_{D'=[0,1] \times [1,e^2]} \frac{1/2 du dv}{v} = \frac{1}{2} \left( \int_0^1 du \right) \cdot \left( \int_1^{e^2} \frac{dv}{v} \right) =$$

(\*)  $x(u,v), y(u,v) \geq 0 \quad \forall (u,v) \in D' = [0,1] \times [1,e^2]$ .

$$\text{Lavors: } |x(u,v)+2x(u,v)y(u,v)| = x(u,v)+2x(u,v)y(u,v) = \frac{1}{2} (\ln e^2 - \ln 1) = \boxed{1}$$

19) Useu coordenades cilíndriques per calcular les següents integrals triples.

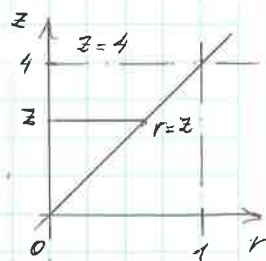
Nota. Coordenades cilíndriques:  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $z = z$ , amb  $r \in [0, +\infty)$ ,  $\theta \in [0, 2\pi]$

$$z \in (-\infty, +\infty), \det \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r \geq 0.$$

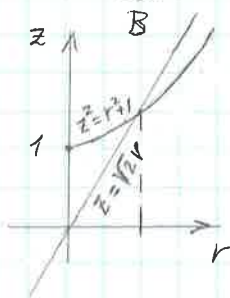
$$(a) I = \iiint_B \sqrt{x^2+y^2+z^2} dx dy dz, B = \{(r,\theta,z) \in \mathbb{R}^3 : \sqrt{x^2+y^2} \leq z \leq 4\}$$

$$\text{Solució. } I = \int_0^{2\pi} d\theta \int_0^4 dz \int_0^z r \sqrt{r^2+z^2} dr = \frac{2\pi}{3} \int_0^4 (z^2+r^2)^{3/2} \Big|_0^z dz$$

$$= \frac{2\pi}{3} (\sqrt{z}-1) \Big|_0^4 = \frac{2\pi}{3} (2\sqrt{2}-1) \cdot \left[ \frac{z^4}{4} \right]_0^4 = \boxed{\frac{128\pi}{3} (2\sqrt{2}-1)}$$



$$(b) I = \iiint_B z e^{-(x^2+y^2)} dx dy dz, \quad B = \{(x,y,z) \in \mathbb{R}^3 : z^2 - 1 \leq x^2 + y^2 \leq \frac{z^2}{2}, z \geq 0\}$$



Solució. En coordenades cilíndriques el domini B es transforma en

$$B': \sqrt{2}r \leq z \leq \sqrt{r^2+1}, \quad 0 \leq r \leq 1.$$

La integral I en coordenades cilíndriques es calcula com:

$$I = \int_0^{2\pi} d\theta \int_0^1 r e^{-r^2} dr \int_{\sqrt{2}r}^{\sqrt{r^2+1}} z dz = \frac{1}{2} 2\pi \int_0^1 r e^{-r^2} (r^2+1-2r^2) dr$$

$$\begin{aligned} z^2 &= r^2 + 1 & \text{intersecció:} \\ z^2 &= 2r^2 & r=1 \quad (r>0) \\ & & z=\pm\sqrt{2} \end{aligned}$$

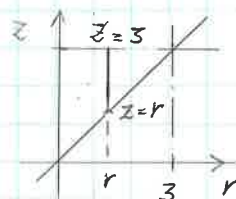
$$= \pi \int_0^1 r(1-r^2) e^{-r^2} dr = \left\{ \begin{array}{l} \text{parts: } f = 1-r^2 \Rightarrow f' = -2r \\ g' = r e^{-r^2} \Rightarrow g = -\frac{e^{-r^2}}{2} \end{array} \right\} =$$

$$= \underbrace{\pi \left( \frac{r^2-1}{2} \right) e^{-r^2}}_{\pi/2} \Big|_0^1 - \underbrace{\pi \int_0^1 r e^{-r^2} dr}_{-\frac{\pi}{2} [e^{-r^2}]_0^1} = \frac{\pi}{2} + \frac{\pi}{2e} - \frac{\pi}{2} = \boxed{\frac{\pi}{2e}}$$

$$(c) I = \iiint_B (x+y-2z) dx dy dz, \quad B = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq z^2, 0 \leq z \leq 3\}$$

Solució. En coordenades cilíndriques el domini B es transforma

$$\text{en } B' = \{(\theta, r, z) \in [0, 2\pi] \times [0, +\infty) \times (-\infty, +\infty), 0 \leq r \leq z, 0 \leq z \leq 3\},$$



d'on la integral es pot calcular com:

$$I = \int_0^{2\pi} d\theta \int_0^3 r dr \int_r^3 (r \cos \theta + r \sin \theta - 2z) dz = \int_0^{2\pi} d\theta \int_0^3 r (r z \cos \theta + r z \sin \theta - z^2) \Big|_{z=r}^{z=3} dr$$

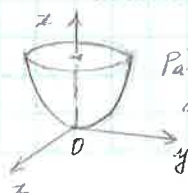
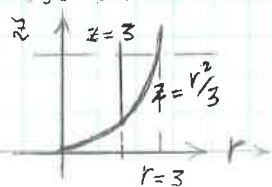
$$= \int_0^{2\pi} d\theta \int_0^3 r (3r \cos \theta + 3r \sin \theta - 9 - r^2 \cos \theta - r^2 \sin \theta + r^2) dr$$

$$= \int_0^{2\pi} d\theta \int_0^3 [-9r + 3(\cos \theta + \sin \theta)r^2 - (\cos \theta + \sin \theta - 1)r^3] dr$$

$$= \int_0^{2\pi} \left[ -\frac{81}{2} + (\cos \theta + \sin \theta) \cdot 27 - (\cos \theta + \sin \theta - 1) \cdot \frac{81}{4} \right] d\theta = 2\pi \left( -\frac{81}{2} + \frac{81}{4} \right) = \boxed{-\frac{81\pi}{2}}$$

$$(d) I = \iiint_B (x^2+y^2) dx dy dz, \quad B = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 3z \leq 9\}$$

Solució. Les coordenades cilíndriques transformen el domini d'integració B, en B':  $\frac{r^2}{3} \leq z \leq 3$ ,



Paraboloida circular.

$$\text{d'on: } I = \int_0^{2\pi} d\theta \int_0^3 r^3 dr \int_{r^2/3}^3 dz = 2\pi \int_0^3 r^3 (3 - \frac{r^2}{3}) dr$$

$$= 2\pi \left( \frac{3}{4} r^4 - \frac{r^6}{18} \right) \Big|_0^3 = 2\pi \left( \frac{3 \cdot 81}{4} - \frac{3^2 \cdot 81}{2 \cdot 3^2} \right) = \frac{2\pi}{4} (3 \cdot 81 - 2 \cdot 81) = \frac{81\pi}{2}$$

$$(e) I = \iiint_B z \, dx \, dy \, dz, \quad B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 6, x^2 + y^2 \leq z, z \geq 0\}.$$

Solució. El domini  $B$  es transforma, per coordenades cilíndriques en

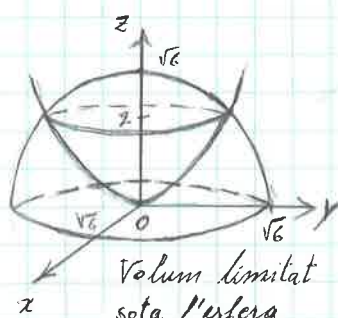
$$B': \quad r^2 \leq z \leq \sqrt{6-r^2}, \quad 0 \leq \theta \leq 2\pi,$$

i la integral  $I$  es pot calcular com

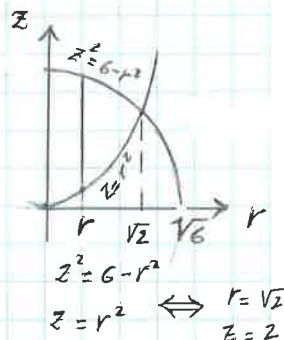
$$I = \int_0^{2\pi} d\theta \int_0^{\sqrt{z}} r \, dr \int_{r^2}^{\sqrt{6-r^2}} z \, dz$$

$$= \frac{2\pi}{2} \int_0^{\sqrt{z}} r(6-r^2-r^4) \, dr = \pi \left( 3r^2 - \frac{r^4}{4} - \frac{r^6}{6} \right) \Big|_0^{\sqrt{z}}$$

$$= \pi \left( 6 - \frac{4}{4} - \frac{1}{3} \right) = \pi \left( 5 - \frac{1}{3} \right) = \boxed{\frac{14\pi}{3}}$$



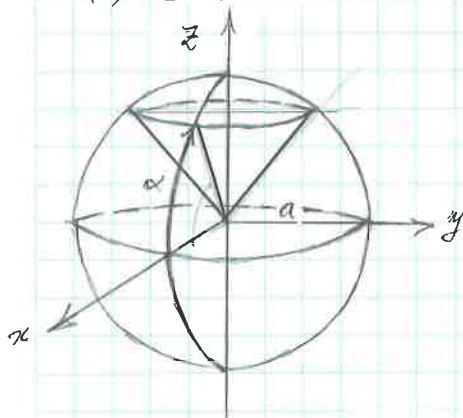
Volum limitat sota l'esfera  $x^2 + y^2 + z^2 = 6$  i sobre el paraboloid circular  $z = x^2 + y^2$ :  $x^2 + y^2 \leq z \leq \sqrt{6-x^2-y^2}$



21) Calculeu els volums dels dominis  $B \subset \mathbb{R}^3$  definits en coordenades esfèriques,  $x = r \cos \varphi \cos \theta$ ,  $y = r \cos \varphi \sin \theta$ ,  $z = r \sin \varphi$  ( $0 \leq \theta \leq 2\pi$ ,  $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$ ,  $r \geq 0$ ) que s'indiquen tot seguit.

Remarca. Recordem que:  $\det \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} = r^2 \cos \varphi$ .

(a)  $B$  domini tallat sobre la bola  $r \leq a$  pel con  $\alpha \leq \varphi \leq \frac{\pi}{2}$  ( $a > 0$ ,  $0 < \alpha < \frac{\pi}{2}$ ).



Solució:

$$V = \int_0^{2\pi} d\theta \int_{\alpha}^{\frac{\pi}{2}} \cos \varphi \, d\varphi \int_0^a r^2 \, dr = \boxed{\frac{2\pi a^3}{3} (1 - \sin \alpha)}$$

(b)  $B$  volum tancat per l'esfera deformada definida per

$r = 1 + 0.2 \sin(8\theta) \sin \varphi$ . (Sòlids d'aquesta mena s'utilitzen com a models de tumors.)

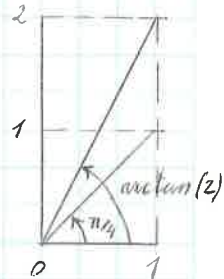
$$\begin{aligned} \text{Solució. } V &= \int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi \, d\varphi \int_0^{1+0.2 \sin(8\theta) \sin \varphi} r^2 \, dr = \frac{1}{3} \int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi (1+0.2 \sin(8\theta) \sin \varphi)^3 \, d\varphi \\ &= \frac{1}{3} \int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \varphi + 0.2 \sin(8\theta) \sin \varphi \cos \varphi + 3 \cdot 0.2^2 \sin^2(8\theta) \sin^2 \varphi \cos \varphi + 0.2^3 \sin^3(8\theta) \sin^3 \varphi \cos \varphi) \, d\varphi \\ &= \frac{1}{3} \int_0^{2\pi} \left( 2 + 3 \cdot 0.04 \cdot \frac{2}{3} \sin^2(8\theta) \right) d\theta = \frac{1}{3} (4\pi + 0.08\pi) = \boxed{1.36\pi} \end{aligned}$$

(c) Anàlogament, calculeu la integral triple  $I = \iiint_B \frac{1}{\sqrt{x^2+y^2+z^2}} dx dy dz$ , on  $B$  és la regió del primer octant de  $\mathbb{R}^3$  acotada pels con  $\varphi = \pi/4$  i  $\varphi = \arctan(2)$  i l'esfera  $r = \sqrt{6}$  (Recordem:  $\sin(\arctan(a)) = \frac{a}{\sqrt{1+a^2}}$ ).

Solució

$$I = \int_0^{\pi/2} d\theta \int_{\pi/4}^{\arctan(2)} \cos\varphi d\varphi \int_0^{\sqrt{6}} \frac{1}{r} r^2 dr = \frac{\pi}{2} \left[ \sin\varphi \right]_{\pi/4}^{\arctan(2)} \cdot \left[ \frac{r^2}{2} \right]_0^{\sqrt{6}}$$

$$= \frac{\pi}{2} \cdot \frac{6}{2} \left( \sin(\arctan 2) - \sin\left(\frac{\pi}{4}\right) \right) \stackrel{(*)}{=} \frac{3\pi}{2} \left( \frac{2}{\sqrt{5}} - \frac{\sqrt{2}}{2} \right)$$

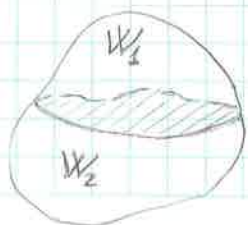


$$(*) \sin(\arctan a) = \frac{\sin(\arctan a) \cos(\arctan a)}{\cos(\arctan a)} = \tan(\arctan a) \frac{1}{\sqrt{1+\tan^2(\arctan a)}} = \frac{a}{\sqrt{1+a^2}}, \quad \left(-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}\right)$$

27) Trobeu la massa total del cilindre  $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 2, 0 \leq z \leq 3\}$  si la seva densitat és  $\rho(x, y, z) = z e^{-z^2} (x^2 + y^2)$ .

Solució.  $m(V) = \iiint_V \rho(x, y, z) dx dy dz = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r^3 dr \int_0^3 z e^{-z^2} dz =$   
coordenades cilíndriques.  
 $= 2\pi \left[ \frac{r^4}{4} \right]_0^{\sqrt{2}} \cdot \left[ -\frac{1}{2} e^{-z^2} \right]_0^3 = \boxed{\pi(1 - e^{-9})}$

29) Sigui  $W \subset \mathbb{R}^3$  un cos amb densitat de masses  $\rho(x, y, z)$ . Si dividim  $W$  en dues parts,  $W = W_1 \cup W_2$ , i denotem per  $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$  i  $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$  els centres de masses de  $W_1$  i  $W_2$  respectivament, demostreu que el centre de masses  $(\bar{x}, \bar{y}, \bar{z})$  de  $W$  és el mateix que si suposem tota la massa de  $W_1$  concentrada en  $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$  i la de  $W_2$  ho està en  $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$ .



Solució. Introduïm la notació següent  $W = W_1 \cup W_2$

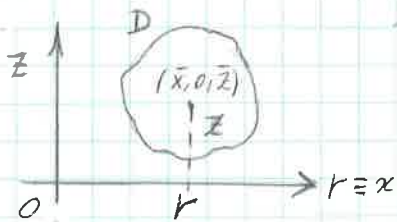
(amb  $\mu(W_1 \cap W_2) = \iiint_{W_1 \cap W_2} dx dy dz = 0$ ),  $\vec{r} = (x, y, z)$ ,  $\rho(\vec{r}) = \rho(x, y, z)$ ,

$d\vec{r} = dx dy dz$ . Així, podem escriure; pel CDM del cos  $W$ :

$$\begin{aligned} \vec{R}(W) &= (X(W), Y(W), Z(W)) = \frac{1}{m(W)} \iiint_W \vec{r} \rho(\vec{r}) d\vec{r} \\ &= \frac{1}{m(W)} \left( m(W_1) \cdot \frac{\iiint_{W_1} \vec{r} \rho(\vec{r}) d\vec{r}}{m(W_1)} + m(W_2) \cdot \frac{\iiint_{W_2} \vec{r} \rho(\vec{r}) d\vec{r}}{m(W_2)} \right) \\ &= \frac{m(W_1) \cdot \vec{R}(W_1) + m(W_2) \cdot \vec{R}(W_2)}{m(W)} \end{aligned}$$

$\vec{R}(W_1)$ : CDM de  $W_1$                        $\vec{R}(W_2)$ : CDM de  $W_2$

30) Sigui  $D$  un recinte pla contingut en el semipla  $\{y=0, x \geq 0\}$  de  $\mathbb{R}^3$ . Si denotem per  $(\bar{x}, 0, \bar{z})$  el centre geomètric de  $D$  (i.e. el seu centre de masses si suposem densitat constant igual a 1), demostreu que el volum del domini de revolució  $W$  que obtenim si fem girar  $D$  entorn de l'eix  $z$  és  $\text{Volum}(W) = 2\pi \bar{x} \cdot \text{Àrea}(D)$ , on  $2\pi \bar{x}$  és la longitud de la circumferència que obtenim en fer girar  $(\bar{x}, 0, \bar{z})$ . (Indicació: Useu coordenades cilíndriques i observeu que  $(\theta, r, z) \in D^* = [0, 2\pi] \times D$ .)



Solució.

$$\{y=0, x > 0\}, \rho(x, 0, z) = \rho \equiv 1 \text{ (densitat const. = 1)}$$

$$V = \int_0^{2\pi} d\theta \underbrace{\iint_D r dr dz}_{\bar{x} \cdot \text{Àrea}(D)} \stackrel{(*)}{=} \int_0^{2\pi} d\theta \cdot \bar{x} \cdot \text{Àrea}(D)$$

$$= 2\pi \bar{x} \cdot \text{Àrea}(D)$$

$$(*) \quad \bar{x} = \frac{\iint_D x \rho(x, 0, z) dx dz}{\iint_D \rho(x, 0, z) dx dz} = \frac{\iint_D x dx dz}{\iint_D dx dz} \quad \rho(x, 0, z) \equiv 1$$

$$\text{d'on:} \quad \iiint_D x dx dz = \bar{x} \cdot \text{Àrea}(D)$$

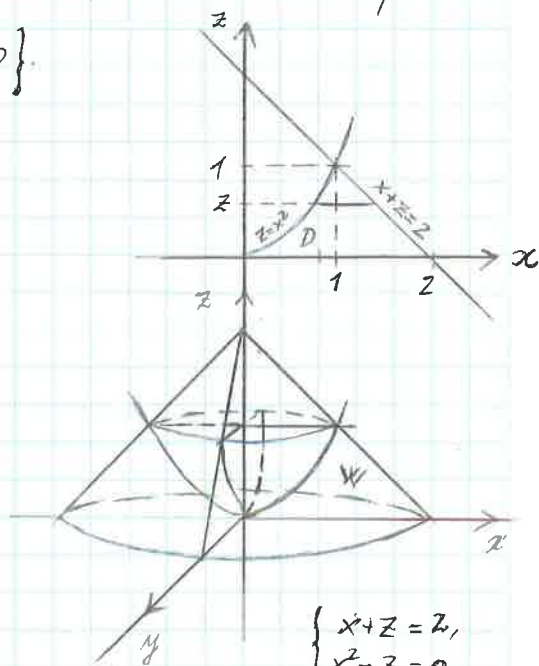
31) Apliquen el resultat del problema anterior al càlcul del volum  $W$  si prenem  
 $D = \{(x, z) \in \mathbb{R}^2; z \leq x^2; x+z \leq 2, x \geq 0, z \geq 0\}$ .

Solució.

$$\begin{aligned} \text{Àrea}(D) \cdot \bar{x} &= \int_0^1 dz \int_{\sqrt{z}}^{2-z} x dx = \frac{1}{2} \int_0^1 [(2-z)^2 - z] dz \\ &= \frac{1}{2} \int_0^1 (4 - 5z + z^2) dz = \frac{1}{2} \left( 4z - \frac{5}{2}z^2 + \frac{z^3}{3} \right) \Big|_0^1 \\ &= \frac{1}{2} \left( 4 - \frac{5}{2} + \frac{1}{3} \right) = \frac{24 - 15 + 2}{12} = \frac{11}{12} \end{aligned}$$

D'on, aplicant la fórmula del problema 30:

$$\text{Volum}(W) = 2\pi \bar{x} \cdot \text{Àrea}(D) = \boxed{\frac{11\pi}{6}}$$

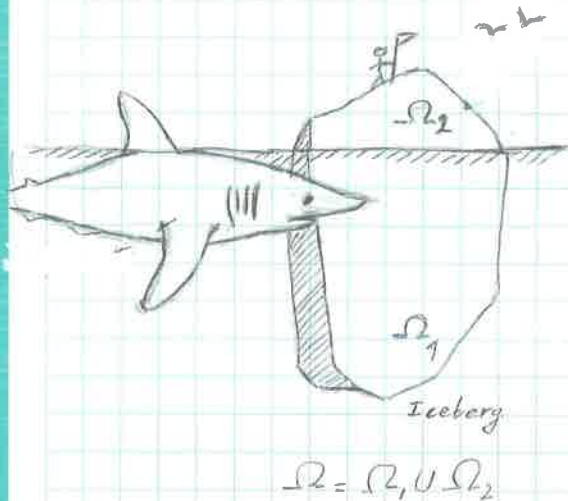


$$\begin{cases} x+z=2, \\ x^2-z=0. \end{cases}$$

Punt de tall:  
 $x^2 + x - 2 = 0 \Leftrightarrow x = -2 \text{ (No)}$   
 $x = 1$

per tant el punt de tall  
resulta:  
 $(x, z) = (1, 1)$

32) La densitat de l'aigua és  $1 \text{ Kg/l}$  i la del gel (aproximadament)  $0.9 \text{ Kg/l}$ . Pel principi d'Arquimedes, si submergim un bloc de gel en l'aigua el volum d'aigua desplaçat per la part submergida del gel té un pes igual al pes total del gel.



a) Vegeu que la part del gel emergent és el 10% del seu volum.

Solució. Si  $\rho = 1 \text{ Kg/l}$  és la densitat de l'aigua i  $\sigma = 0.9 \text{ Kg/l}$  és la densitat del gel, d'acord amb el principi d'Arquimedes, si  $\Omega_1$  és la part submergida,  $\Omega_2$  és la part que sobresurt del bloc de gel  $\Omega = \Omega_1 \cup \Omega_2$  i  $V(\Omega_1), V(\Omega_2), V(\Omega) = V(\Omega_1) + V(\Omega_2)$  ( $V(\Omega_1 \cap \Omega_2) = 0$ ) són els respectius volums, hom té:

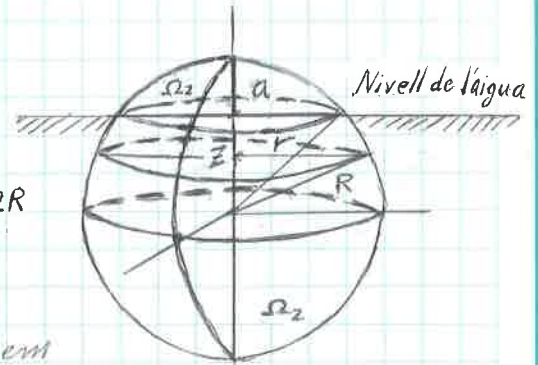
$$V(\Omega_1) \rho = \sigma V(\Omega) \Rightarrow \frac{V(\Omega_1)}{V(\Omega)} = \frac{\sigma}{\rho}$$

$$\text{i per tant: } \frac{V(\Omega_2)}{V(\Omega)} = 1 - \frac{V(\Omega_1)}{V(\Omega)} = 1 - \frac{\sigma}{\rho} = 1 - 0.9 = 0.1 \text{ (10\%)}$$

(b) Si tenim un bloc de gel esfèric, diguem quina és la part submergida del bloc i quina l'emergent.

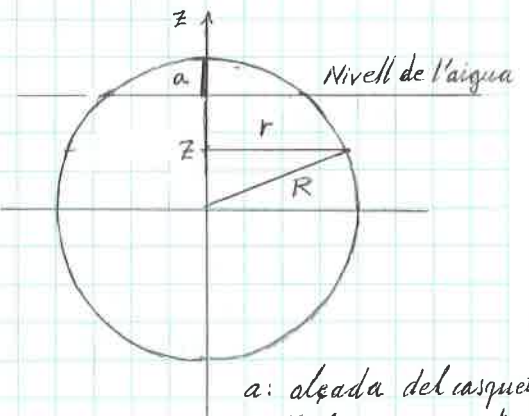
Solució. Suposem un bloc esfèric de radi  $R$  (veure figura). Volem trobar l'alçada,  $0 < a \leq 2R$  del casquet esfèric emergent,  $\Omega_2$ . Per calcular el volum corresponent,  $V(\Omega_2)$ , apliquem el principi de Cavalieri:

$$\begin{aligned} V(\Omega_2) &= \int_{R-a}^R S'(z) dz = \int_{R-a}^R \pi(R^2 - z^2) dz \\ &= \pi \left( R^2 z - \frac{z^3}{3} \right) \Big|_{R-a}^R \\ &= \pi \left( R^3 - \frac{R^3}{3} - R^3 + R^2 a + \frac{R^3}{3} - R^2 a + R a^2 - \frac{a^3}{3} \right) \\ &= \pi R^3 \left( \frac{a^2}{R^2} - \frac{1}{3} \frac{a^3}{R^3} \right). \end{aligned}$$



$$r^2 = R^2 - z^2,$$

$$S'(z) = \pi r^2 = \pi(R^2 - z^2)$$



$a$ : alçada del casquet d'esfera emergent.

llavors, d'acord amb l'apartat anterior:

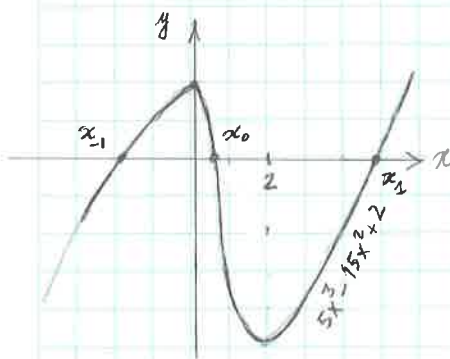
$$\pi R^3 \left( \frac{a^2}{R^2} - \frac{1}{3} \frac{a^3}{R^3} \right) = \frac{1}{10} \frac{4}{3} \pi R^3,$$

i definint  $x = a/R$  s'arriba a l'equació:  $5x^3 - 15x^2 + 2 = 0$ ; la qual té tres solucions reals:  $x_{-1} < 0 < x_0 < 2 < x_1$ . Com  $0 < x = a/R \leq 2$ , la solució que busquem és

$$x_0 = 0.391600211318183^{(*)}.$$

Així doncs, l'alçada del casquet d'esfera emergent, en termes del radi  $R$ , ve donada per:

$$a = 0.391600211318183 R$$



(\*) Càlcul amb MATLAB/Octave:

$\Rightarrow f = \text{inline}('5 * x.^3 - 15 * x.^2 + 2', 'x');$

$\Rightarrow \text{opt} = \text{optimset}('TolFun', 1e-16, 'TolX', 0);$

$\Rightarrow x0 = 1.5$

$\Rightarrow [x0, f0] = \text{fsolve}(f, x0, \text{opt})$

$x0 = 0.391600211318183$

$f0 = -4.44089209850063e-16$

33) La densitat de població d'una certa ciutat pot aproximar-se per la funció  $\rho(x,y) = 4000 e^{-0.01(x^2+y^2)}$  si  $x^2+y^2 \leq 49$  i  $\rho(x,y) = 0$  altrament, on  $x, y$  es mesuren en Km.

(a) Quina és la població de la ciutat.

Solució. Treballant en coordenades polars:  $\tilde{\rho}(r) = \rho(r \cos \theta, r \sin \theta) = 4000 e^{-r^2/100}$ ,

$$P = \iint_{\mathbb{R}^2} \rho(x,y) dx dy = \iint_{B(0,0)} \rho(x,y) dx dy = \int_0^{2\pi} d\theta \int_0^7 r \tilde{\rho}(r) dr$$

$$= \int_0^{2\pi} d\theta \int_0^7 4000 r e^{-r^2/100} dr = 2\pi \cdot 4 \cdot 10^3 \left( -50 e^{-r^2/100} \right) \Big|_0^7 = 4\pi \cdot 10^5 \left( 1 - e^{-49/100} \right) \cong \boxed{486.788'0}$$

(b) Quina és la distància  $0 < R < 7$  al centre geogràfic de la ciutat de forma que el 50% de la població viu a una distància més petita o igual que  $R$  del centre?

Solució.

$$\int_0^{2\pi} d\theta \int_0^R r \tilde{\rho}(r) dr = 8000\pi \int_0^R r e^{-r^2/100} dr = 4 \cdot 10^5 \pi \left( 1 - e^{-R^2/100} \right)$$

$$= 0.5 \cdot 4 \cdot 10^5 \pi \left( 1 - e^{-49/100} \right),$$

d'on, aïllant  $R$ :

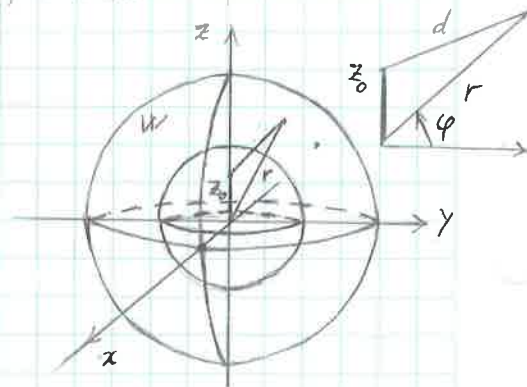
$$R = 10 \sqrt{-\ln \left( 1 - \frac{1 - e^{-49/100}}{2} \right)} = 10 \sqrt{-\ln \left( \frac{1}{2} + \frac{1}{2} e^{-49/100} \right)} \cong \boxed{4.640 \text{ Km}}$$

35) Troben el potencial gravitatori  $V(0,0,z_0)$  generat pel sòlid  $W$  comprès entre dues esferes concèntriques de radis  $a < b$ ,  $W = \{(x,y,z) \in \mathbb{R}^3; a^2 \leq x^2+y^2+z^2 \leq b^2\}$ , suposant densitat constant igual a  $\rho$ . Per simplificar els càlculs considereu només valors  $z_0$  amb  $0 < z_0 < a$  o  $z_0 > b$ .

Solució,  $\rho(x,y,z) \equiv \rho$  cnt.

•  $0 < z_0 < a$

$$V(0,0,z_0) = -G \iiint_W \frac{\rho}{d((0,0,z_0), (x,y,z))} dx dy dz =$$



pàg. següent ...

$$= -G\rho \int_0^{2\pi} d\theta \int_a^b r^2 dr \int_{-\pi/2}^{\pi/2} \frac{\cos\varphi}{\sqrt{z_0^2 + r^2 - 2rz_0 \sin\varphi}} d\varphi = -G\rho \int_0^{2\pi} d\theta \int_a^b r^2 \left[ -\frac{\sqrt{z_0^2 + r^2 - 2rz_0 \sin\varphi}}{rz_0} \right]_{\varphi=-\pi/2}^{\varphi=\pi/2} d\varphi$$

$$= \frac{G\rho}{z_0} 2\pi \int_a^b r(1z_0 - r) dr = -\frac{G\rho}{z_0} 4\pi z_0 \int_a^b r dr = \boxed{-2\pi G(b^2 - a^2)}$$

$$0 < z_0 < a < r < b$$

$$\Rightarrow z_0 - r < a - r < 0$$

$$\forall a < r < b.$$

(b)  $z_0 > b$ :

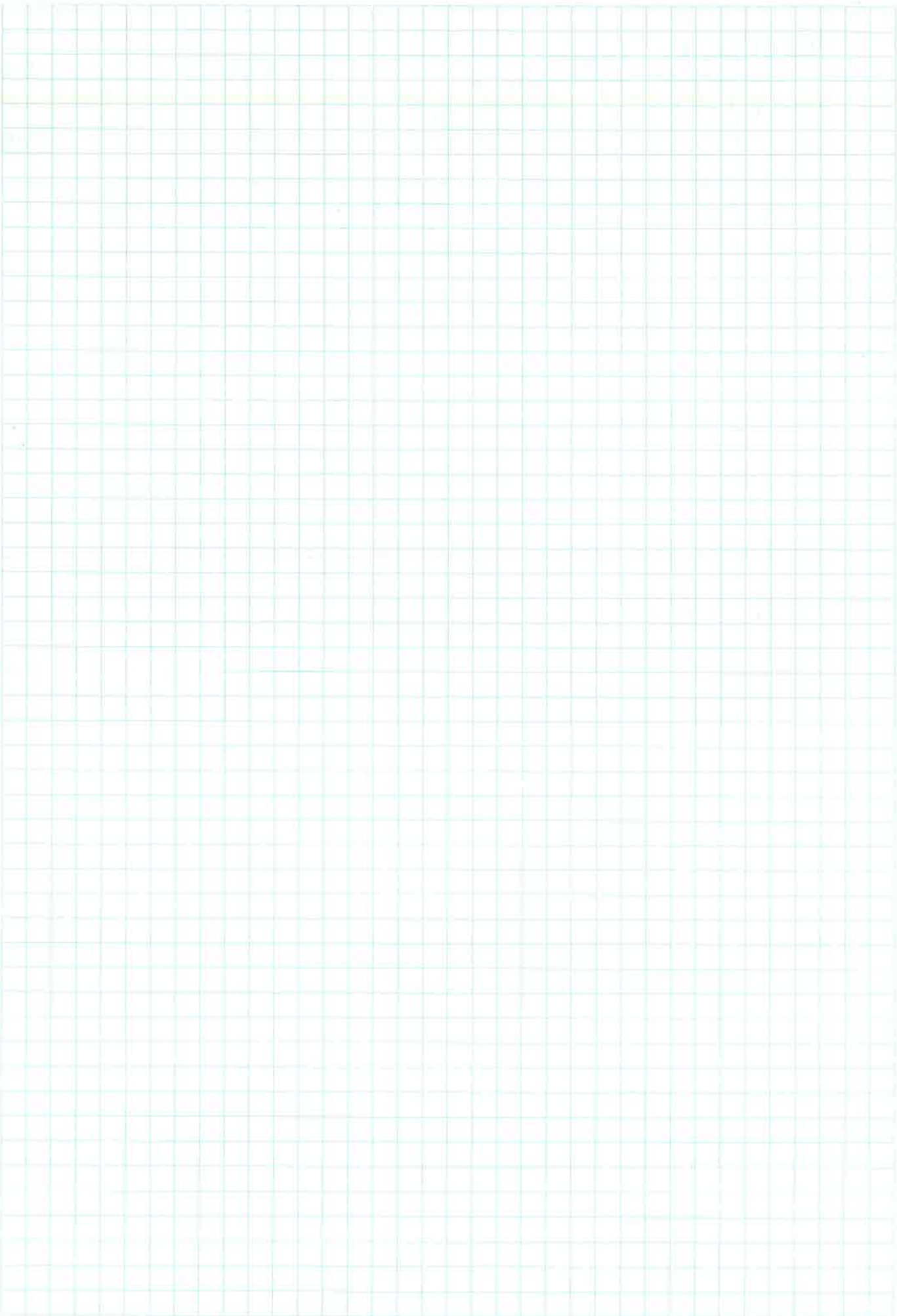
$$V(0,0,z_0) = \dots = \frac{G\rho}{z_0} 2\pi \int_a^b r(1z_0 - r) dr = -\frac{G\rho}{z_0} 2\pi \int_a^b 2r^2 dr = \boxed{-\frac{4\pi G\rho}{3z_0}(b^3 - a^3)}$$

$$0 < a < r < b < z_0$$

$$\Rightarrow z_0 - r > b - r > 0$$

$$\forall a < r < b.$$

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WIDE MIDDLELINE