TEMA 2. Integració de funcions de vàries variables.

3) Apliquem el principi de Cavalieri per calcular els següents volums a partir de l'àrea de seccions amb plans paral·lels als plans coordenats (triades de forma adequada)

(a) Volum envoltat per l'elipsoide: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solució. Fixant $0<x<a$: $\frac{y^2}{b^2\left(1-\frac{x^2}{a^2}\right)}+\frac{z^2}{c^2\left(1-\frac{x^2}{a^2}\right)}=1$, d'on $S(x)=\pi bc\left(1-\frac{x^2}{a^2}\right)$

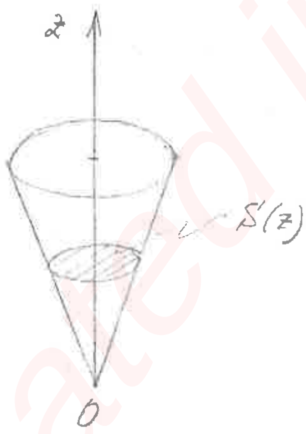
Nota: recordem que l'àrea d'una el·lipse ve donada per πAB on A i B són la longitud dels seus semieixos.

Llavors, aplicant el principi de Cavalieri: ...

$$V=2\int_0^a S(x)\,dx=2\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx=2\pi bc\left(x-\frac{x^3}{3a^2}\right)\Bigg|_0^a=2\pi bc\left(a-\frac{a}{3}\right)$$

$$=\boxed{\frac{4}{3}\pi abc}$$

(b) Volum envoltat pel con de base el·líptica $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z^2$, amb $0\leq z\leq h$



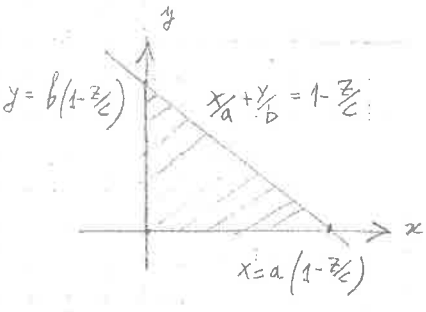
Solució. Fixant $0\leq z\leq h$: $\frac{x^2}{(az)^2}+\frac{y^2}{(bz)^2}=1$, d'on $S(z)=\pi abz^2$

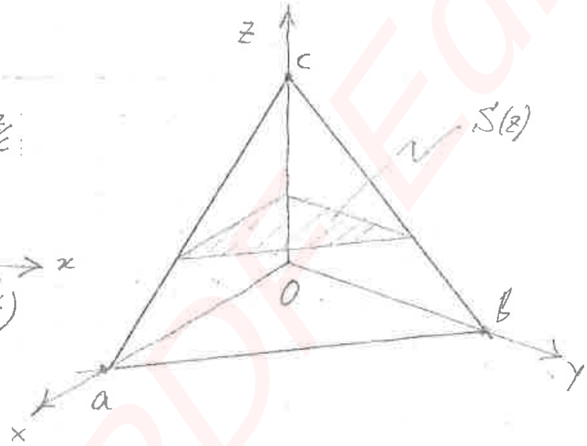
Aleshores, aplicant el principi de Cavalieri:

$$V=\int_0^h S(z)\,dz=\int_0^h \pi abz^2\,dz=\pi ab\,\frac{z^3}{3}\Bigg|_0^h=\boxed{\frac{\pi}{3}abh^3}$$

Nota: $V=\frac{1}{3}\underbrace{\pi(ah)\cdot(bh)}_{\text{Àrea de la base.}}\cdot h$

c) Volum de tetraedre limitat pels plans $x=0,\ y=0,\ z=0,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1\ \ (a,b,c>0)$



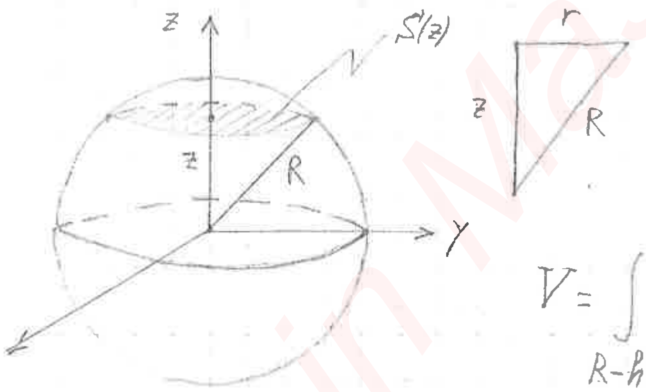


Solució. Fixant z: $S(z)=\frac{1}{2}ab\left(1-\frac{z}{c}\right)^2$,

d'on, aplicant el principi de Cavalieri:

$$V=\int_0^c S(z)\,dz=\frac{1}{2}ab\int_0^c\left(1-\frac{z}{c}\right)^2dz=-\frac{1}{6}abc\left(1-\frac{z}{c}\right)^3\Big|_0^c=\boxed{\frac{abc}{6}}$$

d) Volum envoltat pel casquet esfèric determinat per l'esfera $x^2+y^2+z^2=R^2$ i la condició $R-h\le z\le R$.



Solució.

$$S(z)=\pi r^2(z)=\pi\left(R^2-z^2\right)$$

D'on, aplicant el principi de Cavalieri:

$$V=\int_{R-h}^{R}S(z)\,dz=\pi\int_{R-h}^{R}\left(R^2-z^2\right)dz$$

$$=\pi\left(R^2z-\frac{z^3}{3}\right)\Big|_{R-h}^{R}=\pi\left(R^3-\cancel{\frac{R^3}{3}}-R^3+\cancel{R^2h}+\cancel{\frac{R^3}{3}}-\cancel{R^2h}+Rh^2-\frac{h^3}{3}\right)=\boxed{\frac{\pi h^2}{3}(3R-h)}$$

4) Generalitzeu el principi de Cavalieri al càlcul de volums en $\mathbb{R}^4$ i calculeu el volum de la bola 4-dimensional $B=\{(x,y,z,t)\in\mathbb{R}^4: x^2+y^2+z^2+t^2\le R^2\}$

Solució: Fixem t: $0\le x^2+y^2+z^2\le R^2-t^2$, amb $-R\le t\le R$. Aleshores $V(t)=\frac{4}{3}\pi\left(R^2-t^2\right)^{3/2}$. D'on, aplicant el principi de Cavalieri: $M=\int_{-R}^{R}V(t)\,dt=\frac{4}{3}\pi\int_{-R}^{R}\left(R^2-t^2\right)^{3/2}dt=\frac{8}{3}\pi R^3\int_0^R\left(1-t^2/R^2\right)^{3/2}dt=\left\{\text{c.v. } t/R=\sin\theta\right\}=$

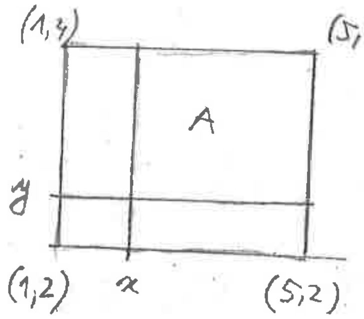
$$=\frac{8}{3}\pi R^4\int_0^{\pi/2}\cos^4\theta\,d\theta\overset{(*)}{=}\frac{8}{3}\pi R^4\int_0^{\pi/2}\left(\frac{1+\cos(2\theta)}{2}-\frac{1-\cos(4\theta)}{8}\right)d\theta=\frac{8}{3}\pi R^4\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\boxed{\frac{\pi^2R^4}{12}}$$

(*) Nota: $\cos^4\theta=\cos^2\theta\left(1-\sin^2\theta\right)=\cos^2\theta-\frac{1}{4}\left(2\sin\theta\cos\theta\right)^2=\cos^2\theta-\frac{1}{4}\sin^2(2\theta)=\frac{1+\cos(2\theta)}{2}-\frac{1-\cos(4\theta)}{8}$ fent servir les fórmules de l'angle doble.

8) Per a les regions $A \subset \mathbb{R}^2$ indicades, escriviu la integral doble $\iint_A f(x,y)\,dx\,dy$ en termes d'integrals iterades preses en diferents ordres $\int\left(\int f\,dx\right)dy$ i $\int\left(\int f\,dy\right)dx$, donats quins són els extrems d'integració per a x i y en cada cas.

a) A rectangle de vèrtexs $(1,2)$, $(5,2)$, $(5,4)$ i $(1,4)$.

Solució.

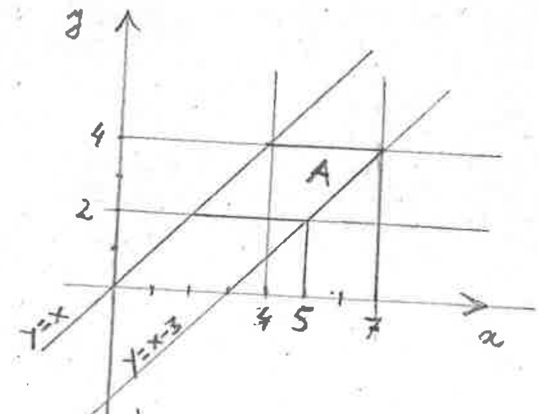


$$\iint_A f(x,y)\,dx\,dy = \int_1^5 dx \int_1^4 f(x,y)\,dy = \int_2^4 dy \int_1^5 f(x,y)\,dx$$

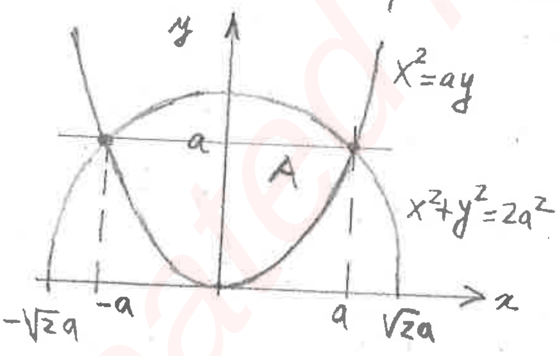
b) A paral·lelogram limitat per les rectes $y=x$, $y=x-3$, $y=2$, $y=4$.

$$I = \iint_A f(x,y)\,dx\,dy = \int_2^4 dx \int_2^x f(x,y)\,dy + \int_4^5 dx \int_2^4 f(x,y)\,dy$$

$$+ \int_5^7 dx \int_{x-3}^4 f(x,y)\,dy = \int_2^4 dy \int_y^{y+3} f(x,y)\,dx$$



c) A regió limitada per les corbes $x^2+y^2=2a^2$, $x^2=ay$ $(y\geq 0,\ a>0)$.



$$I = \iint_A f(x,y)\,dx\,dy = \int_{-a}^{a} dx \int_{\frac{x^2}{a}}^{\sqrt{2a^2-x^2}} f(x,y)\,dy$$

$$= \int_0^a dy \int_{-\sqrt{ay}}^{\sqrt{ay}} f(x,y)\,dx + \int_a^{\sqrt{2}a} dy \int_{-\sqrt{2a^2-y^2}}^{\sqrt{2a^2-y^2}} f(x,y)\,dx$$

$ay+y^2=2a^2 \Leftrightarrow y^2+ay-2a^2=0$

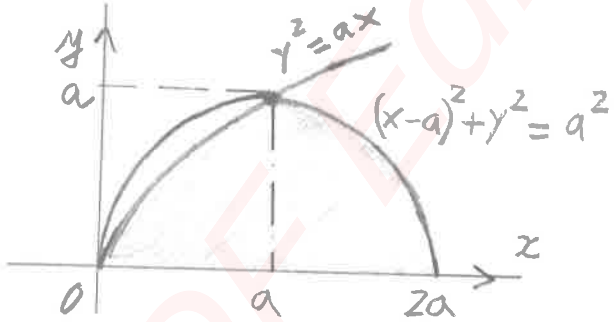
d'on: $y = \dfrac{-a \pm \sqrt{a^2+8a^2}}{2}$

$= \dfrac{-a\pm 3a}{2} = \begin{cases} a \\ -2a \text{ (No)} \end{cases}$

d) A regió limitada per les corbes $y^2=ax$, $x^2+y^2=2ax$ ($y\geq 0$, $a>0$)

$$I=\iint_A f(x,y)\,dx\,dy=\int_0^a\left(\int_0^{\sqrt{ax}} f(x,y)\,dy\right)dx+\int_a^{2a}\left(\int_0^{\sqrt{2ax-x^2}} f(x,y)\,dy\right)dx$$

$$=\int_0^a\left(\int_{y^2/a}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx\right)dy$$

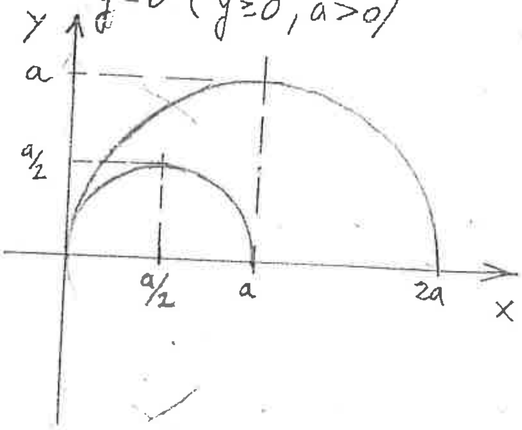


Nota:

$2ax-x^2-ax=ax-x^2=x(a-x)\geq 0$

$\forall\ 0\leq x\leq a$.

e) A regió limitada per les corbes $x^2+y^2=ax$, $x^2+y^2=2ax$, $y=0$ ($y\geq 0$, $a>0$)



$$I=\iint_A f(x,y)\,dx\,dy$$

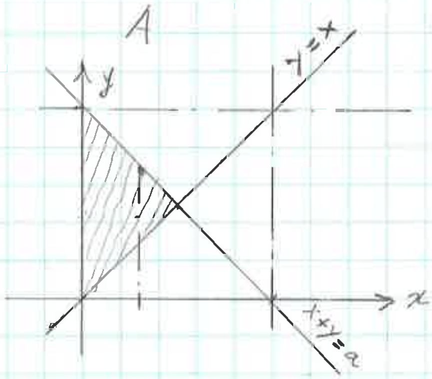
$$=\int_0^a dx\int_{\sqrt{ax-x^2}}^{\sqrt{2ax-x^2}} f(x,y)\,dy+\int_a^{2a} dx\int_0^{\sqrt{2ax-x^2}} f(x,y)\,dy$$

$$=\int_0^{a/2} dy\int_{a-\sqrt{a^2-y^2}}^{a/2-\sqrt{a^2/4-y^2}} f(x,y)\,dx+\int_0^{a/2} dy\int_{a/2+\sqrt{a^2/4-y^2}}^{a} f(x,y)\,dx$$

$$+\int_0^{a/2} dy\int_a^{a+\sqrt{a^2-y^2}} f(x,y)\,dx+\int_{a/2}^{a} dy\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx.\ \square$$

12 Calculeu les següents integrals dobles on els dominis de $\mathbb{R}^2$ que s'indiquen

(a) $I=\iint (x^2+y^2)\,dx\,dy$ A limitat per les rectes $y=x$, $x+y=2a$, $x=0$ $(a>0)$



$$I=\iint_A (x^2+y^2)\,dx\,dy = \int_0^a dx \int_x^{2a-x}(x^2+y^2)\,dy$$

$$=\int_0^a dx\left(x^2y+\frac{y^3}{3}\right)\Bigg|_{y=x}^{y=2a-x} = \int_0^a\left(x^2(2a-x)+\frac13(2a-x)^3-x^3-\frac{x^3}{3}\right)dx$$

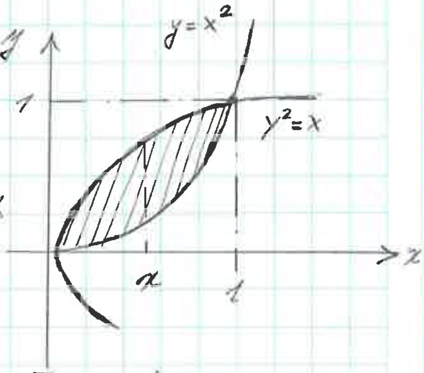
$$\overset{(*)}{=}\int_0^a\left(\frac{8a^3}{3}-4a^2x+4ax^2-\frac83x^3\right)dx = \left[\frac83a^3x-2a^2x^2+\frac43ax^3-\frac23x^4\right]_0^a$$

$$=\boxed{\frac{4a^4}{3}}\ \square$$

(*) $2ax^2-x^3+\frac83a^3-4a^2x+2ax^2-\frac{x^3}{3}-x^3-\frac{x^3}{3}$

$=\frac83a^3-4a^2x+4ax^2-\frac83x^3$

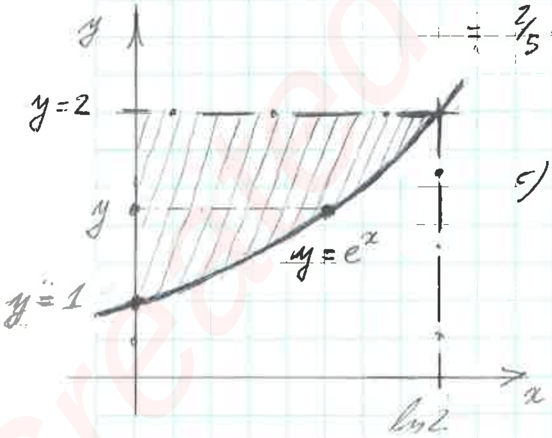
(b) $I=\iint_A (x+2y)\,dx\,dy$, A limitat per les corbes $y=x^2$, $y^2=x$



Solució

$$I=\iint_A (x+2y)\,dx\,dy=\int_0^1 dx\int_{x^2}^{\sqrt{x}}(x+2y)\,dy=\int_0^1 dx\,(xy+y^2)\Bigg|_{y=x^2}^{y=\sqrt{x}}=\int_0^1\left(x^{3/2}+x-x^3-x^4\right)dx$$

$$=\frac25x^{5/2}+\frac{x^2}{2}-\frac{x^4}{4}-\frac{x^5}{5}=\frac{8+10-5-4}{20}=\boxed{\frac{9}{20}}\ \square$$



c) $I=\iint_A e^{x+y}\,dx\,dy$. A limitat per les corbes $y=e^x$, $x=0$, $y=2$

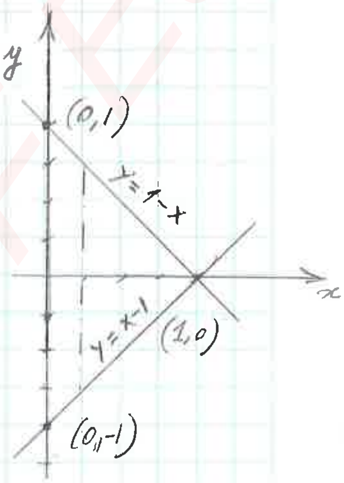
$$I=\int_1^2 e^y\,dy\int_0^{\ln y}e^x\,dx=\int_1^2 e^y\,dy\,\left[e^x\right]_0^{\ln y}=\int_1^2 e^y(y-1)\,dy$$

$$=\left[ye^y-2e^y\right]_1^2=\cancel{2e^2}-\cancel{2e^2}-e+2e=\boxed{e}\ \square$$

Alternativament:

$$I = \int_0^{\ln 2} e^x dx \int_{e^x}^{2} e^y dy = \int_0^{\ln 2} e^x \left[e^y\right]_{e^x}^{2} dx = \int_0^{\ln 2} e^x \left(e^2 - e^{e^x}\right) dx$$

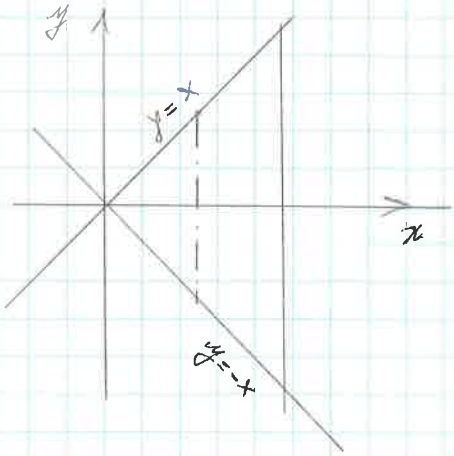
$$= \left(e^{2+x} - e^{e^x}\right|_0^{\ln 2} = 2e^2 - e^2 - e^2 + e = \boxed{e} \quad \square$$



d) $I = \iint_A e^y dx dy$, A triangle de vèrtexs: $(1,0), (0,1), (0,-1)$

$$I = \iint_A e^y dx dy = \int_0^1 dx \int_{x-1}^{1-x} e^y dy = \int_0^1 dx \left(e^{1-x} - e^{x-1}\right)$$

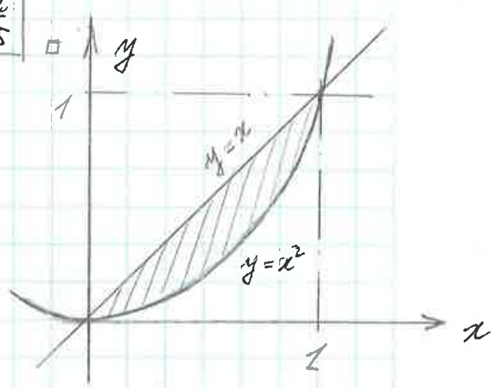
$$= -e^{1-x} - e^{x-1}\Big|_0^1 = \boxed{e + \frac{1}{e} - 2} \quad \square$$



e) $I = \iint_A xy^2 dx dy$, A limitat per les rectes $x=1$, $x=y$, $x+y=0$.

$$I = \iint_A xy^2 dx dy = \int_0^1 x dx \int_{-x}^{x} y^2 dy = \int_0^1 x \left[\frac{y^3}{3}\right]_{y=-x}^{y=x} dx$$

$$= \frac{2}{3}\int_0^1 x^4 dx = \frac{2}{15}\left(x^5\right|_0^1 = \boxed{\frac{2}{15}} \quad \square$$



f) $I = \iint_A xy\, dx dy$. A limitat per les corbes $x=y$, $y=x^2$

$$I = \iint_A xy\, dx dy \int_0^1 x dx \int_{x^2}^{x} y\, dy = \int_0^1 x \left[\frac{y^2}{2}\right]_{x^2}^{x} dx$$

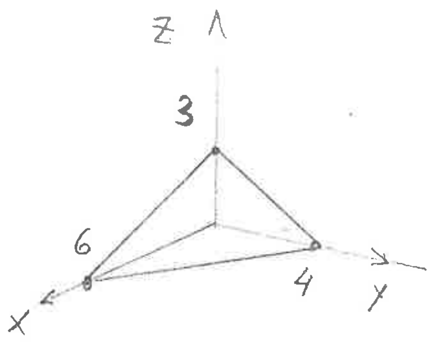
$$= \frac{1}{2}\int_0^1 \left(x^3 - x^5\right) dx = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \boxed{\frac{1}{24}} \quad \square$$

$x=y$, $y=x^2$

$x = x^2 \Leftrightarrow x(x-1)=0$; $x=0$, $x=1$

14) Per les regions de $\mathbb{R}^3$ indicades escriviu la integral triple $\iiint_A f(x,y,z)\,dx\,dy\,dz =: I$ en termes d'integrals iterades preses en diferents ordres.

(a) A tetraedre limitat pels plans $x=0,\ y=0,\ z=0,\ 2x+3y+4z=12$.



$$I = \int_0^6 dx \int_0^{4-\frac{2}{3}x} dy \int_0^{3-\frac{1}{2}x-\frac{3}{4}y} f(x,y,z)\,dz = \int_0^4 dy \int_0^{6-\frac{3}{2}y} dx \int_0^{3-\frac{1}{2}x-\frac{3}{4}y} f(x,y,z)\,dz$$

$$= \int_0^6 dx \int_0^{3-\frac{1}{2}x} dz \int_0^{4-\frac{2}{3}x-\frac{4}{3}z} f(x,y,z)\,dy = \int_0^3 dz \int_0^{6-2z} dx \int_0^{4-\frac{2}{3}x-\frac{4}{3}z} f(x,y,z)\,dy$$

$$= \int_0^4 dy \int_0^{3-\frac{3}{4}y} dz \int_0^{6-\frac{3}{2}y-2z} f(x,y,z)\,dx = \int_0^3 dz \int_0^{4-\frac{4}{3}z} dy \int_0^{6-\frac{3}{2}y-2z} f(x,y,z)\,dx. \ \square$$

- $4z = 12-2x-3y \rightarrow z = 3-\frac{1}{4}x-\frac{3}{4}y$

  $z=0,\ 2x+3y=12 \rightarrow y = 4-\frac{2}{3}x$

  $x = 6-\frac{3}{2}y$

- $3y = 12-2x-4z \rightarrow y = 4-\frac{2}{3}x-\frac{4}{3}z$

  $y=0,\ 2x+4z=12 \rightarrow z = 3-\frac{1}{2}x$

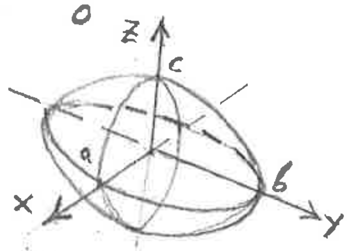
  $x = 6-2z$

- $2x = 12-3y-4z \rightarrow x = 6-\frac{3}{2}y-2z$

  $x=0:\ 3y+4z=12 \rightarrow z = 3-\frac{3}{4}y$

  $y = 4-\frac{4}{3}z$

(b) A interior del el·lipsoide



$$I = \int_{-a}^{a} dx \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} dy \int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} f(x,y,z)\,dz$$

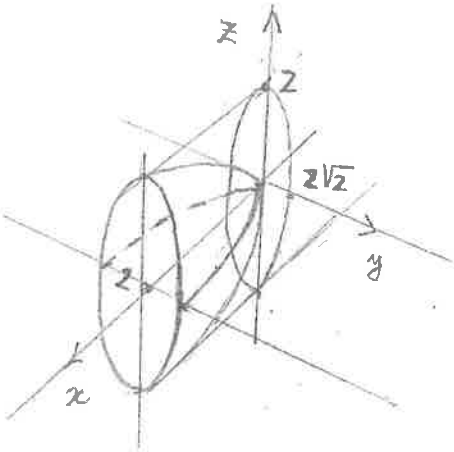
$$= \int_{-b}^{b} dy \int_{-a\sqrt{1-y^2/b^2}}^{a\sqrt{1-y^2/b^2}} dx \int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} f(x,y,z)\,dz$$

$$= \int_{-a}^{a} dx \int_{-c\sqrt{1-x^2/a^2}}^{c\sqrt{1-x^2/a^2}} dz \int_{-b\sqrt{1-x^2/a^2-z^2/c^2}}^{b\sqrt{1-x^2/a^2-z^2/c^2}} f(x,y,z)\,dy = \int_{-c}^{c} dz \int_{-a\sqrt{1-z^2/c^2}}^{a\sqrt{1-z^2/c^2}} dx \int_{-b\sqrt{1-x^2/a^2-z^2/c^2}}^{b\sqrt{1-x^2/a^2-z^2/c^2}} f(x,y,z)\,dy$$

$$= \int_{-b}^{b} dy \int_{-c\sqrt{1-y^2/b^2}}^{c\sqrt{1-y^2/b^2}} dz \int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z)\,dx = \int_{-c}^{c} dz \int_{-b\sqrt{1-z^2/c^2}}^{b\sqrt{1-z^2/c^2}} dy \int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z)\,dx. \ \square$$

$$\frac{y^2}{4}+\frac{z^2}{2}=x$$

(c) A cos limitat per les superficies $y^2+2z^2=4x$, $x=2$.



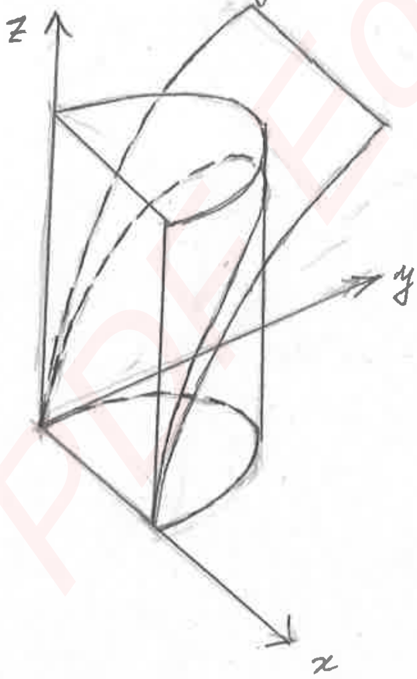
$$I=\int_{-2}^{2}dz\int_{-\sqrt{8-2z^2}}^{\sqrt{8+2z^2}}dy\int_{\frac{y^2}{4}+\frac{z^2}{2}}^{2}f(x,y,z)\,dx=\int_{-2\sqrt{2}}^{2\sqrt{2}}dy\int_{-\sqrt{4-y^2/2}}^{\sqrt{4-y^2/2}}dz\int_{\frac{y^2}{4}+\frac{z^2}{2}}^{2}f(x,y,z)\,dx$$

$$=\int_{0}^{2}dx\int_{-\sqrt{2x}}^{\sqrt{2x}}dz\int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}}f(x,y,z)\,dy=\int_{-2}^{2}dz\int_{z^2/2}^{2}dx\int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}}f(x,y,z)\,dy$$

$$=\int_{-2\sqrt{2}}^{2\sqrt{2}}dy\int_{\frac{1}{4}y^2}^{2}dx\int_{-\sqrt{2x-\frac{1}{2}y^2}}^{\sqrt{2x-\frac{1}{2}y^2}}f(x,y,z)\,dz=\int_{0}^{2}dx\int_{-2\sqrt{x}}^{2\sqrt{x}}dy\int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}}f(x,y,z)\,dz.\ \square$$

15) Calculeu les integrals triples següents en les regions de $\mathbb{R}^3$ que s'indiquen

a) $I=\iiint_A xz\,dx\,dy\,dz$, A limitat pel cilindre de base circular $x^2+y^2-2x=0$ i la superfície: $z^2=2y$ $(y,z\geq 0)$



$$I=\int_0^2 x\,dx\int_0^{\sqrt{2x-x^2}}dy\int_0^{\sqrt{2y}}z\,dz$$

$$=\int_0^2 x\,dx\int_0^{\sqrt{2x-x^2}}y\,dy$$

$$=\int_0^2 x\left(\frac{y^2}{2}\right]_0^{\sqrt{2x-x^2}}dx=\int_0^2\left(x^2-\frac{x^3}{3}\right)dx$$

$$=\left[\frac{x^3}{3}-\frac{x^4}{8}\right]_0^2=\frac{8}{3}-\frac{16}{8}=\frac{8}{3}-2=\boxed{\frac{2}{3}}\ \square$$

b) $I=\iiint_A zy\sqrt{x^2+y^2}\,dz\,dy\,dz$, $A=\left\{(x,y,z)\in\mathbb{R}^3:\ 0\leq z\leq x^2+y^2,\ 0\leq y\leq\sqrt{2x-x^2}\right\}$

$$I=\int_0^2 dx\int_0^{\sqrt{2x-x^2}}y\sqrt{x^2+y^2}\,dy\int_0^{x^2+y^2}z\,dz=\int_0^2 dx\int_0^{\sqrt{2x-x^2}}y\sqrt{x^2+y^2}\left[\frac{z^2}{2}\right]_0^{x^2+y^2}dy$$

$$=\frac{1}{2}\int_0^2 dx\int_0^{\sqrt{2x-x^2}}y\left(x^2+y^2\right)^{5/2}dy=\frac{1}{14}\int_0^2 dx\left[\left(x^2+y^2\right)^{7/2}\right]_0^{\sqrt{2x-x^2}}=\frac{1}{14}\int_0^2 dx\left[\left(x^2+2x-x^2\right)^{7/2}-x^7\right]$$

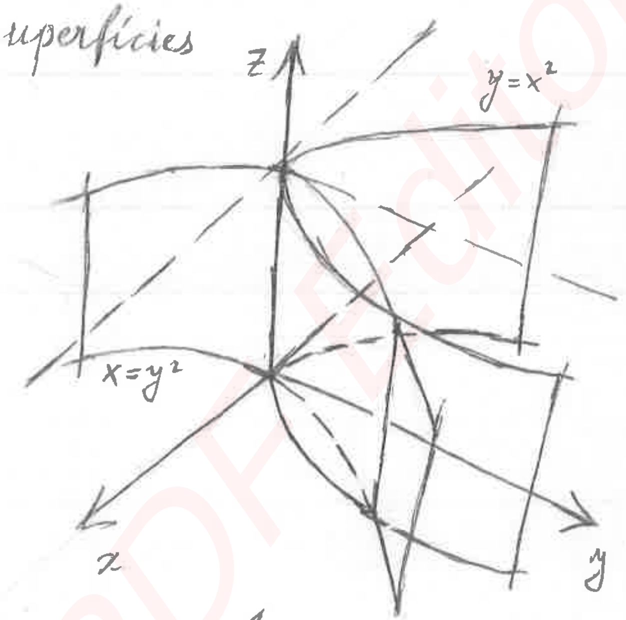
$$=\frac{1}{14}\left[2\cdot 2^{7/2}\frac{x^{9/2}}{9}-\frac{x^8}{8}\right]_0^2=\frac{1}{14}\cdot\frac{1}{72}\cdot\left(16\cdot 2^8-9\cdot 2^8\right)=\frac{1}{14}\cdot\frac{1}{2^3\cdot 9}\cdot 7\cdot 2^8=\boxed{\frac{16}{9}}\ \square$$

c) $I=\iiint_A dx\,dy\,dz$, $A=\left\{(x,y,z)\in\mathbb{R}^3:\ 1\leq x\leq 3,\ 1\leq y\leq 3,\ 0\leq z\leq xy\right\}$

$$I=\int_1^3 dx\int_1^3 dy\int_0^{xy}dz=\int_1^3 x\,dx\int_1^3 y\,dy=\left(\int_1^3 x\,dx\right)^2=\left(\left[\frac{x^2}{2}\right]_1^3\right)^2=\left(\frac{9}{2}-\frac{1}{2}\right)^2=\boxed{16}\ \square$$

(d) $I = \iiint_A xyz\,dx\,dy\,dz$, A limitat per les superfícies

$y = x^2,\ x = y^2,\ z = xy,\ z = 0$

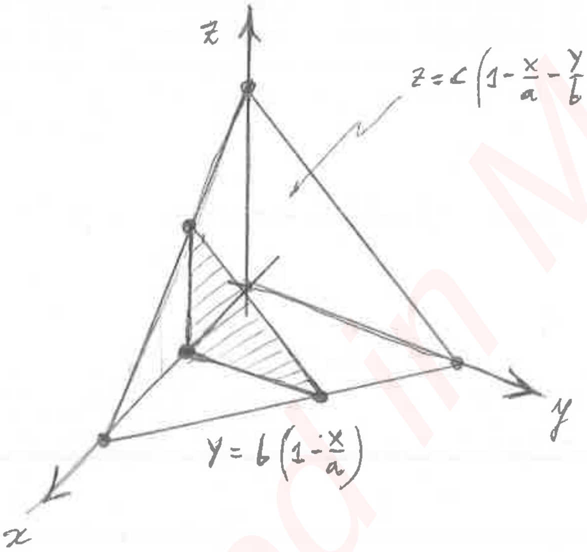


$$I = \int_0^1 x\,dx \int_{x^2}^{\sqrt{x}} y\,dy \int_0^{xy} z\,dz$$

$$= \frac{1}{2}\int_0^1 x^3\,dx \int_{x^2}^{\sqrt{x}} y^3\,dy$$

$$= \frac{1}{8}\int_0^1 dx\, x^3\left(x^2 - x^8\right) = \frac{1}{8}\int_0^1 \left(x^5 - x^{11}\right)dx = \frac{1}{8}\left(\frac{x^6}{6} - \frac{x^{12}}{12}\right)\Big|_0^1 = \frac{1}{8}\left(\frac{1}{6} - \frac{1}{12}\right) = \boxed{\frac{1}{96}} \ \square$$

(e) $I = \iiint_A x\,dx\,dy\,dz$, A tetraedre format pels plans $x=0,\ y=0,\ z=0,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, amb $a,b,c>0$



$$I = \int_a^b x\,dx \int_0^{b(1-\frac{x}{a})} dy \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz$$

$$= c\int_0^a x\,dx \int_0^{b(1-x/a)} dy\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

$$= c\int_0^a dx\, x\left(y - \frac{xy}{a} - \frac{y^2}{2b}\right)\Big|_0^{b(1-\frac{x}{a})}$$

$$= c\int_0^a x\left[b\left(1-\frac{x}{a}\right) - \frac{bx}{a}\left(1-\frac{x}{a}\right) - \frac{b}{2}\left(1-\frac{x}{a}\right)^2\right]dx$$

$$\overset{(*)}{=} \frac{cb}{2}\int_0^a x\left(1-\frac{x}{a}\right)^2 dx = \frac{cb}{2}\int_0^a x\left(1-\frac{2x}{a}-\frac{x^2}{a^2}\right)dx = \frac{cb}{2}\left(\frac{x^2}{2}-\frac{2x^3}{3a}+\frac{x^4}{4a^2}\right)\Big|_0^a$$

$$= \frac{cb}{2}\left(\frac{a^2}{2}-\frac{2a^2}{3}+\frac{a^2}{4}\right) = \boxed{\frac{cba^2}{24}}$$

(*) $\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a}-\frac{1}{2}+\frac{x}{2a}\right) = \frac{b}{2}\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a}\right) = \frac{b}{2}\left(1-\frac{x}{a}\right)^2.$

16) Useu coordenades polars per calcular les següents integrals dobles.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

$$\iint_{T(D)} f(x,y)\,dx\,dy = \iint_{D} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

a) $I = \iint_A (x^2+y^2)\,dx\,dy$, $A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 4\}$

$$I = \int_0^{2\pi} d\theta \int_0^2 r^3\,dr = 2\pi\left[\frac{r^4}{4}\right]_0^2 = \boxed{8\pi}.$$

b) $I = \iint_A \cos(x^2+y^2)\,dx\,dy$, $A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq \frac{\pi}{2}\}$

$$I = \int_0^{2\pi} d\theta \int_0^{\sqrt{\pi/2}} r\cos(r^2)\,dr = 2\pi\,\frac{1}{2}\left[\sin(r^2)\right]_0^{\sqrt{\pi/2}} = \boxed{\pi}.$$

c) $I = \iint_A \frac{(x+y)^2}{x^2+y^2+2}\,dx\,dy$, $A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$

$$I = \int_0^{2\pi} d\theta \int_0^1 \frac{r^3}{r^2+2}\,dr = 2\pi\int_0^1\left(1 - \frac{r}{r^2+2}\right)dr = 2\pi\left[\frac{r^2}{2} - \ln(r^2+2)\right]_0^1$$

$$= 2\pi\left[\frac{1}{2} - \ln 3 + \ln 2\right] = \boxed{2\pi\left[\frac{1}{2} + \ln\left(\frac{2}{3}\right)\right]} \;\square$$

d) $I = \iint_A \frac{dx\,dy}{(1+x^2+y^2)^2\sqrt{x^2+y^2}}$, $A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq R^2\}$. (Indicació: useu propietats elementals del sin i cos per veure que $\sin(\arctan(R)) = \frac{R}{\sqrt{1+R^2}}$ i $\cos(\arctan(R)) = \frac{1}{\sqrt{1+R^2}}$).

$$I=\int_0^{2\pi}d\theta\int_0^R\frac{r\,dr}{(1+r^2)^2\sqrt{r^2}}=2\pi\int_0^R\frac{dr}{(1+r^2)^2}=\begin{cases} r=\tan t\Rightarrow dr=\dfrac{dt}{\cos^2 t}\\ r=0\Rightarrow t=0\\ r=R\Rightarrow t=\arctan(R)\end{cases}$$

$$=2\pi\int_0^{\arctan(R)}\cos^4 t\,\frac{dt}{\cos^2 t}=2\pi\int_0^{\arctan(R)}\frac{1+\cos(2t)}{2}\,dt=2\pi\left(\frac{t}{2}+\frac{\sin(2t)}{4}\right)\Big|_0^{\arctan(R)}$$

$$\overset{(*)}{=}\boxed{\pi\left(\arctan(R)+\frac{R}{1+R^2}\right)}\quad\square$$

(*) Notem que:
$\sin(2t)=2\sin t\cos t=\frac{2\sin t\cos^2 t}{\cos t}=2\tan t\cos^2 t=\frac{2\tan t}{1+\tan^2 t}$

e) $I=\iint_A\sqrt{x^2+y^2-9}\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2:\ 9\le x^2+y^2\le 25\}$.

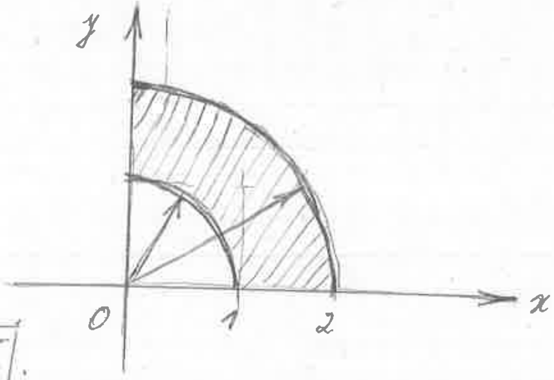
$$I=\int_0^{2\pi}d\theta\int_3^5\sqrt{r^2-9}\,r\,dr=\frac{2\pi}{3}(r^2-9)^{3/2}\Big|_3^5=\frac{2\pi}{3}4^3=\boxed{\frac{128\pi}{3}}\quad\square$$

f) $I=\iint_A xy\,dx\,dy$, A intersecció amb el 1er quadrant de la corona circular de centre (0,0), radi interior 1 i radi exterior 2

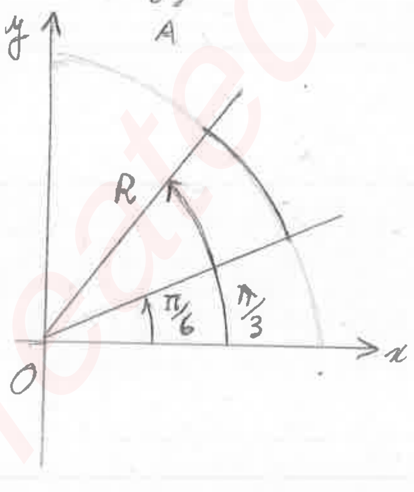
$$I=\int_0^{\pi/2}d\theta\int_1^2 r^3\cos\theta\sin\theta\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\sin(2\theta)\,d\theta\int_1^2 r^3\,dr$$

$$=\left[-\frac{1}{2}\cos(2\theta)\right]_0^{\pi/2}\cdot\left[\frac{r^4}{4}\right]_1^2=\frac{1}{2}\left(4-\frac{1}{4}\right)=\boxed{\frac{15}{8}}\quad\square$$



g) $I=\iint_A x(x^2+y^2)\,dx\,dy$, A sector circular de centre (0,0) i radi R format per angles entre $\frac{\pi}{3}$ i $\frac{\pi}{6}$ amb l'eix x positiu.



$$I=\int_{\pi/6}^{\pi/3}d\theta\int_0^R r^4\cos\theta\,dr=\int_{\pi/6}^{\pi/3}\cos\theta\,d\theta\int_0^R r^4\,dr$$

$$=\left[\sin\theta\right]_{\pi/6}^{\pi/3}\cdot\left[\frac{r^5}{5}\right]_0^R=\boxed{\frac{R^5}{10}(\sqrt{3}-1)}$$

20) Usem coordenades esfèriques per calcular les següents integrals triples

(a) $I=\iiint_B x^4y^2z^3\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3: x^2+y^2+z^2\le a^2\} = B_a(0,0,0)$ $(a>0)$

Solució: $I=\int_0^{2\pi}\cos^4\theta\sin^2\theta\,d\theta\ \overbrace{\int_{-\pi/2}^{\pi/2}\cos^7\varphi\sin^3\varphi\,d\varphi}^{=0}\int_0^a r^{11}\,dr = 0.$ □

(b) $I=\iiint_B z(x^2+y^2)\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3: x^2+y^2+z^2\le a^2,\ z\ge 0\}$

Solució: $I=\int_0^{2\pi}d\theta\int_0^{\pi/2}\cos^3\varphi\sin\varphi\int_0^a r^5\,dr = 2\pi\left[-\frac{\cos^4\varphi}{4}\right]_0^{\pi/2}\cdot\left[\frac{r^6}{6}\right]_0^a = \boxed{\frac{\pi a^6}{12}}$ □

(c) $I=\iiint_B \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$, $B=\{(x,y,z)\in\mathbb{R}^3: a^2\le x^2+y^2+z^2\le b^2\}$

Solució: $I=\int_0^{2\pi}d\theta\int_{-\pi/2}^{\pi/2}\cos\varphi\,d\varphi\int_a^b r^2\frac{dr}{r^3} = 4\pi\ln(b/a)$

(d) $I=\iiint_B \sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}$, B el domini de l'apartat anterior.

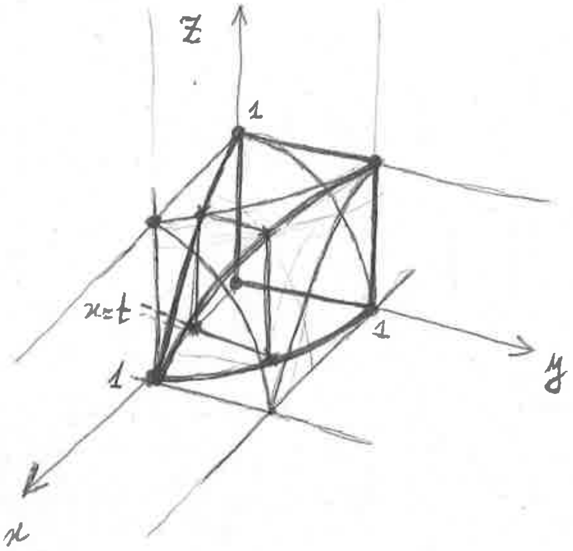
Solució: $I=\int_0^{2\pi}d\theta\int_{-\pi/2}^{\pi/2}\cos\varphi\,d\varphi\int_a^b r^3e^{-r^2}\,dr = 4\pi\int_a^b r^3e^{-r^2}\,dr \overset{(*)}{=} 2\pi\left(e^{-a^2}(a^2+1)-e^{-b^2}(b^2+1)\right)$ □

(*) Parts: $\int_a^b r^3e^{-r^2}\,dr = -\frac{1}{2}\int_a^b r^2\,d(e^{-r^2}) = \left[-\frac{r^2e^{-r^2}}{2}\right]_a^b + \overbrace{\int_a^b re^{-r^2}\,dr}^{\left[-\frac{1}{2}e^{-r^2}\right]_a^b} =$

$= \frac{a^2e^{-a^2}}{2}-\frac{b^2e^{-b^2}}{2}+\frac{e^{-a^2}}{2}-\frac{e^{-b^2}}{2} = \frac{1}{2}(a^2+1)e^{-a^2}-(b^2+1)\frac{e^{-b^2}}{2}$

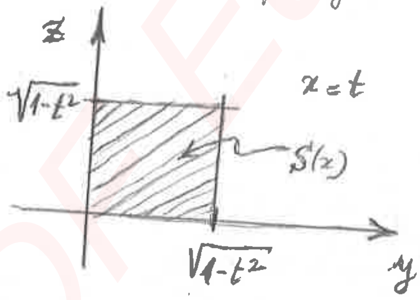
23) Useu coordenades cartesianes, cilíndriques o esfèriques (o bé el principi de Cavalieri per calcular el volum dels dominis de $\mathbb{R}^3$ limitats per les superfícies que s'indiquen.

a) $x^2+z^2=1, \ x^2+y^2=1$



Parametrització de la intersecció per $y\geq 0, z\geq 0$.

$x=t$
$y=\sqrt{1-t^2}$
$z=\sqrt{1-t^2}$



Per tant: $S(x)=1-x^2$

$$V=8\cdot\int_0^1 S(x)\,dx = 8\int_0^1 (1-x^2)\,dx$$

$$=8\left[x-\frac{x^3}{3}\right]_0^1 = 8\left(1-\frac{1}{3}\right)=\boxed{\frac{16}{3}}$$

Tho podem calcular com una integral triple. En efecte, en el 1er octant (veure figura): $0\leq z\leq\sqrt{1-x^2}$, $0\leq y\leq\sqrt{1-x^2}$, $0\leq x\leq 1$. Aleshores

$$V = 8\int_0^1 dx\int_0^{\sqrt{1-x^2}} dy\int_0^{\sqrt{1-x^2}} dz$$

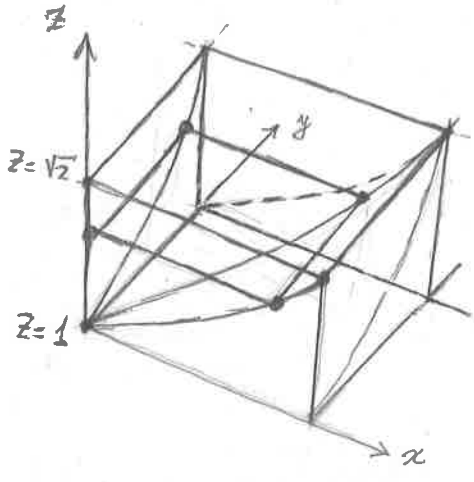
$$= 8\int_0^1 (1-x^2)\,dx$$

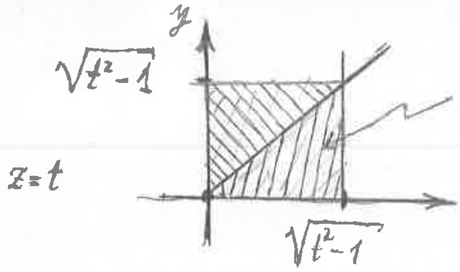
$$= 8\left(x-\frac{x^3}{3}\right)\Big|_{x=0}^{x=1} = \boxed{\frac{16}{3}}$$

b) $z^2-x^2=1$
$z^2-y^2=1$
$z=\sqrt{2}$

Parametrització de la intersecció per $x\geq 0, y\geq 0, z\geq 0$.

$x=\sqrt{t^2-1}, \ y=\sqrt{t^2-1}, \ z=t, \ (t\geq 1)$.





$S(z) = z^2-1\ , \ (1 \le z \le \sqrt{2})$

aplicant el principi de Cavalieri:

$$V = 4\cdot\int_1^{\sqrt{2}} S(z)\,dz = 4\cdot\int_1^{\sqrt{2}}(z^2-1)\,dz = 4\left(\frac{z^3}{3}-z\right|_1^{\sqrt{2}} = 4\left(\frac{2\sqrt{2}}{3}-\sqrt{2}+\frac{2}{3}\right) = \boxed{\frac{4}{3}\left(2-\sqrt{2}\right)} \quad \square$$

Alternativament:

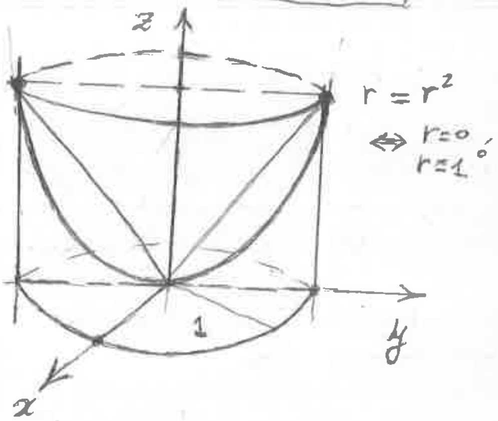
$$\frac{V}{8} = \int_0^1 dx\int_0^x dy\int_{\sqrt{1+x^2}}^{\sqrt{2}} dz = \int_0^1 dx\int_0^x\left(\sqrt{2}-\sqrt{1+x^2}\right)dy = \int_0^1\left(\sqrt{2}\,x - x\sqrt{1+x^2}\right)dx$$

$$= \left[-\frac{1}{3}\left(1+x^2\right)^{3/2}+\frac{\sqrt{2}}{2}x^2\right]_0^1 = -\frac{1}{3}2\sqrt{2}+\frac{\sqrt{2}}{2}+\frac{1}{3} = \frac{1}{6}\left(2-\sqrt{2}\right) \Longrightarrow \boxed{V = \frac{4}{3}\left(2-\sqrt{2}\right)}.$$

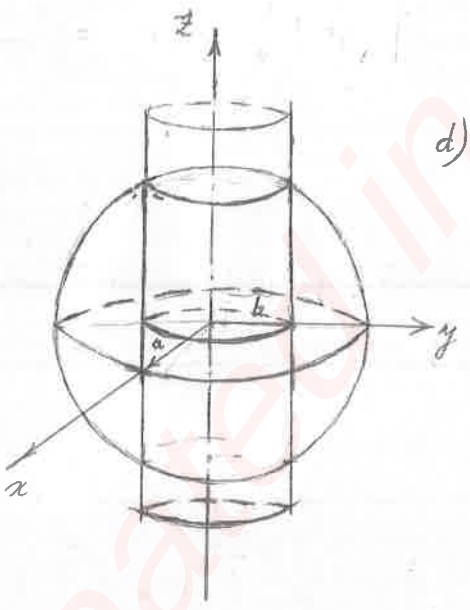
c) $z^2 = x^2+y^2,\ z = x^2+y^2,\ (z\ge 0)$

$$V = \int_0^{2\pi} d\theta\int_0^1 r\,dr\int_{r^2}^{r} dz = 2\pi\int_0^1\left(r^2-r^3\right)dr = 2\pi\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$= \boxed{\frac{\pi}{6}} \quad \square$$



d)



d) Part de l'esfera $x^2+y^2+z^2=a^2$ que és exterior al cilindre $x^2+y^2=b^2$ $(a>b>0)$.

$$V/2 = \int_0^{2\pi} d\theta\int_b^a r\,dr\int_0^{\sqrt{a^2-r^2}} dz = 2\pi\int_b^a r\left(a^2-r^2\right)^{1/2}dr$$

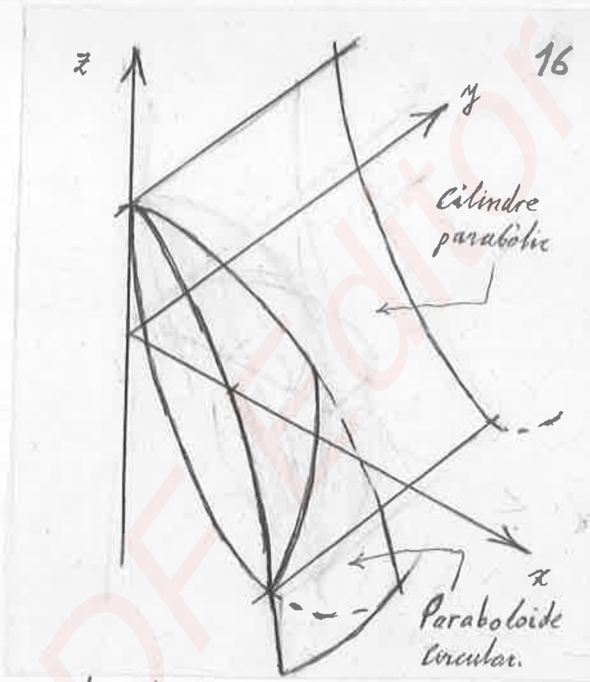
$$= -\frac{2}{3}\left(a^2-r^2\right)^{3/2}\pi\Big|_b^a = \frac{2\pi}{3}\left(a^2-b^2\right)^{3/2} \Rightarrow \boxed{V = \frac{4\pi}{3}\left(a^2-b^2\right)^{3/2}}$$

e) $z = x^2 - 4x + 1, \quad 1 - z = x^2 + y^2$

Intersecció de les superfícies per $y \geq 0$

$1 - x^2 - y^2 = x^2 - 4x + 1 \Rightarrow y^2 = 4x - 2x^2$

quan, $y \geq 0$, llavors $y = \sqrt{4x - 2x^2}$, i tenim la següent parametrització de la intersecció de les superfícies (per $y \geq 0$)



$$\left.\begin{aligned} x &= t \\ y &= \sqrt{4t - 2t^2} \\ z &= t^2 - 4t + 1. \end{aligned}\right\} ; \text{ amb } 0 \leq t \leq 2.$$

La projecció d'aquesta parametrització sobre el pla $z = 0$ ve donada per: $y = \sqrt{4x - 2x^2}$. Amb la qual cosa, per calcular el volum, fem:

Aleshores:

$$V/2 = \int_0^2 dx \int_0^{\sqrt{4x - x^2}} dy \int_{x^2 - 4x + 1}^{1 - x^2 - y^2} dz = \int_0^2 dx \int_0^{\sqrt{4x - 2x^2}} \left(1 - x^2 - y^2 - x^2 + 4x - 1\right) = \int_0^2 dx \int_0^{\sqrt{4x - 2x^2}} dy \left(4x - 2x^2 - y^2\right)$$

$$= \int_0^2 dx \left(4xy - 2x^2 y - \frac{y^3}{3}\right)\Bigg|_0^{\sqrt{4x - 2x^2}} = \int_0^2 dx \left(4x\sqrt{4x - 2x^2} - 2x^2\sqrt{4x - 2x^2} - \frac{1}{3}(4x - 2x^2)\sqrt{4x - 2x^2}\right)$$

$$= \int_0^2 \sqrt{4x - 2x^2}\left(4x - 2x^2 - \frac{4}{3}x + \frac{2}{3}x^2\right) dx = \frac{2}{3}\int_0^2 \left(4x - 2x^2\right)^{3/2} = \frac{2^4}{3}\int_0^2 x^{3/2}\left(1 - \frac{x}{2}\right)^{3/2} dx$$

$$= \left\{\begin{aligned} x &= 2\sin^2 t \\ dx &= 4\sin t \cos t \, dt \\ &\ldots \end{aligned}\right\} = \frac{64}{3} 2^{3/2} \int_0^{\pi/2} \sin^4 t \cos^4 t \, dt = \frac{2^6 \cdot 2^{3/2}}{3 \cdot 2^4} \int_0^{\pi/2} \sin^4(2t) \, dt$$

$$\overset{(*)}{=} \frac{8\sqrt{3}}{3} \cdot \frac{3\pi}{16} = \frac{\pi\sqrt{3}}{2} \Rightarrow \boxed{V = \pi\sqrt{2}}$$

(*) $$\int_0^{\pi/2} \sin^4(2t) \, dt = \int_0^{\pi/2} \sin^2(2t)\left(1 - \cos^2(2t)\right) dt = \int_0^{\pi/2} \left[\sin^2(2t) - \frac{1}{4}\sin^2(4t)\right] dt$$

$$= \int_0^{\pi/2} \left[\frac{1 - \cos(4t)}{2} - \frac{1 - \cos(8t)}{8}\right] dt = \frac{\pi}{4} - \frac{\pi}{16} = \frac{3\pi}{16}$$

f) $x^2 = z$, $y^2 = x$, $z^2 = y$

$x^2 = az$, $y^2 = ax$, $z^2 = ay$. $(a>1)$

$$D: \left.\begin{array}{l} y^2/a \leq x \leq y^2 \\ z^2/a \leq y \leq z^2 \\ x^2/a \leq z \leq x^2 \end{array}\right\};$$

Això suggereix el canvi següent:

$$u = x/y^2, \quad v = y/z^2, \quad w = z/x^2$$

Llavors D': $\frac{1}{a} \leq u \leq 1$, $\frac{1}{a} \leq v \leq 1$, $\frac{1}{a} \leq w \leq 1$ és el domini en

Càlcul del Jacobià de la transformació:

$(x,y,z) \mapsto (u,v,w) = f(x,y,z) = \left(\frac{x}{y^2}\ \frac{y}{z^2}\ \frac{z}{x^2}\right)$, d'on podem calcular $Df(x,y,z)$,

però de fet, necessitem: $(u,v,w) \mapsto (x,y,z) = f^{-1}(u,v,w)$ i el determinant de la corresponent matriu Jacobiana, i.e $\det Df^{-1}(u,v,w)$.

$$f \circ f^{-1}(u,v,w) = (u,v,w) \Rightarrow D f \circ f^{-1}(u,v,w)\, Df^{-1}(u,v,w) = Df(x,y,z) \cdot Df^{-1}(u,v,w) = Id$$

$$\Rightarrow Df^{-1}(u,v,w) = Df(x,y,z)^{-1} \Rightarrow \det Df^{-1}(u,v,w) = \frac{1}{\det Df(x,y,z)}$$

(i.e., apliquem la regla de la cadena a la identitat: $f\, f^{-1} = Id$). Fent els càlculs:

$$\det Df(x,y,z) = \begin{vmatrix} 1/y^2 & -2x/y^3 & 0 \\ 0 & 1/z^2 & -2y/z^3 \\ -2z/x^3 & 0 & 1/x^2 \end{vmatrix} = \frac{1}{x^3y^3z^3}\begin{vmatrix} y & -2x & 0 \\ 0 & z & -2y \\ -2z & 0 & x \end{vmatrix} = \frac{1}{x^3y^3z^3}(xyz - 8xyz) =$$

$$= -\frac{7}{x^2y^2z^2} \Longrightarrow \left|\det Df^{-1}(u,v,w)\right| = \frac{x^2y^2z^2}{7} = \frac{1/7}{u^2v^2w^2}$$

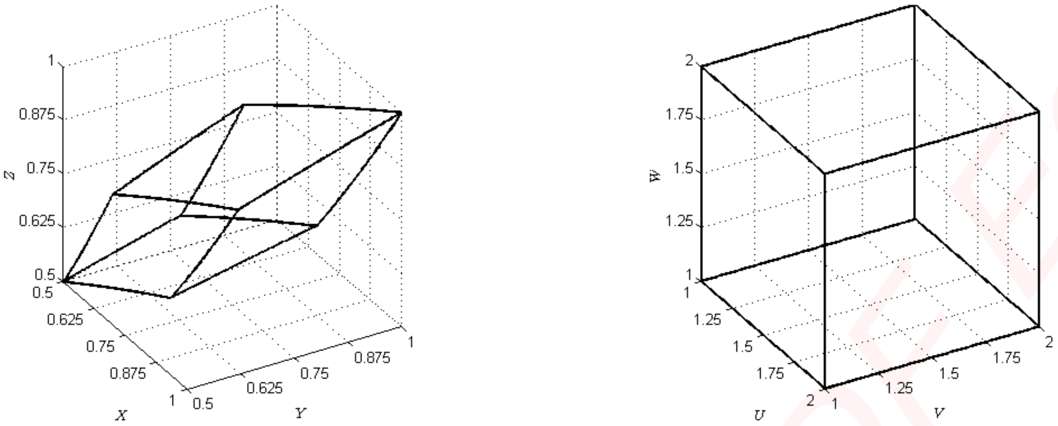
Aleshores:

$$V = \iiint_D dx\,dy\,dz = \iiint_{D'=f(D)} \left|\det Df^{-1}(u,v,w)\right| du\,dv\,dw = \frac{1}{7}\iiint_{D'=[\frac{1}{a},1]\times[\frac{1}{a},1]\times[\frac{1}{a},1]} \frac{du\,dv\,dw}{u^2v^2w^2}$$

$$= \frac{1}{7}\int_{1/a}^{1}\frac{du}{u^2}\int_{1/a}^{1}\frac{dv}{v^2}\int_{1/a}^{1}\frac{dw}{w^2} = \frac{1}{7}\left(\int_{1/a}^{1}\frac{du}{u^2}\right)^3 = \frac{1}{7}\left(\left[-\frac{1}{u}\right]_{1/a}^{1}\right)^3 = \boxed{\frac{(a-1)^3}{7}}$$

$\square$

Nota: amb el canvi: $1 \leq u = y^2/x \leq a$, $1 \leq v = x^2/y \leq a$, $1 \leq w = x^2/z \leq a$ el domini D es transforma en $D' = [1,a]\times[1,a]\times[1,a]$. Llavors el determinant del Jacobià corresponent resulta $\det\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\det\frac{\partial(u,v,w)}{\partial(x,y,z)}} = -\frac{1}{7}$.



(a) $\mathcal{D}$: Domini definit per les superfícies,
$x^2 = z, \quad y^2 = x, \quad z^2 = y,$
$x^2 = az, \quad y^2 = ax, \quad z^2 = ay.$

(b) $\mathcal{D}' : 1 \leq u \leq a,\ 1 \leq v \leq a,\ 1 \leq w \leq a.$

FIGURA 1. Transformació del domini $\mathcal{D}$ en $\mathcal{D}'$ pel canvi (1).

NOTA

Alternativament, es comprova d'immediat que el canvi

$$u = \frac{y^2}{x}, \qquad v = \frac{z^2}{y}, \qquad w = \frac{x^2}{z}, \tag{1}$$

transforma el domini original $\mathcal{D}$, en $\mathcal{D}' = [1,a] \times [1,a] \times [1,a]$. És a dir, en un cub d'aresta $a-1$ (veure Figura 1). Aleshores el Jacobià corresponent surt més senzill. En efecte:

$$\det \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -\dfrac{y^2}{x^2} & 2\dfrac{y}{x} & 0 \\ 0 & -\dfrac{z^2}{y^2} & 2\dfrac{z}{y} \\ 2\dfrac{x}{z} & 0 & -\dfrac{x^2}{z^2} \end{vmatrix} = -7,$$

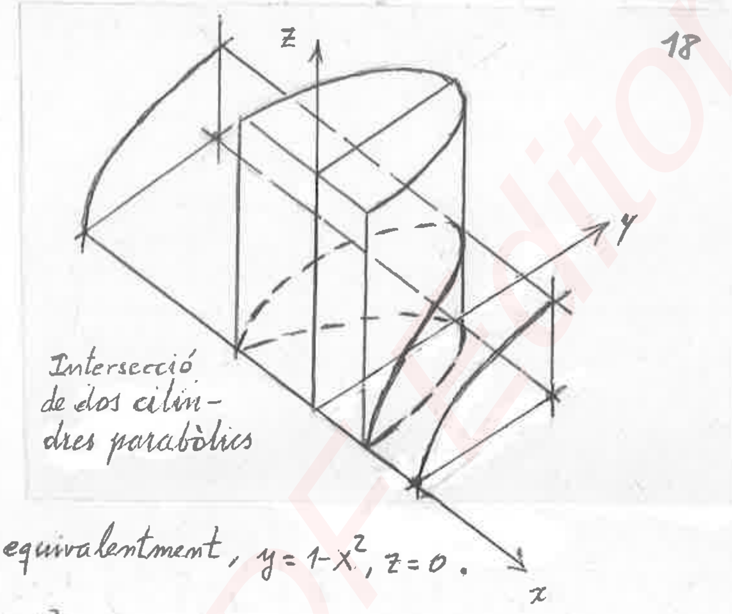
d'on:

$$\left|\det \frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{\left|\det \dfrac{\partial(u,v,w)}{\partial(x,y,z)}\right|} = \frac{1}{7}.$$

Llavors el càlcul del volum es simplifica encara més:

$$\begin{aligned} V = \iiint_{\mathcal{D}} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z &= \iiint_{\mathcal{D}'} \left|\det \frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}w \\ &= \frac{1}{7} \int_1^a \mathrm{d}u \int_1^a \mathrm{d}v \int_1^a \mathrm{d}w = \frac{1}{7} \left(\int_1^a \mathrm{d}u \right)^3 = \frac{(a-1)^3}{7}. \end{aligned}$$

g) $z^2 = y, \quad x^2 = 1-y$



parametrització de la intersecció per a $x \geq 0, z \geq 0$

$$\left.\begin{aligned} x &= \sqrt{1-t} \\ y &= t \\ z &= \sqrt{t} \end{aligned}\right\} : \quad 0 \leq t \leq 1$$

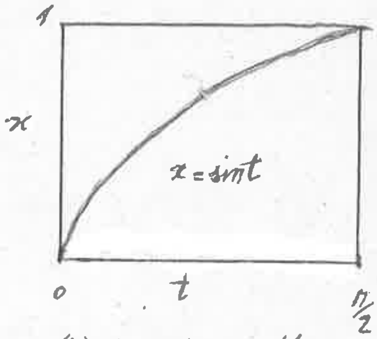
Projecció sobre el pla $z=0$:

$x = \sqrt{1-t}, y = t, z = 0$; o, equivalentment, $y = 1-x^2, z = 0$.

Aleshores:

$$\frac{V}{4} = \int_0^1 dx \int_0^{1-x^2} dy \int_0^{\sqrt{y}} dz = \int_0^1 dx \int_0^{1-x^2} dy\, \sqrt{y} = \frac{2}{3}\int_0^1 dx \left[y^{3/2} \right]_0^{1-x^2}$$

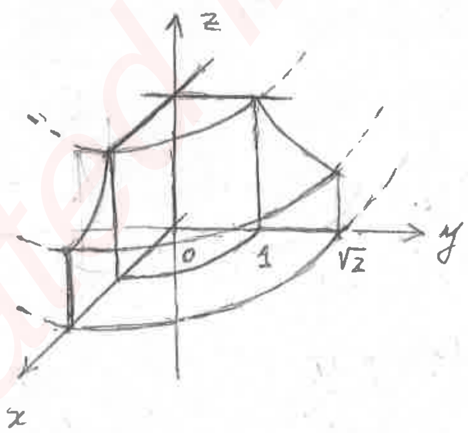
$$= \frac{2}{3} \int_0^1 \left(1-x^2\right)^{3/2} dx = \overset{\text{c.v. } (*)}{\left\{ \begin{aligned} x &= \sin t \\ dx &= \cos t\, dt \end{aligned} \right\}} = \frac{2}{3} \underbrace{\int_0^{\pi/2} \cos^4 t\, dt}_{= 3\pi/16} = \frac{\pi}{8} \Rightarrow \boxed{V = \frac{\pi}{2}}.$$



(*) Canvi de variables a la integral

h) $x^2+y^2 = 1, \quad x^2+y^2 = 2, \quad z(x^2+y^2) = 1, \quad z = 0$.

En coordenades cilíndriques: $\quad 1 \leq r \leq \sqrt{2}, \quad 0 \leq z \leq 1/r^2$

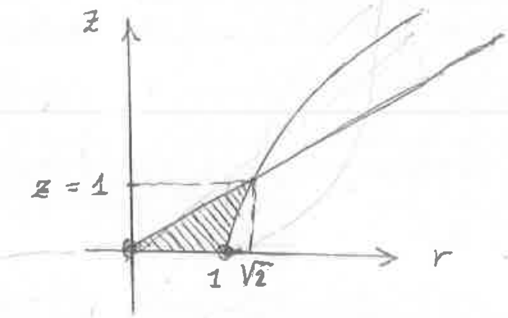


$$V = \int_0^{2\pi} d\theta \int_1^{\sqrt{2}} r\, dr \int_0^{1/r^2} dz = 2\pi \int_1^{\sqrt{2}} \frac{dr}{r}$$

$$= 2\pi \left[\ln r \right]_1^{\sqrt{2}} = \boxed{\pi \ln 2}$$

i) $x^2+y^2 = 2z^2, \quad x^2+y^2 = z^2+1 \quad (x \geq 0, y \geq 0, z \geq 0)$

Apliquem coordenades cilíndriques (pàg. següent)



$$V = \int_0^{\pi/2} d\theta \int_0^1 r\,dr \int_0^{r/\sqrt{2}} dz + \int_0^{\pi/2} d\theta \int_1^{\sqrt{2}} r\,dr \int_{\sqrt{r^2-1}}^{r/\sqrt{2}} dz$$

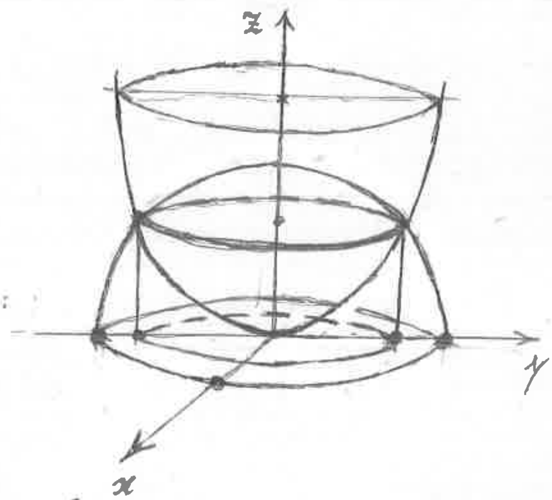
$$= \frac{\pi}{2}\int_0^1 \frac{r^2}{\sqrt{2}}\,dr + \frac{\pi}{2}\int_1^{\sqrt{2}} \left(\frac{r^2}{\sqrt{2}} - r\sqrt{r^2-1}\right) dr$$

$$= \frac{\pi}{4}\sqrt{2}\left[\frac{r^3}{3}\right]_0^1 + \frac{\pi}{2}\left[\frac{\sqrt{2}}{2}\cdot\frac{r^3}{3} - \frac{1}{3}(r^2-1)^{3/2}\right]_1^{\sqrt{2}} = \frac{\sqrt{2}\pi}{12} + \frac{\pi}{2}\left[\frac{2}{3} - \frac{1}{3} - \frac{\sqrt{2}}{6}\right] = \boxed{\frac{\pi}{6}}$$

Alternativament (més 'fàcil'!):

$$V = \int_0^{\pi/2} d\theta \int_0^1 dz \int_{\sqrt{2}z}^{\sqrt{z^2+1}} r\,dr = \frac{\pi}{4}\int_0^1 (z^2+1-2z^2)\,dz = \frac{\pi}{4}\int_0^1 (1-z^2)\,dz$$

$$= \frac{\pi}{4}\left[z - \frac{z^3}{3}\right]_0^1 = \frac{\pi}{4}\left(1-\frac{1}{3}\right) = \boxed{\frac{\pi}{6}}$$

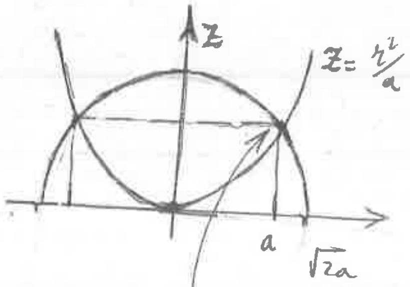


(j) $x^2+y^2+z^2 \le 2a^2$, $z \ge \frac{x^2+y^2}{a}$ $(a>0)$. En cilíndriques:

$$V = \int_0^{2\pi} d\theta \int_0^a r\,dr \int_{r^2/a}^{\sqrt{2a^2-r^2}} dz = 2\pi\int_0^a \left(r\sqrt{2a^2-r^2} - \frac{r^3}{a}\right) dr$$

$$= 2\pi\left[-\frac{1}{3}(2a^2-r^2)^{3/2} - \frac{r^4}{4a}\right]_0^a$$

$$= 2\pi\left[\frac{1}{3}(2\sqrt{2}-1) - \frac{1}{4}\right]a^3 = \boxed{2\pi a^3\left(\frac{2^{3/2}}{3} - \frac{7}{12}\right)}$$



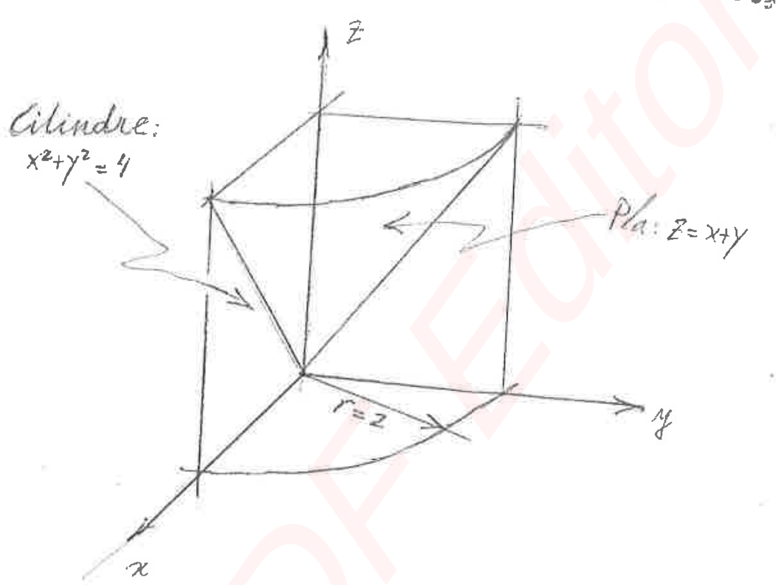
$az + z^2 = 2a^2$

$\Leftrightarrow z^2 + az - 2a^2 = 0$

d'on:

$$z = \frac{-a \pm \sqrt{a^2+8a^2}}{2} = \frac{-a \pm 3a}{2} = \begin{cases} a \to r = a \\ -2a \ (\text{NO}) \end{cases}$$

(k) $x^2+y^2=4$, $z=x+y$ $(x\geq 0, y\geq 0, z\geq 0)$



$$V=\int_0^{\pi/2}d\theta\int_0^2 r\,dr\int_0^{r\cos\theta+r\sin\theta}dz$$

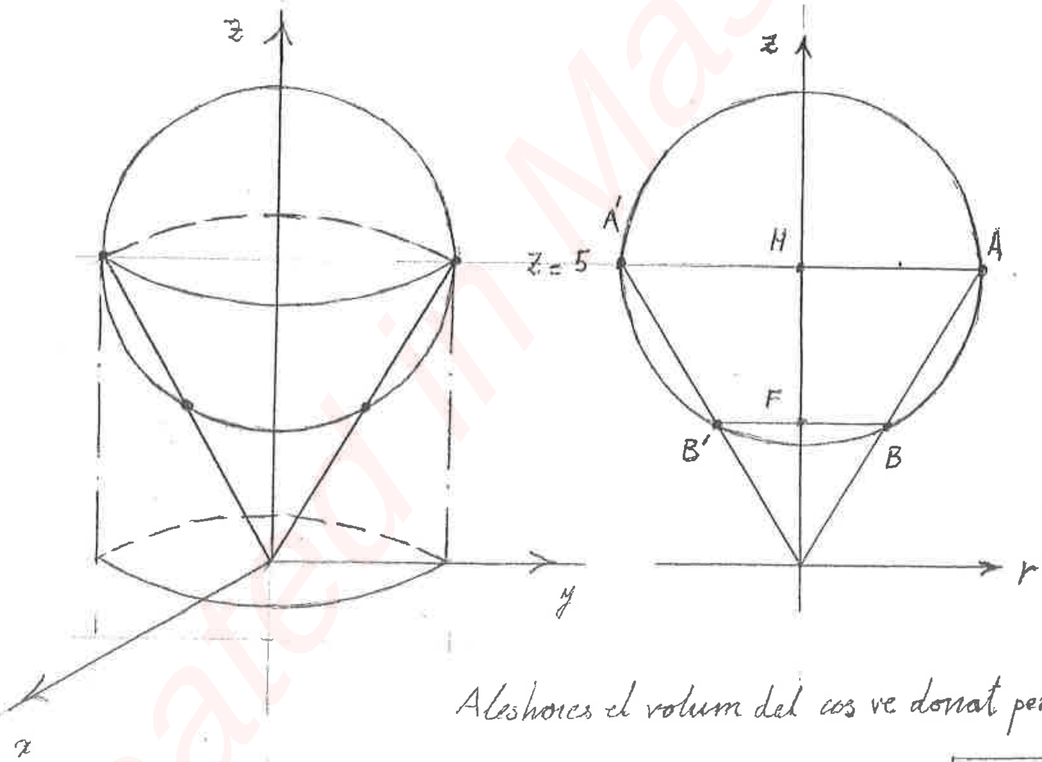
$$=\int_0^{\pi/2}d\theta\int_0^2 r^2(\cos\theta+\sin\theta)\,dr$$

$$=\int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta\int_0^2 r^2\,dr$$

$$=\Big[\sin\theta-\cos\theta\Big]_0^{\pi/2}\cdot\left[\frac{r^3}{3}\right]_0^2=\boxed{\frac{16}{3}}\ \square$$

Exercici: substituir $z=x+y \to z=x+y-1$ i repetir el càlcul...

(l) Con de gelat definit per: $x^2+y^2\leq \frac{z^2}{5}$, $0\leq z\leq 5+\sqrt{5-x^2-y^2}$.



Alçada dels punts d'intersecció A, B, A', B'

$$(z-5)^2=5-\frac{z^2}{5}$$

$$\iff 6z^2-50z+100=0.$$

d'on, $z=5$: alçada de A, A'

$z=\frac{10}{3}$: " " B, B'

i els 'radis' respectius són:

$r=\sqrt{5}$; distància $\overline{HA}=\overline{A'H}$

$r=\frac{2\sqrt{5}}{3}$; " $\overline{FB}=\overline{B'F}$

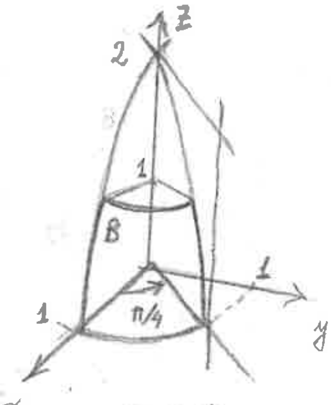
Aleshores el volum del cos ve donat per

$$V=\underbrace{\frac{2}{3}\pi\sqrt{5}^3}_{\text{Volum de la semiesfera}}+\underbrace{\frac{1}{3}\pi\sqrt{5}^2\,5}_{\text{Volum del con.}}=\boxed{\left(\frac{10\sqrt{5}}{3}+\frac{25}{3}\right)\pi}$$

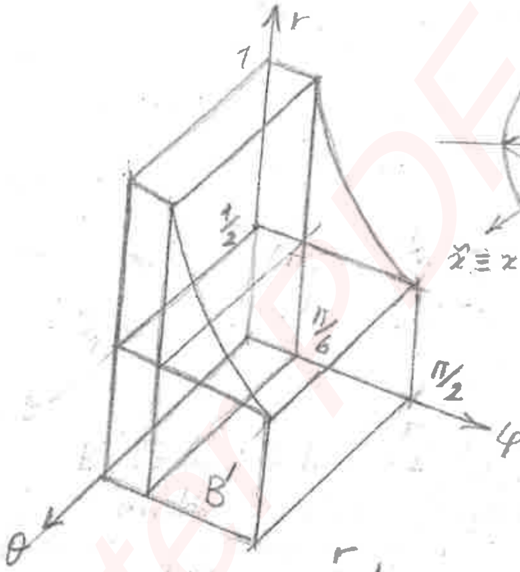
22) Adapten les coordenades esfèriques per calcular les següents integrals triples.

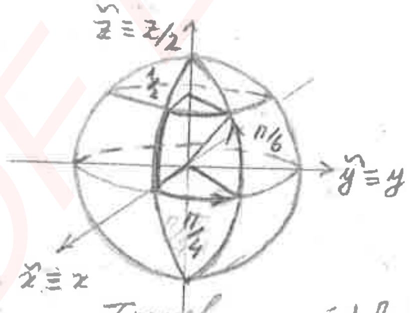
(a) $I = \iiint_B 16z\,dx\,dy\,dz$, $B = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+\frac{z^2}{4} \leq 1,\ 0 \leq z \leq 1,\ 0 \leq y \leq x\}$

Coordenades esfèriques adaptades: $T(r,\theta,\varphi) = (r\cos\theta\cos\varphi,\ r\sin\theta\cos\varphi,\ r\sin\varphi)$.



$B = T(B')$





Transformació del domini B mitjançant les coordenades esfèriques adaptades.

$x^2+y^2+z^2/4 \leq 1 \iff 0 \leq r \leq 1$

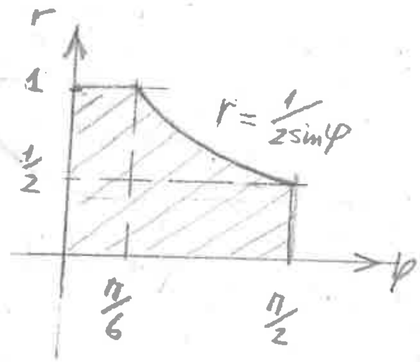
$0 \leq z \leq 1 \iff 0 \leq r \leq \frac{1}{2\sin\varphi}$, amb $0 \leq \varphi \leq \frac{\pi}{2}$

$0 \leq y \leq x \iff 0 \leq r\cos\theta\cos\varphi \leq r\sin\theta\cos\varphi$

amb $r>0$, $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$

$\iff 0 \leq \sin\theta \leq \cos\theta$ amb $0 \leq \theta \leq 2\pi$

$\iff 0 \leq \theta \leq \frac{\pi}{4}$



Aleshores $I = I_1 + I_2$, amb:

$$I_1 \overset{(*)}{=} 64\int_0^{\pi/4} d\theta \int_0^{\pi/6} \sin\varphi\cos\varphi\,d\varphi \int_0^1 r^3\,dr = 64\,\frac{\pi}{4}\left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/6}\left[\frac{r^4}{4}\right]_0^1 = 16\pi\,\frac{1}{8}\cdot\frac{1}{4} = \frac{\pi}{2}$$

$$I_2 = 64\int_0^{\pi/4} d\theta \int_{\pi/6}^{\pi/2} \sin\varphi\cos\varphi\,d\varphi \int_0^{\frac{1}{2\sin\varphi}} r^3\,dr = \frac{64}{16}\cdot\frac{\pi}{16}\int_{\pi/6}^{\pi/2}\frac{\cos\varphi}{\sin^3\varphi}\,d\varphi = \frac{\pi}{8}\left[-\frac{1}{\sin^2\varphi}\right]_{\pi/6}^{\pi/2}$$

$$= \frac{\pi}{8}(4-1) = \frac{3\pi}{8}$$

Llavors: $\boxed{I = I_1 + I_2 = \frac{\pi}{2} + \frac{3\pi}{8} = \frac{7\pi}{8}}$

(*) $\det DT(r,\theta,\varphi) = 2r^2\cos\varphi$.

$z = 2r\sin\varphi$. Aleshores $16\,z\,dx\,dy\,dz = 16\cdot 2r\sin\varphi\cdot 2r^2\cos\varphi\,d\theta\,d\varphi\,dr = 64r^3\sin\varphi\cos\varphi\,d\theta\,d\varphi\,dr$

Alternativament:

$$\frac{1}{8}\int_{1/2}^{1} r\,dr = \frac{1}{16}\left(1-\frac{1}{4}\right) = \frac{3}{64}$$

$I = I_1' + I_2'$,

amb: $$I_1' = 64\int_0^{\pi/4} d\theta \int_{1/2}^{1} r^3\,dr \underbrace{\int_0^{\arcsin\frac{1}{2r}} \cos\varphi \sin\varphi\, d\varphi}_{=\left.\frac{\sin^2\varphi}{2}\right|_0^{\arcsin\frac{1}{2r}}} = 64\int_0^{\pi/4} d\theta \left(\int_{1/2}^{1} r^3\frac{1}{8r^2}\,dr\right) = \cancel{64}\,\frac{3}{\cancel{64}}\cdot\frac{\pi}{4} = \frac{3\pi}{4}$$

$$I_2' = 64\underbrace{\int_0^{\pi/4} d\theta}_{=\frac{\pi}{4}} \underbrace{\int_0^{1/2} r^3\,dr}_{=\frac{1}{64}} \underbrace{\int_0^{\pi/2} \cos\varphi \sin\varphi\, d\varphi}_{=\left.\frac{\sin^2\varphi}{2}\right|_0^{\pi/2}} = \cancel{64}\,\frac{\pi}{4}\cdot\frac{1}{\cancel{64}}\cdot\frac{1}{2} = \frac{\pi}{8}$$

Aleshores: $$\boxed{I = I_1' + I_2' = \frac{3\pi}{4} + \frac{\pi}{8} = \frac{7\pi}{8}}$$

Exercici: intenteu aquest mateix apartat, fent servir coordenades cilíndriques ☺

(b) $I = \iiint_B \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dx\,dy\,dz$; $B = \left\{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}$.

Solució: adaptem les coordenades esfèriques: $x = ar\cos\theta\cos\varphi$, $y = br\sin\theta\cos\varphi$, $z = cr\sin\varphi$, $T:(r,\theta,\varphi) \to (x,y,z) = T(r,\theta,\varphi)$

Aleshores: $B' = \{(r,\theta,\varphi) \in [0,+\infty) \times [0,2\pi] \times [-\frac{\pi}{2},\frac{\pi}{2}] : 0 \leq r \leq 1\} = B_1(0,0,0)$, és el domini transformat i llavors la integral I es pot calcular com:

$$I = \iiint_{B=T(B')} \underbrace{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)}_{=r^2} dx\,dy\,dz = \iiint_{B'} r^2 \underbrace{\left|\det \frac{\partial(x,y,z)}{\partial(\theta,\varphi,r)}\right|}_{=abc\, r^2\cos\varphi} d\theta\, d\varphi\, dr$$

Per què?

$$= abc\int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi\, d\varphi \int_0^1 r^4\,dr = abc\, 2\pi \left[\sin\varphi\right]_{-\pi/2}^{\pi/2} \left[\frac{r^5}{5}\right]_0^1 = \boxed{\frac{4\pi}{5}abc}.$$

24) Sigui B la bola de centre $(0,0,0)$ i radi R. Denotem per $T(x,y,z)$ la temperatura en el punt (x,y,z) i suposem que és proporcional a la distància del punt a l'origen. En quins punts de B la temperatura coincideix amb la temperatura mitjana de la bola?

Solució

$$\langle T \rangle = \frac{\alpha}{\frac{4}{3}\pi R^3} \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi \, d\varphi \int_0^R r^3 dr = \alpha \frac{4\pi \frac{R^4}{4}}{\frac{4}{3}\pi R^3} = \alpha \frac{3}{4} R$$

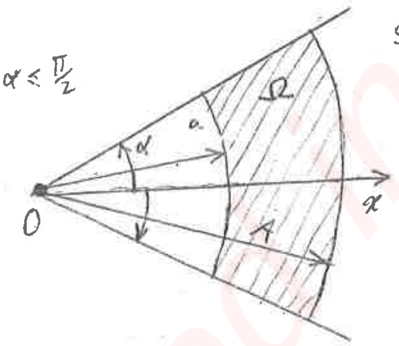
$$T(x,y,z) = \alpha \sqrt{x^2+y^2+z^2}$$
$$= \alpha r$$

$$T(x,y,z) = \alpha \frac{3}{4} R \iff \boxed{x^2+y^2+z^2 = \frac{9}{16} R^2}$$

25) Trobeu el centre de masses de les regions planes amb les densitats que s'indiquen

(a) Sector pla definit per una corona de radi interior a i radi exterior A, un angle d'obertura 2α i que és simètrica respecte l'eix x positiu, suposant densitat constant $\rho(x,y) = 1$.

$0 \le \alpha \le \frac{\pi}{2}$



Solució:

$$M = \iint_\Omega \rho(x,y)\,dx\,dy = \int_{-\alpha}^{\alpha} d\theta \int_a^A r\,dr = \frac{2\alpha}{2}\left(A^2-a^2\right) = \alpha\left(A^2-a^2\right)$$

$$X = \frac{1}{M}\iint_\Omega x\,\rho(x,y)\,dx\,dy = \frac{1}{M}\int_{-\alpha}^{\alpha} \cos\theta\, d\theta \int_a^A r^2 dr = \frac{2}{3M}(\sin\alpha)\cdot\left(A^3-a^3\right)$$

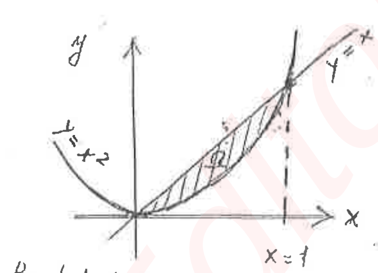
$$= \frac{2}{3}\cdot\frac{\sin\alpha}{\alpha}\cdot\frac{A^3-a^3}{A^2-a^2}$$

$$Y = \frac{1}{M}\iint_\Omega y\,\rho(x,y)\,dx\,dy = \frac{1}{M}\int_{-\alpha}^{\alpha} \sin\theta\, d\theta \int_a^A r^2 dr = 0$$

Posició del Centre de Masses:

$$(X,Y) = \left(\frac{2}{3}\,\frac{\sin\alpha}{\alpha}\,\frac{A^3-a^3}{A^2-a^2},\ 0\right) \square$$

b) Regió entre $y=x^2$ i $y=x$ amb $\rho(x,y)=x+y$



Punt de tall: $x^2=x \iff x(x-1)=0$

$\iff x=0,\ x=1.$

$$M=\iint_{\Omega}\rho(x,y)\,dx\,dy=\int_0^1 dx\int_{x^2}^{x}(x+y)\,dy$$

$$=\int_0^1 dx\left(xy+\frac{y^2}{2}\right)\Bigg|_{x^2}^{x}=\int_0^1 dx\left(x^2+\frac{x^2}{2}-x^3-\frac{x^4}{2}\right)$$

$$=\int_0^1\left(\frac{3}{2}x^2-x^3-\frac{x^4}{2}\right)dx=\frac{x^3}{2}-\frac{x^4}{4}-\frac{x^5}{10}\Bigg|_0^1=\frac{1}{2}-\frac{1}{4}-\frac{1}{10}=\frac{10-5-2}{20}=\frac{3}{20}$$

$$X=\frac{1}{M}\int_0^1 dx\int_{x^2}^{x}x(x+y)\,dy=\frac{1}{M}\int_0^1 dx\left[x^2y+x\frac{y^2}{2}\right]_{x^2}^{x}=\int_0^1\left(\frac{3x^3}{2}-x^4-\frac{x^5}{2}\right)dx$$

$$=\frac{1}{M}\left(\frac{3x^4}{8}-\frac{x^5}{5}-\frac{x^6}{12}\right)\Bigg|_0^1=\frac{1}{M}\left(\frac{3}{8}-\frac{1}{5}-\frac{1}{12}\right)=\frac{45-24-10}{120M}=\frac{11}{120M}=\frac{11}{120}\cdot\frac{20}{3}=\frac{11}{18}$$

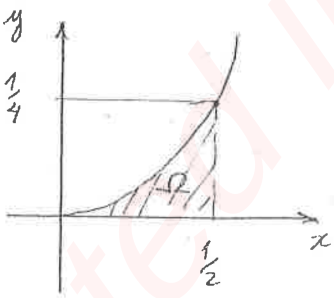
$$Y=\frac{1}{M}\int_0^1 dx\int_{x^2}^{x}y(x+y)\,dy=\frac{1}{M}\int_0^1 dx\left[\frac{y^2}{2}x+\frac{y^3}{3}\right]_{x^2}^{x}=\frac{1}{M}\int_0^1\left(\frac{5x^3}{2}-\frac{x^5}{2}-\frac{x^6}{3}\right)dx$$

$$=\frac{1}{M}\left(\frac{5x^4}{24}-\frac{x^6}{12}-\frac{x^7}{21}\right)\Bigg|_0^1=\frac{1}{M}\left(\frac{1}{24}-\frac{1}{12}-\frac{1}{21}\right)=\frac{1}{M}\cdot\frac{35-14-8}{168}=\frac{13}{168}\cdot\frac{20}{3}=\frac{65}{126}$$

$\therefore$ Posició del CDM:

$$(X,Y)=\left(\frac{11}{8},\frac{65}{126}\right).\ \square$$

c) Regió entre $y=0$ i $y=x^2$ $\left(0\le x\le\frac{1}{2}\right)$, amb $\rho(x,y)=1$.



$$M=\iint_{\Omega}\rho(x,y)\,dx\,dy=\int_0^{1/2}dx\int_0^{x^2}dy=\int_0^{1/2}x^2\,dx=\left[\frac{x^3}{3}\right]_0^{1/2}=\frac{1}{24}$$

$$X=\frac{1}{M}\iint_{\Omega}x\,\overset{1}{\rho(x,y)}\,dx\,dy=\frac{1}{M}\int_0^{1/2}x\,dx\int_0^{x^2}dy=\frac{1}{M}\int_0^{1/2}x^3\,dx=\frac{1}{M}\left[\frac{x^4}{4}\right]_0^{1/2}=\frac{3}{8}$$

$$Y=\frac{1}{M}\iint_{\Omega}y\,\overset{1}{\rho(x,y)}\,dx\,dy=\frac{1}{M}\int_0^{1/2}dx\int_0^{x^2}y\,dy=\frac{1/2}{M}\int_0^{1/2}x^4\,dx=\frac{1}{M}\left[\frac{x^5}{10}\right]_0^{1/2}=\frac{3}{40}$$

$\therefore$ Posició del CDM:

$$(X,Y)=\left(\frac{3}{8},\frac{3}{40}\right)\ \square$$

26) Em cadascun dels casos següents, trobeu el centre de masses del sòlid A suposant distribució de masses homogènia.

(a) $A = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq R^2, x \geq 0, y \geq 0, z \geq 0\}$.

(*) Recordem que $\cos^2\varphi = \frac{1+\cos 2\varphi}{2}$, per tant:

$$\int_0^{\pi/2} \cos^2\varphi\, d\varphi = \int_0^{\pi/2} \frac{1+\cos(2\varphi)}{2} d\varphi = \left[\frac{\varphi}{2} + \frac{\sin(2\varphi)}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

Solució. $\rho = \rho_0 = \text{ctnt}$

$$m(A) = \frac{4}{3}\pi R^3 \rho_0 \cdot \frac{1}{8} = \frac{\rho_0}{6}\pi R^3$$

$$X = \frac{1}{M}\iiint_A x \underbrace{\rho(x,y,z)}_{\rho_0} dx\,dy\,dz = \frac{\rho_0}{M} \underbrace{\int_0^{\frac{\pi}{2}} \cos\theta\, d\theta}_{1} \underbrace{\int_0^{\pi/2} \cos^2\varphi\, d\varphi}_{(*)} \underbrace{\int_0^R r^3 dr}_{R^4/4}$$

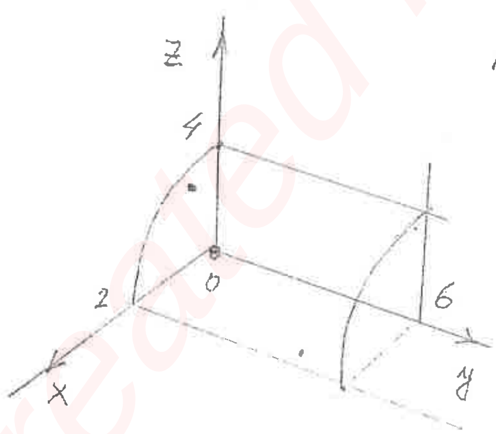
$$= \frac{\rho_0}{M} \cdot \frac{\pi}{4} \cdot \frac{R^4}{4} = \frac{\rho_0/16\, \pi R^4}{\rho_0/6\, \pi R^3} = \boxed{\frac{3}{8}R}$$

$$Y = \frac{1}{M}\iiint_A y\, \rho(x,y,z)\, dx\,dy\,dz = \frac{\rho_0}{M}\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{\frac{\pi}{2}} \cos^2\varphi\, d\varphi \int_0^R r^3 dr = \dots = \boxed{\frac{3}{8}R}$$

$$Z = \frac{1}{M}\iiint_A z\, \rho(x,y,z)\, dx\,dy\,dz = \frac{\rho_0}{M} \underbrace{\int_0^{\pi/2} d\theta}_{\frac{\pi}{2}} \underbrace{\int_0^{\pi/2} \sin\varphi\cos\varphi\, d\varphi}_{\left[\frac{\sin^2\varphi}{2}\right]_0^{\frac{\pi}{2}}} \underbrace{\int_0^R r^3 dr}_{\frac{R^4}{4}} = \frac{\rho_0 \frac{\pi}{16} R^4}{\rho_0 \pi R^4} = \boxed{\frac{3}{8}R}$$

∴ Posició del CDM: $(X,Y,Z) = \left(\frac{3}{8}R, \frac{3}{8}R, \frac{3}{8}R\right)$

(b) $A = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 2, 0 \leq y \leq 6, 0 \leq z \leq 4-x^2\}$, $\rho(x,y,z) = \rho_0 = \text{ctnt}$.



$$M = \iiint_\Omega \underbrace{\rho(x,y,z)}_{\rho_0} dx\,dy\,dz = \rho_0 \int_0^2 dx \int_0^6 dy \int_0^{4-x^2} dz = 6\rho_0 \int_0^2 (4-x^2)dx =$$

$$= 6\rho_0 \left(4x - \frac{x^3}{3}\right)\Big|_0^2 = 6\rho_0\left(8 - \frac{8}{3}\right) = 32\rho_0$$

$$X = \frac{1}{M}\iiint_\Omega x \underbrace{\rho(x,y,z)}_{\rho_0} dx\,dy\,dz = \frac{1}{M}\int_0^2 x\,dx \int_0^6 dy \int_0^{4-x^2} dz$$

$$= \frac{6\rho_0}{M}\int_0^2 x(4-x^2)dx = \frac{6\rho_0}{M}\left(2x^2 - \frac{x^4}{4}\right)\Big|_0^2 = \frac{6\rho_0}{M}(8-4) = \frac{24\rho_0}{32\rho_0} = \boxed{\frac{3}{4}}$$

$$Y = \frac{1}{M}\iiint_{\Omega} y\,\rho(x,y,z)\,dx\,dy\,dz = \frac{\rho_0}{M}\int_0^2 dx\int_0^6 y\,dy\int_0^{4-x^2} dz = \frac{\rho_0}{M}\left[\frac{y^2}{2}\right]_0^6\cdot\left[4x-\frac{x^3}{3}\right]_0^2$$

$$= \frac{\rho_0}{M}\,18\cdot\frac{16}{3} = \frac{6\cdot 16\cdot\rho_0}{32\cdot\rho_0} = \boxed{3}$$

$$Z = \frac{1}{M}\iiint_{M} z\,\rho(x,y,z)\,dx\,dy\,dz = \frac{\rho_0}{M}\int_0^2 dx\int_0^6 dy\int_0^{4-x^2} z\,dz = \frac{6\rho_0}{M}\int_0^2 \frac{(4-x^2)^2}{2}\,dx$$

$$= \frac{3\rho_0}{M}\int_0^2 (16-8x^2+x^4)\,dx = \frac{3\rho_0}{M}\left(16x-\frac{8}{3}x^3+\frac{x^5}{5}\right)\Big|_0^2 = \frac{3\rho_0}{M}\left(32-\frac{64}{3}+\frac{32}{5}\right) = \frac{3\rho_0}{M}\cdot\frac{380-320+96}{15}$$

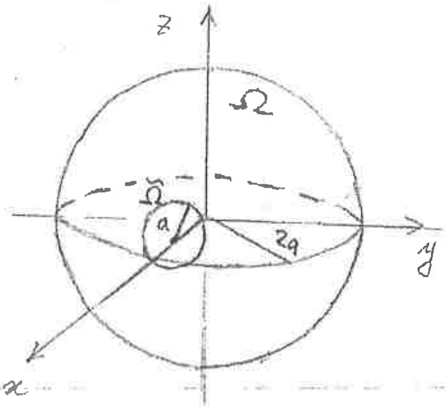
$$= \frac{3\rho_0}{32\rho_0}\cdot\frac{256}{15} = \boxed{\frac{8}{5}}$$

Posició del CDM: $(X,Y,Z) = \left(\frac{3}{4}, 3, \frac{8}{5}\right)$. □

(c) $A = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 \le 4a^2,\ (x-a)^2+y^2+z^2 \ge a^2\}$, $\rho(x,y,z) = \rho_0 = \text{ct}$.

Solució. Tenim: $\tilde{\Omega} = B_a(a,0,0)$, $\Omega = B_{2a}(0,0,0)\setminus\tilde{\Omega}$, amb:

$$m(\tilde{\Omega}) = \frac{4}{3}\pi a^3\rho_0$$



$$m(\Omega) = m(B_{2a}(0,0,0)) - m(\tilde{\Omega}) = \frac{4}{3}\pi 8a^3\rho_0 - \frac{4}{3}\pi a^3\rho_0$$

$$= \frac{4}{3}\pi 7a^3\rho_0.$$

$$m(\Omega)X + m(\tilde{\Omega})\underbrace{\tilde{X}}_{a} = \frac{4}{3}\pi\rho_0 7a^3 X + \frac{4}{3}\pi\rho_0 a^4 = 0 \Rightarrow X = -\frac{\frac{4}{3}\pi\rho_0 a^4}{\frac{4}{3}\pi\rho_0 7a^3} = -\frac{a}{7}$$

$$m(\Omega)Y + m(\tilde{\Omega})\underbrace{\tilde{Y}}_{0} = 0 \Rightarrow Y = 0.$$

$$m(\Omega)Z + m(\tilde{\Omega})\tilde{Z} = 0 \Rightarrow Z = 0.$$

Posició del CDM: $(X,Y,Z) = (-a/7, 0, 0)$. □

28) Troben el centre de masses de la semiesfera definida per $x^2+y^2+z^2 \leq R^2$ i $z \geq 0$, si la densitat en cada punt és proporcional a la distància d'aquest punt al centre.

Solució.

$$x^2+y^2+z^2 \leq R_1^2$$

$$z \geq 0$$

$$\rho(x,y,z) = \sqrt{x^2+y^2+z^2} =: r$$

$$M = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \cos\varphi \, d\varphi \int_0^R r^3 dr = 2\pi \frac{R^4}{4} = \boxed{\frac{\pi R^4}{2}}$$

$$X = \frac{1}{M} \underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{0} \int_0^{\frac{\pi}{2}} \cos\varphi \, d\varphi \int_0^R r^4 dr = 0$$

$$Y = \frac{1}{M} \underbrace{\int_0^{2\pi} \sin\theta \, d\theta}_{0} \int_0^{\frac{\pi}{2}} \cos\varphi \, d\varphi \int_0^R r^4 dr = 0$$

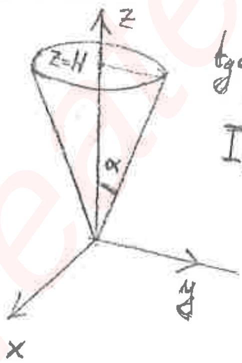
$$Z = \frac{1}{M} \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \cos\varphi \sin\varphi \, d\varphi \int_0^R r^4 dr = \frac{\pi R^5/5}{\pi R^4/2} = \boxed{\frac{2}{5}R}$$

Posició del CDM: $(X,Y,Z) = (0,0,\frac{2R}{5})$. □

34) Pels sòlids següents, calculeu els moments d'inèrcia que es demanen en cada cas tot suposant densitat homogènia igual a 1.

(a) Calculeu I_z pel sòlid limitat pel con de revolució d'alçada H i radi de base R donat per $x^2+y^2 \leq \frac{R^2}{H^2} z^2$ $(0 \leq z \leq H)$.

Solució.



$$\operatorname{tg}\alpha = \frac{R^2}{H^2}, \quad \rho = 1 \text{ ctnt.}$$

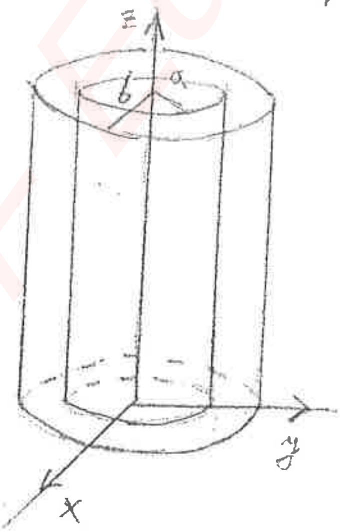
$$I_z = \int_0^{2\pi} d\theta \int_0^R r^3 dr \int_{\frac{H}{R}r}^{H} \rho \, dz \overset{\rho=1}{=} 2\pi \int_0^R \left(Hr^3 - \frac{H}{R} r^4\right) dr = 2\pi H R^4 \left(\frac{1}{4} - \frac{1}{5}\right)$$

$$= \boxed{\frac{\pi H R^4}{10}} \; □$$

(b) Calculeu I_z pel sòlid limitat per dos cilindres d'alçada h, $a^2 \leq x^2+y^2 \leq b^2$, $0 \leq z \leq h$

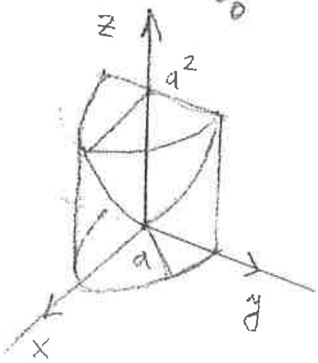
Solució.

$$I_z = \int_0^{2\pi} d\theta \int_a^b r^3 dr \int_0^h \underset{\overset{\parallel}{1}}{\rho}\, dz = 2\pi h \left[\frac{r^4}{4}\right]_a^b = \frac{\pi h}{4}(b^4-a^4)$$



(c) Calculeu I_z pel sòlid limitat pel paraboloide $z=x^2+y^2$, el cilindre $x^2+y^2=a^2$ $(z \geq 0)$

$$I_z = \int_0^{2\pi} d\theta \int_0^a r^3 dr \int_0^{r^2} \overset{=1}{\rho}\, dz = 2\pi \int_0^a r^5 dr = 2\pi \left(\frac{r^6}{6}\right|_0^a = \boxed{\frac{\pi a^6}{3}}$$



(d) Calculeu I_x, I_y, I_z pel sòlid tancat pel paraboloide $z=x^2+y^2$ i el pla $z=a$ $(a>0)$

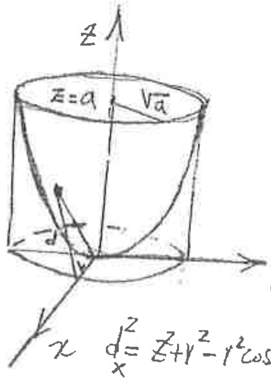
d_x: distància a l'eix x.

$$I_x = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r\,dr \int_{r^2}^{a} \rho\, d_x^2\, dz$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r\,dr \int_{r^2}^{a} \overset{\overset{1}{\parallel}}{\rho}(z^2+r^2\sin^2\theta)\, dz$$

$$= \underbrace{\int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r\,dr \int_{r^2}^{a} z^2 dz}_{=:I_1} + \underbrace{\int_0^{2\pi} \sin^2\theta\, d\theta \int_0^{\sqrt{a}} r^3 dr \int_{r^2}^{a} dz}_{=:I_2}$$

$$= I_1 + I_2$$



$$d_x^2 = z^2 + r^2 - r^2\cos^2\theta$$
$$= z^2 + r^2(1-\cos^2\theta)$$
$$= z^2 + r^2\sin^2\theta.$$

$$I_1 := \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r\,dr \int_{r^2}^{a} z^2 dz = \frac{2\pi}{3}\int_0^{\sqrt{a}} r(a^3-r^6)\,dr = \frac{2\pi}{3}\left(\frac{a^4}{2}-\frac{a^4}{8}\right) = \frac{\pi a^4}{4}.$$

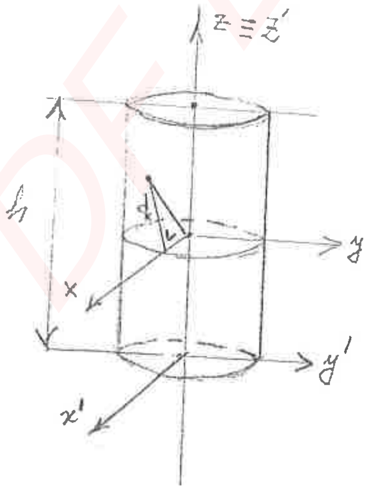
$$I_2 := \underbrace{\int_0^{2\pi} \sin^2\theta\, d\theta}_{\pi} \int_0^{\sqrt{a}} r^3 dr \int_{r^2}^{a} dz = \pi \int_0^{\sqrt{a}} r^3(a-r^2)\,dr = \pi a^3\left(\frac{1}{4}-\frac{1}{6}\right) = \frac{\pi a^3}{12}.$$

$$\therefore\ I_x = I_1 + I_2 = \frac{\pi a^3}{12}(1+3a). \ \square$$

$$I_y = I_x = \frac{\pi a^3}{12}(1+3a) \quad (\text{per simetria}). \square$$

$$I_z = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r^3 dr \int_{r^2}^{a} \underset{\substack{\| \\ 1}}{\rho}\, dz = 2\pi \int_0^{\sqrt{a}} r^3(a-r^2)\,dr = 2\pi\left(a\frac{r^4}{4} - \frac{r^6}{6}\right)\Bigg|_0^{\sqrt{a}} = \boxed{\frac{\pi a^3}{6}}. \square$$

(e) Calculeu I_x, I_y, I_z pel cilindre $x^2+y^2 \le R^2$, $-\frac{h}{2} \le z \le h/2$



$$I_x = \int_0^{2\pi} d\theta \int_0^R r\,dr \int_{-h/2}^{h/2} \rho d_x^2\, dz$$

d_x: distància a l'eix x:
$$d_x^2 = z^2 + r^2 - r^2\cos^2\theta = z^2 + r^2(1-\cos^2\theta) = z^2 + r^2\sin^2\theta.$$

$$= \int_0^{2\pi} d\theta \int_0^R r\,dr \int_{-h/2}^{h/2} \underset{\substack{\| \\ 1}}{\rho}\,(z^2 + r^2\sin^2\theta)\, dz$$

$$= \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_0^R r\,dr}_{R^2/2} \underbrace{\int_{-h/2}^{h/2} z^2\,dz}_{h^3/12} + \underbrace{\int_0^{2\pi} \sin^2\theta\, d\theta}_{\pi} \underbrace{\int_0^R r^3 dr}_{R^4/4} \underbrace{\int_{-h/2}^{h/2} dz}_{h} = 2\pi \frac{R^2}{2}\frac{h^3}{12} + \pi\frac{R^4}{4}h = \frac{\pi R^2 h}{12}(h^2+3R^2)$$

$$I_y = I_x = \frac{\pi R^2 h}{12}(h^2+3R^2) \quad (\text{per simetria})$$

Teorema de Steiner $I_r = I_s + m(W)\, d^2(CM, r)$

$$\text{Nota: } I_{x'} = I_{y'} = \frac{\pi R^2 h}{12}(h^2+3R^2) + \pi R^2 h \cdot \frac{h^2}{4} = \pi R^2 h\left(\frac{h^2}{12} + \frac{h^2}{4} + 3R^2\right) = \frac{\pi R^2 h}{12}(4h^2+3R^2)$$

Moment d'inèrcia respecte de l'eix x, que passa pel CDM

Massa del cos ($\rho=1$)

$d^2(x,x')$

$$I_z = \int_0^{2\pi} d\theta \int_0^R r^3 dr \int_{-h/2}^{h/2} dz = 2\pi\frac{R^4}{4}h = \frac{\pi R^2 h}{2} h^2. \square$$

Nota: posant $M := \pi R^2 h$ (la "massa" del cos):

$$I_x = \frac{M}{12}(h^2+3R^2) = I_y, \quad I_{x'} = \frac{M}{12}(4h^2+3R^2) = I_{y'}, \quad I_z = \frac{M}{2}h^2. \square$$

TEMA 2

Problemes: 1, 2, 5, 6, 7, 9, 10, 11, 13, 17, 18, 19, 21, 27, 29, 30, 31, 32, 33, 35.

1. Usant el teorema del valor mig per integrals, proveu les següents desigualtats

(a) $4e^5 \iint_A e^{x^2+y^2}\,dx\,dy \leq 4e^{25}$, on $A = [1,3]\times[2,4]$

S. Recordem el Teorema del Valor Mig (veure apunts de teoria, tema 2):

▷ Sigui $f: \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, Ω acotat i arc-connex, i $f \in C^0(\bar{\Omega})$, llavors, $\exists\; c \in \bar{\Omega}$ tal que:

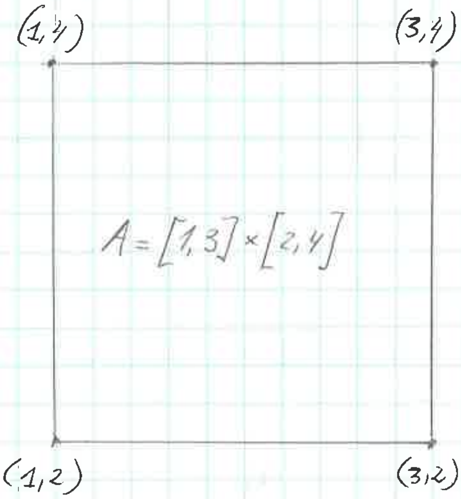
$$\int_\Omega f(x) = f(c)\, m(\Omega),$$

i on $m(\Omega)$ és la mesura del conjunt Ω (longitud en dimensió 1, àrea en dimensió 2, volumen en dimensió 3, ...). Llavors tenim les acotacions següents:

$$m(\Omega) \inf_{x\in\Omega} f(x) \leq \int_\Omega f(x) \leq m(\Omega) \sup_{x\in\Omega} f(x). \qquad (1)$$

(a) $f(x,y) = e^{x^2+y^2}$, d'on: $f_x(x,y) = 2xe^{x^2+y^2} = 0 \Leftrightarrow x=0$, $f_y(x,y) = 2ye^{x^2+y^2} = 0$. Llavors l'únic punt crític de la funció és $(x,y) = (0,0) \notin A = [1,3]\times[2,4]$

- $x=1$: $f(1,y) = e^{1+y^2}$: $f'(1,y) = 2y\,e^{1+y^2} = 0 \Leftrightarrow y=0$: $(1,0)\notin A$.
- $x=3$: $f(3,y) = e^{9+y^2}$: $f'(3,y) = 2y\,e^{9+y^2} = 0 \Leftrightarrow y=0$: $(3,0)\notin A$.
- $y=2$: $f(x,2) = e^{x^2+4}$: $f'(x,2) = 2x\,e^{x^2+4} = 0 \Leftrightarrow x=0$: $(0,2)\notin A$.
- $y=4$: $f(x,4) = e^{x^2+16}$: $f'(x,4) = 2x\,e^{x^2+16} = 0$ $\Leftrightarrow x=0$, $(0,4)\notin A$.



Veiem que el punt crític de f no pertany a A i que les restriccions de f sobre els costats d'A no tenen punts crítics. Avaluem f als vèrtexs d'A:

$f(1,2) = e^5$, $f(3,2) = e^{13}$, $f(1,4) = e^{17}$, $f(3,4) = e^{25}$

d'on: $\max\limits_{(x,y)\in A} e^{x^2+y^2}=e^{25}$ i $\min\limits_{(x,y)\, A} e^{x^2+y^2}=e^{5}$; s'atanyen als punts

$(3,4)$ i $(1,2)$ respectivament. D'altra banda: $m(A)=\iint_A dxdy=(3-1)\cdot(4-2)=4$.

Així doncs, aplicant el Teorema del Valor Mig (fórmula (1)),

$$4e^5 \le \iint_{A=[1,3]\times[2,4]} e^{x^2+y^2}\,dxdy \le 4e^{25}. \quad \square$$

(b) $\dfrac{1}{e} \le \dfrac{1}{4\pi^2}\iint_A e^{\sin(x+y)}\,dxdy \le e$, $A=[-\pi,\pi]\times[-\pi,\pi]$

Solució: clarament: $\max\limits_{(x,y)\in A} e^{\sin(x+y)}=e$, $\min\limits_{(x,y)\in A} e^{\sin(x+y)}=1/e$, mentre que

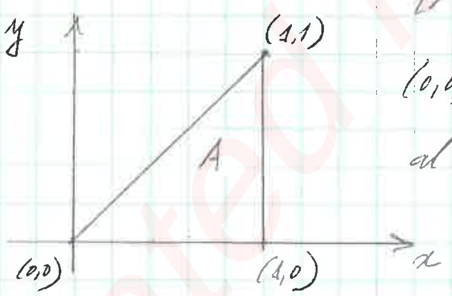
$m(A)=4\pi^2$. Aleshores, aplicant (1) resulta:

$$\frac{4\pi^2}{e} \le \iint_{A=[-\pi,\pi]\times[-\pi,\pi]} e^{\sin(x+y)}\,dxdy \le 4\pi^2 e,$$

i dividint les dues desigualtats per $4\pi^2$ s'obtenen les acotacions buscades. $\square$

(c) $\dfrac{1}{6} \le \iint_A \dfrac{dxdy}{y-x+3} \le \dfrac{1}{4}$, on A és el triangle de vèrtexs $(0,0)$, $(1,1)$ i $(1,0)$

Solució.



El màxim de $y-x+3$ sobre el triangle A s'atany als vèrtexs $(0,0)$ i $(1,1)$ i val 3, mentre que el mínim té lloc al vèrtex $(1,0)$ i val 2. Per tant:

$$\frac{1}{3} \le \frac{1}{y-x+3} \le \frac{1}{2}$$

per tot punt del triangle donat, mentre que l'àrea corresponent és $m(A)=\frac{1}{2}$.

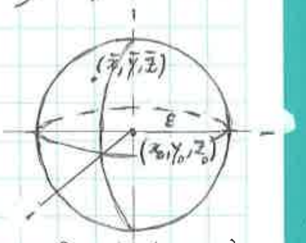
Finalment doncs, aplicant la fórmula (1) del Teorema del Valor Mitjà, obtenim:

$$\frac{1}{6} \le \iint_A \frac{dxdy}{y-x+3} \le \frac{1}{4},$$ que són les desigualtats buscades. $\square$

2) Signi $f(x,y,z)$ una funció contínua i B_ε la bola de centre (x_0,y_0,z_0) i radi ε. Proveu que se satisfà que $f(x_0,y_0,z_0) = \lim_{\varepsilon \to 0^+} \frac{1}{\text{volum}(B_\varepsilon)} \iiint_{B_\varepsilon} f(x,y,z)\,dx\,dy\,dz$

Solució: Pel Teorema del Valor Mig, existeix $(\bar{x},\bar{y},\bar{z}) \in B_\varepsilon = B_\varepsilon(x_0,y_0,z_0)$ t.q.:

$$\iiint_{B_\varepsilon(x_0,y_0,z_0)} f(x,y,z)\,dx\,dy\,dz = f(\bar{x},\bar{y},\bar{z}) \cdot \text{Volum}\, B_\varepsilon(x_0,y_0,z_0).$$



$B_\varepsilon := B_\varepsilon(x_0,y_0,z_0)$

D'altra banda, com que f és contínua i $(\bar{x},\bar{y},\bar{z}) \longrightarrow (x_0,y_0,z_0)$ quan $\varepsilon \to 0$, prenent límits a l'expressió de dalt hom obté:

$$\lim_{\varepsilon \to 0} \frac{1}{\text{Volum}(B_\varepsilon(x_0,y_0,z_0))} \iiint_{B_\varepsilon(x_0,y_0,z_0)} f(x,y,z)\,dx\,dy\,dz = \lim_{(\bar{x},\bar{y},\bar{z}) \to (x_0,y_0,z_0)} f(\bar{x},\bar{y},\bar{z}) = f(x_0,y_0,z_0),$$

que és el resultat que es buscava. □

---

5) Troben les següents integrals dobles en els rectangles que s'indiquen

(a) $I = \iint_R x^2 y\,dx\,dy$, $R = [0,1] \times [0,1]$.

Solució: $I = \iint_R x^2 y\,dx\,dy = \left(\int_0^1 x^2\,dx\right)\left(\int_0^1 y\,dy\right) = \left[\frac{x^3}{3}\right]_0^1 \cdot \left[\frac{y^2}{2}\right]_0^1 = \boxed{\frac{1}{6}}$

(b) $I = \iint_R \frac{x^2}{1+y^2}\,dx\,dy$, $R = [0,1] \times [0,1]$

Solució: $I = \iint_{R=[0,1]\times[0,1]} \frac{x^2}{1+y^2}\,dx\,dy = \int_0^1 x^2\,dx \int_0^1 \frac{dy}{1+y^2} = \left[\frac{x^3}{3}\right]_0^1 \cdot \left[\arctan y\right]_0^1 = \boxed{\frac{\pi}{12}}$

(c) $I = \iint_R y \ln x \, dx\,dy$, $R = [1,e]\times[1,e]$.

Solució. $I = \iint_{R=[1,e]\times[1,e]} y \ln x \, dx\,dy = \left(\int_1^e \ln x \, dx\right)\cdot\left(\int_1^e y\,dy\right) \overset{(*)}{=} \boxed{\dfrac{e^2-1}{2}}$

(*) $\int_1^e \ln x \, dx = \Big[x\ln x - x\Big]_1^e = \cancel{e}-\cancel{e}+1$ (primitivització per parts + Barrow).

$\int_1^e y\,dy = \left[\frac{y^2}{2}\right]_1^e = \frac{1}{2}(e^2-1)$

(d) $\iint_R (x^2+y)\,dx\,dy$, $R=[0,1]\times[0,2]$

Solució $I = \iint_{R=[0,1]\times[0,2]} (x^2+y)\,dx\,dy = \int_0^1 dx \int_0^2 (x^2+y)\,dy = \int_0^1 dx \left[x^2 y + \frac{y^2}{2}\right]_0^2 =$

$= \int_0^1 (2x^2+2)\,dx = \left[2\frac{x^3}{3}+2x\right]_0^1 = \boxed{8/3}$

(e) $I = \iint_R \frac{1}{(x+2y)^2}\,dx\,dy$, $R=[2,5]\times[1,3]$

Solució. $I = \iint_R \frac{dx\,dy}{(x+2y)^2} = \int_2^5 dx \int_1^3 \frac{dy}{(x+2y)^2} = \int_2^5 \left[\frac{-1/2}{x+2y}\right]_{x=1}^{x=3} dx = \frac{1}{2}\int_2^5 \left(\frac{1}{x+2} - \frac{1}{x+6}\right) dx$

$= \frac{1}{2}\cdot \ln\left(\frac{x+2}{x+6}\right)\Bigg|_2^5 = \frac{1}{2}\left(\ln\frac{7}{11} - \ln\frac{4}{8}\right) = \boxed{\frac{1}{2}\ln\frac{14}{11}}$

(f) $I = \iint_R e^y \sin\left(\frac{x}{y}\right) dx\,dy$, $R = [-\frac{\pi}{2}, \frac{\pi}{2}]\times[1,2]$

Solució. $I = \iint_{R=[-\frac{\pi}{2},\frac{\pi}{2}]\times[1,2]} e^y \sin\left(\frac{x}{y}\right) dx\,dy = \int_1^2 e^y dy \int_{-\pi/2}^{\pi/2} \sin\left(\frac{x}{y}\right) dx = -\int_1^2 e^y y \left[\cos\left(\frac{x}{y}\right)\right]_{x=-\pi/2}^{x=\pi/2} dx \overset{(*)}{=} \boxed{0}$

(*) $\Big[\dots\Big|_{x=-\pi/2}^{x=\pi/2} = \cos\left(\frac{\pi}{2y}\right) - \cos\left(-\frac{\pi}{2y}\right) = 0.$

(g) $I = \iint_R (x+y)^{27}\,dx\,dy$, $R=[-1,1]\times[-1,1]$.

Solució. $I = \iint_{R=[-1,1]\times[-1,1]} (x+y)^{27}dx\,dy = \int_{-1}^{1} dx \int_{-1}^{1} (x+y)^{27} dy = \int_{-1}^{+1}\left[\frac{(x+y)^{28}}{28}\right]_{y=-1}^{y=1} dx$

$= \int_{-1}^{1}\left[\frac{(x+1)^{28}}{28} - \frac{(x-1)^{28}}{28}\right]dx = \left[\frac{(x+1)^{29}}{29\cdot 28} - \frac{(x-1)^{29}}{29\cdot 28}\right]_{-1}^{1} = \frac{1}{29\cdot 28}\left(2^{29} + (-2)^{29}\right) = \boxed{0}$.

6) Calculeu $I = \iint_R x^y\,dx\,dy$ on $R=[0,1]\times[a,b]$, essent $0<a<b$, i deduïu el valor de la integral $\int_0^1 \frac{x^b - x^a}{\ln x}dx$.

Solució. $I = \iint_{R=[0,1]\times[a,b]} x^y\,dx\,dy = \int_a^b dy \int_0^1 x^y dx = \int_a^b \left.\frac{x^{y+1}}{y+1}\right|_{x=0}^{x=1} dy = \int_a^b \frac{dy}{y+1} = \ln\frac{b+1}{a+1}$.

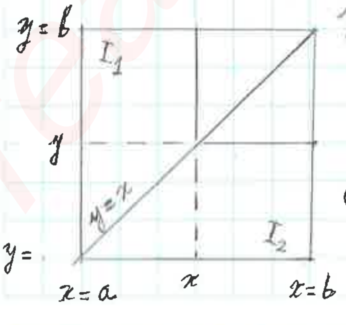
D'altra banda, invertint l'ordre d'integració (Fubini):

$I = \iint_{R=[0,1]\times[a,b]} x^y\,dx\,dy = \int_0^1 dx \int_a^b e^{y\ln x}dy = \int_0^1 \frac{1}{\ln x}\left[x^y\right]_a^b dx = \int_0^1 \frac{x^b - x^a}{\ln x}$,

d'on es segueix que: $\boxed{\int_0^1 \frac{x^b - x^a}{\ln x}dx = I = \ln\frac{b+1}{a+1}}$

7) Proveu que $2\int_a^b\int_x^b f(x)f(y)\,dy\,dx = \left(\int_a^b f(x)dx\right)^2$. (Indicació $\left(\int_a^b f(x)\,dx\right)^2 = \iint_{[a,b]\times[a,b]} f(x)f(y)\,dx\,dy$).

Solució. $2\int_a^b\int_x^b f(x)f(y)\,dx\,dy = \int_a^b f(x)\,dx\int_x^b f(y)dy + \overbrace{\int_a^b f(y)dy\int_y^b f(x)dx}^{\text{canviant les variables } x\leftrightarrow y}$



$\overset{(*)}{=} \iint_{I_1} f(x)f(y)\,dx\,dy + \iint_{I_2} f(x)f(y)\,dx\,dy = \iint_{I_1\cup I_2} f(x)f(y)\,dx\,dy =$ pàgina següent

(*) On $I_1 = \{(x,y)\in\mathbb{R}^2 : a\le x\le y\le b\}$, $I_2 = \{(x,y)\in\mathbb{R}^2: a\le y\le x\le b\}$, i notem que $m(I_1\cap I_2)=0$, per tant $\iint_{I_1} + \iint_{I_2} = \iint_{I_1\cup I_2}$

$$= \iint_{I_1 \cup I_2 = [a,b]\times[a,b]} f(x) f(y)\, dx\, dy = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b f(y)\,dy\right) = \left(\int_a^b f(x)\,dx\right)^2$$

---

9) Calculeu les següents integrals dobles en els dominis de $\mathbb{R}^2$ que s'indiquen

(a) $I = \iint_A y^3\,dx\,dy$, $A = \left\{(x,y)\in\mathbb{R}^2 : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2},\ 0 \leq y \leq 2\cos x\right\}$

Solució. $I = \int_{-\pi/2}^{\pi/2} dx \int_0^{2\cos x} y^3\,dy = 4\int_{-\pi/2}^{\pi/2} \cos^4 x\,dx = 8\int_0^{\pi/2} \cos^4 x\,dx = 8\int_0^{\pi/2} \cos^2 x\,(1-\sin^2 x)\,dx$

$$= 8\cdot\int_0^{\pi/2}\left(\cos^2 x - \frac{1}{4}\,4\sin^2 x\cos^2 x\right)dx = 8\int_0^{\pi/2}\left(\cos^2 x - \frac{1}{4}\sin^2(2x)\right)dx$$

$$= 8\cdot\int_0^{\pi/2}\left(\frac{1+\cos(2x)}{2} - \frac{1-\cos(4x)}{8}\right)dx = 8\left(\frac{\pi}{4}-\frac{\pi}{16}\right) = 8\,\frac{3\pi}{16} = \boxed{\frac{3\pi}{2}}$$

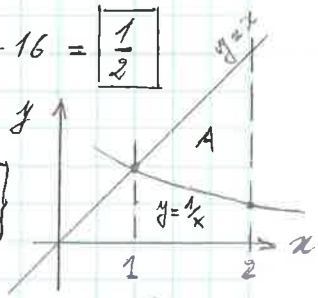
(b) $I = \iint_A x\,dx\,dy$, $A = \left\{(x,y)\in\mathbb{R}^2 : 0\leq x\leq 1,\ 0\leq y\leq e^x\right\}$

integració per parts

Solució. $I = \iint_A x\,dx\,dy = \int_0^1 x\,dx \int_0^{e^x} dy = \int_0^1 x e^x\,dx = \left(xe^x - e^x\right)\Big|_0^1 = \boxed{1}$.

(c) $I = \iint_A xy\,dx\,dy$, $A = \left\{(x,y)\in\mathbb{R}^2 : 0\leq x\leq 2,\ 0\leq y\leq \frac{x}{2}\right\}$

Solució. $I = \iint_A xy\,dx\,dy = \int_0^2 x\,dx\int_0^{x/2} y\,dy = \frac{1}{8}\int_0^2 x^3\,dx = \frac{1}{32}\cdot 16 = \boxed{\frac{1}{2}}$



(d) $I = \iint_A \frac{x^2}{y^2}\,dx\,dy$, $A = \left\{(x,y)\in\mathbb{R}^2 : 1\leq x\leq 2,\ \frac{1}{x}\leq y\leq x\right\}$

Solució. $I = \iint_A \frac{x^2}{y^2}\,dx\,dy = \int_1^2 x^2\,dx\int_{1/x}^{x}\frac{dy}{y^2} = \int_1^2 x^2\left(x-\frac{1}{x}\right)dx = \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_1^2 = 4-2-\frac{1}{4}+\frac{1}{2}$

$= \boxed{\frac{9}{4}}$

e) $I = \iint_A (x+2y)\,dx\,dy,\quad A=\left\{(x,y)\in\mathbb{R}^2:\ -3\le y\le 3,\ y^2-4\le x\le 5\right\}$

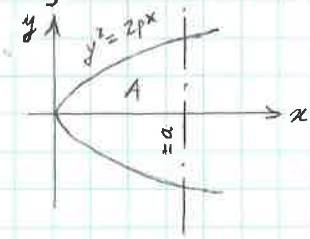
Solució. $I=\int_{-3}^{3}dy\int_{y^2-4}^{5}(x+2y)\,dx = \int_{-3}^{3}dy\left(\frac{x^2}{2}+2yx\right)\Bigg|_{y^2-4}^{5} = \int_{-3}^{3}\left(\frac{9}{2}+2y+4y^2+2y^3-\frac{1}{2}y^4\right)dy$

$$= 2\cdot\int_0^3\left(\frac{9}{2}+4y^2-\frac{1}{2}y^4\right)dy = 2\cdot\left(\frac{27}{2}+\frac{4}{3}27-\frac{9}{10}27\right) = \frac{54}{30}(15+40-27)$$

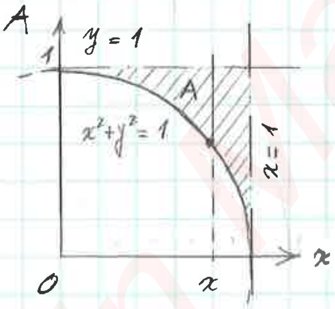
$$= \frac{54}{30}\cdot 28 = \boxed{\frac{252}{5}}$$

f) $I=\iint_A y^3\,dx\,dy,\quad A=\left\{(x,y)\in\mathbb{R}^2:\ 0\le x\le a,\ y^2\le 2px\right\},\ a>0,\ p>0$

Solució. $I=\int_0^a dx\int_{-\sqrt{2px}}^{\sqrt{2px}} y^3\,dy = \int_0^a \underbrace{\left[\frac{y^4}{4}\right]_{y=-\sqrt{2px}}^{y=\sqrt{2px}}}_{=0} dy = \boxed{0.}$



g) $I=\iint_A \frac{y}{1+x^3}\,dx\,dy,\quad A=\left\{(x,y)\in\mathbb{R}^3:\ 0\le x\le 1,\ 0\le\ \le 1,\ x^2+y^2\ge 1\right\}$



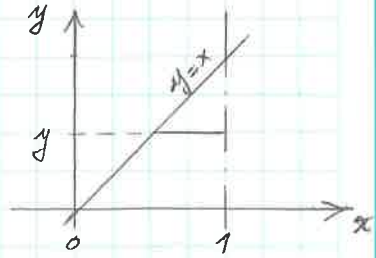
Solució:

$$I=\iint_A\frac{y}{1+x^3}\,dx\,dy=\int_0^1\frac{dx}{1+x^3}\int_{\sqrt{1-x^2}}^{1}y\,dy$$

$$=\frac{1}{2}\int_0^1\frac{x^2}{1+x^3}\,dx=\frac{1}{6}\ln(1+x^3)\Big|_0^1=\frac{\ln 2}{6}$$

h) $I=\iint_A x^2\sin(xy)\,dx\,dy,\quad A=\left\{(x,y)\in\mathbb{R}^2:\ 0\le y\le 1,\ y\le x\le 1\right\}$

Solució. $I=\iint_A x^2\sin(xy)\,dx\,dy = \int_0^1 x^2\,dx\int_0^x\sin(xy)\,dy$

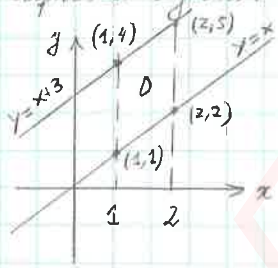


$$= -\int_0^1 x^2\cdot\left[\frac{\cos(xy)}{x}\right]_{y=0}^{y=x}dx = -\int_0^1 x\left(\cos(x^2)-1\right)dx$$

$$= \left[-\frac{\sin(x^2)}{2}+\frac{x^2}{2}\right]_0^1 = \boxed{\frac{1-\sin(1)}{2}}$$

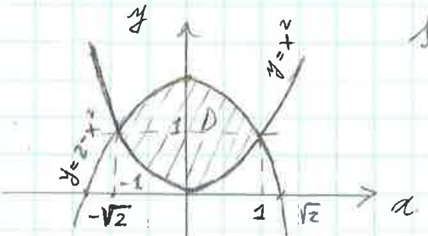
10) Per a les integrals iterades següents escriviu les equacions de les corbes que limiten les regions d'integració i dibuixeu aquestes regions.

(a) $\displaystyle\int_1^2\left(\int_x^{x+3} f(x,y)\,dy\right)dx.$



Solució: $y=x$, $y=x+3$, $x=1$, $x=2$

(b) $\displaystyle\int_{-1}^1\left(\int_{x^2}^{2-x^2} f(x,y)\,dy\right)dx$

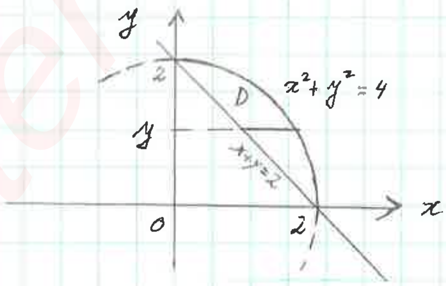


Solució.

$y=x^2$, $y=2-x^2$, $x=1$, $x=-1$ (veure figura).

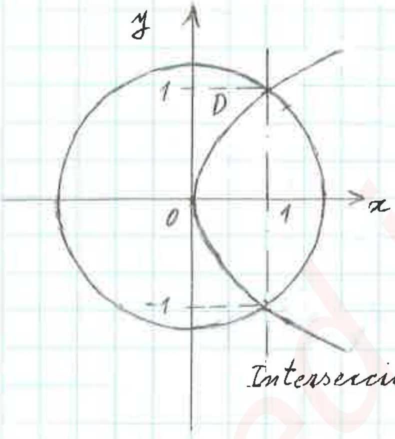
Intersecció: $2-x^2=x^2 \Leftrightarrow x=\pm 1$, $y=1$

(c) $\displaystyle\int_0^2\left(\int_{2-y}^{\sqrt{4-y^2}} f(x,y)\,dx\right)dy.$



Solució:

$x+y=2$, $x^2+y^2=4$, $y=0$, $y=2$ (veure figura).



(d) $\displaystyle\int_0^1\left(\int_{\sqrt{x}}^{\sqrt{2-x^2}} f(x,y)\,dy\right)dx$

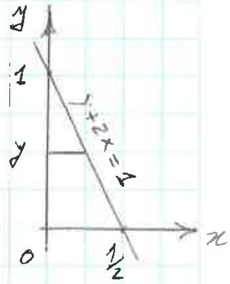
Solució.

$y^2=x$, $x^2+y^2=2$, $x=0$, $x=1$. (veure figura).

Intersecció: $\begin{matrix} x^2+y^2=2 \\ -x+y^2=0 \end{matrix}$ d'on: $x^2+x-2=0$, i llavors $x=\dfrac{-1\pm\sqrt{1+8}}{2}$; $\begin{matrix} -2 \text{ (No)} \\ 1 \text{ d'on } y=\pm 1 \end{matrix}$

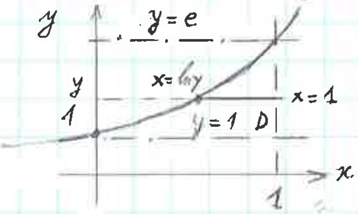
11) Invertiu l'ordre d'integració en les integrals iterades següents

(a) $\displaystyle I=\int_0^{1/2}\left(\int_0^{1-2x} f(x,y)\,dy\right)dx.$



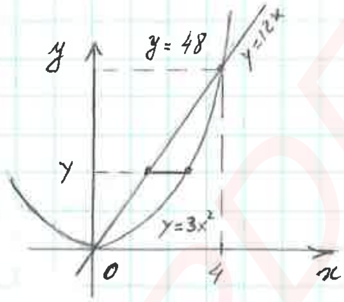
Solució $\displaystyle I=\int_0^{1/2}\left(\int_0^{1-2x} f(x,y)\,dy\right)dx=\int_0^1\left(\int_0^{\frac{1-y}{2}} f(x,y)\,dx\right)dy.\ \square$

(b) $I = \int_0^1 \left(\int_1^{e^x} f(x,y)\,dy \right) dx$



Solució. $I = \int_0^1 \left(\int_1^{e^x} f(x,y)\,dy \right) dx = \int_1^e \left(\int_{\ln y}^1 f(x,y)\,dx \right) dy$. □

c) $I = \int_0^4 \left(\int_{3x^2}^{12x} f(x,y)\,dy \right) dx$.

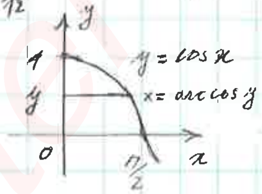


Intersecció: $3x^2 = 12x$
$\Leftrightarrow x = 0,\ x = 4$
$y = 0,\ y = 48$

Solució.

$$I = \int_0^4 \left(\int_{3x^2}^{12x} f(x,y)\,dy \right) dx = \int_0^{48} \left(\int_{y/12}^{\sqrt{y/3}} f(x,y)\,dx \right) dy \quad \square$$

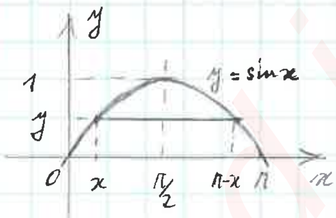
d) $I = \int_0^{\pi/2} \left(\int_0^{\cos x} f(x,y)\,dy \right) dx$



Solució

$$I = \int_0^{\pi/2} \left(\int_0^{\cos x} f(x,y)\,dy \right) dx = \int_0^1 \left(\int_0^{\arccos y} f(x,y)\,dx \right) dy. \square$$

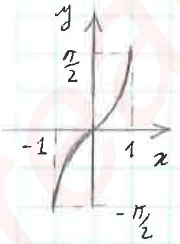
e) $\int_0^{\pi} \left(\int_0^{\sin x} f(x,y)\,dy \right) dx$



Solució.

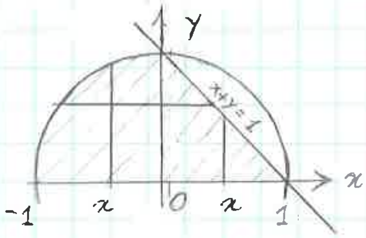
$$I = \int_0^{\pi} \left(\int_0^{\sin x} f(x,y)\,dy \right) dx \overset{(*)}{=} \int_0^1 \left(\int_{\arcsin y}^{\pi - \arcsin y} f(x,y)\,dx \right) dy \quad \square$$

(*) Definim: $\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ de manera que $v = \arcsin u$, amb $-1 \le u \le 1$, si i només si $u = \sin v$, amb $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$.



funció arc sin

f) $I = \int_0^1 \left(\int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\,dx \right) dy$

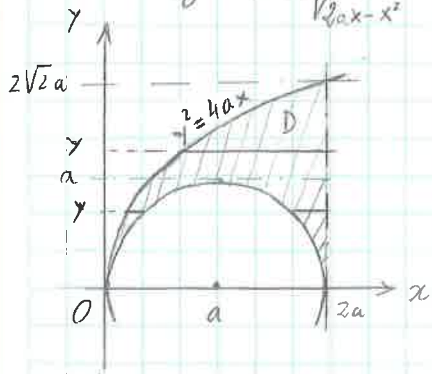


Solució.

$$I = \int_0^1 \left(\int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\,dx \right) dy = \int_{-1}^0 \left(\int_0^{\sqrt{1-x^2}} f(x,y)\,dy \right) dx + \int_0^1 \left(\int_0^{1-x} f(x,y)\,dy \right) dx$$

□

g) $I=\int_0^{2a}\left(\int_{\sqrt{2ax-x^2}}^{\sqrt{4ax}} f(x,y)\,dy\right)dx\ ,\ a>0.$



Solució.

$$I=\int_0^{2a}\left(\int_{\sqrt{2ax-x^2}}^{\sqrt{4ax}} f(x,y)\,dy\right)dx$$

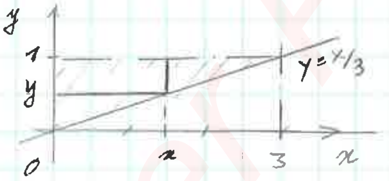
$$=\int_0^{a}\left(\int_{y^2/4a}^{a-\sqrt{a^2-y}} f(x,y)\,dx\right)dy+\int_0^{a}\left(\int_{a+\sqrt{a^2-y}}^{2a} f(x,y)\,dx\right)dy+\int_a^{2\sqrt{2}a}\left(\int_{y^2/4a}^{2a} f(x,y)\,dx\right)dy \quad \square$$

Intersecció:

$(x-a)^2+y^2=a^2$ d'on: $x^2-2ax+a^2+4ax=a^2 \Leftrightarrow x(x+2a)=0 \Leftrightarrow \begin{matrix} x=0 \\ x=-2a<0 \text{ (No)}\end{matrix}$

$y^2=4ax$

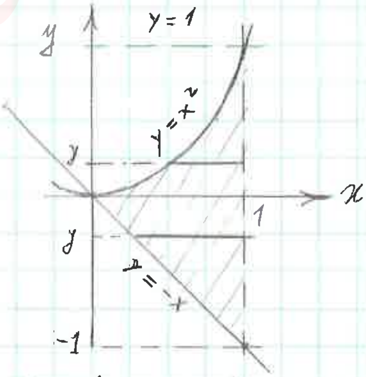
h) $I=\int_0^{3}\left(\int_{x/3}^{1} f(x,y)\,dy\right)dx$



Solució.

$$I=\int_0^{3}\left(\int_{x/3}^{1} f(x,y)\,dy\right)dx=\int_0^{1}\left(\int_0^{3y} f(x,y)\,dx\right)dy \quad \square$$

i) $I=\int_0^{1}\left(\int_{-x}^{x^2} f(x,y)\,dy\right)dx.$



Solució.

$$I=\int_0^{1}\left(\int_{-x}^{x} f(x,y)\,dy\right)dx$$

$$=\int_{-1}^{0}\left(\int_{-y}^{1} f(x,y)\,dx\right)dy+\int_0^{1}\left(\int_{\sqrt{y}}^{1} f(x,y)\,dx\right)dy \quad \square$$

13) Calculeu les següents integrals iterades.

(a) $I = \int_1^2 \int_0^1 \int_0^{\pi/2} x^2 y^3 \sin z \, dz\, dy\, dx$

Solució.

$$I = \int_1^2 \int_0^1 \int_0^{\pi/2} x^2 y^3 \sin z \, dz\, dy\, dx = \int_0^1 x^2 dx \int_0^1 y^3 dy \int_0^{\pi/2} \sin z \, dz$$

$$= \left[\frac{x^3}{3}\right]_1^2 \cdot \left[\frac{y^4}{4}\right]_0^1 \cdot \left[-\cos z\right]_0^{\pi/2} = \left(\frac{8}{3} - \frac{1}{3}\right) \cdot \frac{1}{4} = \boxed{\frac{7}{12}}$$

(b) $I = \int_0^1 \int_0^x \int_0^{\sqrt{x^2+y^2}} z \, dz\, dy\, dx$

Solució.

$$I = \int_0^1 \int_0^x \int_0^{\sqrt{x^2+y^2}} z \, dz\, dy\, dx = \frac{1}{2}\int_0^1 dx \int_0^x (x^2+y^2)\, dy = \frac{1}{2}\int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_{y=0}^{y=x} dx$$

$$= \frac{1}{2} \cdot \frac{4}{3} \int_0^1 x^3 dx = \boxed{\frac{1}{6}}$$

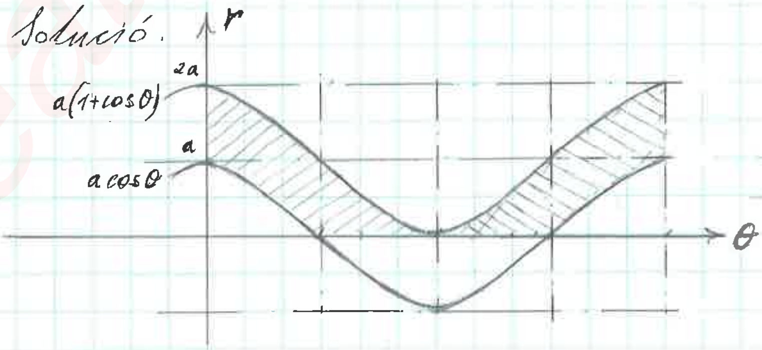
(c) $I = \int_0^3 \int_0^{2x} \int_0^{\sqrt{xy}} z \, dz\, dy\, dx = \int_0^3 dx \int_0^{2x} dy \int_0^{\sqrt{xy}} z\, dz = \frac{1}{2}\int_0^3 x\, dx \int_0^{2x} y\, dy$

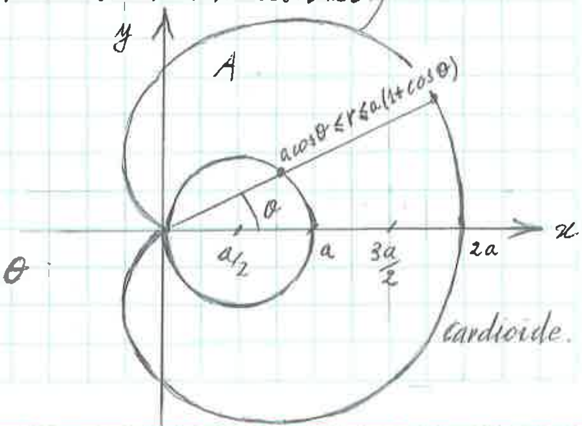
$$= \int_0^3 x^3 dx = \boxed{\frac{81}{4}}$$

17) Calculeu les àrees dels dominis $A \subset \mathbb{R}^2$ definits en coordenades polars, $x = r\cos\theta$, $y = r\sin\theta$, que s'indiquen tot seguit.

(a) A figura definida per $a\cos\theta \le r \le a(1+\cos\theta)$ $(a>0)$. (Indicació: Observeu que l'expressió té sentit quan $\cos\theta \ge 0$. Dibuixar les gràfiques de $a\cos\theta$ i $a(1+\cos\theta)$ pot ajudar a veure els valors de r admissibles.)

Solució.





Dibuix amb Matlab/Octave

```
>> a=2; % per exemple
>> theta = linspace(0, 2*pi, 201);
>> rho1= a*cos(theta);
>> rho2= a*(1+cos(theta));
>> polar (theta, rho1, 'b-');
>> axis equal
>> hold on
>> polar(thetha, rho2, 'r-')
>> hold off
```

$$I_1 = \int_0^{\pi/2} d\theta \int_{a\cos\theta}^{a(1+\cos\theta)} r\,dr + \int_{3\pi/2}^{\pi/2} d\theta \int_{a\cos\theta}^{a(1+\cos\theta)} r\,dr$$

$$= 2\int_0^{\pi/2} d\theta \int_{a\cos\theta}^{a(1+\cos\theta)} r\,dr = a^2\int_0^{\pi/2}(1+2\cos\theta)\,d\theta$$

$$= a^2\left(2+\frac{\pi}{2}\right).$$

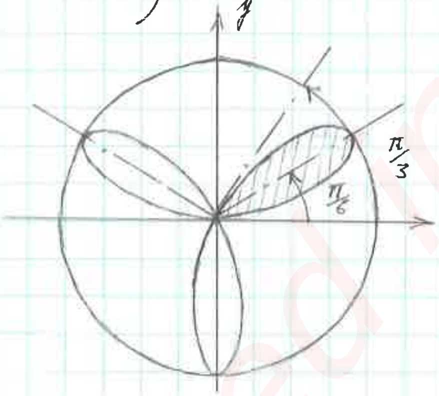
$$I_2 = 2\int_{\pi/2}^{\pi} d\theta \int_0^{a(1+\cos\theta)} r\,dr = a^2\int_{\pi/2}^{\pi}(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_{\pi/2}^{\pi}\left(1+2\cos\theta+\cos^2\theta\right)d\theta = a^2\left(\frac{\pi}{2}-2+\frac{\pi}{2}-\frac{\pi}{4}\right)$$

$$= a^2\left(\frac{3\pi}{4}-2\right)$$

$$\therefore\quad I = \iint_A dx\,dy = I_1+I_2 = a^2\left(2+\frac{\pi}{2}+\frac{3\pi}{4}-2\right) = \boxed{a^2\,\frac{5\pi}{4}}.$$

(b) A regió limitada per un pètal de la rosa definit per $r = a\sin 3\theta$ $(0\le\theta\le\pi/3,\ a>0)$.

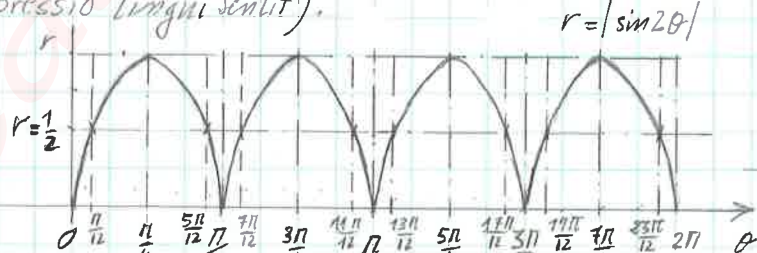


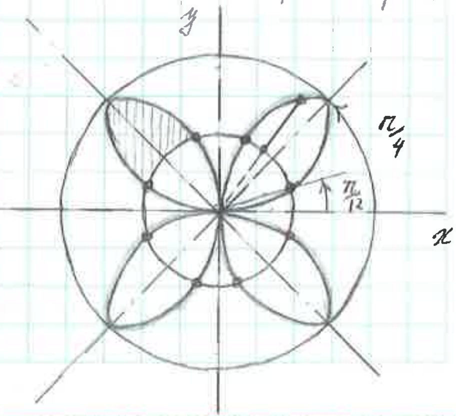
Solució.

$$A = \int_0^{\pi/3} d\theta \int_0^{a\sin(3\theta)} r\,dr = \frac{a^2}{2}\int_0^{\pi/3}\frac{1-\cos(6\theta)}{2}\,d\theta$$

$$= \boxed{\frac{a^2\pi}{12}}.$$

(c) A regió definida per $\frac{1}{2}\le r\le|\sin(2\theta)|$. (Indicació: Cal $|\sin(2\theta)|\ge\frac{1}{2}$ perquè l'expressió tingui sentit).





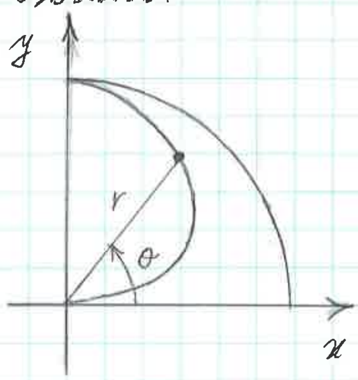
$$A = 4\int_{\pi/12}^{5\pi/12} d\theta \int_{1/2}^{|\sin 2\theta|} r\,dr = 2\cdot\int_{\pi/12}^{5\pi/12}\left(\sin^2 2\theta - \frac{1}{4}\right)d\theta = 2\int_{\pi/12}^{5\pi/12}\left(\frac{1-\cos 4\theta}{2} - \frac{1}{4}\right)d\theta$$

$$= 2\int_{\pi/12}^{5\pi/12} \frac{1-2\cos 4\theta}{4}\,d\theta = 2\left[\frac{\theta}{4} - \frac{\sin 4\theta}{8}\right|_{\pi/12}^{5\pi/12} = 2\left(\frac{5\pi}{48} - \frac{1}{8}\sin\left(\frac{5\pi}{3}\right) - \frac{\pi}{48} + \frac{1}{8}\sin\frac{\pi}{3}\right)$$

$$= 2\left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right) = \frac{2\pi + 3\sqrt{3}}{12} = \boxed{\frac{\pi}{6} + \frac{\sqrt{3}}{4}}$$

(d) Anàlogament, calculeu la integral doble $\iint_A \arcsin(x^2+y^2)\,dx\,dy$, on A és la regió limitada per la corba $r = \sqrt{\sin\theta}$ $(0 \leq \theta \leq \pi/2)$.

Solució.



$$A = \int_0^{\pi/2} d\theta \int_0^{\sqrt{\sin\theta}} r\arcsin(r^2)\,dr = \left\{\begin{array}{l}\text{Integració per parts:}\\ f = \arcsin r^2 \Rightarrow f' = \dfrac{2r}{\sqrt{1-r^4}}\\ g' = r \Rightarrow g = \dfrac{r^2}{2}\end{array}\right.$$

$$= \int_0^{\pi/2} d\theta \left\{\left[\frac{r^2}{2}\arcsin(r^2)\right]_0^{\sqrt{\sin\theta}} - \int_0^{\sqrt{\sin\theta}} \frac{r^3}{\sqrt{1-r^4}}\,dr\right\}$$

$$\overset{(*)}{=} \int_0^{\pi/2} \frac{1}{2}\left(\theta\sin\theta + \cos\theta - 1\right)d\theta = \left\{\begin{array}{l}\text{parts:}\\ f = \theta \Rightarrow f' = 1\\ g' = \sin\theta \Rightarrow g = -\cos\theta\end{array}\right.$$

$$= \frac{1}{2}\left[-\theta\cos\theta\right]_0^{\pi/2} + \int_0^{\pi/2}\left(\cos\theta - \frac{1}{2}\right)d\theta = \left[\sin\theta - \frac{\theta}{2}\right|_0^{\pi/2} = \boxed{1 - \frac{\pi}{4}}.$$

(*) Nota:

$$\int_0^{\sqrt{\sin\theta}} \frac{r^3}{\sqrt{1-r^4}}\,dr = -\frac{1}{2}\sqrt{1-r^4}\Bigg|_0^{\sqrt{\sin\theta}} = \frac{1}{2} - \frac{1}{2}|\cos\theta| = \frac{1}{2} - \frac{1}{2}\cos\theta$$

18) Calculeu les integrals dobles, fent servir el canvi de variable adequat en cada cas.

(a) $\iint_D xy\,dx\,dy$, $D = \{(x,y) \in \mathbb{R}^2 : 6 \leq 2y - x \leq 12,\ 0 \leq x \leq 4\}$.

Solució. Introduïm el canvi: $\begin{array}{l} x = 4u \\ 6v = 2y - x \end{array} \Leftrightarrow y = 3v + 2u$,

és a dir: $(u,v) \in \mathbb{R}^2 \xmapsto{T} (x,y) = T(u,v) = (4u, 3v+2u)$, d'on $\det DT(u,v) = \begin{vmatrix} 4 & 0 \\ 2 & 3 \end{vmatrix} = 12$.

D'altra banda: $6 \leq 6v = 2y - x \leq 12$, $0 \leq 4u \leq 4 \Leftrightarrow 0 \leq u \leq 1$, $1 \leq v \leq 2$, és a dir el domini transformat és, $D' = T^{-1}(D) = [0,1]\times[1,2]$. Aleshores:

$$I=\iint_D xy\,dx\,dy=\iint_{D'=T^{-1}(D)} x(u,v)\cdot y(u,v)\,\left|\det DT(u,v)\right|du\,dv = 12\iint_{D'=[0,1]\times[1,2]} 4u(3v+2u)\,du\,dv$$

$$=48\int_1^2 dv\int_0^1 (3uv+2u^2)\,du = 48\int_1^2\left[\tfrac{3}{2}u^2v+\tfrac{2}{3}u^3\right]_{u=0}^{u=1}dv = 48\int_1^2\left(\tfrac{3}{2}v+\tfrac{2}{3}\right)dv$$

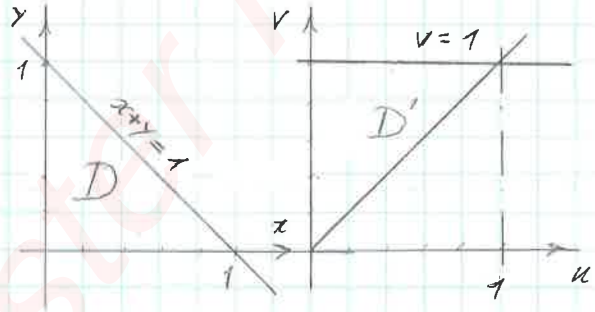
$$=48\left[\tfrac{3}{4}v^2+\tfrac{2}{3}v\right]_1^2 = 48\left(3+\tfrac{4}{3}-\tfrac{3}{4}-\tfrac{2}{3}\right)=48\,\frac{36+8-9}{12}=4\cdot 35=\boxed{140}$$

(b) $I=\iint_D \frac{1}{(1+x+y)^5}\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2:\ x\geq 0,\ y\geq 0,\ x+y\leq 1\}$

Solució. Fem el canvi: $\begin{matrix} u=x \\ v=x+y\end{matrix} \Leftrightarrow \begin{matrix} x=u \\ y=v-u\end{matrix}$, i.e.: $(u,v)\in\mathbb{R}^2 \xmapsto{T} (x,y)=T(u,v)=(u,v-u)$.

$DT(u,v)=\begin{vmatrix}1 & 0\\ -1 & 1\end{vmatrix}=1$,

d'altra banda, el domini D es transforma (veure figura) en:



$$D'=T^{-1}(D)=\{(u,v)\in\mathbb{R}^2:\ 0\leq u\leq 1,\ u\leq v\leq 1\}.$$

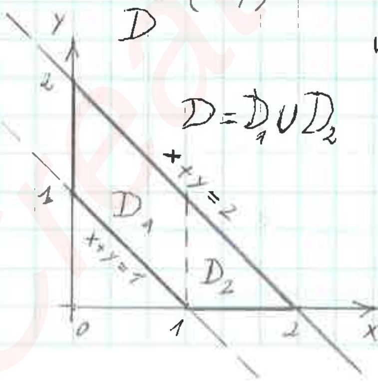
I podem calcular la integral,

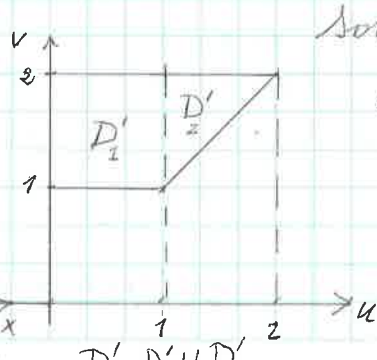
$$I=\iint_D \frac{1}{(1+x+y)^5}\,dx\,dy = \iint_{D'=T^{-1}(D)} \frac{1}{(1+x(u,v)+y(u,v))^5}\left|\det DT(u,v)\right|$$

$$=\iint_{D'}\frac{1}{(1+v)^5}\,du\,dv=\int_0^1 du\int_u^1\frac{dv}{(1+v)^5}=\int_0^1\left[\frac{(1+v)^{-4}}{-4}\right]_u^1 du$$

$$=\frac{1}{4}\int_0^1\left[\frac{1}{(1+u)^4}-\frac{1}{16}\right]du=\frac{1}{4}\left[\frac{(1+u)^{-3}}{-3}-\frac{u}{16}\right]_0^1=\frac{1}{4}\left(\frac{1}{3}-\frac{1}{24}-\frac{1}{16}\right)=\boxed{\frac{11}{192}}$$

(c) $I=\iint_D\frac{dx\,dy}{(x+y)^{m+1}}$, $D=\{(x,y)\in\mathbb{R}^2:\ 1\leq x+y\leq 2,\ x\geq 0,\ y\geq 0\}$.





$D'=D'_1\cup D'_2$

Solució.

Considerem el canvi: $\begin{matrix} u=x \\ v=x+y\end{matrix} \Leftrightarrow \begin{matrix} x=u \\ y=v-u,\end{matrix}$

i.e.: $(u,v)\in\mathbb{R}^2\xmapsto{T}(x,y)=T(u,v)=(u,v-u)$.

$\det DT(u,v)=\begin{vmatrix}1&0\\-1&1\end{vmatrix}=1$, mentre que els dominis es transformen com: $D' = T^{-1}(D) = \{(u,v)\in\mathbb{R}^2 :\ u\le v,\ 1\le v\le 2\}$, i podem calcular I fent:

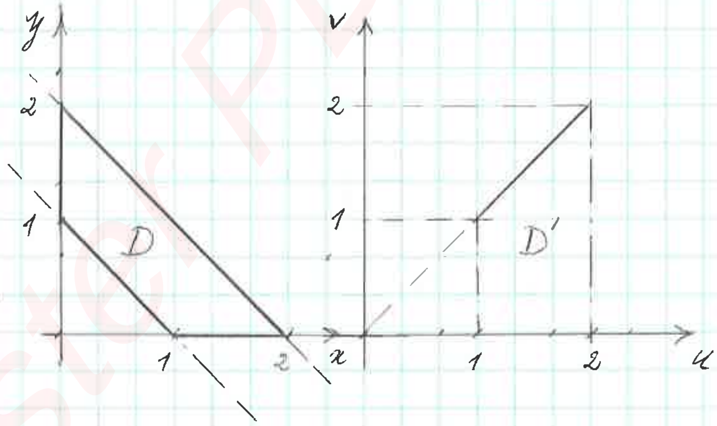
$$I = \iint_D \frac{dx\,dy}{(x+y)^{n+1}} = \iint_{D'=T^{-1}(D)} \frac{1}{\left(x(u,v)+y(u,v)\right)^{n+1}} \left|\det DT(u,v)\right| \cdot du\,dv$$

$$= \int_1^2 \frac{dv}{v^{n+1}} \int_0^v du = \int_1^2 \frac{dv}{v^{n}} = \left[\frac{v^{-n+1}}{-n+1}\right]_1^2 = \boxed{\frac{1}{n-1}\left(1-\frac{1}{2^{n-1}}\right)}$$

Alternativament, podem fer el canvi:

$$(u,v)\in\mathbb{R}^2 \xrightarrow{T} (x,y) = T(u,v) = (v,\ u-v);$$

el qual transforma el domini D en $D' = T^{-1}(D) = \{(u,v)\in\mathbb{R}^2 :\ v\le u,\ 1\le u\le 2\}$, (veure figura al costat).



Tanmateix:

$$\det DT(u,v) = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1.$$

i llavors el càlcul de la integral resulta:

$$I = \iint_D \frac{dx\,dy}{(x+y)^{n+1}} = \int_{D'=T^{-1}(D)} \frac{1}{\left(x(u,v)+y(u,v)\right)^{n+1}} \left|\det DT(u,v)\right| du\,dv$$

$$= \int_1^2 \frac{du}{u^{n+1}} \int_0^u dv = \int_1^2 \frac{du}{u^{n}} = \left[\frac{u^{-n+1}}{-n+1}\right]_1^2 = \frac{1}{n-1} - \frac{2^{-n+1}}{n-1} = \boxed{\frac{1}{n-1}\left(1-\frac{1}{2^{n-1}}\right)}$$

(d) $I = \iint_D \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{3/2} dx\,dy$, $D = \{(x,y)\in\mathbb{R}^2 :\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}$.

Solució. Fem servir coordenades polars adaptades: $\begin{array}{l} x = |a|\, r\cos\theta \\ y = |b|\, r\sin\theta \end{array}$, que transformen el domini D en D': $0\le r\le 1$, $0\le\theta\le 2\pi$, mentre que $\det\frac{\partial(x,y)}{\partial(r,\theta)} = |ab|\, r$. Aleshores:

$$I = \iint_D \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{3/2} dx\,dy = \iint_{D'} \left(1 - x^2(r,\theta)/a^2 - y^2(r,\theta)/b^2\right)^{3/2} \left|\det\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= |ab| \int_0^{2\pi} d\theta \int_0^1 r\left(1-r^2\right)^{3/2} dr = 2\pi|ab| \left[\left(-\tfrac{1}{5}\right)\left(1-r^2\right)^{5/2}\right]_0^1 = \boxed{\frac{2\pi}{5}|ab|}$$

(e) $I = \iint_D \arctan\left(x^2 + \frac{y^2}{2}\right) dx\,dy$, $D = \left\{(x,y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{2} \leq 1,\ x \geq 0,\ y \geq 0\right\}$.

Solució. Apliquem, de nou, coordenades polars adaptades, aquest cop de la forma $(r,\theta) \in [0,+\infty) \times [0,\frac{\pi}{2}] \xrightarrow{T} (x,y) = T(r,\theta) = (r\cos\theta, \sqrt{2}\, r\sin\theta)$, amb la qual cosa el domini es transforma en: $D' = T^{-1}(D) = [0,1] \times [0,\frac{\pi}{2}]$, i podem calcular I tot fent:

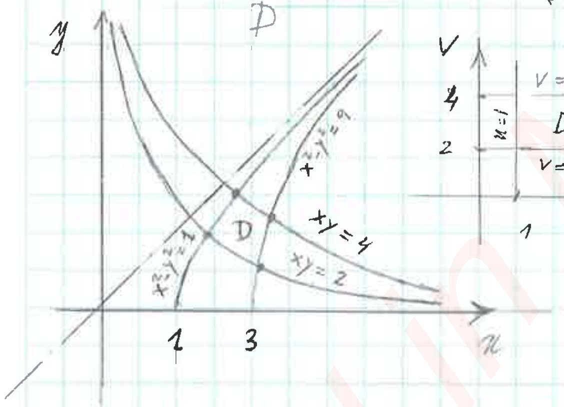
$$I = \iint_D \arctan\left(x^2 + \frac{y^2}{2}\right) dx\,dy = \iint_{D' = T^{-1}(D)} \arctan\left(x^2(r,\theta) + y^2(r,\theta)/2\right) \underbrace{\left|\det \frac{\partial(x,y)}{\partial(r,\theta)}\right|}_{\sqrt{2}\, r} dr\,d\theta =$$

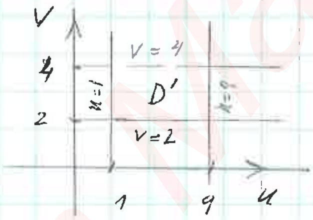
$$\sqrt{2} \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r \arctan(r^2)\, dr = \frac{\pi\sqrt{2}}{2} \int_0^1 r\arctan(r^2)\, dr = \left\{ \begin{array}{l} f = \arctan(r^2) \Rightarrow f' = \frac{2r}{1+r^4} \\ g' = r \Rightarrow g = r^2/2 \end{array} \right.$$

$$= \frac{\sqrt{2}\pi}{4} \left[r^2 \arctan(r^2)\right]_0^1 - \frac{\sqrt{2}\pi}{2} \int_0^1 \frac{r^3}{1+r^4} dr = \frac{\sqrt{2}\pi^2}{16} - \frac{\pi\sqrt{2}}{8} \ln(1+r^4)\Big|_0^1 =$$

$$= \frac{\sqrt{2}\pi}{2} - \frac{\sqrt{2}\pi}{8} \ln 2 = \boxed{\frac{\pi\sqrt{2}}{8}\left(\frac{\pi}{2} - \ln 2\right)}.$$

(f) $I = \iint_D (x^2+y^2)\, dx\,dy$, $D = \left\{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 - y^2 \leq 9,\ 2 \leq xy \leq 4,\ x \geq 0,\ y \geq 0\right\}$





Solució. Fem el canvi T, on T^{-1} ve donat per: $\left\{\begin{array}{l} u = x^2 - y^2 \\ v = xy \end{array}\right.$, d'on: $\left\{\begin{array}{l} 1 \leq u = x^2 - y^2 \leq 9 \\ 2 \leq v = xy \leq 4 \end{array}\right.$

Llavors tenim, pel domini transformat:

$$D' = T^{-1}(D) = [1,9] \times [2,4],$$

(veure figura).

D'altra banda, pel determinant del Jacobià, tenim:

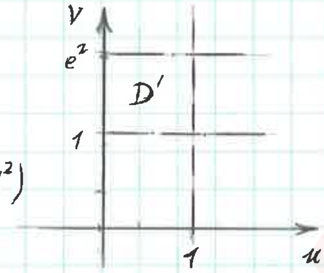
$$\det \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2+y^2), \text{ d'on: } \det \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\det \frac{\partial(u,v)}{\partial(x,y)}} = \frac{1/2}{x^2+y^2},$$

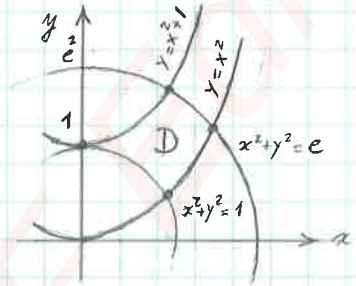
i podem calcular la integral fent,

$$I = \iint_D (x^2+y^2)\, dx\,dy = \iint_{D' = T(D)} \left(x^2(u,v) + y^2(u,v)\right) \left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{2} \iint_{D' = [1,9]\times[2,4]} \frac{x^2(u,v)+y^2(u,v)}{x^2(u,v)+y^2(u,v)} du\,dv$$

$$= \frac{1}{2} \left(\int_1^9 du\right) \cdot \left(\int_2^4 dv\right) = \frac{1}{2}(9-1)\cdot(4-2) = \boxed{8}.$$

$$\iint_D \frac{x+2xy}{x^2+y^2}\,dx\,dy,\ D=\{(x,y)\in\mathbb{R}^2:\ x^2\le y\le x^2+1,\ 1\le x^2+y^2\le e^2,\ x\ge 0\}.$$





Introduïm el canvi:

$$(x,y)\in\mathbb{R}^2 \xrightarrow{T^{-1}} (u,v)=T^{-1}(x,y)=(y-x^2,\ x^2+y^2)$$

Llavors:

$0\le u=y-x^2\le 1$

$1\le v=x^2+y^2\le e^2$ d'on, clarament: $D'=T^{-1}(D)=[0,1]\times[1,e^2]$, mentre que el determinant del Jacobià de T es pot calcular a partir del de T^{-1}. En efecte:

$$\det DT^{-1}(x,y)=\det\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}-2x & 1\\ 2x & 2y\end{vmatrix}=-4xy-2x=-2(x+2xy),$$

d'on:

$$\det DT(u,v)=\det\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\det\frac{\partial(u,v)}{\partial(x,y)}}=\frac{1}{\det DT^{-1}(x,y)}=\frac{-1/2}{x+2xy},$$

i el càlcul de la integral I es redueix a:

$$I=\iint_D\frac{x+2xy}{x^2+y^2}\,dx\,dy=\iint_{D'=T^{-1}(D)}\frac{x(u,v)+2x(u,v)y(u,v)}{x^2(u,v)+y^2(u,v)}\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

$$=\iint_{D'=T(D)=[0,1]\times[1,e^2]}\frac{x(u,v)+2x(u,v)y(u,v)}{x^2(u,v)+y^2(u,v)}\cdot\frac{1/2}{|x(u,v)+2x(u,v)y(u,v)|}\,du\,dv$$

$$\overset{(*)}{=}\iint_{D'=[0,1]\times[1,e^2]}\frac{\frac12\,du\,dv}{v}=\frac12\left(\int_0^1 du\right)\cdot\left(\int_1^{e^2}\frac{dv}{v}\right)=$$

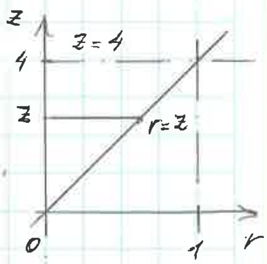
(*) $x(u,v),\ y(u,v)\ge 0\quad\forall (u,v)\in D'=[0,1]\times[1,e^2]$.

Llavors: $|x(u,v)+2x(u,v)y(u,v)|=x(u,v)+2x(u,v)y(u,v)$.

$$=\frac12\left(\ln e^2-\ln 1\right)=\boxed{1}$$

19) Useu coordenades cilíndriques per calcular les següents integrals triples.

Nota. Coordenades cilíndriques: $x=r\cos\theta$, $y=r\sin\theta$, $z=z$, amb $r\in[0,+\infty)$, $\theta\in[0,2\pi]$, $z\in(-\infty,+\infty)$, $\det\frac{\partial(x,y,z)}{\partial(r,\theta,z)}=r\ge 0$.

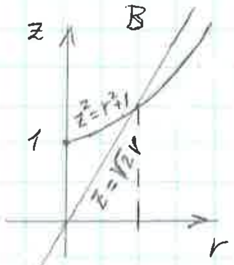


(a) $I=\iiint_B\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3:\ \sqrt{x^2+y^2}\le z\le 4\}$.

Solució. $I=\int_0^{2\pi}d\theta\int_0^4 dz\int_0^z r\sqrt{r^2+z^2}\,dr=\frac{2\pi}{3}\int_0^4\left(z^2+r^2\right)^{3/2}\Big|_0^z dz$

$$=\frac{2\pi}{3}(\sqrt2-1)\int_0^4 z^3\,dz==\frac{2\pi}{3}(2\sqrt2-1)\left[\frac{z^4}{4}\right]_0^4=\boxed{\frac{128\pi}{3}(2\sqrt2-1)}.$$

(b) $I = \iiint_B z\, e^{-(x^2+y^2)}\, dx\,dy\,dz\,,\quad B = \{(x,y,z) \in \mathbb{R}^3 :\ z^2 - 1 \le x^2+y^2 \le \frac{z^2}{2},\ z \ge 0\}$



Solució. En coordenades cilíndriques el domini B es transforma en

B': $\sqrt{2}\,r \le z \le \sqrt{r^2+1}\,,\quad 0 \le r \le 1.$

$z^2 = r^2+1$; intersecció : $r = 1\ (r>0)$
$z^2 = 2r^2$; $z = \pm\sqrt{2}$

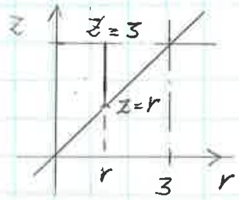
La integral I en coordenades cilíndriques es calcula com:

$$I = \int_0^{2\pi} d\theta \int_0^1 r\, e^{-r^2} dr \int_{\sqrt{2}r}^{\sqrt{r^2+1}} z\, dz = \frac{1}{2}\, 2\pi \int_0^1 r e^{-r^2}(r^2+1-2r^2)\, dr$$

$$= \pi \int_0^1 r(1-r^2)\, e^{-r^2} dr = \left\{ \text{parts: } \begin{array}{l} f = 1-r^2 \Rightarrow f' = -2r \\ g' = r e^{-r^2} \Rightarrow g = -e^{-r^2}/2 \end{array} \right\} =$$

$$= \underbrace{\pi\left(\frac{r^2-1}{2}\right) e^{-r^2}\Big|_0^1}_{\pi/2} - \underbrace{\pi \int_0^1 r e^{-r^2} dr}_{-\frac{\pi}{2}\left[e^{-r^2}\right]_0^1 \,=\, -\frac{\pi}{2}\left(\frac{1}{e}-1\right)} = \cancel{\frac{\pi}{2}} + \frac{\pi}{2e} - \cancel{\frac{\pi}{2}} = \boxed{\frac{\pi}{2e}}$$

(c) $I = \iiint_B (x+y-2z)\, dx\,dy\,dz. \quad B = \{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2 \le z^2,\ 0 \le z \le 3\}.$



Solució. En coordenades cilíndriques el domini B es transforma

en $B' = \{(\theta,r,z) \in [0,2\pi]\times[0,+\infty)\times(-\infty,+\infty),\ 0 \le r \le z,\ 0 \le z \le 3\}$,

d'on la integral es pot calcular com:

$$I = \int_0^{2\pi} d\theta \int_0^3 r\, dr \int_r^3 (r\cos\theta + r\sin\theta - 2z)\, dz = \int_0^{2\pi} d\theta \int_0^3 r\left(rz\cos\theta + rz\sin\theta - z^2\right)\Big|_{z=r}^{z=3} dr$$

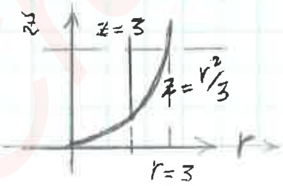
$$= \int_0^{2\pi} d\theta \int_0^3 r\left(3r\cos\theta + 3r\sin\theta - 9 - r^2\cos\theta - r^2\sin\theta + r^2\right) dr$$

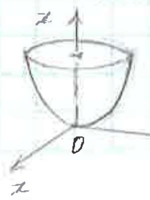
$$= \int_0^{2\pi} d\theta \int_0^3 \left[-9r + 3(\cos\theta+\sin\theta)r^2 - (\cos\theta+\sin\theta-1)r^3\right] dr$$

$$= \int_0^{2\pi} \left[-\frac{81}{2} + (\cos\theta+\sin\theta)\cdot 27 - (\cos\theta+\sin\theta-1)\cdot\frac{81}{4}\right] d\theta = 2\pi\left(-\frac{81}{2}+\frac{81}{4}\right) = \boxed{-\frac{81\pi}{2}}$$

(d) $I = \iiint_B (x^2+y^2)\, dx\,dy\,dz,\quad B = \{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2 \le 3z \le 9\}$

Solució. Les coordenades cilíndriques transformen el domini d'integració B, en B': $\frac{r^2}{3} \le z \le 3$,



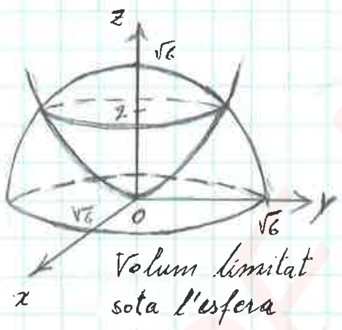


Paraboloide circular.

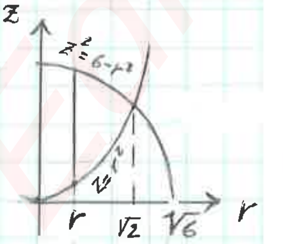
d'on: $I = \int_0^{2\pi} d\theta \int_0^3 r^3 dr \int_{r^2/3}^3 dz = 2\pi \int_0^3 r^3\left(3 - \frac{r^2}{3}\right) dr$

$$= 2\pi\left(\frac{3}{4}r^4 - \frac{r^6}{18}\right)\Big|_0^3 = 2\pi\left(\frac{3\cdot 81}{4} - \frac{3^2\cdot 81}{2\cdot 3^2}\right) = \frac{2\pi}{4}(3\cdot 81 - 2\cdot 81) = \frac{81\pi}{2}$$

(e) $I = \iiint_B z\,dx\,dy\,dz$, $B = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 6,\ x^2+y^2 \leq z,\ z \geq 0\}$.



Volum limitat sota l'esfera $x^2+y^2+z^2=6$ i sobre el paraboloide circular $z=x^2+y^2$:
$x^2+y^2 \leq z \leq \sqrt{6-x^2-y^2}$



$z^2 = 6-r^2$, $z = r^2 \iff r=\sqrt{2},\ z=2$

Solució. El domini B es transforma, per coordenades cilíndriques en

B' : $r^2 \leq z \leq \sqrt{6-r^2}$, $0 \leq \theta \leq 2\pi$,

i la integral I es pot calcular com

$$I = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r\,dr \int_{r^2}^{\sqrt{6-r^2}} z\,dz$$

$$= \frac{2\pi}{2}\int_0^{\sqrt{2}} r(6-r^2-r^4)\,dr = \pi\left(3r^2 - \frac{r^4}{4} - \frac{r^6}{6}\right)\Big|_0^{\sqrt{2}}$$

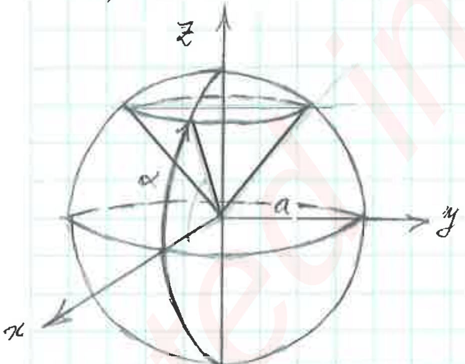
$$= \pi\left(6 - \frac{4}{4} - \frac{4}{3}\right) = \pi\left(5 - \frac{4}{3}\right) = \boxed{\frac{11\pi}{3}}.$$

---

21) Calculeu els volums dels dominis $B \subset \mathbb{R}^3$ definits en coordenades esfèriques, $x = r\cos\varphi\cos\theta$, $y = r\cos\varphi\sin\theta$, $z = r\sin\varphi$ ($0 \leq \theta \leq 2\pi$, $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$, $r \geq 0$) que s'indiquen tot seguit.

Remarca. Recordem que: $\det \dfrac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2\cos\varphi$.

(a) B domini tallat sobre la bola $r \leq a$ pel con $\alpha \leq \varphi \leq \frac{\pi}{2}$ ($a>0$, $0<\alpha<\frac{\pi}{2}$).



Solució:

$$V = \int_0^{2\pi} d\theta \int_\alpha^{\pi/2} \cos\varphi\,d\varphi \int_0^a r^2\,dr = \boxed{\frac{2\pi a^3}{3}(1-\sin\alpha)}$$

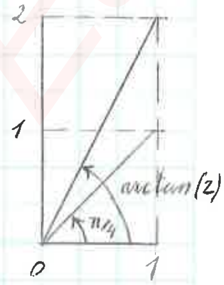
(b) B volum tancat per l'esfera deformada definida per $r = 1 + 0{\cdot}2\sin(8\theta)\sin\varphi$. (Sòlids d'aquesta mena s'utilitzen com a models de tumors.)

Solució. $$V = \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi\,d\varphi \int_0^{1+0.2\sin(8\theta)\sin\varphi} r^2\,dr = \frac{1}{3}\int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi\,(1+0{\cdot}2\ \ (8\theta)\sin\varphi)^3\,d\varphi$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \left(\cos\varphi + 0{\cdot}2\sin(8\theta)\sin\varphi\cos\varphi + 3\cdot 0'2^2\sin^2(8\theta)\sin^2\varphi\cos\varphi + 0'2^3\sin^3(8\theta)\cdot \sin^3\varphi\cos\varphi\right) d\varphi$$

$$= \frac{1}{3}\int_0^{2\pi}\left(2 + 3\cdot 0'04\cdot\frac{2}{3}\sin^2(8\theta)\right) d\theta = \frac{1}{3}\left(4\pi + 0'08\pi\right) = \boxed{1'36\pi}.$$

(c) Analogament calculeu la integral triple $I = \iiint_B \frac{1}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$, on B és la regió del primer octant de $\mathbb{R}^3$ acotada pels cons $\varphi = \pi/4$ i $\varphi = \arctan(2)$ i l'esfera $r = \sqrt{6}$ (Recordeu: $\sin(\arctan(a)) = \frac{a}{\sqrt{1+a^2}}$.)



Solució

$$I = \int_0^{\pi/2} d\theta \int_{\pi/4}^{\arctan(2)} \cos\varphi\, d\varphi \int_0^{\sqrt{6}} \frac{1}{r} r^2\, dr = \frac{\pi}{2}\left[\sin\varphi\right]_{\pi/4}^{\arctan(2)} \cdot \left[\frac{r^2}{2}\right]_0^{\sqrt{6}}$$

$$= \frac{\pi}{2}\cdot\frac{6}{2}\left(\sin(\arctan 2) - \sin\left(\frac{\pi}{4}\right)\right) \overset{(*)}{=} \underline{\overline{\frac{3\pi}{2}\left(\frac{2}{\sqrt{5}} - \frac{\sqrt{2}}{2}\right)}}$$

(*) $\sin(\arctan a) = \frac{\sin(\arctan a)\cos(\arctan a)}{\cos(\arctan a)} = \tan(\arctan a)\,\frac{1}{\sqrt{1+\tan^2(\arctan a)}} = \frac{a}{\sqrt{1+a^2}}$,
$\left(-\frac{\pi}{2} \le a \le \frac{\pi}{2}\right)$

---

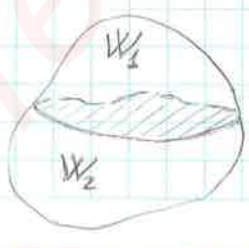
27) Trobeu la massa total del cilindre $V = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le 2,\ 0 \le z \le 3\}$ si la seva densitat és $\rho(x,y,z) = z e^{-z^2}(x^2+y^2)$.

Solució. $m(V) = \iiint_V \rho(x,y,z)\,dx\,dy\,dz \underset{\substack{\text{coordenades}\\ \text{cilíndriques}}}{=} \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r^3\, dr \int_0^3 z e^{-z^2}\, dz =$

$$= 2\pi\left[\frac{r^4}{4}\right]_0^{\sqrt{2}} \cdot \left[-\frac{1}{2}e^{-z^2}\right]_0^3 = \boxed{\pi(1-e^{-9})}$$

---

29) Sigui $W \subset \mathbb{R}^3$ un cos amb densitat de masses $\rho(x,y,z)$. Si dividim W en dues parts, $W = W_1 \cup W_2$, i denotem per $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$ i $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$ els centres de masses de W_1 i W_2 respectivament, demostreu que el centre de masses $(\bar{x}, \bar{y}, \bar{z})$ de W és el mateix que si suposem tota la massa de W_1 concentrada en $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$ i la de W_2 ho està en $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$.



Solució. Introduïm la notació següent $W = W_1 \cup W_2$ (amb $\mu(W_1 \cap W_2) = \iiint_{W_1 \cap W_2} dx\,dy\,dz = 0$), $\vec{r} = (x,y,z)$, $\rho(\vec{r}) = \rho(x,y,z)$,

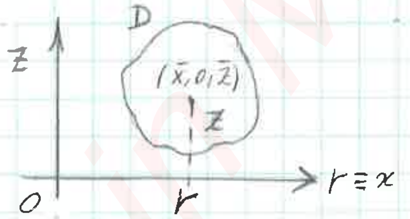
$d\vec{r} = dx\,dy\,dz$. Així, podem escriure, pel CDM del cos W:

$$\vec{R}(W) = \left(X(W), Y(W), Z(W)\right) = \frac{1}{m(W)} \iiint_W \vec{r}\,\rho(\vec{r})\,d\vec{r}$$

$$= \frac{1}{m(W)} \left(m(W_1) \cdot \underbrace{\frac{\iiint_{W_1} \vec{r}\,\rho(\vec{r})\,d\vec{r}}{m(W_1)}}_{\vec{R}(W_1)\,:\ \text{CDM de } W_1} + m(W_2) \cdot \underbrace{\frac{\iiint_{W_2} \vec{r}\,\rho(\vec{r})\,d\vec{r}}{m(W_2)}}_{\vec{R}(W_2)\,:\ \text{CDM de } W_2} \right)$$

$$= \boxed{\frac{m(W_1)\,\vec{R}(W_1) + m(W_2)\,\vec{R}(W_2)}{m(W)}}$$

30) Sigui D un recinte pla contingut en el semipla $\{y=0, x \geq 0\}$ de $\mathbb{R}^3$. Si denotem per $(\bar{x}, 0, \bar{z})$ el centre geomètric de D (i.e. el seu centre de masses si suposem densitat constant igual a 1), demostreu que el volum del domini de revolució W que obtenim si fem girar D entorn de l'eix z és $\text{Volum}(W) = 2\pi\bar{x}\,\text{Àrea}(D)$, on $2\pi\bar{x}$ és la longitud de la circumferència que obtenim en fer girar $(\bar{x}, 0, \bar{z})$. (Indicació: Useu coordenades cilíndriques i observeu que $(\theta, r, z) \in D^* = [0, 2\pi] \times D$.)



Solució.

$\{y=0, x>0\}$, $\rho(x,0,z) = \rho \equiv 1$ (densitat ctnt. = 1)

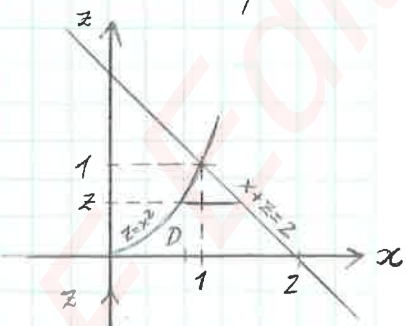
$$V = \int_0^{2\pi} d\theta \underbrace{\iint_D r\,dr\,dz}_{\bar{x}\cdot\text{Àrea}(D)} \overset{(*)}{=} \int_0^{2\pi} d\theta\ \bar{x}\ \text{Àrea}(D) = \boxed{2\pi\,\bar{x}\cdot\text{Àrea}(D)}$$

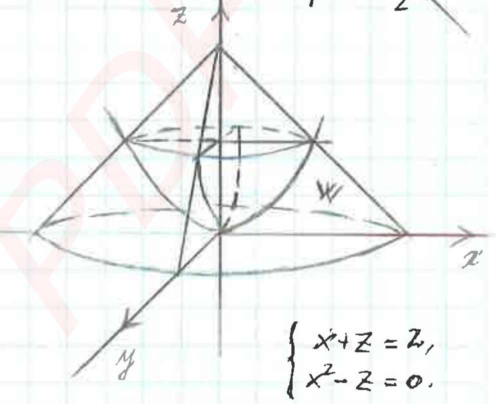
(*) $$\bar{x} = \frac{\iint_D x\,\rho(x,0,z)\,dx\,dz}{\iint_D \rho(x,0,z)\,dx\,dz} \underset{\rho(x,0,z)\equiv 1}{=} \frac{\iint_D x\,dx\,dz}{\iint_D dx\,dz}$$

d'on:
$$\iint_D x\,dx\,dz = \bar{x}\cdot\text{Àrea}(D).$$

31) Apliquem el resultat del problema anterior al càlcul del volum W si prenem

$$D = \{ (x,z) \in \mathbb{R}^2 : z \leq x^2 ;\ x+z \leq 2,\ x \geq 0,\ z \geq 0 \}.$$





Solució.

$$\text{Àrea}(D)\cdot\bar{x} = \int_0^1 dz \int_{\sqrt{z}}^{2-z} x\,dx = \frac{1}{2}\int_0^1 \left[(2-z)^2 - z\right] dz$$

$$= \frac{1}{2}\int_0^1 (4-5z+z^2)\,dz = \frac{1}{2}\left(4z - \frac{5}{2}z^2 + \frac{z^3}{3}\right)\Bigg|_0^1$$

$$= \frac{1}{2}\left(4 - \frac{5}{2} + \frac{1}{3}\right)\Bigg|_0^1 = \frac{24-15+2}{12} = \frac{11}{12}$$

D'on, aplicant la fórmula del problema 30:

$$\text{Volum}(W) = 2\pi\bar{x}\cdot\text{Àrea}(D) = \boxed{\frac{11\pi}{6}}$$

$$\begin{cases} x+z = 2, \\ x^2 - z = 0. \end{cases}$$

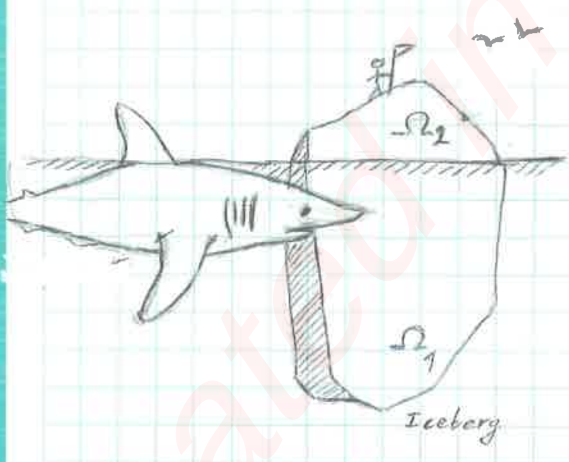
Punt de tall:

$$x^2 + x - 2 = 0 \Leftrightarrow \begin{array}{l} x=-2 \text{ (No)} \\ x = 1 \end{array}$$

per tant el punt de tall resulta:

$$(x,z) = (1,1)$$

32) La densitat de l'aigua és 1 Kg/l i la del gel (aproximadament) 0'9 Kg/l. Pel principi d'Arquimedes, si submergim un bloc de gel en l'aigua el volum d'aigua desplaçat per la part submergida del gel té un pes igual al pes total del gel.



$\Omega = \Omega_1 \cup \Omega_2$

a) Vegeu que la part del gel emergent és el 10% del seu volum.

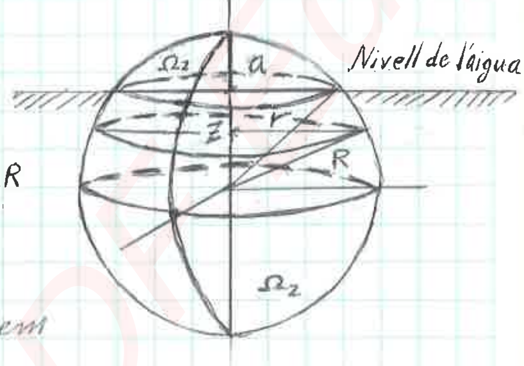
Solució. Si $\rho = 1$ Kg/l és la densitat de l'aigua i $\sigma = 0.9$ Kg/l és la densitat del gel, d'acord amb el principi d'Arquimedes, si Ω_1 és la part submergida, Ω_2 és la part que sobresurt del bloc de gel $\Omega = \Omega_1 \cup \Omega_2$ i $V(\Omega_1), V(\Omega_2), V(\Omega) = V(\Omega_1)+V(\Omega_2)$ ($V(\Omega_1 \cap \Omega_2) = 0$) són els respectius volums, hom té:

$$V(\Omega_1)\,\rho = \sigma\, V(\Omega) \Rightarrow \frac{V(\Omega_1)}{V(\Omega)} = \frac{\sigma}{\rho}$$

i per tant:

$$\frac{V(\Omega_2)}{V(\Omega)} = 1 - \frac{V(\Omega_1)}{V(\Omega)} = 1 - \frac{\sigma}{\rho} = 1 - 0'9 = 0'1 \text{ (10\%)}.$$

(b) Si tenim un bloc de gel esfèric, digueu quina és la part submergida del bloc i quina l'emergent.



Solució. Suposem un bloc esfèric de radi R (veure figura). Volem trobar l'alçada, $0<a<2R$ del casquet esfèric emergent, Ω_2. Per calcular el volum corresponent, $V(\Omega_2)$, apliquem el principi de Cavalieri:

$$r^2 = R^2 - z^2,$$

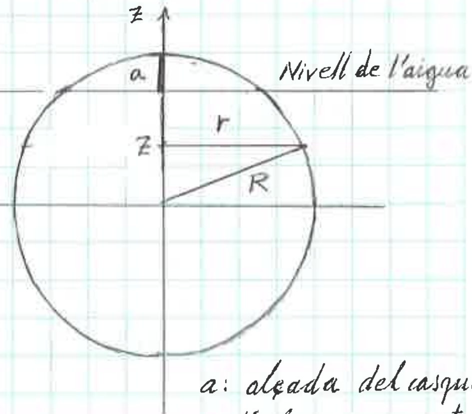
$$S(z) = \pi r^2 = \pi (R^2 - z^2)$$

$$V(\Omega_2) = \int_{R-a}^{R} S(z)\,dz = \int_{R-a}^{R} \pi (R^2 - z^2)\,dz$$

$$= \pi \left(R^2 z - \frac{z^3}{3} \right) \Big|_{R-a}^{R}$$

$$= \pi \left(R^3 - \frac{R^3}{3} - R^3 + R^2 a + \frac{R^3}{3} - R^2 a + R a^2 - \frac{a^3}{3} \right)$$

$$= \pi R^3 \left(\frac{a^2}{R^2} - \frac{1}{3}\frac{a^3}{R^3} \right).$$



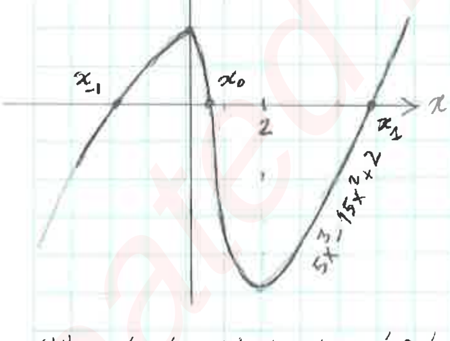
a: alçada del casquet d'esfera emergent.

Llavors, d'acord amb l'apartat anterior:

$$\cancel{\pi R^3} \left(\frac{a^2}{R^2} - \frac{1}{3}\frac{a^3}{R^3} \right) = \frac{1}{10}\,\frac{4}{3}\,\cancel{\pi R^3},$$

i definint $x \equiv a/R$ s'arriba a l'equació: $5x^3 - 15x^2 + 2 = 0$; la qual té tres solucions reals: $x_{-1} < 0 < x_0 < 2 < x_1$. Com $0 < x = a/R < 2$, la solució que busquem és

$$x_0 = 0.391600211318183^{(*)}.$$



Així doncs, l'alçada del casquet d'esfera emergent, en termes del radi R ve donada per:

$$\boxed{a = 0.391600211318183R}$$

(*) Càlcul amb MATLAB/Octave:

```
>> f=inline('5*x.^3 - 15*x.^2 +2','x');
>> opt=optimset('TolFun',1e-16,'TolX',0);
>> x0 = 1.5
>> [x0,f0] = fsolve(f,x0,opt)
x0 = 0.391600211318183
f0 = -4.440892098500063e-16
```

33) La densitat de població d'una certa ciutat pot aproximar-se per la funció $\rho(x,y) = 4000\, e^{-0.01(x^2+y^2)}$ si $x^2+y^2 \leq 49$ i $\rho(x,y) = 0$ altrament, on x, y es mesuren en Km.

(a) Quina és la població de la ciutat.

Solució. Treballant en coordenades polars: $\tilde{\rho}(r) = \rho(r\cos\theta, r\sin\theta) = 4000\, e^{-r^2/100}$;

$$P = \iint_{\mathbb{R}^2} \rho(x,y)\,dx\,dy = \iint_{\substack{B(0,0)\\ R=7}} \rho(x,y)\,dx\,dy = \int_0^{2\pi} d\theta \int_0^7 r\tilde{\rho}(r)\,dr$$

$$= \int_0^{2\pi} d\theta \int_0^7 4000\, r\, e^{-r^2/100}\,dr = 2\pi \cdot 4\cdot 10^3 \left(-50\, e^{-r^2/100}\right)\Big|_0^7 = 4\pi\cdot 10^5\left(1-e^{-\frac{49}{100}}\right) \cong \boxed{486.788'0}$$

(b) Quina és la distància $0 < R < 7$ al centre geogràfic de la ciutat de forma que el 50 % de la població viu a una distància més petita o igual que R del centre?

Solució.

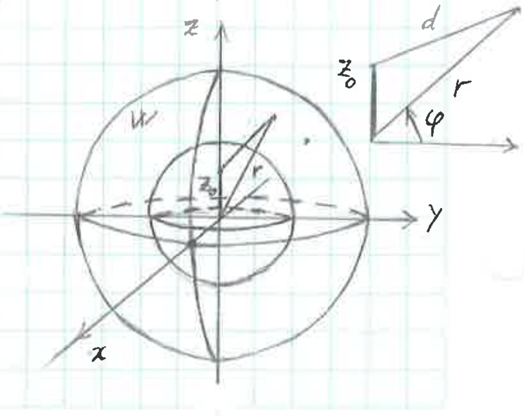
$$\int_0^{2\pi} d\theta \int_0^R r\tilde{\rho}(r)\,dr = 8000\pi \int_0^R r\, e^{-r^2/100}\,dr = 4\cdot 10^5\pi\left(1-e^{-R^2/100}\right)$$

$$= 0'5\cdot 4\cdot 10^5\pi\left(1-e^{-49/100}\right),$$

d'on, aïllant R:

$$R = 10\sqrt{-\ln\left(1-\frac{1-e^{-49/100}}{2}\right)} = 10\sqrt{-\ln\left(\tfrac{1}{2}+\tfrac{1}{2}e^{-49/100}\right)} \cong \boxed{4.640 \text{ Km}}$$

35) Trobeu el potencial gravitatori $V(0,0,z_0)$ generat pel sòlid W comprès entre dues esferes concèntriques de radis $a<b$, $W = \{(x,y,z)\in\mathbb{R}^3 : a^2 \leq x^2+y^2+z^2 \leq b^2\}$, suposant densitat constant igual a ρ. Per simplificar els càlculs considereu només valors z_0 amb $0 < z_0 < a$ o $z_0 > b$.



Solució. $\rho(x,y,z) \equiv \rho$ ctnt.

- $0 < z_0 < a$

$$V(0,0,z_0) = -G \iiint_W \frac{\overset{\rho}{\overline{\rho(x,y,z)}}\,dx\,dy\,dz}{d\big((0,0,z_0),(x,y,z)\big)} =$$

pàg. següent ...

$$= -G\rho \int_0^{2\pi} d\theta \int_a^b r^2 dr \int_{-\pi/2}^{\pi/2} \frac{\cos\varphi}{\sqrt{z_0^2+r^2-2rz_0\sin\varphi}}\, d\varphi = -G\rho \int_0^{2\pi} d\theta \int_a^b r^2 \left[-\frac{\sqrt{z_0^2+r^2-2rz_0\sin\varphi}}{rz_0}\right]_{\varphi=-\pi/2}^{\varphi=\pi/2} d\varphi$$

$$= \frac{G\rho}{z_0} 2\pi \int_a^b r\left(|z_0-r| - z_0 - r\right) dr = -\frac{G\rho}{\cancel{z_0}} 4\pi \cancel{z_0} \int_a^b r\,dr = \boxed{-2\pi G(b^2-a^2)}$$

$0 < z_0 < a < r < b$

$\Rightarrow z_0 - r < a - r < 0$

$\forall\, a < r < b.$

(b) $z_0 > b$:

$$V(0,0,z_0) = \dots = \frac{G\rho}{z_0} 2\pi \int_a^b r\left(|z_0-r| - z_0 - r\right) dr = -\frac{G\rho}{z_0} 2\pi \int_a^b 2r^2 dr = \boxed{-\frac{4\pi G\rho}{3z_0}(b^3-a^3)}$$

$0 < a < r < b < z_0$

$\Rightarrow z_0 - r > b - r > 0$

$\forall\, a < r < b.$