Ideas/Defs

Let:

- $E \subset \mathbb{R}$
- $p \in E'$ (i.e. $\forall \delta > 0$, $N'_\delta(p) \cap E \neq \emptyset$)
- $f \colon E \to \mathbb{R}$ be a function.

Define functions $s, i \colon (0, \infty) \to \mathbb{R}$ by

$$s(\delta) = \sup \left\{ f(x) : x \in N'_\delta(p) \cap E \right\} \qquad \text{which is an increasing function of } \delta$$
$$i(\delta) = \inf \left\{ f(x) : x \in N'_\delta(p) \cap E \right\} \qquad \text{which is an decreasing function of } \delta \ .$$

Now define (and examine)

$$\limsup_{x\to p} f(x) \overset{\text{notation}}{=} \overline{\lim_{x\to p}}\, f(x) \overset{\text{def}}{=} \lim_{\delta\to 0^+} s(\delta) = \lim_{\delta\to 0^+} \sup_{x\in N'_\delta(p)\cap E} f(x) \overset{\text{Thm 2.1.9}}{=} \inf_{\delta>0} \sup_{x\in N'_\delta(p)\cap E} f(x) \in \widehat{\mathbb{R}}$$
$$\liminf_{x\to p} f(x) \overset{\text{notation}}{=} \underline{\lim}_{x\to p}\, f(x) \overset{\text{def}}{=} \lim_{\delta\to 0^+} i(\delta) = \lim_{\delta\to 0^+} \inf_{x\in N'_\delta(p)\cap E} f(x) \overset{\text{Thm 2.1.9}}{=} \sup_{\delta>0} \inf_{x\in N'_\delta(p)\cap E} f(x) \in \widehat{\mathbb{R}} \ .$$

Claim 1. $\underline{\lim}_{x\to p} f(x) \le \overline{\lim}_{x\to p} f(x)$

Claim 2. If f is bounded on $N'_\delta(p) \cap E$ for some $\delta > 0$, then $\underline{\lim}_{x\to p} f(x)$, $\overline{\lim}_{x\to p} f(x) \in \mathbb{R}$.

Claim 3. Let $\overline{\lim}_{x\to p} f(x) \in \mathbb{R}$.

$$\overline{\lim_{x\to p}}\, f(x) \le \beta \iff \forall \varepsilon > 0 \ \ \exists \delta > 0 \ \ \forall x \in N'_\delta(p) \cap E \ , \ f(x) < \beta + \varepsilon$$

Claim 4. Let $\overline{\lim}_{x\to p} f(x) \in \mathbb{R}$.

$$\overline{\lim_{x\to p}}\, f(x) \ge \beta \iff \forall \varepsilon > 0 \ \ \forall \delta > 0 \ \ \exists x \in N'_\delta(p) \cap E \ , \ \beta - \varepsilon < f(x)$$

Claim 5. Let $\underline{\lim}_{x\to p} f(x) \in \mathbb{R}$.

$$\underline{\lim}_{x\to p}\, f(x) \ge \alpha \iff \forall \varepsilon > 0 \ \ \exists \delta > 0 \ \ \forall x \in N'_\delta(p) \cap E \ , \ \alpha - \varepsilon < f(x)$$

Claim 6. Let $\underline{\lim}_{x\to p} f(x) \in \mathbb{R}$.

$$\underline{\lim}_{x\to p}\, f(x) \le \alpha \iff \forall \varepsilon > 0 \ \ \forall \delta > 0 \ \ \exists x \in N'_\delta(p) \cap E \ , \ f(x) < \alpha + \varepsilon$$

Claim 7. $\underline{\lim}_{x\to p} f(x) = -\overline{\lim}_{x\to p} (-f)(x)$

Claim 8. $\lim_{x\to p} f(x)$ exists in the extended sense (i.e. in $\widehat{\mathbb{R}}$) $\iff$ $\underline{\lim}_{x\to p} f(x) = \overline{\lim}_{x\to p} f(x)$.
In this case, $\underline{\lim}_{x\to p} f(x) = \lim_{x\to p} f(x) = \overline{\lim}_{x\to p} f(x)$.

Corollary. Let $\overline{\lim}_{x\to p} f(x) \in \mathbb{R}$. Then $\overline{\lim}_{x\to p} f(x) = \beta$ if and only if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in N'_\delta(p) \cap E \quad , \quad f(x) < \beta + \varepsilon \tag{3}$$

and

$$\forall \varepsilon > 0 \quad \forall \delta > 0 \quad \exists x \in N'_\delta(p) \cap E \quad , \quad \beta - \varepsilon < f(x) \tag{4}$$

Corollary. Let $\underline{\lim}_{x\to p} f(x) \in \mathbb{R}$. Then $\underline{\lim}_{x\to p} f(x) = \alpha$ if and only if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in N'_\delta(p) \cap E \quad , \quad \alpha - \varepsilon < f(x) \tag{5}$$

and

$$\forall \varepsilon > 0 \quad \forall \delta > 0 \quad \exists x \in N'_\delta(p) \cap E \quad , \quad f(x) < \alpha + \varepsilon \tag{6}$$

Claim 9. There exists a sequence $\{b_k\}_{k=1}^{\infty}$ s.t.

(1) $b_k \in E \setminus \{p\} \qquad \forall k \in \mathbb{N}$

(2) $\lim_{k\to\infty} b_k = p$

(3) $\lim_{k\to\infty} f(b_k) \;=\; \overline{\lim}_{x\to p} f(x)$.

Also, there exists a sequence $\{a_k\}_{k=1}^{\infty}$ s.t.

(1) $a_k \in E \setminus \{p\} \qquad \forall k \in \mathbb{N}$

(2) $\lim_{k\to\infty} a_k = p$

(3) $\lim_{k\to\infty} f(a_k) \;=\; \underline{\lim}_{x\to p} f(x)$.

Claim 10. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence s.t.

(1) $x_k \in E \setminus \{p\} \qquad \forall k \in \mathbb{N}$

(2) $\lim_{k\to\infty} x_k = p$

(3) $\lim_{k\to\infty} f(x_k)$ exists in the extended sense.

Then

$$\underline{\lim}_{x\to p} f(x) \overset{(2)}{\leq} \lim_{k\to\infty} f(x_k) \overset{(1)}{\leq} \overline{\lim}_{x\to p} f(x) \ .$$

Claim 11.

$$\overline{\lim}_{x\to p} f(x) \;=\; \sup\left\{\lim_{k\to\infty} f(x_k) : x_k \in E \setminus \{p\} \ , \ \lim_{k\to\infty} x_k = p \ , \ \lim_{k\to\infty} f(x_k) \ \text{exists in } \widehat{R}\right\}$$

$$\underline{\lim}_{x\to p} f(x) \;=\; \inf\left\{\lim_{k\to\infty} f(x_k) : x_k \in E \setminus \{p\} \ , \ \lim_{k\to\infty} x_k = p \ , \ \lim_{k\to\infty} f(x_k) \ \text{exists in } \widehat{R}\right\}$$

Claim 12.

$$\underline{\lim}_{x\to p} f(x) + \underline{\lim}_{x\to p} g(x) \overset{(2)}{\leq} \underline{\lim}_{x\to p} (f+g)(x) \overset{(\text{Claim 1})}{\leq} \overline{\lim}_{x\to p} (f+g)(x) \overset{(1)}{\leq} \overline{\lim}_{x\to p} f(x) + \overline{\lim}_{x\to p} g(x)$$

even more ....

$$\underline{\lim}_{x\to p} f(x) + \underline{\lim}_{x\to p} g(x) \overset{(2)}{\leq} \underline{\lim}_{x\to p} (f+g)(x) \overset{(4)}{\leq} \underline{\lim}_{x\to p} f(x) + \overline{\lim}_{x\to p} g(x) \overset{(3)}{\leq} \overline{\lim}_{x\to p} (f+g)(x) \overset{(1)}{\leq} \overline{\lim}_{x\to p} f(x) + \overline{\lim}_{x\to p} g(x)$$