On Guillotine Cuts of Boundary Rectangles

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Abstract

Let $\mathcal{R}$ be a set of n pairwise disjoint axis-aligned rectangles, each having at least one side contained in the boundary of a given rectangular domain. In this note, we prove that any set $\mathcal{R}$ as above has a subset of at least $\frac{3}{4}n - O(1)$ rectangles that are separable by a straight guillotine partition of the plane, and that this bound is tight.

1 Introduction

A *guillotine partition* of the plane is a subdivision of the plane into (possibly unbounded) cells using the following recursive strategy: We start with a single subdivision, consisting of the whole plane as the unique cell. At each step, we take a cell C of the current subdivision and a straight line ℓ through the interior of C, and subdivide C into the two new subcells $C_1 = C \cap \ell^+$ and $C_2 = C \cap \ell^-$, losing cell C, where ℓ^+ and ℓ^- are the two closed half-planes bounded by ℓ, respectively. A set $\mathcal{R}^*$ of k pairwise disjoint rectangles is *guillotine-separable* if there exists a guillotine partition P of k cells that separates $\mathcal{R}^*$. That is, each cell of P contains one rectangle of $\mathcal{R}^*$ (see Figure 1a and Figure 1b). In this paper, all rectangles are considered axis-aligned and close.

Pach and Tardos [8] wrote that "it seems plausible" that there exists a constant c such that any set $\mathcal{R}$ of n pairwise disjoint rectangles in the plane has a guillotine-separable subset of at least $c \cdot n$ rectangles. They proved this statement for $c = \Omega(1/\log n)$, not a constant, and considered in the proof axis-aligned guillotine partitions. We say that a guillotine partition is *axis-aligned* if it is built with axis-parallel lines, which subdivide the plane into (possibly unbounded) rectangular cells. Abed et al. [1] proved that the statement is true for axis-aligned guillotine

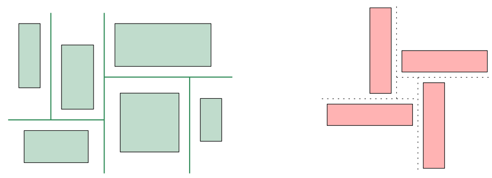

Figure 1: (a) 6 guillotine-separable rectangles. (b) 4 rectangles that are not guillotine-separable.

partitions when $\mathcal{R}$ is a set of squares (with $c = 1/81$, recently improved to $1/40$ [5]). More importantly, they noted that the existence of such a constant c for any set of pairwise disjoint rectangles (when restricted to axis-aligned guillotine partitions) implies a $O(n^5)$-time c-approximation algorithm for the Maximum Independent Set of Rectangles (MISR) problem. The MIRS problem is a fundamental NP-hard combinatorial optimization problem defined as: Given a set of n rectangles, finding a subset of them of maximum cardinality such that the rectangles in the subset are pairwise disjoint.

Recently, Mitchell [7] proved that any set $\mathcal{R}$ of n pairwise disjoint rectangles in a rectangular domain, has a subset $\mathcal{R}'$ of size at least $n/10$, for which there exists a hierarchical rectilinear cut of the domain such that any leaf cell contains exactly one rectangle of $\mathcal{R}'$, and every segment of every cut penetrates at most two rectangles of $\mathcal{R}'$. Every cut is a polyline made of $O(1)$ axis-aligned segments, so that every cell is an orthogonal polygon obtained by removing a set of up to four disjoint subrectangles from a given rectangle R, with each subrectangle containing exactly one of the four corners of R. Base on this, Mitchell [7] gave the first polynomial-time constant approximation algorithm to the MIRS problem, with $O(n^{34})$ running time and $1/10$ approximation factor.

We consider that all guillotine partitions are axis-aligned, and study the next open question:

Question 1 *Do there exist constants $c \in (0, 1]$ and $n_0 > 0$, such that any set of $n \geq n_0$ pairwise disjoint rectangles in the plane has a guillotine-separable subset of at least $c \cdot n$ rectangles?*

*Email: pablo.perez.l@usach.cl. Research supported by DICYT 041933PL Vicerrectoría de Investigación, Desarrollo e Innovación USACH (Chile), and Programa Regional STICAMSUD 19-STIC-02.

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This work has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 734922.

We have several motivations to study Question 1: First, if it is true, then we will directly obtain a $O(n^5)$-time c-approximation algorithm for the MISR problem. Second, we consider this question interesting by itself, because it explores the combinatorial distribution of finite sets of pairwise disjoint rectangles. Finally, as the topic of this paper, we can impose restrictions to the set of rectangles such as: The rectangles are contained in a rectangular region with a side in the boundary of the region [2, 6], or the rectangles intersect the same diagonal line [3, 4].

Ahmadinejad and Zarrabi-Zadeh [2], and Kong et al. [6], studied the MISR problem when given a rectangular domain $\mathcal{D}$, each of the n rectangles of the input has at least one side contained in the boundary of $\mathcal{D}$, solving the problem exactly in $O(n^4)$ time. We consider Question 1 in this setting. That is, we have n pairwise disjoint rectangles, each having at least one side contained in the boundary of the domain $\mathcal{D}$. We call such a set a *boundary* set of rectangles (see Figure 2a). First, we observe that Question 1 is true for $c = 1/4$. Namely, there always exists one of the four sides of $\mathcal{D}$ such that at least $n/4$ of the rectangles have a side contained in that side of $\mathcal{D}$. This subset of $k \geq n/4$ rectangles is guillotine-separable by the guillotine partition induced by $k-1$ parallel lines separating the k rectangles. Unless otherwise specified, all lines in this paper are considered axis-parallel. Second, we prove in Section 2 that there always exists a subset of at least $\frac{3}{4}n - O(1)$ rectangles that are guillotine-separable, and that this bound is tight.

2 Boundary rectangles

Let $\mathcal{D}$ be a rectangular domain, and let $\mathcal{R}$ be a boundary set of n rectangles in $\mathcal{D}$. Let $\mathcal{R}_t$ (resp. $\mathcal{R}_b$, $\mathcal{R}_\ell$, and $\mathcal{R}_r$) be the subset of $\mathcal{R}$ with the rectangles whose top (resp. bottom, left, and right) side is contained in the top (resp. bottom, left, and right) side of $\mathcal{D}$. Note that $\mathcal{R}_t$, $\mathcal{R}_b$, $\mathcal{R}_\ell$, and $\mathcal{R}_r$ are all guillotine-separable by taking the guillotine partition induced by parallel lines separating the rectangles. For every rectangle R, let $top(R)$, $bottom(R)$, $left(R)$, and $right(R)$ denote the lines through the top, bottom, left, and right sides of R, respectively.

We say that $\mathcal{R}$ is a *corner* set of rectangles if each rectangle of $\mathcal{R}$ has a side contained in the union of two fixed adjacent sides of $\mathcal{D}$. We say that $\mathcal{R}$ is a *parallel* set of rectangles if each rectangle of $\mathcal{R}$ has a side contained in the union of two parallel sides of $\mathcal{D}$.

Lemma 2 *If $\mathcal{R}$ is a corner set of rectangles, then $\mathcal{R}$ is guillotine-separable.*

Proof. Assume w.l.o.g. that $\mathcal{R} = \mathcal{R}_\ell \cup \mathcal{R}_t$. We proceed by induction. If $|\mathcal{R}_\ell| = 0$ or $|\mathcal{R}_t| = 0$, then all rectangles of $\mathcal{R}$ are trivially guillotine-separable. So, consider that $|\mathcal{R}_\ell| \geq 1$ and $|\mathcal{R}_t| \geq 1$. Let A be the bottommost rectangle of $\mathcal{R}_\ell$. If $top(A)$ does not cut (the interior of) any rectangle of $\mathcal{R}_t$ (see Figure 2b), then by the induction hypothesis we can assume that $\mathcal{R}\setminus\{A\}$ is guillotine-separable by some guillotine partition G. Since $top(A)$ separates A from all rectangles in $\mathcal{R}\setminus\{A\}$, by combining $top(A)$ with G we can create a guillotine partition that separates $\mathcal{R}$. Otherwise, assume that $top(A)$ does cut some rectangle of $\mathcal{R}_t$, and let B be the leftmost rectangle of $\mathcal{R}_t$ cut by $top(A)$ (see Figure 2c). Let $\mathcal{R}' \subseteq \mathcal{R}_t$ be the set of the rectangles (including B) to the right of $left(B)$. By the induction hypothesis, we have that $\mathcal{R} \setminus (\mathcal{R}' \cup \{A\})$ is guillotine-separable by some guillotine partition G'. Since $left(B)$ separates all rectangles in $\mathcal{R}'$ from all in $\mathcal{R} \setminus \mathcal{R}'$, and also $top(A)$ separates A from all rectangles in $\mathcal{R} \setminus (\mathcal{R}' \cup \{A\})$, by combining $left(B)$, the part of $top(A)$ to the left of $left(B)$, and G' we can create a guillotine partition that separates $\mathcal{R}$. $\square$

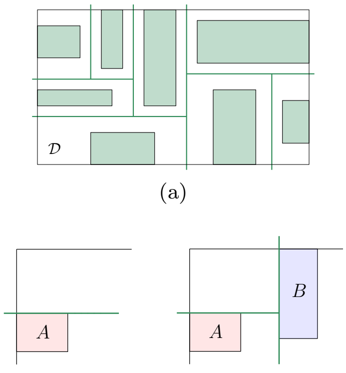

Figure 2: (a) 8 guillotine-separable boundary rectangles. (b-c) Proof of Lemma 2.

Lemma 3 *If $\mathcal{R}$ is a parallel set of rectangles, then $\mathcal{R}$ has a guillotine-separable subset of cardinality at least $\frac{3}{4}|\mathcal{R}|$. Furthermore, this lower bound is tight.*

Proof. Assume w.l.o.g. that $\mathcal{R} = \mathcal{R}_t \cup \mathcal{R}_b$. We proceed by induction, based on the idea of sweeping the elements of $\mathcal{R}$ from left to right. If $|\mathcal{R}_t| = 0$ or $|\mathcal{R}_b| = 0$, then all rectangles of $\mathcal{R}$ are trivially guillotine-separable. So, consider that $|\mathcal{R}_t| \geq 1$ and $|\mathcal{R}_b| \geq 1$. Let $A \in \mathcal{R}$ be the rectangle such that $right(A)$ is leftmost among all rectangles of $\mathcal{R}$, and assume w.l.o.g. that $A \in \mathcal{R}_b$. Let $n = |\mathcal{R}|$. The proof follows into the following seven disjoint cases, enumerated from (a) to (g) (see Figure 3):

(a) $right(A)$ does not cut any rectangle in $\mathcal{R}_t$ (see Figure 3a): All rectangles in $\mathcal{R} \setminus \{A\}$ are to the right of $right(A)$. Then, by the inductive hypothesis, we have that $\mathcal{R}\setminus\{A\}$ has a guillotine-separable subset $\mathcal{R}'$ such that $|\mathcal{R}'| \geq \frac{3}{4}|\mathcal{R} \setminus \{A\}| = \frac{3}{4}(n-1)$. Combining $right(A)$ with the guillotine partition that separates

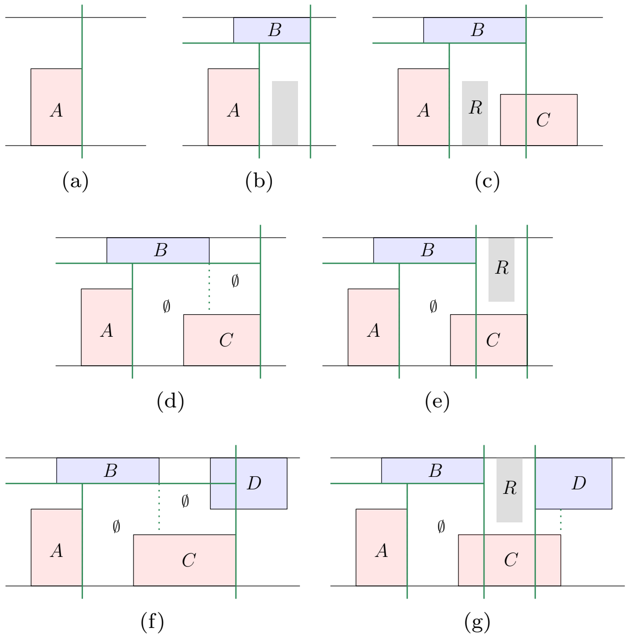

Figure 3: Proof of Lemma 3.

$\mathcal{R}'$, we have that $\{A\} \cup \mathcal{R}'$ is guillotine-separable, and $|\{A\} \cup \mathcal{R}'| = 1 + |\mathcal{R}'| \geq 1 + \frac{3}{4}(n-1) \geq \frac{3}{4}n$.

(b) $right(A)$ cuts a rectangle $B \in \mathcal{R}_t$, and $right(B)$ does not cut any rectangle in $\mathcal{R}_b$ (see Figure 3b): Let $\mathcal{R}' \subseteq \mathcal{R}$ denote the rectangles to the left of $right(B)$, which includes A and B. Let $t = |\mathcal{R}'| \geq 2$. Observe that B must be the leftmost rectangle in $\mathcal{R}_t$, and also the unique rectangle of $\mathcal{R}_t$ that is in $\mathcal{R}'$. Furthermore, $bottom(B)$ separates B from all rectangles in $\mathcal{R}' \setminus \{B\}$. Hence, the t rectangles of $\mathcal{R}'$ are guillotine-separable. By the inductive hypothesis, the $n - t$ rectangles of $\mathcal{R} \setminus \mathcal{R}'$ have a guillotine-separable subset $\mathcal{R}_2$ of at least $\frac{3}{4}(n - t)$ rectangles. Combining $right(B)$ with the guillotine partitions of $\mathcal{R}'$ and $\mathcal{R}_2$, respectively, we obtain that $\mathcal{R}' \cup \mathcal{R}_2$ is guillotine-separable, and satisfies $|\mathcal{R}' \cup \mathcal{R}_2| = |\mathcal{R}'| + |\mathcal{R}_2| \geq t + \frac{3}{4}(n-t) \geq \frac{3}{4}n$.

(c) $right(A)$ cuts a rectangle $B \in \mathcal{R}_t$, $right(B)$ cuts a rectangle $C \in \mathcal{R}_b$, and there is a rectangle $R \in \mathcal{R}_b$ located between $right(A)$ and $left(C)$ (see Figure 3c): Let $\mathcal{R}' \subset \mathcal{R}$ be the set of the rectangles to the left of $right(B)$, which satisfies $A, B, R \in \mathcal{R}'$ and are guillotine-separable (using arguments similar to those of previous cases), and let $t = |\mathcal{R}'| \geq 3$. By the inductive hypothesis, the $n - t - 1$ rectangles to the right of $right(B)$ has a guillotine-separable subset $\mathcal{R}_c$ such that $|\mathcal{R}_c| \geq \frac{3}{4}(n - t - 1)$. Note that $\mathcal{R}' \cup \mathcal{R}_c$ is guillotine-separable, since $right(B)$ separates all rectangles in $\mathcal{R}'$ from all rectangles in $\mathcal{R}_c$, and satisfies $|\mathcal{R}' \cup \mathcal{R}_c| = |\mathcal{R}'| + |\mathcal{R}_c| \geq t + \frac{3}{4}(n-t-1) = \frac{3}{4}n + t/4 - 3/4 \geq \frac{3}{4}n$.

(d) $right(A)$ cuts a rectangle $B \in \mathcal{R}_t$, $right(B)$ cuts a rectangle $C \in \mathcal{R}_b$, and A, B, C are the only rectangles of $\mathcal{R}$ to the left of $right(C)$ (see Figure 3d): $\{A, B, C\}$ is guillotine-separable, and by the inductive hypothesis the $n - 3$ rectangles to the right of $right(C)$ have a guillotine-separable subset $\mathcal{R}_d$ such that $|\mathcal{R}_d| \geq \frac{3}{4}(n-3)$. We have that $\{A, B, C\} \cup \mathcal{R}_d$ is guillotine-separable and satisfies $|\{A, B, C\} \cup \mathcal{R}_d| \geq 3 + \frac{3}{4}(n-3) \geq \frac{3}{4}n$.

(e) $right(A)$ cuts a rectangle $B \in \mathcal{R}_t$, $right(B)$ cuts a rectangle $C \in \mathcal{R}_b$, there is no rectangle in $\mathcal{R}_b$ located between A and C from left to right, $right(C)$ does not cut any rectangle in $\mathcal{R}_t$, and there exists at least a rectangle $R \in R_t$ located between $right(B)$ and $right(C)$ (see Figure 3e): Let $\mathcal{R}' \subseteq \mathcal{R}$ be the set of the rectangles to the left of $right(C)$ without including C, and let $t = |\mathcal{R}'|$. We have that $A, B, R \in \mathcal{R}'$, thus $t \geq 3$. Furthermore, $\mathcal{R}'$ is guillotine-separable. By the inductive hypothesis, the $n - t - 1$ rectangles of $\mathcal{R}$ to the right of $right(C)$ have a guillotine-separable subset $\mathcal{R}_e$ such that $|\mathcal{R}_e| \geq \frac{3}{4}(n - t - 1)$. We finally have that $\mathcal{R}' \cup \mathcal{R}_e$ is guillotine-separable and satisfies $|\mathcal{R}' \cup \mathcal{R}_e| \geq t + \frac{3}{4}(n-t-1) = \frac{3}{4}n + t/4 - 3/4 \geq \frac{3}{4}n$.

(f) $right(A)$ cuts a rectangle $B \in \mathcal{R}_t$, $right(B)$ cuts a rectangle $C \in \mathcal{R}_b$, $right(C)$ cuts a rectangle $D \in \mathcal{R}_t$, and A, B, C are the only rectangles of $\mathcal{R}$ to the left of $right(C)$ (see Figure 3f): $\{A, B, C\}$ is guillotine-separable, and by the inductive hypothesis, the $n - 4$ rectangles to the right of $right(C)$ have a guillotine-separable subset $\mathcal{R}_f$ such that $|\mathcal{R}_f| \geq \frac{3}{4}(n-4)$. We have that $\{A, B, C\} \cup \mathcal{R}_f$ is guillotine-separable and satisfies $|\{A, B, C\} \cup \mathcal{R}_f| \geq 3 + \frac{3}{4}(n-4) = \frac{3}{4}n$.

(g) $right(A)$ cuts a rectangle $B \in \mathcal{R}_t$, $right(B)$ cuts a rectangle $C \in \mathcal{R}_b$, $right(C)$ cuts a rectangle $D \in \mathcal{R}_t$, there is no rectangle of $\mathcal{R}_b$ located between $right(A)$ and $left(C)$, and there is at least a rectangle $R \in \mathcal{R}_t$ located between $right(B)$ and $left(D)$ (see Figure 3g): Let $\mathcal{R}' \subset \mathcal{R}$ be the set of rectangles to the left of $left(D)$, and let $t = |\mathcal{R}'|$. Since $A, B, R \in \mathcal{R}'$, we have $t \geq 3$. Using $right(B)$ as the first cut, we have that $\mathcal{R}'$ is guillotine-separable. Furthermore, by the inductive hypothesis, the $n-t-1$ rectangles to the right of $left(D)$ have a guillotine-separable subset $\mathcal{R}_g$ such that $|\mathcal{R}_g| \geq \frac{3}{4}(n-t-1)$. We have that $\mathcal{R}' \cup \mathcal{R}_g$ is guillotine-separable, and satisfies $|\mathcal{R}' \cup \mathcal{R}_g| \geq t + \frac{3}{4}(n-t-1) \geq \frac{3}{4}n$.

Hence, the first part of the lemma follows. To see that the bound $\frac{3}{4}n$ is tight, refer to Figure 4. We have $n/4$ groups of 4 rectangles each. In each group, at most 3 rectangles are separated by any guillotine partition. Thus, any guillotine partition will separate at most $\frac{3}{4}n$ rectangles. □

Theorem 4 *If $\mathcal{R}$ is a boundary set of rectangles, then $\mathcal{R}$ has a guillotine-separable subset of size at least $\frac{3}{4}|\mathcal{R}| - O(1)$. Furthermore, this lower bound in the cardinality is tight, up to the additive term $O(1)$.*

Proof. Assume w.l.o.g. that none of the sets $\mathcal{R}_t$, $\mathcal{R}_b$, $\mathcal{R}_\ell$, and $\mathcal{R}_r$ is empty. So, let L and R be the rectan-

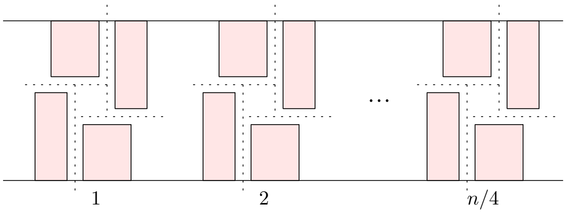

Figure 4: Proof of Lemma 3.

gles of maximum width in $\mathcal{R}_\ell$ and $\mathcal{R}_r$, respectively. Similarly, let T and B be the rectangles of maximum height in $\mathcal{R}_t$ and $\mathcal{R}_b$, respectively. Further assume w.l.o.g. that $L \neq R$ and $T \neq B$. The proof is divided into the following two disjoint cases:

(a) $right(L)$ is to the left of $left(R)$, or $bottom(T)$ is above $top(B)$: By symmetry, assume w.l.o.g. that $right(L)$ is to the left of $left(R)$ (see Figure 5a). Note that this case generalizes the cases in which $\mathcal{R}_t$, $\mathcal{R}_b$, $\mathcal{R}_\ell$, or $\mathcal{R}_r$ is empty, $L = R$ or $T = B$. Let $\mathcal{R}_1 \subset \mathcal{R}$ be the rectangles to the left of $right(L)$, where $L \in \mathcal{R}_1$. The rectangles of $\mathcal{R}_1$ above L are a corner set of rectangles, then guillotine-separable by Lemma 2. Similarly, the rest of the rectangles of $\mathcal{R}_1$, those below L, are a corner set of rectangles, then guillotine-separable. Hence, $\mathcal{R}_1$ is guillotine-separable. Similarly, $\mathcal{R}_3$ is guillotine-separable, where $\mathcal{R}_3 \subset \mathcal{R}$ is the set of the rectangles to the right of $left(R)$. Let $\mathcal{R}_2$ be the set of rectangles between $right(L)$ and $left(R)$, which is a parallel set of rectangles, and then has a guillotine-separable subset $\mathcal{R}'_2$ of cardinality at least $\frac{3}{4}|\mathcal{R}_2|$, by Lemma 3. By the lines $right(L)$ and $left(R)$, we have that $\mathcal{R}_1 \cup \mathcal{R}'_2 \cup \mathcal{R}_3$ is guillotine-separable. Let $t = |\mathcal{R}_1 \cup \mathcal{R}_3| \geq 2$. Note that $|\mathcal{R}_2| \geq n - t - 4$, since lines $right(L)$ and $left(R)$ cut at most four rectangles of $\mathcal{R}$. We then obtain that $|\mathcal{R}_1 \cup \mathcal{R}'_2 \cup \mathcal{R}_3| = |\mathcal{R}_1 \cup \mathcal{R}_3| + |\mathcal{R}'_2| \geq t + \frac{3}{4}|\mathcal{R}_2| \geq t + \frac{3}{4}(n - t - 4) = \frac{3}{4}n + t/4 - 3 \geq \frac{3}{4}n - 5/2$.

(b) $right(L)$ is to the right of $left(R)$, and $bottom(T)$ is below $top(B)$ (see Figure 5b): The main observation is that there are at most 4 rectangles of $\mathcal{R}$ which can be cut by more than one line among $right(L)$, $left(R)$, $bottom(T)$, and $top(B)$ (the rectangles denoted L', R', T', and B' in the figure). Furthermore, each of these lines cuts at most two rectangles among L', R', T', and B'. Hence, there must be a line among these four ones that cuts at most $2 + (n-4)/4 = n/4 + 1$ rectangles of $\mathcal{R}$. Assume w.l.o.g. that this line is $right(L)$, and let $\mathcal{R}' \subset \mathcal{R}$ be the rectangles not cut by $right(L)$, which satisfies $|\mathcal{R}'| \geq \frac{3}{4}n - 1$. Using arguments similar to those used in case (a), we have that the rectangles of $\mathcal{R}'$ to the left of $right(L)$ are guillotine-separable (since $L \in \mathcal{R}'$ separates two corner sets of rectangles), and the rectangles of $\mathcal{R}'$ to the right of $left(R)$ are also guillotine-separable (since $R \notin \mathcal{R}'$ separates two corner sets of rectangles). Hence, $\mathcal{R}'$ is guillotine-separable.

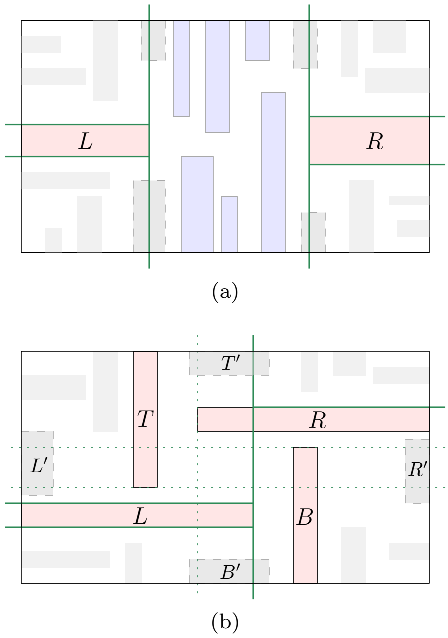

Figure 5: Proof of Theorem 4.

The tightness of the bound follows from the second part of Lemma 3. □

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