

Lecture 8

Design for performance

This lecture follows Chapter 10 of Doyle-Francis-Tannenbaum, skipping the material about 2-norm minimization.

Design for performance

- The performance criterion (the ability to follow a set of signals with prescribed error) is

$$||W_1 S||_\infty < 1.$$

- In this lecture we attack the problem of finding a proper C so that the criterion is met.
- This could be done using the loopshaping method with $W_2 = 0$, but we will present an analytical method that works even when P or P^{-1} are unstable.
- For P^{-1} stable the problem is very easy. For P^{-1} unstable (with zeros in the open right half-plane) nonclassical (model matching) methods will be required.

P^{-1} stable (I)

- First we assume that P^{-1} is stable, *i.e.* P has no zeros in $\Re(s) \geq 0$. The weighting function W_1 is assumed to be stable and strictly proper. The later condition is not restrictive at all since trying to follow signals of arbitrary high frequency makes no sense.
- Let $k = 1, 2, 3, \dots$ and $\tau > 0$, and consider the transfer function

$$J_\tau(s) = \frac{1}{(\tau s + 1)^k}.$$

- The Bode plot of J_τ is as follows. The magnitude starts out at 0, is flat up to the corner frequency $1/\tau$, and then rolls out to $-\infty$ with slope $-k$. The phase starts at 0, is flat up to $0.1/\tau$, and afterwards rolls off to $-k\pi/2$.

P^{-1} stable (II)

- The behavior of J_τ that we have sketched shows that for low frequencies ($< 0.1/\tau$) it approximates the transfer function 1. This means that for $\tau \rightarrow 0$ it approximates 1 to all frequencies. This can be formulated in the following

Lemma 1. If G is stable and strictly proper, then

$$\lim_{\tau \rightarrow 0} \|G(1 - J_\tau)\|_\infty = 0.$$

- Now we will develop the design procedure, first with the additional assumption that P is stable. From Lecture 4, the set of all internally stabilizing controllers is in this case

$$C = \frac{Q}{1 - PQ}, \quad Q \in \mathcal{Q}.$$

P^{-1} stable (III)

- Using this we get $W_1S = W_1(1 - PQ)$, and we are tempted to set $Q = P^{-1}$ to get $\|W_1S\|_\infty = 0 < 1$. This Q is indeed, by assumption, stable, but it is not proper.
- Let us try

$$Q = P^{-1}J_\tau$$

with k large enough to make Q proper (k must be equal to, or greater than, the relative degree of P).

- Then

$$W_1S = W_1(1 - J)$$

whose ∞ -norm is less than 1 for sufficiently small τ , by Lemma 1. This solves the problem.

P^{-1} stable (IV)

- If P is unstable, the Youla-Kucera parametrization yields all the internally stabilizing controllers:

$$C = \frac{X + MQ}{Y - NQ}, \quad S = M(Y - NQ).$$

- First do a coprime factorization of P in $\mathcal{Q}$:

$$P = \frac{N}{M}, \quad NX + MY = 1.$$

- Set $k \geq$ the relative degree of P and choose τ so small that

$$\|W_1 MY(1 - J_\tau)\|_\infty < 1.$$

This can be done, by Lemma 1, because $W_1 MY$ is stable and strictly proper (W_1 is strictly proper).

P^{-1} stable (V)

- Set

$$Q = N^{-1}YJ_\tau.$$

Then

$$W_1S = W_1M(Y - NN^{-1}YJ_\tau) = W_1MY(1 - J_\tau)$$

and $\|W_1S\|_\infty < 1$.

- Finally, substitute Q in the Youla-Kucera parametrization to get the controller.
- In general, τ must be found numerically, going down from an initial guess.

P^{-1} stable (VI)

Example. Let

$$P(s) = \frac{1}{(s-2)^2}, \quad W_1(s) = \frac{100}{s+1}.$$

This weight has bandwidth 1 rad/s and should give less than 1% tracking error over that range.

- The coprime factorization of P over $\mathcal{Q}$ yields

$$N(s) = \frac{1}{(s+1)^2}, \quad M(s) = \frac{(s-2)^2}{(s+1)^3}, \quad X(s) = 27 \frac{s-1}{s+1}, \quad Y(s) = \frac{s+7}{s+1}.$$

- We set $k = 2$ and choose τ so that

$$W_1 M Y (1 - J_\tau) = \frac{100(s-2)^2(s+7)}{(s+1)^4} \left(1 - \frac{1}{(\tau s + 1)^2} \right)$$

has ∞ -norm less than 1. One gets

τ	10^{-1}	10^{-2}	10^{-3}	10^{-4}
∞ -norm	199.0	19.97	1.997	0.1997

so we choose $\tau = 10^{-4}$.

P^{-1} stable (VII)

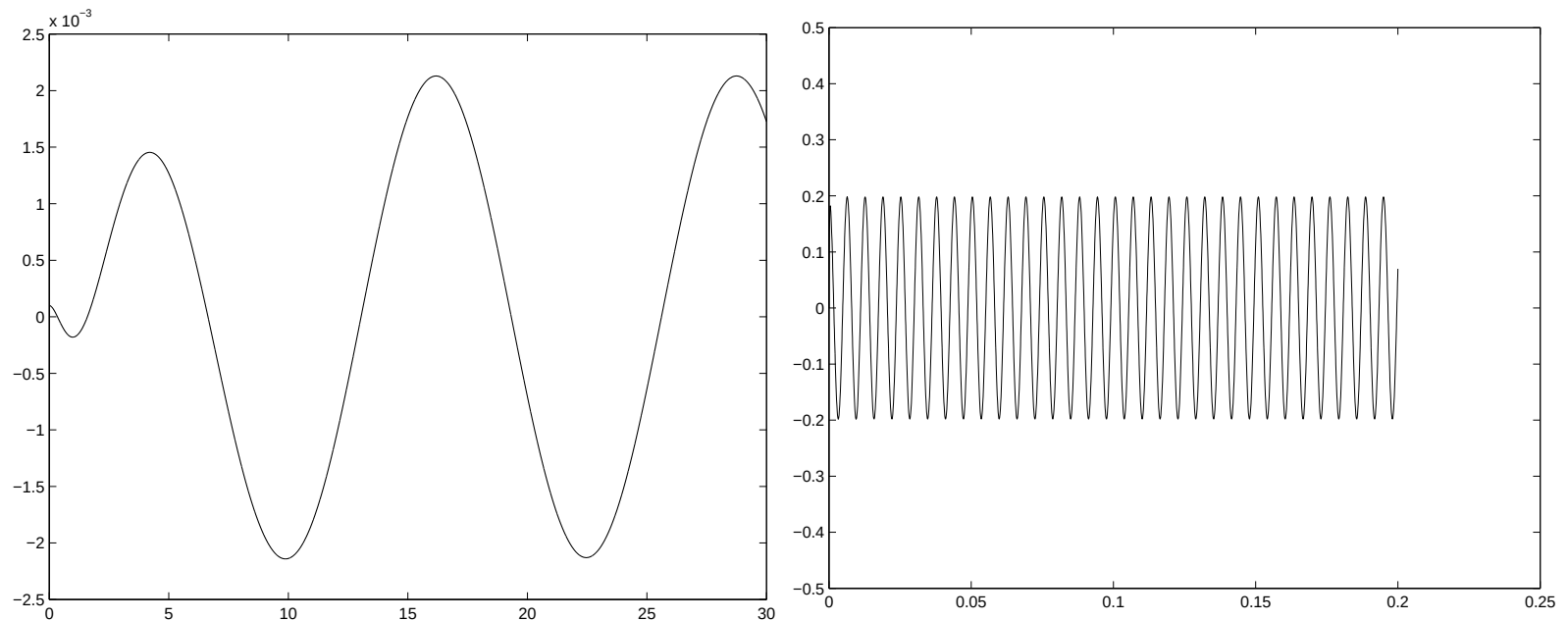
- Now

$$Q(s) = N^{-1}(s)Y(s)J_{10^{-4}}(s) = \frac{(s+1)(s+7)}{(10^{-4}s+1)^2}$$

and the controller is

$$C(s) = \frac{X(s) + M(s)Q(s)}{Y(s) - N(s)Q(s)} = 10^4 \frac{(s+1)^3}{s(s+7)(10^{-4}s+2)}.$$

- The figure shows the tracking error for unit sinusoidal inputs with $\omega = 0.5$ rad/s and $\omega = 1000$ rad/s. The performance degradation for the later is clearly seen.



P^{-1} unstable (I)

- The interpolation theory (NP problem) presented in the previous lecture will be used here to solve the performance design problem for P^{-1} unstable.
- To simplify, we will assume that
 - P has no poles or zeros on the imaginary axis, only distinct poles and zeros in the right half-plane, and at least one zero in the right half-plane.
 - W_1 is stable and strictly proper.

These assumptions could be relaxed, but the development would be messier.

- To motivate the presentation, remember from Lecture 2 that C is internally stabilizing iff PC has no right half-plane pole-zero cancellations and $1 + PC$ has no poles in $\Re(s) \geq 0$. Since

$$S = \frac{1}{1 + PC}$$

we must have

- $S(z) = 1$ for z a zero of P in $\Re(s) > 0$.
- $S(p) = 0$ for p a pole of P in $\Re(s) > 0$.

P^{-1} unstable (II)

- In terms of the weighted sensitivity function $G = W_1 S$ this translates to
 - $G(z) = W_1(z)$ for z a zero of P in $\Re(s) > 0$.
 - $G(p) = 0$ for p a pole of P in $\Re(s) > 0$.
- The performance test $\|W_1 S\|_\infty < 1$ is $\|G\|_\infty < 1$, and the stability of S requires G to be analytic in the right half-plane. Hence we have a NP problem.
- Now we could solve this NP problem, get S , and finally obtain C . However, this does not guarantee that the final C is proper, and so we must bring again coprime factorization into the problem.
- Remember that, using the Youla-Kucera parametrization,

$$W_1 S = W_1 M(Y - NQ).$$

The parameter Q must be both stable and proper. Our approach is to drop first the properness requirement and find a suitable (improper) Q_{im} , and then roll it off at high frequency. This works because W_1 is strictly proper, and hence there is no performance requirement at high frequency.

P^{-1} unstable (III)

The algorithm goes as follows. Given P, W_1 :

Step 1. Do a coprime factorization of P in $\mathcal{Q}$:

$$P = \frac{M}{N}, \quad NX + MY = 1.$$

Step 2. Find a stable function Q_{im} such that

$$\|W_1 M(Y - NQ_{\text{im}})\|_{\infty} < 1.$$

Step 3. Set

$$J_{\tau}(s) = \frac{1}{(\tau s + 1)^k},$$

where k is large enough that $Q_{\text{im}}J_{\tau}$ is proper and τ is small enough that

$$\|W_1 M(Y - NQ_{\text{im}}J_{\tau})\|_{\infty} < 1.$$

Step 4. Set $Q = Q_{\text{im}}J_{\tau}$ and $C = (X + MQ)/(Y - NQ)$.

P^{-1} unstable (IV)

- Step 2 is the model-matching problem: find a stable function Q_{im} to minimize

$$\|T_1 - T_2 Q_{\text{im}}\|_{\infty}$$

where $T_1 = W_1 M Y$ and $T_2 = W_1 M N$. Step 2 is feasible iff the minimum model-matching error γ_{opt} is less than 1.

- Step 3 is feasible because

$$W_1 M (Y - N Q_{\text{im}} J_{\tau}) = W_1 M (Y - N Q_{\text{im}}) J_{\tau} + W_1 M Y (1 - J_{\tau}).$$

The first term on the right-hand side has ∞ -norm less than 1 from Step 2 and the fact that $\|J_{\tau}\|_{\infty} \leq 1$, while the ∞ -norm of the second term goes to zero as $\tau \rightarrow 0$ because of Lemma 1.

Flexible beam (I)

- Here we apply the preceding theory to a real experimental setup. It describes a flexible beam, one end fixed to the shaft of a high torque dc motor, and the other free, where a sensor is situated.
- The full model is infinite dimensional due to the continuous nature of the beam. A truncation of the model, taking into account only the first mode, yields for the transfer function from motor torque to deflection of the sensed tip

$$P(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{s(5s^3 + 3.5682s^2 + 139.5021s + 0.0929)},$$

where the sensor, the motor and the amplifier have been incorporated.

- This $P(s)$ has a pole on the imaginary axis, so we perturb it to

$$P(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{5s^4 + 3.5682s^3 + 139.5021s^2 + 0.0929s + 10^{-6}}.$$

Flexible beam (II)

- For a system like this, it does not make much sense to make it follow sinusoidal signals. Rather, we would like it to follow step inputs with specified transient, for instance

$$\text{settling time} = 8 \text{ s}, \quad \text{overshoot} \leq 10\%.$$

- We will accomplish this by shaping $T(s)$, the transfer function from r to y , so that it approximates a standard second order model

$$T_{\text{id}}(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

- A settling time of 8 seconds requires

$$\frac{4.6}{\xi\omega_n} = 8$$

and the overshoot of ten percent is

$$\exp\left(\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right) = 0.1.$$

Flexible beam (III)

- The solution is $\xi = 0.5912$ and $\omega_n = 0.9583$, so we take $\xi = 0.6$, $\omega_n = 1$ and

$$T_{\text{id}}(s) = \frac{1}{s^2 + 1.2s + 1}.$$

The ideal sensitivity function is then

$$S_{\text{id}}(s) = 1 - T_{\text{id}}(s) = \frac{s(s + 1.2)}{s^2 + 1.2s + 1}.$$

- Now we will argue that the sensible weighting function for this setup is

$$W_1(s) = S_{\text{id}}^{-1}(s) = \frac{s^2 + 1.2s + 1}{s(s + 1.2)}.$$

Flexible beam (IV)

- The argument goes as follows. From Step 2 of the algorithm, one gets

$$F = W_1 M(Y - NQ_{\text{im}})$$

which is a constant times an all-pass function. The procedure then rolls off Q_{im} at high frequency to result in the weighted sensitivity function

$$W_1 S = W_1 M(Y - NQ_{\text{im}} J_\tau).$$

This means that, except at high frequencies, $W_1 S \sim F$. With the form of W_1 that we propose, this is

$$S \sim F S_{\text{id}}.$$

- Now F , being an all-pass function, behaves like a time delay, except at high frequencies, so we get

$$S \sim (\text{time delay}) S_{\text{id}}.$$

Hence, the actual step response equals the ideal step response plus a time delay, which does not change the overshoot and, hopefully, is small and does not change the settling time too much.

Flexible beam (V)

- One further adjustment is required: $W_1(s)$ must be stable and strictly proper, so we change it to

$$W_1(s) = \frac{s^2 + 1.2s + 1}{(s + 0.001)(s + 1.2)(0.001s + 1)}.$$

- First, since $P \in \mathcal{Q}$, no coprime factorization is required and we take $N = P$, $M = 1$, $X = 0$ and $Y = 1$, and the controller parametrization is then $C = \frac{Q}{1 - PQ}$.
- The model matching problem is to minimize

$$\|W_1 M(Y - NQ_{\text{im}})\|_{\infty} = \|W_1(1 - PQ_{\text{im}})\|_{\infty}.$$

Since $T_2 = W_1 P$ has only one right half-plane zero, that of P , at $s = 5.5308$, the model-matching problem is next to trivial and

$$\min \|W_1(1 - PQ_{\text{im}})\|_{\infty} = |W_1(5.5308)| = 1.0210.$$

This means that $\|W_1 S\|_{\infty} < 1$ is not achievable with this W_1 , so we scale it down by

$$W_1 \rightarrow \frac{0.9}{1.0210} W_1 \quad \text{and now} \quad |W_1(5.5308)| = 0.9.$$

Flexible beam (VI)

- The optimal Q_{im} is then

$$Q_{\text{im}} = \frac{T_1 - T_1(5.5308)}{T_2} = \frac{W_1 - 0.9}{W_1 P},$$

that is

$$Q_{\text{im}}(s) = \frac{0.0008s^6 + 0.0221s^5 + 0.1768s^4 + 0.7007s^3 + 3.8910s^2 + 0.0026s}{s^3 + 6.1081s^2 + 6.8897s + 4.9801}.$$

- To get a proper Q , we take

$$J_\tau(s) = \frac{1}{(\tau s + 1)^3}$$

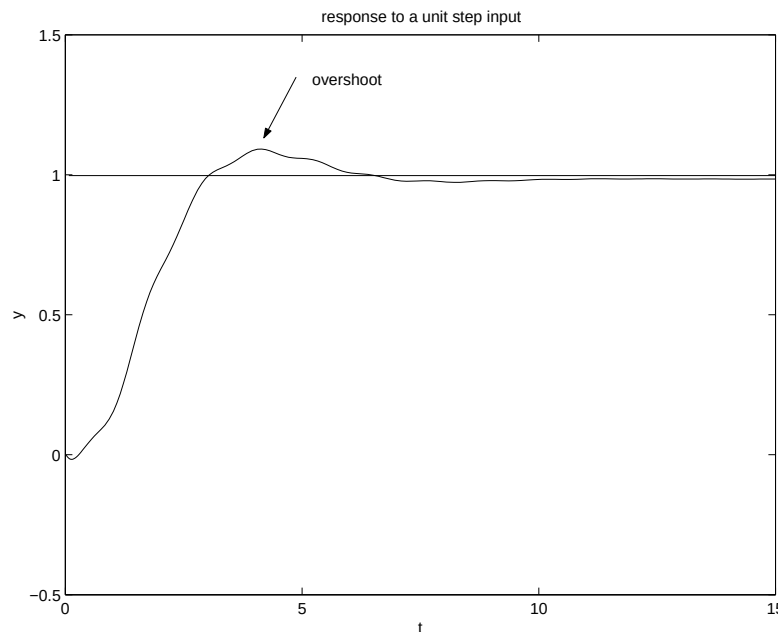
and compute $\|W_1(1 - PQ_{\text{im}}J_\tau)\|_\infty$ for decreasing values of τ until we go below 1:

τ	0.1	0.05	0.04
∞ -norm	1.12	1.01	0.988

We take $\tau = 0.04$.

Flexible beam (VII)

- Finally, $Q = Q_{\text{im}} J_{0.04}$ and we compute the controller. The controller C so obtained is of very high order (quotient of two order 10 polynomials). The common way to alleviate this is to approximate Q by a lower-order transfer function (it cannot be done directly with C due to the internal stability constraint, while any proper and stable Q resembling the computed one will do).
- The output to a step input for the controller just computed is



and it clearly satisfies the specifications, in spite of all the approximations and *ad hoc* changes performed.