

Lecture 7

Model matching

This lecture follows Chapter 9 of Doyle-Francis-Tannenbaum.

Model matching (I)

- We study an hypothetical problem, the model matching problem, which will be used in later lectures.
- Let $T_1(s)$ and $T_2(s)$ be stable, proper, real-rational transfer functions (*i.e.* in $\mathcal{Q}$).
- The **model matching problem** is to find a stable transfer function $Q(s)$ that minimizes $\|T_1 - T_2Q\|_\infty$.
- Interpretation: T_1 is a model, T_2 is a real plant, and Q is a cascade controller such that T_2Q approximates T_1 .
- Q is required to be stable but not necessarily proper (this makes the problem easier).
- We will assume that T_2 has no zeros on the imaginary axis.

Model matching (II)

- Define the minimum model-matching error

$$\gamma_{\text{opt}} = \min \|T_1 - T_2 Q\|_{\infty},$$

where the minimum is taken over all the stable Q s. A Q achieving the minimum is said to be **optimal**.

- If T_1/T_2 is stable, the optimal Q is

$$Q = T_1/T_2$$

and $\gamma_{\text{opt}} = 0$.

- Next in order of complexity is when T_2 has only one zero in the (open) right-half plane, say at $s = s_0$, $\Re(s_0) > 0$. Now T_1/T_2 is not valid because it is not stable.

Model matching (III)

- If Q is stable and T_2Q has finite ∞ -norm (i.e. $T_2Q \in \mathcal{Q}$; this means that T_2 restores the properness of Q , if Q was not proper), then by the maximum modulus theorem

$$\|T_1 - T_2Q\|_\infty \geq |T_1(s_0) - T_2(s_0)Q(s_0)| = |T_1(s_0)|,$$

so $\gamma_{\text{opt}} \geq |T_1(s_0)|$.

- The function

$$Q = \frac{T_1 - T_1(s_0)}{T_2}$$

is stable (there is pole-zero cancellation at $s = s_0$) and yields the value $|T_1(s_0)|$ for the model-matching error.

- Hence $\gamma_{\text{opt}} = |T_1(s_0)|$ and the given Q is optimal.

Model matching (IV)

- **Example.** If

$$T_1(s) = \frac{4}{s+3}, \quad T_2(s) = \frac{s-2}{(s+1)^3}$$

we have $\gamma_{\text{opt}} = T_1(2) = 4/5$ and the optimal Q (notice that it is not proper) is

$$Q(s) = \frac{4(s+1)^3}{5(s+3)}.$$

This is the best we can do to approximate T_1 by T_2Q uniformly over the imaginary axis ($s = j\omega$).

- To solve the problem in general, we need the mathematics of rational interpolation theory.

Nevanlinna-Pick problem (I)

- Let $\mathcal{Q}_c$ denote the set of stable, proper, **complex-rational** functions. For instance,

$$\frac{(1 + j)s + 2j}{s + 1 + j}$$

belongs to $\mathcal{Q}_c$.

- The ∞ -norm, maximum magnitude on the imaginary axis, can be defined as we did for real-rational functions.
- Let $\{a_1, a_2, \dots, a_n\}$ be a set of distinct points in the $\Re(s) > 0$ region, and let $\{b_1, b_2, \dots, b_n\}$ be any set of points in $\Re(s) < 0$.

Nevanlinna-Pick problem (II)

- The **Nevanlinna-Pick interpolation problem**, or the NP problem for short, is to find a function $G \in \mathcal{Q}_c$ such that
 1. $\|G\|_\infty \leq 1$.
 2. $G(a_i) = b_i, i = 1, 2, \dots, n$.
- The problem is not solvable for all data $\{a, b\}$. An obvious necessary condition is $|b_i| \leq 1, i = 1, 2, \dots, n$, because the maximum module theorem can be applied to $G \in \mathcal{Q}_c$ and $\|G\|_\infty \leq 1$ would be violated if $|b_i| > 1$ for a given i .
- To state precisely when the NP problem is solvable, we need some facts about complex matrices.

Nevanlinna-Pick problem (III)

- Let M be a square complex matrix and let $M^\dagger$ denote its complex-conjugate transpose. Notice that $(M^\dagger)^\dagger = M$.
- If $M = M^\dagger$, M is said to be **Hermitian**.
- The eigenvalues of an Hermitian matrix are all real.
- If M is Hermitian, it is said to be **positive semidefinite** if $x^\dagger M x \geq 0$ for all complex vectors x , and **positive definite** if $x^\dagger M x > 0$ for all nonzero complex vectors x . This is denoted as $M \geq 0$ and $M > 0$ respectively.
- One has that $M \geq 0$ iff all the eigenvalues are ≥ 0 , and $M > 0$ iff all the eigenvalues are > 0 (this makes sense because the eigenvalues are real).

Nevanlinna-Pick problem (IV)

- To the data of a given NP problem we can associate a complex matrix Q whose elements Q_{ij} are

$$Q_{ij} = \frac{1 - b_i \bar{b}_j}{a_i + \bar{a}_j}.$$

This is called the **Pick matrix**.

- Q is Hermitian:

$$\bar{Q}_{ji} = \frac{1 - \bar{b}_j b_i}{\bar{a}_j + a_i} = \frac{1 - b_i \bar{b}_j}{a_i + \bar{a}_j} = Q_{ij}.$$

- **Theorem (Pick's theorem)**

The NP problem is solvable iff $Q \geq 0$

Nevanlinna-Pick problem (V)

- Pick's theorem gives an easy way to check whether the NP problem is solvable for given data: just construct Q and see if its smallest eigenvalue is nonnegative.
- It is easy to see that $Q \geq 0$ implies $|b_i| \leq 1$ for all i , which we saw was a necessary condition.
- The proof of the sufficiency part illustrates in system-theoretic terms how the Pick matrix arises. It has to do with **dissipative** systems, *i.e.* systems for which the outgoing energy is less than or equal to the incoming energy. See DFT for details.

Nevanlinna's algorithm (I)

- We will see a procedure to construct a solution of the NP problem when it is solvable. First the case $n = 1$ will be solved, and the case with n points will be reduced to the $n - 1$ points case.
- Let $\mathcal{D}$ denote the open unit disk in $\mathbb{C}$, $|z| < 1$, and let $\bar{\mathcal{D}}$ stand for the closed unit disk, $|a| \leq 1$.
- A **Möbius function** is a complex-rational first-order function of the form

$$M_b(z) = \frac{z - b}{1 - z\bar{b}}$$

for any $b \in \mathbb{C}$ satisfying $|b| < 1$.

Nevanlinna's algorithm (II)

- Möbius functions have the following properties:
 1. M_b is analytic in $\mathcal{D}$.
 2. If $|z| = 1$ then $|M_b(z)| = 1$.
 3. M_b maps $\mathcal{D}$ onto $\mathcal{D}$, *i.e.* if $|z| < 1$ then $|M_b(z)| < 1$.
 4. The inverse map (in the sense of composition) is

$$M_b^{-1}(z) = \frac{z + b}{1 + z\bar{b}} = M_{-b}(z),$$

and hence it is also a Möbius function.

- We will also use the all-pass function

$$A_a(s) = \frac{s - a}{s + \bar{a}}, \quad \Re(a) > 0.$$

Nevanlinna's algorithm (III)

- We will first solve the NP problem for the data a_1, b_1 .
- If $|b_1| = 1$, a solution is given by the constant function $G(s) = b_1$. Using the maximum modulus theorem it can be seen that this solution is unique.
- If $|b_1| < 1$, there are an infinite number of solutions, which are characterized by the following

Lemma 1 *The set of all solutions is*

$$\{G : G(s) = M_{-b_1}(G_1(s)A_{a_1}(s)), G_1 \in \mathcal{Q}_c, \|G_1\|_\infty \leq 1\}$$

If G_1 is all-pass, so is G .

Nevanlinna's algorithm (IV)

- Now we turn to the NP problem with n points. The problem is assumed to be solvable.
- If $|b_1| = 1$, we have again that the unique solution must be $G(s) = b_1$ and, since the problem is solvable, the data *must* satisfy $b_1 = b_2 = \dots = b_n$.
- If $|b_1| < 1$, consider a new NP problem (call it NP'), with the $n - 1$ data points $a_2, a_3, \dots, a_n$ and $b'_2, b'_3, \dots, b'_n$, where

$$b'_i = \frac{M_{b_1}(b_i)}{A_{a_1}(a_i)}, \quad i = 2, \dots, n.$$

Nevanlinna's algorithm (V)

- Now we have the main result

Lemma 2 *The set of all solutions to the NP problem is given by the formula*

$$G(s) = M_{-b_1}(G_1(s)A_{a_1}(s)),$$

where G_1 ranges over all solutions of the NP' problem. If G_1 is all-pass, so is G .

- Since the solution for $n = 1$ can be chosen to be all-pass, the NP problem always has an all-pass solution.

Nevanlinna's algorithm (VI)

- In our application of the NP theory to the model-matching problem, the data $\{a, b\}$ will have conjugate symmetry; that is, if a_i, b_i appears, so will the conjugate pair $\bar{a}_i, \bar{b}_i$. Then we will want a real-rational solution instead of complex.
- The solution G to the NP problem can be written uniquely as

$$G(s) = G_R(s) + jG_I(s)$$

where both G_R and G_I are real-rational. We have then

Lemma 3 *For conjugate-symmetric data, $G_R \in \mathcal{Q}$ and G_R is a solution to the NP problem.*

Model-matching solved (I)

- Let us use now NP theory to solve the model-matching problem. For simplicity we will assume that T_2 has no repeated zeros in $\Re(s) > 0$.
- The minimum model-matching error, γ_{opt} , equals the minimum γ such that

$$\|T_1 - T_2 Q\|_{\infty} \leq \gamma, \quad \text{or} \quad \left\| \frac{T_1 - T_2 Q}{\gamma} \right\|_{\infty} \leq 1$$

for some stable Q .

- Fix $\gamma > 0$ and consider all the stable functions of the form

$$G = \frac{1}{\gamma}(T_1 - T_2 Q).$$

- Now the problem is to find the minimum γ such that $\|G\|_{\infty} < 1$ with $G \in \mathcal{Q}$. However, there are G s of the above form in $\mathcal{Q}$ such that Q is not stable, and we must exclude them when computing this minimum. This is where the NP problem enters the scene.
- If $z_i, i = 1, \dots, n$ are the (non repeated) zeros of T_2 in $\Re(s) > 0$ then, if

$$G(z_i) = \frac{1}{\gamma}T_1(z_i), \quad i = 1, \dots, n,$$

one has that Q is stable iff G is stable.

Model-matching solved (II)

- Summing up, we have to find the minimum $\gamma > 0$ so that $G \in \mathcal{Q}$ and
 - $\|G\|_\infty \leq 1$,
 - $G(z_i) = \frac{1}{\gamma} T_1(z_i)$.
- This is a NP problem with data $(a_i = z_i, b_i = \gamma^{-1} T_1(z_i))$.
- The corresponding Pick matrix can be written as

$$A - \gamma^{-2} B$$

where A and B have elements

$$\frac{1}{a_i + \bar{a}_j} \quad \text{and} \quad \frac{b_i \bar{b}_j}{a_i + \bar{a}_j}$$

respectively.

Model-matching solved (III)

- From Pick's theorem, we know that γ_{opt} equals the minimum γ such that

$$A - \gamma^{-2}B \geq 0.$$

- It can be shown that the matrix A is positive definite (this is because the a_i are distinct), and that a square root matrix $A^{1/2}$ such that $A^{1/2}A^{1/2} = A$ can be computed. Denote its inverse by $A^{-1/2}$. Then we have the following lemma, which is a simple result in matrix theory:

Lemma 3 γ_{opt} equals the square root of the largest eigenvalue of the matrix

$$A^{-1/2}BA^{-1/2}.$$

The algorithm (I)

Tracing backwards, we get the following procedure for solving the model-matching problem:

Input: T_1, T_2 .

Step 1: Determine $\{z_i, i = 1, \dots, n\}$, the (distinct) zeros of T_2 in $\Re(s) > 0$.

Step 2: Define $b_i = T_1(z_i)$ and form the matrices

$$A = \left(\frac{1}{z_i + \bar{z}_j} \right) \quad \text{and} \quad B = \left(\frac{b_i \bar{b}_j}{z_i + \bar{z}_j} \right).$$

Step 3: Compute γ_{opt} as the square root of the largest eigenvalue of $A^{-1/2} B A^{-1/2}$.

The algorithm (II)

Step 4: Solve the NP problem with data $\{z_i, \gamma_{\text{opt}}^{-1} b_i\}$. Denote the solution by G .

Step 5: Set

$$Q = \frac{T_1 - \gamma_{\text{opt}} G}{T_2}.$$

Since the solution G in Step 4 can always be chosen to be all-pass, for the optimal Q the error transfer function $T_1 - T_2 Q$ equals γ_{opt} times an all-pass function.