

Lecture 6

Loopshaping

This short lecture follows Chapter 7 of Doyle-Francis-Tannenbaum.

The basic idea (I)

- Remember again the robust performance test (RPC)

$$\| |W_1 S| + |W_2 T| \|_\infty < 1,$$

with W_1 , describing the performance goal, W_2 , describing the set of plants, and the nominal plant P , are given, and C is to be found so as to satisfy the RPC.

- In the previous lecture we saw that this is not always achievable.
- Necessary and sufficient conditions on P , W_1 and W_2 are unknown.
- Loopshaping** is a trial-and-error technique to solve the RPC.

The basic idea (II)

- In terms of $L = PC$, the RPC is

$$\Gamma(j\omega) \equiv \left| \frac{W_1(j\omega)}{1 + L(j\omega)} \right| + \left| \frac{W_2(j\omega)L(j\omega)}{1 + L(j\omega)} \right| < 1, \quad \forall \omega \in \mathbb{R}.$$

- If L satisfying this can be found, then C can be obtained as

$$C = \frac{L}{P}.$$

- The nominal feedback system must be internally stable and C must be proper, so L is not freely assignable.
- Since internal stability forbids pole-zero cancellation in $\Re(s) \geq 0$ in $L = PC$, L must contain any unstable zero or pole of P . This is an additional constraint which makes loopshaping quite difficult, so we will assume that all the zeros and poles of P are stable.

The basic idea (III)

- From the last lecture we know that a necessary condition for the RPC is

$$\min\{|W_1|, |W_2|\} < 1$$

for all ω . We drop the argument $j\omega$ for the rest of this discussion.

- From the definition of Γ we have

$$\begin{aligned} (|W_1| - |W_2|)|S| + |W_2| &\leq \Gamma \leq (|W_1| + |W_2|)|S| + |W_2|, \\ (|W_2| - |W_1|)|T| + |W_1| &\leq \Gamma \leq (|W_2| + |W_1|)|T| + |W_1|, \end{aligned}$$

where

$$|a + b| \leq |a| + |b|, \quad |a - b| \geq |a| - |b|,$$

have been used.

The basic idea (IV)

Either $|W_1| < 1$ or $|W_2| < 1$. Assume first that $|W_2| < 1$.

$$\begin{aligned}\frac{|W_1| + |W_2|}{1 - |W_2|} |S| < 1 &\Leftrightarrow (|W_1| + |W_2|)|S| < 1 - |W_2| \\ &\Leftrightarrow (|W_1| + |W_2|)|S| + |W_2| < 1 \\ &\Rightarrow \Gamma < 1.\end{aligned}$$

$$\begin{aligned}\Gamma < 1 &\Rightarrow (|W_1| - |W_2|)|S| + |W_2| < 1 \\ &\Leftrightarrow (|W_1| - |W_2|)|S| < 1 - |W_2| \\ &\Leftrightarrow \frac{|W_1| - |W_2|}{1 - |W_2|} |S| < 1.\end{aligned}$$

The basic idea (V)

Assume now that we are in a range of frequencies where $|W_1| \gg 1$ (low frequencies). We have, approximately,

$$\frac{|W_1|}{1 - |W_2|} |S| < 1 \Leftrightarrow \Gamma < 1, \quad \Gamma < 1 \Leftrightarrow \frac{|W_1|}{1 - |W_2|} |S| < 1,$$

so we may approximate the RPC $\Gamma < 1$ by

$$\frac{|W_1|}{1 - |W_2|} |S| < 1.$$

In terms of L this is $\frac{|W_1|}{1 - |W_2|} < |1 + L|$, but since the left-hand side is very large, this can (again!) be approximated by

$$\frac{|W_1|}{1 - |W_2|} < |L|.$$

The basic idea (VI)

Suppose now that $|W_1| < 1$. In the range where $|W_2| \gg 1$ (high frequencies), the RPC can now be approximated by

$$\frac{|W_2|}{1 - |W_1|} |T| < 1.$$

In terms of L this is $\frac{|W_2|}{1 - |W_1|} |L| < |1 + L| \leq 1 + |L|$, or

$$\frac{|W_2|}{1 - |W_1|} < |1 + L| \leq \frac{1}{|L|} + 1.$$

Again, the left-hand side is very large and this is, approximately,

$$|L| < \frac{1 - |W_1|}{|W_2|}.$$

The basic idea (VII)

The preceding discussion can be summarized as follows:

- In the region where $|W_1| \gg 1 > |W_2|$, we should impose

$$|L| > \frac{|W_1|}{1 - |W_2|},$$

- while, in the region where $|W_2| \gg 1 > |W_1|$,

$$|L| < \frac{1 - |W_1|}{|W_2|}.$$

Due to the approximations involved, **the above conditions do not guarantee that the RPC holds**, not even for the range where $|W_1|$ or $|W_2|$ are very large.

Practical matters (I)

Taking into account the previous results, loopshaping goes like this:

- Plot two curves on log-log scale, magnitude versus frequency: first, the graph of

$$\frac{|W_1|}{1 - |W_2|}$$

over the low-frequency range where $|W_1| > 1 > |W_2|$;
second, the graph of

$$\frac{1 - |W_1|}{|W_2|}$$

over the high-frequency range where $|W_1| < 1 < |W_2|$.

Practical matters (II)

- On this plot fit another curve, which is going to be the graph of $|L|$. At low frequency, let it lie above the first curve and also be $\gg 1$; at high frequency let it lie below the second curve and also be $\ll 1$. Also, at very high frequency let it roll off at least as fast as does $|P|$, so that $C = L/P$ is proper.
- Do a smooth transition from low to high frequency, keeping the slope as gentle as possible near crossover (the frequency where the magnitude equals 1; more on this later).
- Get a stable, minimum-phase transfer function L whose Bode magnitude plot is the curve just constructed, normalizing so that $L(0) > 0$.

The phase formula (I)

It is a fundamental fact that if L is stable and minimum phase and normalized so that $L(0) > 0$, then its magnitude Bode plot uniquely determines its phase plot. The normalization is necessary, for

$$\frac{1}{s+1} \quad \text{and} \quad \frac{-1}{s+1}$$

are stable, minimum phase, and have the same magnitude plot, but they have different phase plots.

Theorem (the phase formula). For every frequency ω_0

$$\arg L(\omega_0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d \ln |L|}{d\nu} \ln \coth \frac{|\nu|}{2} d\nu,$$

where the integration variable is $\nu = \ln \frac{\omega}{\omega_0}$.

The phase formula (II)

Notice that, since $|L|$ is a decreasing function of frequency, we have

$$\frac{d \ln |L|}{d\nu} < 0.$$

This is multiplied by the function

$$\ln \coth \frac{|\nu|}{2}.$$

This function is infinite at $\nu = 0$, *i.e.* $\omega = \omega_0$, and decreases very steeply to the left and to the right. Hence, the value of the integral in the phase formula is dominated by the value of $\frac{d \ln |L|}{d\nu}$

at $\omega = \omega_0$.

the steeper the graph of $|L|$ near the frequency ω_0 , the smaller (more negative) the value of $\arg L$.

The phase formula (III)

- Now, with $L(0) > 0$ and with $|L|$ decreasing, the Nyquist plot of L starts out on the positive real axis and begins to move clockwise (since the argument of L decreases if $|L|$ decreases).
- If $|L| = 1$ on the fourth or third quadrant, it is clear that, since L decreases, we will have nominal plant internal stability. But if $|L|$ drops off too rapidly at crossover frequency, $\arg L$ will be very negative (near 180°) and hence the system will be on the verge of instability.
- A rule of thumb is that the slope of $|L|$, in log-log scale, should be no more than 2.

An example (I)

- Let

$$W_2(s) = \frac{s + 1}{20(0.01s + 1)}.$$

We have that $|W_2(j\omega)|$ starts at 0.05 and increases monotonically up to 5, crossing 1 at $\omega \sim 20.4 \text{ rad s}^{-1}$.

- Performance objective: to track sinusoidal reference signals of amplitude less than one in the frequency range from 0 to 1 rad s^{-1} .
- Remember from Lecture 2 that for such input

$$|e(t)| \leq \|S\|_\infty$$

so, if we want $|e(t)| < \epsilon \forall t$ we may impose

$$\|SW_1\|_\infty < 1$$

with

$$|W_1(j\omega)| < \frac{1}{\epsilon} \quad \forall \omega.$$

An example (II)

- Let us take

$$|W_1(j\omega)| = \begin{cases} a & \text{if } 0 \leq \omega \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

with a unspecified so that $\epsilon = 1/a$ and let us see how small we can get it.

- Let us first try a first-order, low-pass loop transfer function

$$L(s) = \frac{b}{cs + 1}.$$

It is reasonable to take $c = 1$ so that $|L|$ starts rolling off near the upper end of the operating band $[0, 1]$.

- We want b as large as possible for good tracking, but we must enforce the high-frequency constraint

$$|L| < \frac{1 - |W_1|}{|W_2|}.$$

Since the crossover frequency is $\omega \sim 20$, high frequency means $\omega > 20$, and since there $W_1 = 0$, the above condition reduces to $|L| < \frac{1}{|W_2|}$ for $\omega > 20$.

An example (III)

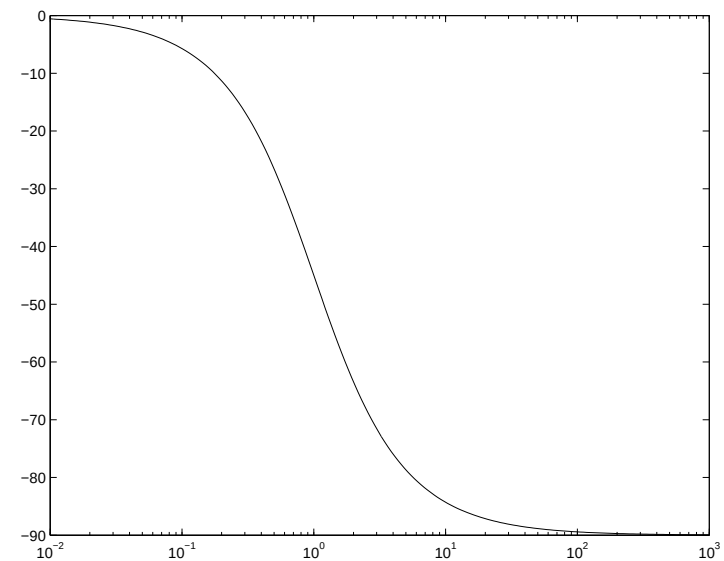
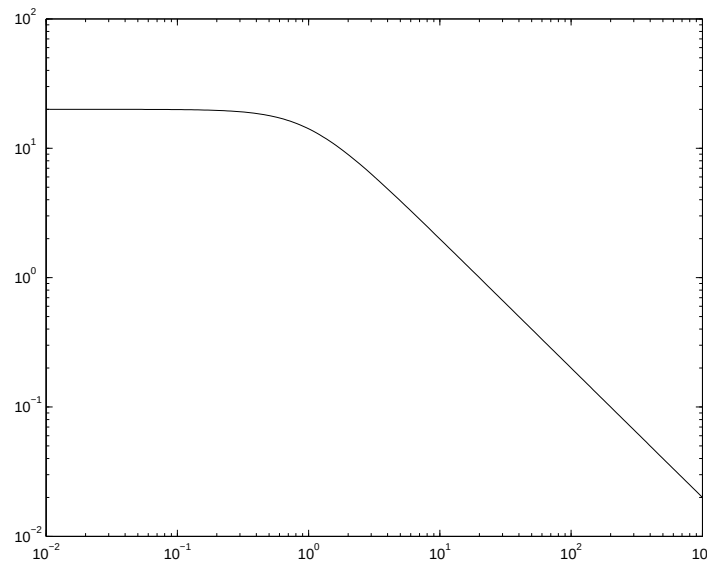
- The later condition is

$$b < 20|0.01\omega + 1| = 20\sqrt{10^{-4}\omega^2 + 1} \quad \text{for } \omega > 20.$$

Evaluating at $\omega = 20$ we get that b must verify $b < 20\sqrt{1.04}$ so we take $b = 20$ and

$$L(s) = \frac{20}{s + 1}.$$

- The Bode plot of L is



An example (IV)

- It remains to check what performance level we have achieved. We must see what is the largest value of a so that the low-frequency constraint

$$|L| > \frac{|W_1|}{1 - |W_2|} = \frac{a}{1 - |W_2|}, \quad \omega \leq 1$$

is enforced.

- While $|L(j\omega)|$ is an decreasing function, the right-hand side is an increasing function of ω , so the worst case in the range is when $\omega = 1$.
- Solving

$$|L(j1)| = \frac{a}{1 - |W_2(j1)|}$$

yields $a = 19/\sqrt{2} \sim 13.4$, and hence the maximum relative error is

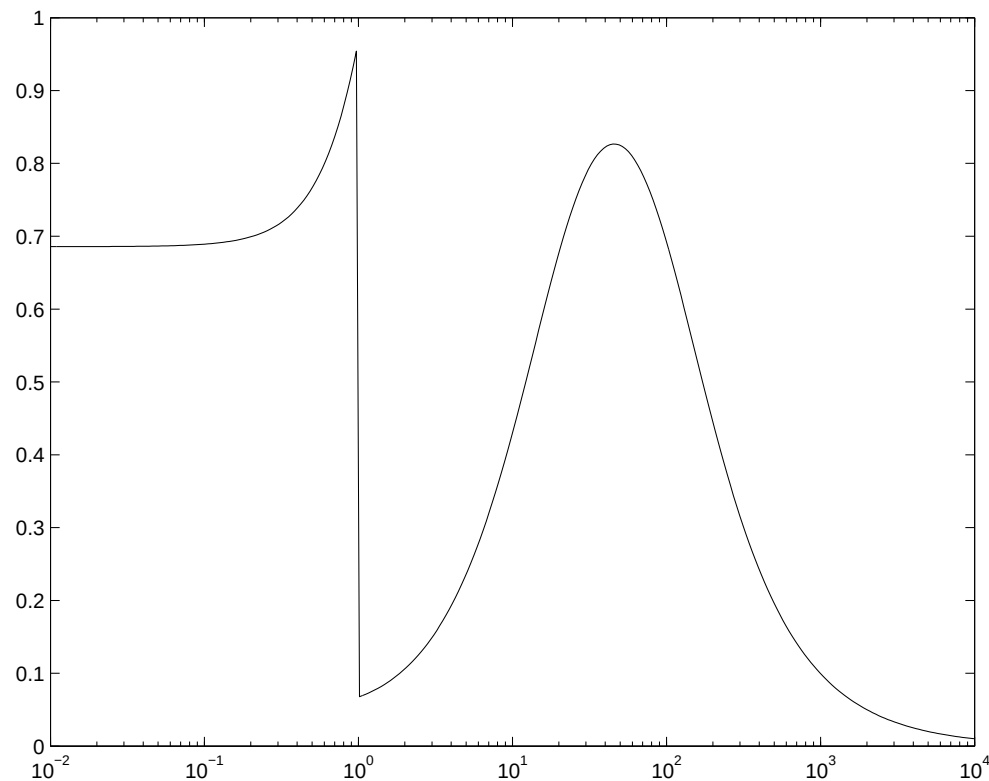
$$\epsilon = 1/a \sim 7.4\%.$$

An example (V)

- Remember that loopshaping does not guarantee the RPC. Now we must check that

$$|W_1(j\omega)S(j\omega)| + |W_2(j\omega)T(j\omega)| < 1 \quad \forall \omega.$$

- We get, in semilog scale,



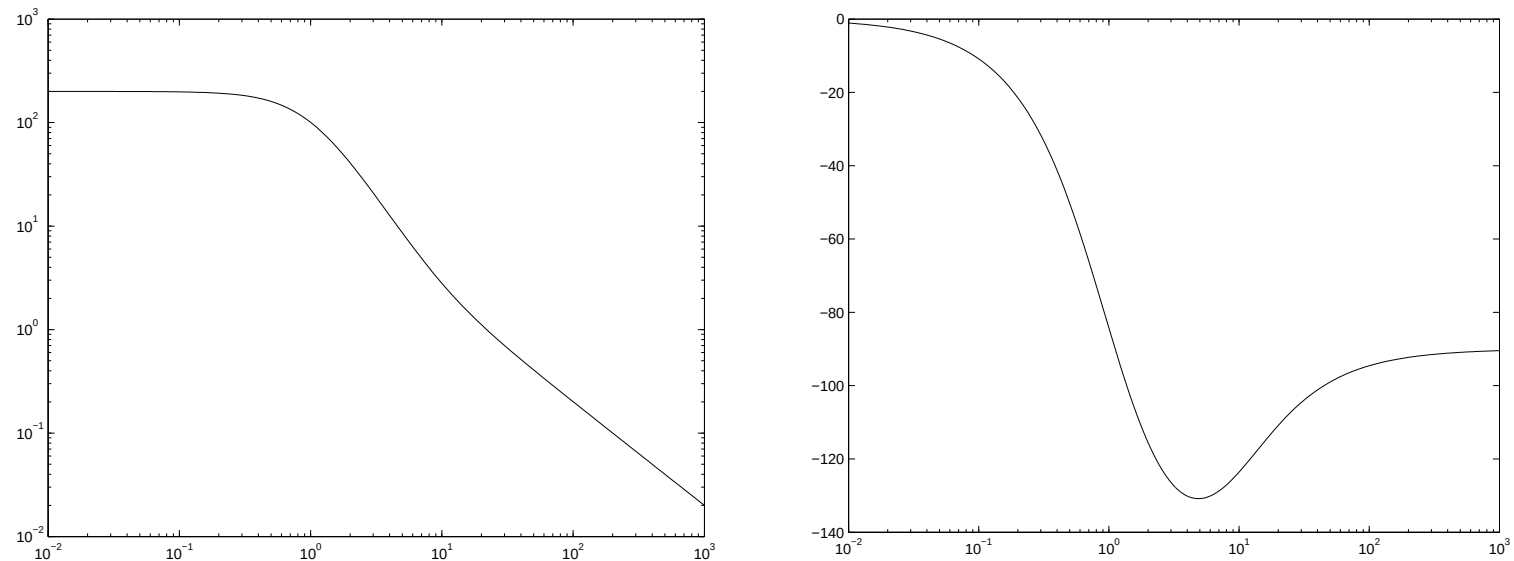
so the RPC holds.

An example (VI)

- Suppose now that the 7.4% error is too large. The way to reduce the error is to make $|L|$ large over the tracking range, but without changing it too much at high frequencies.
- We can try

$$L(s) = \frac{s + 10}{s + 1} \frac{20}{s + 1}.$$

- The Bode plot of L is



- It can be seen that the RPC is enforced, and that now $a \sim 93.5$, giving $\epsilon \sim 1.07$.