

Lecture 1

Norms for signals and systems

This is essentially Chapter 2 of Doyle-Francis-Tannenbaum, with optional sections mostly omitted.

Norms

$PC(\mathbb{R})$, vector space of piecewise continuous functions.

For most signals $u \in PC(\mathbb{R})$, $u(t) = 0$ if $t < 0$.

A norm is map $\|\cdot\|$ from a vector space in $\mathbb{R}$ such that

$$(N1) \ 0 \leq \|u\| < \infty.$$

$$(N2) \ \|u\| = 0 \text{ if and only if } u(t) = 0 \ \forall t.$$

$$(N3) \ \|\alpha u\| = |\alpha| \|u\|, \forall \alpha \in \mathbb{R}.$$

$$(N4) \ \|u + v\| \leq \|u\| + \|v\|.$$

In general we will have to restrict the definition to a subset of $PC(\mathbb{R})$ for $\|u\| < \infty$ to hold.

Norms for signals (I)

- 1-norm

$$\|u\|_1 = \int_{-\infty}^{+\infty} |u(t)| \, dt.$$

- 2-norm

$$\|u\|_2 = \left(\int_{-\infty}^{+\infty} u^2(t) \, dt \right)^{1/2}.$$

- ∞ -norm

$$\|u\|_\infty = \sup_{t \in \mathbb{R}} |u(t)|.$$

These define norms in $PC(\mathbb{R}) \cap L_1(\mathbb{R})$, $PC(\mathbb{R}) \cap L_2(\mathbb{R})$ and $PC(\mathbb{R}) \cap L_\infty(\mathbb{R})$, respectively.

Norms for signals (II)

- For $u(t) = \theta(t)$ we have $u \notin L_1(\mathbb{R})$, $u \notin L_2(\mathbb{R})$, but $u \in L_\infty(\mathbb{R})$ and $\|u\|_\infty = 1$.
- For $u(t) = (1 - e^{-t})\theta(t)$ we have $\|u\|_\infty = 1$.
- For $u(t) = \frac{1}{\sqrt{t}}\theta(1 - t)\theta(t)$, $\|u\|_1 = 2$ but $u \notin L_2(\mathbb{R})$, $u \notin L_\infty(\mathbb{R})$ (actually, this is not a $PC(\mathbb{R})$ function).

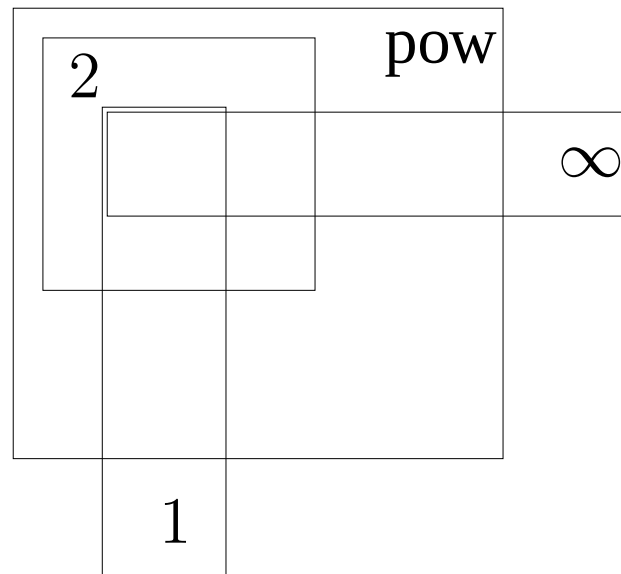
We say that u is a **power signal** if its average power is finite or zero:

$$\text{pow}(u) = \left(\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T u^2(t) \, dt \right)^{1/2} < +\infty.$$

This is **not** a norm, since $\text{pow}(u) = 0$ does not imply $u = 0$. For instance, take $u(t) = e^{-t}\theta(t)$.

Norms for signals (III)

The sets for which the norms and pow exist are related as follows



For instance, if $\|u\|_2$ exists, then u is a power signal and furthermore $\text{pow}(u) = 0$:

$$\frac{1}{2T} \int_{-T}^T u^2(t) \, dt \leq \frac{1}{2T} \int_{-\infty}^{+\infty} u^2(t) \, dt = \frac{1}{2T} \|u\|_2^2 \xrightarrow{T \rightarrow \infty} 0.$$

Norms for systems (I)

A LTI SISO system can be described by its **impulse response** function, h , such that the output y is given in terms of the input u by means of a convolution

$$y = h \star u,$$

i.e.

$$y(t) = \int_{-\infty}^{+\infty} h(t - \tau) u(\tau) \, d\tau.$$

Causality means that $h(t) = 0$ for $t < 0$. The **transfer function** of the system, H , is the Laplace transform of h :

$$H(s) = \int_{0^-}^{+\infty} h(t) e^{-st} \, dt.$$

Norms for systems (II)

The system is

- **stable** if H is analytic in the right half plane $\Re(s) \geq 0$.
- **proper** if H is a rational function and $H(j\infty)$ is finite.
- **strictly proper** if it is proper and $H(j\infty) = 0$.

Norms are given with H evaluated on the imaginary axis:

- The 2-norm

$$||H||_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} |H(j\omega)|^2 d\omega \right)^{1/2}.$$

- The ∞ -norm

$$||H||_\infty = \sup_{\omega \in \mathbb{R}} |H(j\omega)|.$$

Norms for systems (III)

- If the system is causal and stable then $H(j\omega)$ is the Fourier transform of h and Parseval's theorem yields

$$\|H\|_2 = \left(\int_{-\infty}^{+\infty} h^2(t) dt \right)^{1/2}.$$

- The 2-norm of an appropriate signal can be computed as the 2-norm of its Laplace transform.
- The ∞ -norm is **submultiplicative**
 $\|HG\|_\infty \leq \|H\|_\infty \|G\|_\infty.$
- The 2-norm of H exists iff H is strictly proper and has no poles on the imaginary axis.
- The ∞ -norm of H exists iff H is proper and has no poles on the imaginary axis.

How to compute $\|H\|_2$ (I)

Assume $\|H\|_2$ exists. Then

$$\begin{aligned}\|H\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |H(j\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(-j\omega)H(j\omega) d\omega \\ &= \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} H(-s)H(s) ds \\ &= \frac{1}{2\pi j} \oint_{\Gamma} H(-s)H(s) ds\end{aligned}$$

where Γ is the closed curve up the imaginary axis and around an infinite semicircle in the left half-plane.

Notice that

- The integral along the imaginary axis exists due to the absence of poles there and the strict properness of H .
- The integral along the infinite semicircle is zero due to the strict properness of H .

How to compute $\|H\|_2$ (II)

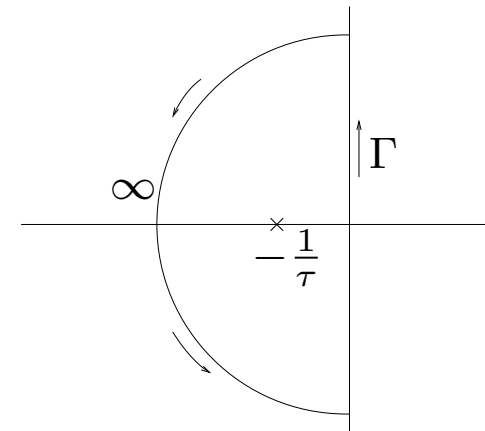
Consider the transfer function of a simple lag,

$$H(s) = \frac{1}{\tau s + 1}, \quad \tau > 0.$$

We have $H(-s)H(s) = \frac{-1}{\tau s - 1} \frac{1}{\tau s + 1}$. The pole in the left half plane is at $s = -1/\tau$ and, using the residue theorem,

$$\begin{aligned} \|H\|_2^2 &= 2\pi j \frac{1}{2\pi j} \text{Res}(s = -1/\tau, H(-s)H(s)) \\ &= \text{Res}(s = -1/\tau, H(-s)H(s)) \\ &= \lim_{s \rightarrow -1/\tau} \left(s - \frac{-1}{\tau} \right) \frac{-1}{\tau s - 1} \frac{1}{\tau s + 1} \\ &= \frac{1}{2\tau} \end{aligned}$$

and hence $\|H\|_2 = \frac{1}{\sqrt{2\tau}}$.



How to compute $\|H\|_\infty$ (I)

- **Numerically:** Set up a grid of frequency points $\{\omega_1, \dots, \omega_N\}$ and **estimate** $\|H\|_\infty$ by

$$\max_{k=1, \dots, N} |H(j\omega_k)|.$$

- **Analytically:** Find the supremum of $|H|^2(j\omega)$ by computing local extrema

$$\frac{d|H|^2}{d\omega}(j\omega) = 0$$

and comparing with the asymptotic values of $|H|^2(j\omega)$ when $|\omega| \rightarrow \infty$. We use $|H|^2$ to simplify computations. The critical points must be substituted in $|H(j\omega)|$ to compute the actual norm.

How to compute $\|H\|_\infty$ (II)

Consider the transfer function of a phase-lag,

$$H(s) = \frac{as + 1}{bs + 1}$$

with $a, b > 0$. One has $|H(j\omega)|^2 = \frac{a^2\omega^2+1}{b^2\omega^2+1}$. If $a \neq b$ this has a local maximum at $\omega = 0$, and $|H(j\omega)|_{\omega=0}^2 = 1$. Furthermore,

$$\lim_{\omega \rightarrow \infty} |H(j\omega)|^2 = \frac{a^2}{b^2}.$$

Hence

$$\|H\|_\infty = \begin{cases} \frac{a}{b} & \text{if } a \geq b, \\ 1 & \text{if } a < b. \end{cases}$$

Input-output relationships (I)

Consider a causal, stable and strictly proper system (so that both $\|H\|_\infty$ and $\|H\|_2$ exist).

If we know how big is the input, how big is the output going to be?

To answer this, it is convenient to consider inputs of norm less than or equal to 1.

The **2-norm/2-norm system gain** is defined as

$$\gamma_{2,2} = \sup_{\|u\|_2 \leq 1} \{\|y\|_2\}.$$

This means that, for any input signal with 2-norm,

$$\|y\|_2 \leq \gamma_{2,2} \|u\|_2.$$

Input-output relationships (II)

- The last relation comes from the fact that, the system being linear, if the output to u is y , then the output to $u/||u||_2$ is $y/||u||_2$. By the definition of $\gamma_{2,2}$, since $u/||u||_2$ has 2-norm ≤ 1 ,

$$\left\| \frac{y}{||u||_2} \right\|_2 \leq \gamma_{2,2},$$

and the relation follows from property (N3).

- It can be shown that, actually,

$$\sup_{||u||_2 \leq 1} \{||y||_2\} = \sup_{||u||_2 = 1} \{||y||_2\}.$$

Input-output relationships (III)

Proposition: The 2-norm/2-norm system gain is given by

$$\gamma_{2,2} = \|H\|_{\infty}.$$

Similarly, one can define the 2-norm/ ∞ -norm system gain, the power/2-norm system gain, ...

Results are summarized in the following table

	$\ u\ _2$	$\ u\ _{\infty}$	pow(u)
$\ y\ _2$	$\ H\ _{\infty}$	∞	∞
$\ y\ _{\infty}$	$\ H\ _2$	$\ h\ _1$	∞
pow(y)	0	$\leq \ H\ _{\infty}$	$\ H\ _{\infty}$

Infinites are valid as long as H is not identically zero.

Input-output relationships (IV)

Consider a system such that the disturbance-to-output transfer function is given by

$$H(s) = \frac{1}{10s + 2}.$$

Assume that the only thing we know about the disturbance, d , is that its 2-norm does not exceed 0.6. We would like to get an estimate of the maximum value that the output is going to have.

From the preceeding table we know that

$$|y(t)| \leq \|y\|_{\infty} \leq \|H\|_2 \|d\|_2 \leq 0.6 \|H\|_2.$$

Since $\|H\|_2 = \frac{1}{\sqrt{40}}$, we can be sure that

$$|y(t)| \leq \frac{0.6}{\sqrt{40}} \sim 0.095 \quad \forall t.$$

State space computations

Norms for systems can be computed from a state space representation of the transfer function.

For instance, assume that H is strictly proper and that we have the representation

$$H(s) = C(s\mathbb{I} - A)^{-1}B.$$

Construct the matrix

$$\Gamma = \begin{pmatrix} A & BB' \\ -C'C & -A' \end{pmatrix}.$$

Theorem: $\|H\|_{\infty} < 1$ if and only if Γ has no eigenvalues on the imaginary axis.

This can be used in an iterative way to compute $\|H\|_{\infty}$. Select $\gamma > 0$ and test if $\|H\|_{\infty} < \gamma$, that is, if $\|\gamma^{-1}H\|_{\infty} < 1$, by using the above theorem for a representation of $\gamma^{-1}H$; increase or decrease γ and repeat with a bisection-like algorithm.