

Partial Solution Set for **Feedback Control Theory**

2.1 (a) $\sup |\dot{u}|$ is not a norm

Proof Set $u(t) \equiv 1$. Then $\sup |\dot{u}| = 0$, but u is not zero.

(b) $\|u\| := |u(0)| + \sup |\dot{u}(t)|$ is a norm

Proof We must verify all four properties.

(i) $\|u\| \geq 0$ is obvious

(ii) If $u(t) = 0 \forall t$, then $\|u\| = 0$. Conversely, suppose $\|u\| = 0$, i.e.,

$$u(0) = 0, \quad \dot{u}(t) = 0 \quad \forall t$$

The second equality shows that $u(t)$ is a constant, i.e., $u(t) = u(0) \forall t$. Then the first equality gives $u(t) = 0 \forall t$.

(iii) $\|au\| = |a|\|u\|$ is easy

(iv)

$$\begin{aligned} \|u + v\| &= |u(0) + v(0)| + \sup |\dot{u}(t) + \dot{v}(t)| \\ &\leq |u(0)| + |v(0)| + \sup |\dot{u}(t)| + \sup |\dot{v}(t)| \\ &= \|u\| + \|v\| \end{aligned}$$

(c) $\|u\| := \max\{\sup |u(t)|, \sup |\dot{u}(t)|\}$ is a norm

Proof Properties (i) – (iii) are easy.

(iv)

$$\begin{aligned} \|u + v\| &= \max\{\sup |u(t) + v(t)|, \sup |\dot{u}(t) + \dot{v}(t)|\} \\ &\leq \max\{\sup |u(t)| + \sup |v(t)|, \sup |\dot{u}(t)| + \sup |\dot{v}(t)|\} \\ &\leq \max\{\sup |u(t)|, \sup |\dot{u}(t)|\} + \max\{\sup |v(t)|, \sup |\dot{v}(t)|\} \\ &\quad (\text{since } \max\{a + b, c + d\} \leq \max\{a, c\} + \max\{b, d\} \text{ for } a, b, c, d \geq 0) \\ &= \|u\| + \|v\| \end{aligned}$$

(d) $\|u\| := \sup |u(t)| + \sup |\dot{u}(t)|$ is a norm — similar proof

2.2

$$\|u_8\|_1 = 1 + 1/2^2 + 1/3^2 + \cdots < \infty$$

$$\|u_8\|_2^2 = 1 + 1/2 + 1/3 + \cdots = \infty$$

$$\|u_8\|_\infty = \infty \text{ (not bounded)}$$

To show that u_8 is a power signal, we must show that

$$\frac{1}{2T} \int_0^T u_8^2(t) dt$$

has a limit as $T \rightarrow \infty$. In fact, the limit equals zero. For each $T > 0$, let k denote the largest integer less than or equal to T . Thus

$$\begin{aligned} \frac{1}{2T} \int_0^T u_8^2(t) dt &\leq \frac{1}{2k} \int_0^{k+1} u_8^2(t) dt \\ &= \frac{1}{2k} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k} \right) \end{aligned}$$

So it remains to show that this series converges to 0. But

$$\frac{1}{2} + \cdots + \frac{1}{k} \leq \int_1^k \frac{dt}{t} = \ln k$$

So

$$\frac{1}{2k} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k} \right) \leq \frac{1}{2k} (1 + \ln k) \rightarrow 0$$

2.3 False. The two transfer functions

$$\frac{1}{s+1}, \quad \frac{-1}{s+1}$$

are both stable, they have stable inverses, they have identical Bode magnitude plots, but their phase plots differ by 180° .

2.4 Both the 2-norm and the ∞ -norm are time-delay invariant. This follows immediately from the fact that

$$|\hat{D}(j\omega)| = 1 \quad \forall \omega$$

2.5 Since

$$G(t) = \frac{1}{\tau} \exp\left(-\frac{t}{\tau}\right) 1(t)$$

there follows

$$\|G\|_1 = \frac{1}{\tau} \int_0^\infty \exp\left(-\frac{t}{\tau}\right) dt = 1$$

2.6 Assume $\hat{G}(s)$ is stable and strictly proper and consider its partial-fraction expansion. If $\hat{G}$ has no repeated poles, the expansion looks like

$$\hat{G}(s) = \sum \frac{c}{s-a}$$

where the summation is over all poles a . Here c , the residue at the pole a , is a complex number. Also, since $\hat{G}$ is stable, $\operatorname{Re} a < 0$. When $\hat{G}$ has repeated poles, the general form of the expansion is

$$\hat{G}(s) = \sum \frac{c}{(s-a)^k}$$

and the summation is over the poles a and certain integers k . For example, the expansion of

$$\hat{G}(s) = \frac{s+2}{(s+1)^2}$$

has the form

$$\frac{c_1}{s+1} + \frac{c_2}{(s+1)^2}$$

In view of the triangle inequality, to prove that $\|G\|_1$ is finite, it suffices to prove finiteness of the 1-norm of the inverse transform of

$$\frac{1}{(s-a)^k}$$

This inverse transform is a constant times

$$t^{k-1}e^{at}1(t)$$

So it remains to show that the latter function has finite 1-norm.

Let $\sigma := \operatorname{Re} a$. Then

$$\int_0^\infty |t^{k-1}e^{at}| dt = \int_0^\infty t^{k-1}e^{\sigma t} dt$$

Integrate by parts to get

$$\begin{aligned} \int_0^\infty t^{k-1}e^{\sigma t} dt &= \frac{1}{\sigma} t^{k-1}e^{\sigma t} \Big|_0^\infty - \frac{k-1}{\sigma} \int_0^\infty t^{k-2}e^{\sigma t} dt \\ &= -\frac{k-1}{\sigma} \int_0^\infty t^{k-2}e^{\sigma t} dt \end{aligned}$$

Continuing in this way we find that

$$\int_0^\infty t^{k-1}e^{\sigma t} dt = \text{constant} \times \int_0^\infty e^{\sigma t} dt = \text{constant} \times \frac{1}{\sigma}$$

Finally, the relationship between the two norms is $\|\hat{G}\|_\infty \leq \|G\|_1$. This follows directly from the definition of $\hat{G}$.

2.7 The inverse Laplace transform of

$$c + \hat{G}_1(s)$$

is

$$c\delta(t) + G_1(t)1(t)$$

and then the input-output convolution equation is

$$y(t) = cu(t) + \int_{-\infty}^\infty G_1(t-\tau)u(\tau) d\tau$$

Suppose $\|u\|_\infty \leq 1$. Then

$$\begin{aligned}
 |y(t)| &\leq |cu(t)| + \int_{-\infty}^{\infty} |G_1(t-\tau)u(\tau)| d\tau \\
 &\leq \left(|c| + \int_{-\infty}^{\infty} |G_1(t-\tau)| d\tau \right) \|u\|_\infty \\
 &\leq |c| + \int_{-\infty}^{\infty} |G_1(t-\tau)| d\tau \\
 &= |c| + \|G_1\|_1
 \end{aligned}$$

Hence

$$\sup\{\|y\|_\infty : \|u\|_\infty \leq 1\} \leq |c| + \|G_1\|_1$$

Conversely, fix t and set

$$u(t-\tau) = \begin{cases} |c|/c, & \tau = 0 \\ |G_1(\tau)|/G_1(\tau), & \tau \neq 0 \end{cases}$$

Then $\|u\|_\infty = 1$ and

$$\begin{aligned}
 y(t) &= cu(t) + \int_{-\infty}^{\infty} G_1(\tau)u(t-\tau) d\tau \\
 &= |c| + \|G\|_1
 \end{aligned}$$

and so

$$\sup\{\|y\|_\infty : \|u\|_\infty \leq 1\} \geq |c| + \|G_1\|_1$$

2.9 True:

$$\begin{aligned}
 y(t) &= \int_{-\infty}^{\infty} G(t-\tau)u(\tau)d\tau \\
 \|y\|_1 &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} G(t-\tau)u(\tau)d\tau \right| dt \\
 &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G(t-\tau)||u(\tau)|d\tau dt \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G(t-\tau)|dt|u(\tau)|d\tau \\
 &= \|G\|_1 \|u\|_1
 \end{aligned}$$

2.10 From Table 2.1

$$\begin{aligned}
 \text{pow}(y) &= \frac{1}{\sqrt{2}} |\hat{G}(0.1j)| \\
 &= \frac{1}{\sqrt{2}} \left| \frac{\omega_n^2}{\omega_n^2 - 0.01 + 0.2\zeta\omega_n j} \right|
 \end{aligned}$$

2.11 We have

$$\frac{s+2}{4s+1} = 0.25 + \frac{1.75}{4s+1} =: c + \hat{G}_1(s).$$

So from Exercise 2.7

$$\begin{aligned} \sup_{\|u\|_\infty \leq 1} \|y\|_\infty &= |c| + \|G_1\|_1 \\ &= 0.25 + \frac{1.75}{4} \int_0^\infty e^{-0.25t} dt. \end{aligned}$$

A worse input is

$$u(-\tau) = \begin{cases} \operatorname{sgn}(c), & \tau = 0 \\ \operatorname{sgn}[G_1(\tau)], & \tau \neq 0, \end{cases}$$

that is, $u(\tau) = 1$.

2.12 Define

$$s_1 = \sup_{\|u\| \leq 1} \|y\|, \quad s_2 = \sup_{\|u\|=1} \|y\|.$$

Since

$$\{u : \|u\| = 1\} \subset \{u : \|u\| \leq 1\},$$

so $s_1 \geq s_2$.

For the other way, let $\epsilon > 0$. Choose u_1 so that

$$\|u_1\| \leq 1, \quad \|y\| > s_1 - \epsilon.$$

Set $u_2 = u_1/\|u_1\|$, which has $\|u_2\| = 1$. Then the output is

$$y_2 = \frac{1}{\|u_1\|} y_1.$$

Therefore

$$\|y_2\| = \frac{1}{\|u_1\|} \|y_1\| \geq \|y_1\| > s_1 - \epsilon,$$

so $s_2 > s_1 - \epsilon$. Since ϵ was arbitrary, so $s_2 \geq s_1$.

2.13 Define

$$G_1(s) = \frac{1}{s+a}, \quad G_2(s) = \frac{1}{s+b}, \quad a, b > 0, \quad a \neq b.$$

You can compute using the residue method that

$$\|G_a\|_2^2 = \frac{1}{2a}, \quad \|G_b\|_2^2 = \frac{1}{2b}, \quad \|G_a G_b\|_2^2 = \frac{1}{2ab(a+b)}.$$

So $\|G_a G_b\|_2 \leq \|G_a\|_2 \|G_b\|_2$ iff

$$\frac{1}{2ab(a+b)} \leq \frac{1}{4ab}$$

or equivalently $a+b \geq 2$. So for a counterexample, just take $a+b < 2$.

3.1 When $F = 1$ the 9 transfer functions are

$$\frac{1}{1+PC} \begin{bmatrix} 1 & -P & -1 \\ C & 1 & -C \\ PC & P & 1 \end{bmatrix}$$

Modulo signs, there are 4 distinct ones:

$$\frac{1}{1+PC}, \quad \frac{P}{1+PC}, \quad \frac{C}{1+PC}, \quad \frac{PC}{1+PC}$$

So internal stability is equivalent to stability of these 4.

None of these 4 transfer functions alone is sufficient. For example, $PC/(1+PC)$ may be stable and yet $P/(1+PC)$ is unstable (see the example in Section 3.2). So the minimum number to test is at least 2. In fact

$$\frac{P}{1+PC}, \quad \frac{C}{1+PC}$$

suffice, and hence the minimum number is 2. To prove this, assume these two transfer functions are stable, and bring in coprime numerator-denominator pairs:

$$P = \frac{N_P}{M_P}, \quad C = \frac{N_C}{M_C},$$

Then

$$\frac{N_P M_C}{M_P M_C + N_P N_C}, \quad \frac{N_C M_P}{M_P M_C + N_P N_C}$$

are stable. Since the greatest common divisor of the 3 polynomials

$$N_P M_C, \quad N_C M_P, \quad M_P M_C + N_P N_C$$

is 1 (easily checked), it follows that the characteristic polynomial

$$M_P M_C + N_P N_C$$

has no roots in $\text{Re } s \geq 0$. By Theorem 3.1, the feedback system is internally stable.

3.2 The characteristic poly is $10s + 1 + k$. So internal stability holds iff $k > -1$.

The transfer function from r to e is

$$\frac{10s + 1}{10s + 1 + k}$$

The dc gain of this is $1/(1+k)$. Thus condition 2 holds iff

$$\frac{1}{1+k} \leq 0.1 \Leftrightarrow k \geq 9$$

The transfer function from d to y is

$$\frac{1}{10s + 1 + k}$$

Hence

$$\sup_{\|d\|_2 \leq 1} \|y\|_\infty = \left\| \frac{1}{10s + 1 + k} \right\|_2 = 1/\sqrt{20(1+k)}$$

So condition 3 holds iff

$$1/\sqrt{20(1+k)} \leq 0.1 \Leftrightarrow k \geq 4$$

The least k satisfying all 3 conditions is 9.

3.3 We have

$$\hat{y}(s) = \frac{\hat{P}(s)}{1 + \hat{P}(s)\hat{C}(s)}\hat{d}(s)$$

With the assumption that the feedback system is internally stable,

$$\frac{\hat{P}}{1 + \hat{P}\hat{C}}$$

is stable. But $\hat{d}(s)$ has a pole at $j\omega$. Application of the final value theorem shows that $y(t) \rightarrow 0$ as $t \rightarrow \infty$ iff

$$\frac{\hat{P}}{1 + \hat{P}\hat{C}}(j\omega) = 0$$

This holds iff

$$\hat{P}(j\omega) = 0 \text{ or } \hat{C}(j\omega) = \infty$$

3.4 To avoid triviality, assume P and F are $\neq 0$. Let $e = r - y$. Then

$$e = \left(1 - \frac{PC}{1 + PCF}\right)r = \frac{1 + PCF - PC}{1 + PCF}r$$

The desired conditions are

- (1) internal stability
- (2) $(1 + PCF - PC)/(1 + PCF)$ has a zero at $s = 0$
- (3) $P/(1 + PCF)$ has a zero at $s = 100j$

Note that (1) is feasible iff PF has no pole-zero cancellations in $\text{Re } s \geq 0$. Assume this and let C be internally stabilizing. Do coprime factorizations with polynomial numerators and denominators. Then

$$\frac{1 + PCF - PC}{1 + PCF} = \frac{D_P D_C D_F + N_P N_C (N_F - D_F)}{\text{char poly}}$$

$$\frac{P}{1 + PCF} = \frac{N_P D_C D_F}{\text{char poly}}$$

So (2) and (3) hold iff respectively

- (2') $D_P D_C D_F + N_P N_C (N_F - D_F)$ has a zero at $s = 0$
- (3') $N_P D_C D_F$ has a zero at $s = 100j$

Look at (2'):

- (i) If $D_P D_F(0) = 0$ and $N_P(N_F - D_F)(0) = 0$, any C suffices.

- (ii) If $D_P D_F(0) = 0$ and $N_P(N_F - D_F)(0) \neq 0$, we must have $N_C(0) = 0$; such a C is not stabilizing, so this case is not feasible.
- (iii) If $D_P D_F(0) \neq 0$ and $N_P(N_F - D_F)(0) = 0$, we must have $D_C(0) = 0$; this is feasible iff $N_P N_F(0) \neq 0$.
- (iv) If $D_P D_F(0) \neq 0$ and $N_P(N_F - D_F)(0) \neq 0$, we must have

$$C(0) = -\frac{1}{P(0)[F(0) - 1]}$$

This is always feasible.

Look at (3'):

- (i) If $N_P D_F(100j) = 0$, any C suffices.
- (ii) If $N_P D_F(100j) \neq 0$, we must take $D_C(100j) = 0$; this is feasible iff $N_F(100j) \neq 0$.

In summary, a suitable controller exists iff the following four conditions hold:

PF has no pole-zero cancellations in $\text{Re } s \geq 0$

it's not true that $D_P D_F(0) = 0, N_P(N_F - D_F)(0) \neq 0$

if $D_P D_F(0) \neq 0, N_P(N_F - D_F)(0) = 0$ then $N_P N_F(0) \neq 0$

if $N_P D_F(100j) \neq 0$ then $N_F(100j) \neq 0$

4.1 False. For example $C(s) = 2$ stabilizes both

$$\frac{1}{s+1} \text{ and } \frac{1}{s-1}$$

4.2 False. For example $C(s) = 2$ stabilizes

$$P_\alpha(s) = \frac{1}{s+1-2\alpha}$$

for all α in $[0, 1]$.

4.3 No such controller exists.

To prove this, let C be a controller transfer function and write $C = N_C/M_C$, a ratio of coprime polynomials. Then the characteristic polynomial is

$$sM_C(s) + kN_C(s)$$

The lowest degree term, i.e., the constant, is $kN_C(0)$ while the highest degree term equals that of $sM_C(s)$; this is because $\text{degree } sM_C(s) > \text{degree } kN_C(s)$. If $N_C(0) = 0$, then C has a zero at $s = 0$, where the plant pole is. So such a controller would not achieve internal stability. Now suppose $N_C(0) \neq 0$. Then the lowest degree term, $kN_C(0)$, and the highest degree term have unequal signs either for $k = +1$ or for $k = -1$. By Routh-Hurwitz, internal stability fails for this k .

4.4 Sketch the Bode magnitude plot of P ; it's a monotonically decreasing function of frequency. In particular, $|P|$ equals 1 at a unique frequency, say ω_1 , called the gain crossover frequency.

Solve for ω_1 :

$$\begin{aligned} |P(j\omega)| &= 1 \\ \Rightarrow \frac{\omega_n^2}{\omega|j\omega + 2\zeta\omega_n|} &= 1 \\ \Rightarrow \omega_n^4 &= \omega^2(\omega^2 + 4\zeta^2\omega_n^2) \\ \Rightarrow \omega^4 + 4\zeta^2\omega^2 - \omega_n^4 &= 0 \\ \Rightarrow \omega_1 &= \left[\sqrt{4\zeta^4 + 1} - 2\zeta^2 \right]^{1/2} \omega_n \end{aligned}$$

Then

$$P(j\omega_1) = \frac{\omega_n^2}{j\omega_1(j\omega_1 + 2\zeta\omega_n)}$$

So the phase margin, ϕ_{max} , is given by

$$\begin{aligned} \phi_{max} &= \frac{\pi}{2} - \tan^{-1} \frac{\omega_1}{2\zeta\omega_n} \\ &= \frac{\pi}{2} - \tan^{-1} \frac{\left[\sqrt{4\zeta^4 + 1} - 2\zeta^2 \right]^{1/2}}{2\zeta} \end{aligned}$$

4.5 The characteristic polynomial equals

$$s(s+1)(s+\alpha)+1=s^3+(1+\alpha)s^2+\alpha s+1$$

By Routh-Hurwitz, the feedback system is internally stable iff

$$\alpha > 0 \text{ and } \alpha(1+\alpha) > 1$$

that is,

$$\alpha > \frac{\sqrt{5}-1}{2} \approx .618$$

Assume α satisfies this inequality. A glance at the Nyquist plot of $P(s)C(s)$ shows that it crosses the negative real axis at only one point. To find the frequency of crossover (called the phase crossover frequency—when the phase crosses 180°), solve the equation

$$\text{Im } PC(j\omega) = 0$$

This gives $\omega = \sqrt{\alpha}$. At this frequency,

$$PC(j\omega) = \frac{-1}{\alpha(1+\alpha)}$$

So the upper gain margin is obtained via

$$\frac{-1}{k_{max}} = \frac{-1}{\alpha(1+\alpha)}$$

i.e., $k_{max} = \alpha(1+\alpha)$. The lower gain margin k_{min} equals 0.

4.6 (a) The feedback system is internally stable iff $a < 10$.

(b) Note that the feedback system with $C(s) = 10$ and $P(s) = 1/s$ is internally stable. First, suppose $a \geq 0$. Allow Δ to be a variable real number in the interval $[0, 1]$. Then

1. P and $\tilde{P}$ have the same number of poles in $\text{Re } s \geq 0$, i.e., $\tilde{P}$ has exactly one pole there
2. $\|\Delta\|_\infty \leq 1$

From problem 4.3, robust stability for such perturbations is guaranteed if

$$\|W_2PS\|_\infty < 1$$

i.e., $a < 10$.

Second, suppose $a < 0$. Allow Δ to be a variable real number in $[-1, 0]$. The same argument shows that robust stability is guaranteed if $a > -10$.

Putting the two cases together, we get that the feedback system is internally stable for $-10 < a < 10$.

(c) Again, the feedback system with $C(s) = 10$ and $P(s) := 1/(s+100)$ is internally stable. We have

$$\tilde{P}(s) = \frac{1}{s-a}$$

if $W_2(s) = -(a+100)$ and $\Delta(s) = 1$. Fix W_2 as just written. First, suppose $a+100 \geq 0$. Allow Δ to be a variable real number in the interval $[-1, 0]$. Then

1. P and $\tilde{P}$ have the same number of poles in $\text{Re } s \geq 0$, namely, none

2. $\|\Delta\|_\infty \leq 1$

so robust stability for such perturbations is guaranteed if

$$\|W_2PS\|_\infty < 1$$

i.e., $a + 100 < 110$, i.e., $a < 10$.

Second, suppose $a + 100 < 0$. Allow Δ to be a variable real number in $[0, 1]$. The same argument shows that robust stability is guaranteed if $a + 100 > -110$, i.e., $a > -210$.

Putting the two cases together, we get that the feedback system is internally stable for $-210 < a < 10$.

The conclusions of this exercise are these: for real parameter variations, robust stability theory is conservative; the results are dependent on what the nominal plant is taken to be.

4.7 Draw the block diagram:

Now redraw in this way:

Assuming internal stability for the nominal plant P and controller C , all the 16 transfer functions G_{ij} are stable. We want conditions for stability of the transfer functions from r and d to e and u .

Write the equations for x_1 and x_3 :

$$\begin{aligned} x_1 &= G_{31}r + G_{32}d + G_{33}\Delta_2x_1 + G_{34}\Delta_1x_3 \\ x_3 &= G_{41}r + G_{42}d + G_{43}\Delta_2x_1 + G_{44}\Delta_1x_3. \end{aligned}$$

It follows that internal stability will be guaranteed if the following matrix has a stable inverse:

$$\begin{bmatrix} 1 - G_{33}\Delta_2 & -G_{34}\Delta_1 \\ -G_{43}\Delta_2 & 1 - G_{44}\Delta_1 \end{bmatrix}.$$

The determinant of this matrix equals

$$(1 - G_{33}\Delta_2)(1 - G_{44}\Delta_1) - G_{43}\Delta_2G_{34}\Delta_1.$$

We conclude that internal stability will be guaranteed if this determinant has a stable inverse. Bring in T , the complementary sensitivity function. It can be derived that, in terms of T , the determinant equals

$$1 + WT\Delta_1 + VT\Delta_2 + VWT\Delta_1\Delta_2.$$

From this we can conclude the following robust stability result: Assume V and W are stable and proper; assume internal stability for the nominal P and C . Then internal stability holds for all stable perturbations Δ_1 and Δ_2 satisfying $\|\Delta_1\|_\infty < 1$ and $\|\Delta_2\|_\infty < 1$ provided the following small-gain condition holds:

$$\|(|W| + |V| + |VW|)T\|_\infty \leq 1.$$

4.8 Similar to Project #2.

4.9 Assume $\|W_2PS\|_\infty \leq 1$ and let Δ be a stable perturbation satisfying $\|\Delta\|_\infty < 1$. Draw the block diagram:

Now redraw in this way:

Robust stability follows.

4.10 The robust stability condition is $\|W_2T\|_\infty < 0.5$. We have

$$T(s) = \frac{k}{s + k - 1}.$$

Internal stability holds iff $k > 1$. Assuming this, we have

$$\|W_2 T\|_\infty = |W_2(0)T(0)| = \frac{k}{5(k-1)}.$$

Since $\|W_2 T\|_\infty < 0.5$ iff $k > 5/3$, this latter is the range of k for robust stability.

5.1 The algorithm goes like this:

$$\begin{aligned}
n(\lambda) &= (1 - \lambda)^3 \\
m(\lambda) &= 3\lambda^3 - 3\lambda^2 + \lambda \\
q_1(\lambda) &= -\frac{1}{3} \\
r_1(\lambda) &= 2\lambda^2 - \frac{8}{3}\lambda + 1 \\
q_2(\lambda) &= \frac{3}{2}\lambda + \frac{1}{2} \\
r_2(\lambda) &= \frac{5}{6}\lambda - \frac{1}{2} \\
q_3(\lambda) &= \frac{12}{5}\lambda + \frac{6 \times 22}{5 \times 15} \\
r_3(\lambda) &= \frac{3}{25} \\
x(\lambda) &= 30\lambda^2 - 12\lambda + 1 \\
y(\lambda) &= 10\lambda^2 - 24\lambda + 15 \\
N(s) &= \frac{s^3}{(s+1)^3} \\
M(s) &= \frac{s^2 - s + 1}{(s+1)^3} \\
X(s) &= \frac{s^2 - 10s + 19}{(s+1)^2} \\
Y(s) &= \frac{15s^2 + 6s + 1}{(s+1)^2}
\end{aligned}$$

5.2 Do coprime factorization of P :

$$N(s) = \frac{3}{s+1}, \quad M(s) = \frac{s-4}{s+1}$$

$$X(s) = \frac{5}{3}, \quad Y(s) = 1$$

Then $S = M(Y - NQ)$. The tracking condition holds iff S has a zero at $s = 0$ of multiplicity at least 2. Try a Q in $\mathcal{S}$ with 2 variables, say

$$Q(s) = \frac{as+b}{s+1}$$

Then

$$(Y - NQ)(s) = 1 - \frac{3}{s+1} \frac{as+b}{s+1} = \frac{s^2 + (2-3a)s + (1-3b)}{(s+1)^2}$$

So take $a = 2/3$, $b = 1/3$. The resulting C is

$$C(s) = \frac{7s^2 + 3s + 1}{3s^2}$$

5.3 The first step is to do a coprime factorization of P . Following the procedure in Section 5.2 gives

$$n(\lambda) = \lambda^3, \quad m(\lambda) = -(\lambda - 1)(1.8\lambda^2 - 1.8\lambda + 1)$$

Procedure A terminates at step 3. Then

$$x(\lambda) = 7.6320\lambda^2 - 10.2240\lambda + 3.5920, \quad y(\lambda) = 4.24\lambda^2 + 2.8\lambda + 1$$

Conversion back to s gives

$$\begin{aligned} N(s) &= \frac{1}{(s+1)^3} \\ M(s) &= \frac{s(s^2 + 0.2s + 1)}{(s+1)^3} \\ X(s) &= \frac{3.592s^2 - 3.04s + 1}{(s+1)^2} \\ Y(s) &= \frac{s^2 + 4.8s + 8.04}{(s+1)^2} \end{aligned}$$

The two performance specs translate to

$$M(Y - NQ) \text{ has two zeros at } s = 0$$

$$N(Y - NQ)(2j) = 0$$

Since M already has one zero at the origin, the two specs become

$$(Y - NQ)(0) = 0$$

$$(Y - NQ)(2j) = 0$$

i.e.,

$$Q(0) = 8.04 \quad Q(2j) = -15.16 + 17.68j$$

The rest of the solution goes exactly as in Section 5.5.

5.4 From Exercise 3.1 we know that internal stability holds iff

$$\frac{P}{1+PC}, \quad \frac{C}{1+PC}$$

are stable. Substituting in $C = Q/(1 - PQ)$, we get that internal stability holds iff

$$P(1 - PQ), \quad Q$$

are stable. Now

$$P(s)[1 - P(s)Q(s)] = \frac{s - Q(s)}{s^2}.$$

We conclude that the functions Q for which internal stability holds is precisely

$$\{Q : Q \in \mathcal{S} \text{ and } s - Q(s) \text{ has two zeros at } s = 0\}.$$

5.5 If N and M are coprime, there exist functions X and Y in $\mathcal{S}$ such that

$$NX + MY = 1.$$

Since

$$NM^{-1}X + Y = M^{-1},$$

if $NM^{-1} \in \mathcal{S}$, then $M^{-1} \in \mathcal{S}$. The result is not true without the coprimeness assumption; for example,

$$N(s) = M(s) = \frac{s-1}{s+1}.$$

5.6 Parametrize all internally stabilizing C s. Then the transfer function from r to e equals $M(Y - NQ)$. If $M(0) = 0$ (i.e., P has a pole at $s = 0$), then every internally stabilizing controller achieves step tracking. Now assume $M(0) \neq 0$. Then a step-tracking controller exists iff $N(0) \neq 0$, in which case all step-tracking controllers are given by

$$\left\{ \frac{X + MQ}{Y - NQ} : Q \in \mathcal{S}, Q(0) = Y(0)/N(0) \right\}.$$

5.7 Again, do a coprime factorization:

$$N(s) = \frac{s}{(s+1)^2} \quad M(s) = \frac{(s-1)(10s+1)}{(s+1)^2}$$

$$X(s) = \frac{139s-59}{10(s+1)} \quad Y(s) = \frac{s-10}{10(s+1)}$$

We can take $C = X/Y$:

$$C(s) = \frac{139s-59}{s-10}$$

Then the characteristic poly is

$$(s-10)(s-1)(10s+1) + s(139s-59) = 10(s+1)^3$$

So the closed-loop poles are $\{-1, -1, -1\}$. Finally, the dc gain from d to y equals 0, because P has a zero at $s = 0$.

6.1 First, note that the signs of the factors are not unique, for example, if $G = G_{ap}G_{mp}$, then $G = (-G_{ap})(-G_{mp})$. We want to show that this is the only type of nonuniqueness.

Suppose we have two factorizations $G = G_{ap}G_{mp}$, $G = F_{ap}F_{mp}$. Then G_{ap} and F_{ap} have exactly the same zeros, hence the same poles (the poles of an all-pass function are the mirror images of the zeros). Thus they must differ by a gain, that is, $G_{ap} = cF_{ap}$ for some constant real c . Since G_{ap} and F_{ap} both have magnitude 1 on the imaginary axis, it must be that $|c| = 1$. But the only real numbers with magnitude 1 are $c = \pm 1$. Thus $G_{ap} = \pm F_{ap}$.

6.2 False. Take

$$P(s) = \frac{s-1}{(s-2)(s+1)}$$

and let C be any internally stabilizing controller. Then

$$T(1) = 0, \quad T(2) = 1$$

From the first equation, T must have the form

$$T(s) = \frac{s-1}{s+1}T_1(s)$$

for some T_1 in $\mathcal{S}$. Then

$$\|T\|_\infty = \|T_1\|_\infty$$

But

$$1 = T(2) = \frac{1}{3}T_1(2)$$

and by the maximum modulus theorem

$$\|T_1\|_\infty \geq |T_1(2)| = 3$$

So δ can't be less than 3.

6.4 From the proof of Theorem 6.1

$$\begin{aligned} 0 \leq \log |S_{ap}(2)^{-1}| &= \frac{1}{\pi} \int_{-\infty}^{\infty} \log |S(j\omega)| \frac{2}{4+\omega^2} d\omega \\ &\leq \log M_1 \frac{2}{\pi} \int_0^{0.1} \frac{2}{4+\omega^2} d\omega + (\log 1.5) \frac{2}{\pi} \int_{0.1}^{\infty} \frac{2}{4+\omega^2} d\omega \\ &= \log M_1 \frac{2}{\pi} 0.025 + (\log 1.5) \frac{2}{\pi} 0.7604 \end{aligned}$$

Thus $\log M_1 \geq -5.356$, so

$$\max_{\omega \leq 0.1} |S| \geq 10^{-5.356}$$

6.5 Since $S(1 + PC) = 1$ we have

$$W_1(s_0) = W_1S(1 + PC)(s_0) = W_1S(s_0) + W_1PCS(s_0)$$

and hence

$$|W_1(s_0)| \leq |W_1S(s_0)| + |W_1P(s_0)| |CS(s_0)|$$

By the maximum modulus theorem this implies that

$$|W_1(s_0)| \leq \|W_1S\|_\infty + |W_1P(s_0)| \|CS\|_\infty$$

6.6 Evaluating the following functions at $s = j\omega$ but dropping the argument, we have

$$\epsilon = 1/|1 + PC|$$

Thus

$$\frac{1}{\epsilon} = |1 + PC| \leq 1 + |P||C|$$

and so

$$|C| \geq \frac{1}{|P|} \frac{1 - \epsilon}{\epsilon}$$

The right-hand side blows up as $\epsilon \rightarrow 0$.

6.7 Since P has relative degree 2, so will PC ; hence the area formula applies. The poles of P are $0.5 \pm 1.9365j$. So we have

$$\int_0^\infty \log |S| \geq \pi \log e.$$

From the given specs,

$$\int_0^\infty \log |S| \leq 0.1 \log \epsilon + 4.9 \log 2.$$

Thus

$$0.1 \log \epsilon + 4.9 \log 2 \geq \pi \log e,$$

which gives $\epsilon \geq 0.0782$.