

Chapter 9

Model Matching

This chapter studies a hypothetical control problem called the model-matching problem which will be used in later chapters for control system design. The mathematics of interpolation theory is used to solve the model-matching problem.

9.1 The Model-Matching Problem

Let $T_1(s)$ and $T_2(s)$ be stable proper transfer functions (i.e., functions in $\mathcal{S}$). The *model-matching problem* is to find a stable transfer function $Q(s)$ to minimize the ∞ -norm of $T_1 - T_2Q$. The interpretation is this: T_1 is a model, T_2 is a plant, and Q is a cascade controller to be designed so that T_2Q approximates T_1 . Thus $T_1 - T_2Q$ is the error transfer function. The transfer function Q is required to be stable but not necessarily proper (this makes the problem easier). We will *assume* that T_2 has no zeros on the imaginary axis. Define the minimum model-matching error

$$\gamma_{\text{opt}} := \min \|T_1 - T_2Q\|_{\infty},$$

where the minimum is taken over all stable Q s. It turns out that the minimum is achieved by virtue of the assumption on T_2 ; a Q achieving the minimum is said to be *optimal*.

The trivial case of the problem is when T_1/T_2 is stable, for then the unique optimal Q is $Q = T_1/T_2$ and $\gamma_{\text{opt}} = 0$. The simplest nontrivial case is when T_2 has only one zero in the right half-plane, say at $s = s_0$. If Q is stable and T_2Q has finite ∞ -norm (i.e., $T_2Q \in \mathcal{S}$), then by the maximum modulus theorem

$$\|T_1 - T_2Q\|_{\infty} \geq |T_1(s_0)|,$$

so $\gamma_{\text{opt}} \geq |T_1(s_0)|$. On the other hand, the function

$$Q = \frac{T_1 - T_1(s_0)}{T_2} \tag{9.1}$$

is stable and yields the value $|T_1(s_0)|$ for the model-matching error. The conclusion is that $\gamma_{\text{opt}} = |T_1(s_0)|$ and (9.1) is an optimal Q , in fact, the unique optimal Q .

Example For

$$T_1(s) = \frac{4}{s+3}, \quad T_2(s) = \frac{s-2}{(s+1)^3}$$

$\gamma_{\text{opt}} = T_1(2) = 4/5$ and the optimal Q is

$$Q(s) = -\frac{4(s+1)^3}{5(s+3)}.$$

To solve the problem in the general case we will need the mathematics developed in the next two sections.

9.2 The Nevanlinna-Pick Problem

Recall that $\mathcal{S}$ stands for the space of stable, proper, real-rational functions. Let $\mathcal{S}_c$ stand for the space of stable, proper, complex-rational functions (i.e., the coefficients are permitted to be complex numbers). For example, the function

$$\frac{(1-j)s + (2+3j)}{(0.1+j)s + (-3+j)}$$

is in $\mathcal{S}_c$ because it is proper and its pole is at $s = -0.6931 - 3.0693j$ in the left half-plane. The ∞ -norm, maximum magnitude on the imaginary axis, is defined for such functions too.

Let $\{a_1, \dots, a_n\}$ be a set of points in the open right half-plane, $\text{Re } s > 0$, and $\{b_1, \dots, b_n\}$ a set of points in $\mathbb{C}$. For simplicity we shall *assume* that the points $a_1, \dots, a_n$ are distinct. The *Nevanlinna-Pick interpolation problem*, or the NP problem for short, is to find a function G in $\mathcal{S}_c$ satisfying the two conditions

$$\begin{aligned} \|G\|_\infty &\leq 1, \\ G(a_i) &= b_i, \quad i = 1, \dots, n. \end{aligned}$$

The latter equation says that G is to interpolate the value b_i at the point a_i , or in other words the graph of G is to pass through the point (a_i, b_i) . The constraints are important: G must be stable, proper, and satisfy $\|G\|_\infty \leq 1$. The NP problem is said to be *solvable* if such a function G exists. It will be convenient to write the problem data as an array, like this:

$$\begin{array}{ccc} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{array}$$

In fact, the NP problem is not solvable for all data. An obvious necessary condition for solvability is $|b_i| \leq 1$, $i = 1, \dots, n$. This follows from the maximum modulus theorem: If G belongs to $\mathcal{S}_c$ and satisfies $G(a_i) = b_i$, then its magnitude equals $|b_i|$ at the point $s = a_i$, so its maximum magnitude in the right half-plane is $\geq |b_i|$ (i.e., $\|G\|_\infty \geq |b_i|$); but if it is also true that $\|G\|_\infty \leq 1$, then $|b_i| \leq 1$.

To state precisely when the NP problem is solvable, we need some elementary concepts and facts about complex matrices. Let M be a square complex matrix. Its complex-conjugate transpose is denoted by M^* . If $M = M^*$, M is said to be a *Hermitian matrix*. If M is real, it is Hermitian iff it is symmetric. It can be shown that the eigenvalues of a Hermitian matrix are all real. If M is Hermitian, it is said to be *positive semidefinite* if $x^* M x \geq 0$ for all complex vectors x , and *positive definite* if $x^* M x > 0$ for all nonzero complex vectors x . The notation is $M \geq 0$ and $M > 0$, respectively. It is a fact that $M \geq 0$ (respectively, $M > 0$) iff all its eigenvalues are ≥ 0 (respectively, > 0).

Example 1 The matrix

$$\begin{bmatrix} 2 & 1+j \\ 1-j & 4 \end{bmatrix}$$

is Hermitian; notice that the diagonal elements must be real. The eigenvalues are 1.2679 and 4.7321. Since these are both positive, the matrix is positive definite.

Associated with the NP problem data

$$\begin{array}{ccc} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{array}$$

is the $n \times n$ matrix Q , whose ij^{th} element is

$$\frac{1 - b_i \overline{b_j}}{a_i + \overline{a_j}}.$$

This is called the *Pick matrix*. Notice that Q is Hermitian.

Example 2 For the data

$$\begin{array}{cc} 6+j & 6-j \\ 0.1-0.1j & 0.1+0.1j \end{array}$$

the Pick matrix is

$$\begin{bmatrix} 0.0817 & 0.0814 - 0.0119j \\ 0.0814 + 0.0119j & 0.0817 \end{bmatrix},$$

whose eigenvalues are -0.0005 and 0.1639 . Since one is negative, it turns out that the NP problem is not solvable for these data.

Solvability of the NP problem is completely determined by the Pick matrix. The result is Pick's famous theorem:

Theorem 1 *The NP problem is solvable iff $Q \geq 0$.*

Pick's theorem shows that it is an easy matter to check solvability of the NP problem by computer: Input the data

$$\begin{array}{ccc} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{array}$$

form the Pick matrix; compute its eigenvalues; see if the smallest one is nonnegative.

We saw above that a necessary condition for solvability is $|b_i| \leq 1$ for all i . So it must be that this condition is implied by the condition $Q \geq 0$. This is indeed the case: If $Q \geq 0$, then each diagonal element of Q is ≥ 0 , that is,

$$\frac{1 - |b_i|^2}{2\text{Re}a_i} \geq 0.$$

Since $\text{Re}a_i > 0$, this implies that

$$1 - |b_i|^2 \geq 0$$

(i.e., $|b_i| \leq 1$).

In the next section is given a procedure for constructing a solution to the NP problem when it is solvable. The remainder of this section contains a proof of the necessity part of Pick's theorem, a

proof that illustrates in system-theoretic terms how the Pick matrix arises. A system is said to be *dissipative* if it dissipates energy—the outgoing energy (2-norm squared) is less than or equal to the incoming energy. The following proof shows that Pick's theorem says something about dissipative systems.

Since both time and frequency domains appear, the $\hat{\cdot}$ -convention is in force. Also, complex-valued signals and complex-rational transfer functions are used. There is no obstacle to extending the material of Chapter 2 to the complex case.

The proof is separated into three lemmas.

Lemma 1 *Consider a linear system with input signal $u(t)$ of finite 2-norm, output signal $y(t)$, and transfer function $\hat{G}(s)$ in $\mathcal{S}_c$. If $\|\hat{G}\|_\infty \leq 1$, then*

$$\int_{-\infty}^0 |y(t)|^2 dt \leq \int_{-\infty}^0 |u(t)|^2 dt.$$

Proof Define a new input

$$u_1(t) := \begin{cases} u(t), & \text{if } t \leq 0 \\ 0, & \text{if } t > 0 \end{cases}$$

and let the corresponding output be $y_1(t)$. It follows from entry (1,1) in Table 2.2 that

$$\int_{-\infty}^{\infty} |y_1(t)|^2 dt \leq \int_{-\infty}^{\infty} |u_1(t)|^2 dt.$$

Since $u_1 = 0$ for positive time, this implies that

$$\int_{-\infty}^{\infty} |y_1(t)|^2 dt \leq \int_{-\infty}^0 |u_1(t)|^2 dt$$

and hence that

$$\int_{-\infty}^0 |y_1(t)|^2 dt \leq \int_{-\infty}^0 |u_1(t)|^2 dt.$$

But $y = y_1$ and $u = u_1$ for negative time. ■

The second lemma shows that complex exponentials are eigenfunctions for linear systems.

Lemma 2 *Consider a linear system with transfer function $\hat{G}(s)$ in $\mathcal{S}_c$. Apply the input signal*

$$u(t) = e^{at}, \quad -\infty < t \leq 0$$

with $\operatorname{Re} a > 0$. Then the output signal is

$$y(t) = \hat{G}(a)u(t), \quad -\infty < t \leq 0.$$

Proof Use the convolution equation: For every $t \leq 0$,

$$\begin{aligned} y(t) &= \int_0^{\infty} G(\tau)u(t-\tau)d\tau \\ &= \int_0^{\infty} G(\tau)e^{a(t-\tau)}d\tau \\ &= \hat{G}(a)e^{at}. \quad \blacksquare \end{aligned}$$

The final lemma is the necessity part of Pick's theorem.

Lemma 3 *If the NP problem is solvable, then $Q \geq 0$.*

Proof To simplify notation, assume there are only two interpolation points (i.e., $n = 2$). Let $\hat{G}$ be a solution to the NP problem. For arbitrary complex numbers c_1 and c_2 apply the input signal

$$u(t) = c_1 e^{a_1 t} + c_2 e^{a_2 t}, \quad -\infty < t \leq 0$$

to the system with transfer function $\hat{G}$. By Lemma 2 and linearity the output signal is

$$\begin{aligned} y(t) &= c_1 \hat{G}(a_1) e^{a_1 t} + c_2 \hat{G}(a_2) e^{a_2 t} \\ &= c_1 b_1 e^{a_1 t} + c_2 b_2 e^{a_2 t}. \end{aligned}$$

Starting with Lemma 1, we get in succession

$$\begin{aligned} \int_{-\infty}^0 |y(t)|^2 dt &\leq \int_{-\infty}^0 |u(t)|^2 dt, \\ \int_{-\infty}^0 |c_1 b_1 e^{a_1 t} + c_2 b_2 e^{a_2 t}|^2 dt &\leq \int_{-\infty}^0 |c_1 e^{a_1 t} + c_2 e^{a_2 t}|^2 dt, \end{aligned}$$

and thus

$$\begin{aligned} \int_{-\infty}^0 &\left[\overline{c_1} c_1 (1 - \overline{b_1} b_1) e^{(\overline{a_1} + a_1)t} + \overline{c_1} c_2 (1 - \overline{b_1} b_2) e^{(\overline{a_1} + a_2)t} \right. \\ &\left. + \overline{c_2} c_1 (1 - \overline{b_2} b_1) e^{(\overline{a_2} + a_1)t} + \overline{c_2} c_2 (1 - \overline{b_2} b_2) e^{(\overline{a_2} + a_2)t} \right] dt \geq 0. \end{aligned}$$

This integral can be evaluated to give

$$\overline{c_1} c_1 \frac{1 - \overline{b_1} b_1}{\overline{a_1} + a_1} + \overline{c_1} c_2 \frac{1 - \overline{b_1} b_2}{\overline{a_1} + a_2} + \overline{c_2} c_1 \frac{1 - \overline{b_2} b_1}{\overline{a_2} + a_1} + \overline{c_2} c_2 \frac{1 - \overline{b_2} b_2}{\overline{a_2} + a_2} \geq 0,$$

which is equivalent to

$$x^* Q x \geq 0,$$

where

$$x := \begin{pmatrix} \overline{c_1} \\ \overline{c_2} \end{pmatrix}.$$

Since c_1 and c_2 were arbitrary, it must be that $Q \geq 0$. ■

9.3 Nevanlinna's Algorithm

This section presents a procedure to construct a solution of the NP problem when it is solvable. The procedure is developed inductively: First, the case $n = 1$ is solved; then the case of n points is reduced to the case of $n - 1$ points.

Let us begin by letting $\mathcal{D}$ denote the open unit disk, $|z| < 1$, and $\overline{\mathcal{D}}$ the closed unit disk, $|z| \leq 1$. A *Möbius function* has the form

$$M_b(z) = \frac{z - b}{1 - \overline{z} \overline{b}},$$

where $|b| < 1$. Following is a list of some properties of Möbius functions (you should check these):

1. M_b has a zero at $z = b$ and a pole at $z = 1/\overline{b}$. Thus M_b is analytic in $\mathcal{D}$.

2. The magnitude of M_b equals 1 on the unit circle.
3. M_b maps $\mathcal{D}$ onto $\mathcal{D}$ and the unit circle onto the unit circle.
4. The inverse map is

$$M_b^{-1}(z) = \frac{z + b}{1 + z\bar{b}}$$

(i.e., $M_b^{-1} = M_{-\bar{b}}$). So the inverse map is a Möbius function too.

We will also need the all-pass function

$$A_a(s) := \frac{s - a}{s + \bar{a}}, \quad \operatorname{Re} a > 0.$$

With the aid of these functions we can solve the NP problem for the data

$$\begin{array}{c} a_1 \\ b_1 \end{array}$$

There are two cases.

Case 1 $|b_1| = 1$ A solution is $G(s) = b_1$. By the maximum modulus theorem this solution is unique.

Case 2 $|b_1| < 1$ There are an infinite number of solutions:

Lemma 4 *The set of all solutions is*

$$\{G : G(s) = M_{-b_1}[G_1(s)A_{a_1}(s)], G_1 \in \mathcal{S}_c, \|G_1\|_\infty \leq 1\}.$$

If G_1 is an all-pass function, so is G .

Proof Let $G_1 \in \mathcal{S}_c$, $\|G_1\|_\infty \leq 1$, and define G as

$$G(s) = M_{-b_1}[G_1(s)A_{a_1}(s)].$$

Thus G equals the composition of the two functions

$$\begin{array}{ll} s & \mapsto G_1(s)A_{a_1}(s), \\ z & \mapsto M_{-b_1}(z). \end{array}$$

The first is analytic in the closed right half-plane and maps it into the closed disk $\overline{\mathcal{D}}$; the second is analytic in $\overline{\mathcal{D}}$ and maps it back into $\overline{\mathcal{D}}$. It follows that $G \in \mathcal{S}_c$ and $\|G\|_\infty \leq 1$. Also, G interpolates b_1 at a_1 :

$$G(a_1) = M_{-b_1}[G_1(a_1)A_{a_1}(a_1)] = M_{-b_1}(0) = b_1.$$

Thus G solves the NP problem. Moreover, if G_1 is an all-pass function, then so is $G_1A_{a_1}$, hence so is G (because M_{-b_1} maps the unit circle onto itself).

Conversely, suppose that G solves the NP problem. Define G_1 so that

$$G(s) = M_{-b_1}[G_1(s)A_{a_1}(s)],$$

that is,

$$G_1(s) = \frac{M_{b_1}[G(s)]}{A_{a_1}(s)}.$$

The function $M_{b_1}[G(s)]$ belongs to $\mathcal{S}_c$, has ∞ -norm ≤ 1 , and has a zero at $s = a_1$. Therefore, $G_1 \in \mathcal{S}_c$ and $\|G_1\|_\infty \leq 1$. ■

Example 1 For the interpolation data

$$\begin{array}{c} 2 \\ 0.6 \end{array}$$

the formula in the lemma gives

$$G(s) = \frac{G_1(s) \frac{s-2}{s+2} + 0.6}{1 + 0.6G_1(s) \frac{s-2}{s+2}}.$$

The all-pass function $G_1(s) = (s-1)/(s+1)$ results in

$$G(s) = \frac{s^2 - 0.75s + 2}{s^2 + 0.75s + 2}.$$

Now we turn to the NP problem with n data points, the problem being assumed solvable, and see how to reduce it to the case of $n-1$ points. Again, there are two cases.

Case 1 $|b_1| = 1$ Since the problem is solvable, by the maximum modulus theorem it must be that $G(s) = b_1$ is the unique solution (and hence that $b_1 = \dots = b_n$).

Case 2 $|b_1| < 1$ Pose a new problem, labeled the NP' problem, with the $n-1$ data points

$$\begin{array}{ccc} a_2 & \cdots & a_n \\ b'_2 & \cdots & b'_n \end{array}$$

where $b'_i := M_{b_1}(b_i)/A_{a_1}(a_i)$.

Lemma 5 *The set of all solutions to the NP problem is given by the formula*

$$G(s) = M_{-b_1}[G_1(s)A_{a_1}(s)],$$

where G_1 ranges over all solutions to the NP' problem. If G_1 is all-pass, so is G .

Proof G solves the NP problem iff

$$G \in \mathcal{S}_c, \quad \|G\|_\infty \leq 1, \quad G(a_1) = b_1, \quad \text{and} \quad G(a_i) = b_i, \quad i = 2, \dots, n.$$

From Lemma 4, the set of all G s satisfying the first three conditions is

$$\{G : G(s) = M_{-b_1}[G_1(s)A_{a_1}(s)], G_1 \in \mathcal{S}_c, \|G_1\|_\infty \leq 1\}.$$

Then G satisfies the fourth condition iff

$$G_1(a_i) = \frac{M_{b_1}(b_i)}{A_{a_1}(a_i)}, \quad i = 2, \dots, n$$

(i.e., G_1 solves the NP' problem). ■

It follows by induction that the NP problem always has an all-pass solution.

Example 2 Consider the NP problem for the data

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{array} = \begin{array}{ccc} 1 & 2 & 3 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{array}$$

The Pick matrix is

$$\begin{bmatrix} 0.3750 & 0.2778 & 0.2188 \\ 0.2778 & 0.2222 & 0.1833 \\ 0.2188 & 0.1833 & 0.1563 \end{bmatrix}.$$

The smallest eigenvalue equals 0.0004. Since this is positive, the NP problem is solvable.

A solution can be obtained by reduction to one interpolation point by applying Lemma 5 twice. First, reduce to the NP' problem with two points:

$$\begin{array}{cc} a_2 & a_3 \\ b'_2 & b'_3 \end{array} = \begin{array}{cc} 2 & 3 \\ -0.6 & -0.5714 \end{array}$$

Here $b'_i := M_{b_1}(b_i)/A_{a_1}(a_i)$, $i = 2, 3$. Second, reduce to the NP'' problem with only one point:

$$\begin{array}{c} a_3 \\ b''_3 \end{array} = \begin{array}{c} 3 \\ 0.2174 \end{array}$$

Here $b''_3 := M_{b'_2}(b'_3)/A_{a_2}(a_3)$.

Now solve the problems in reverse order. By Lemma 4 the solution of the NP'' problem is

$$G_2(s) = M_{-b'_3}[G_3(s)A_{a_3}(s)],$$

where G_3 is an arbitrary function in $\mathcal{S}_c$ of ∞ -norm ≤ 1 . Let's take $G_3(s) = 1$, the simplest all-pass function. Then

$$G_2(s) = \frac{1.2174s - 2.3478}{1.2174s + 2.3478}.$$

The induced solution to the NP' problem is

$$G_1(s) = M_{-b'_2}[G_2(s)A_{a_2}(s)] = \frac{0.4870s^2 - 7.6522s + 1.8783}{0.4870s^2 + 7.6522s + 1.8783}.$$

Finally, the solution to the NP problem is

$$G(s) = M_{-b_1}[G_1(s)A_{a_1}(s)] = \frac{0.7304s^3 - 4.0696s^2 + 14.2957s - 0.9391}{0.7304s^3 + 4.0696s^2 + 14.2957s + 0.9391}.$$

Notice that the degree of the numerator and denominator of G equals 3, the number of data points. In general, there always exists an all-pass solution of degree $\leq n$.

In our application of NP theory to the model-matching problem, the data

$$\begin{array}{ccc} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{array}$$

will have conjugate symmetry; that is, if (a_i, b_i) appears, so will the conjugate pair $(\overline{a_i}, \overline{b_i})$. Then we will want the solution G to be real-rational instead of complex. Suppose that the data do have conjugate symmetry and that G is a solution in $\mathcal{S}_c$. It can be written uniquely as

$$G(s) = G_R(s) + jG_I(s),$$

where G_R and G_I both are real-rational. Then G_R belongs to $\mathcal{S}$ and is also a solution to the NP problem (the proof is left as an exercise).

Example 3 For the data

$$\begin{array}{cc} 5 + 2j & 5 - 2j \\ 0.1 - 0.1j & 0.1 + 0.1j \end{array}$$

the NP problem is solvable (the smallest eigenvalue of the Pick matrix equals 0.0051). Starting from the all-pass function 1, Nevanlinna's algorithm produces the all-pass solution

$$G(s) = \frac{(0.5268 + 0.1213j)s^2 - (9 + j)s + (47.1073 + 1.1410j)}{(0.5268 - 0.1213j)s^2 + (9 - j)s + (47.1073 - 1.1410j)}.$$

Let $\underline{G}$ denote the function obtained from G by conjugating all coefficients. The function G_R is then

$$G_R = \frac{1}{2}(G + \underline{G}),$$

that is,

$$G_R(s) = \frac{0.2628s^4 - 30.6418s^2 + 2217.7975}{0.2923s^4 + 9.7255s^3 + 131.9119s^2 + 850.2137s + 2220.4012}.$$

This solution is not all-pass.

9.4 Solution of the Model-Matching Problem

Now let's see how to use NP theory to solve the model-matching problem. For simplicity we will *assume* that T_2 has no repeated zeros in the right half-plane. The minimum model-matching error, γ_{opt} , equals the minimum γ such that

$$\|T_1 - T_2 Q\|_{\infty} \leq \gamma$$

for some stable Q . Fix $\gamma > 0$ and consider the mapping $Q \mapsto G$ defined by

$$G = \frac{1}{\gamma}(T_1 - T_2 Q).$$

If Q is stable, so is G , but the converse is not always true. A stable function G must satisfy certain conditions in order that Q be stable. To see what they are, let $\{z_i : i = 1, \dots, n\}$ denote the zeros of T_2 in $\text{Res} > 0$. If Q is stable, then G satisfies the interpolation conditions

$$G(z_i) = \frac{1}{\gamma}T_1(z_i), \quad i = 1, \dots, n.$$

You can check that, conversely, if G is stable and satisfies these interpolation conditions, then Q is stable.

Therefore, γ_{opt} equals the minimum γ so that there exists a function G in $\mathcal{S}$ satisfying the conditions

$$\begin{aligned} \|G\|_{\infty} &\leq 1, \\ G(z_i) &= \frac{1}{\gamma} T_1(z_i), \quad i = 1, \dots, n. \end{aligned}$$

This is precisely a Nevanlinna-Pick problem with data

$$\begin{array}{ccc} a_1 & \cdots & a_n \\ \gamma^{-1}b_1 & \cdots & \gamma^{-1}b_n \end{array}$$

where $a_i := z_i$ and $b_i := T_1(z_i)$. The associated Pick matrix equals

$$A - \gamma^{-2}B,$$

where the ij^{th} elements of A and B are, respectively,

$$\frac{1}{a_i + \bar{a}_j}, \quad \frac{b_i \bar{b}_j}{a_i + \bar{a}_j}.$$

From Pick's theorem we can now conclude that γ_{opt} equals the minimum γ such that $A - \gamma^{-2}B \geq 0$. Both A and B are Hermitian. Furthermore, it can be proved that A is positive definite because the a_i s are distinct. Such a matrix has a positive definite *squareroot* (i.e., a matrix $A^{1/2}$ satisfying $A^{1/2}A^{1/2} = A$). The inverse of this squareroot is denoted $A^{-1/2}$.

The next lemma, a simple result in matrix theory, gives an explicit way to compute γ_{opt} .

Lemma 6 γ_{opt} equals the squareroot of the largest eigenvalue of the matrix $A^{-1/2}BA^{-1/2}$.

Tracing backwards, we get the following procedure for solving the model-matching problem.

Procedure

Input: T_1, T_2

Step 1 Determine $\{z_i : i = 1, \dots, n\}$, the zeros of T_2 in $\text{Res} > 0$.

Step 2 Define

$$b_i := T_1(z_i), \quad i = 1, \dots, n$$

and form the matrices

$$A := \left(\frac{1}{z_i + \bar{z}_j} \right), \quad B := \left(\frac{b_i \bar{b}_j}{z_i + \bar{z}_j} \right).$$

Step 3 Compute γ_{opt} as the squareroot of the largest eigenvalue of $A^{-1/2}BA^{-1/2}$.

Step 4 Solve the NP problem with data

$$\begin{array}{ccc} z_1 & \cdots & z_n \\ \gamma_{\text{opt}}^{-1}b_1 & \cdots & \gamma_{\text{opt}}^{-1}b_n \end{array}$$

Denote the solution by G .

Step 5 Set

$$Q := \frac{T_1 - \gamma_{\text{opt}} G}{T_2}.$$

The NP problem in Step 4 has a unique, all-pass solution. Thus, for the optimal Q the error transfer function $T_1 - T_2 Q$ equals γ_{opt} times an all-pass function.

Example 4 The procedure applied to

$$T_1(s) = \frac{s+1}{10s+1}, \quad T_2(s) = \frac{(s-1)(s-5)}{(s+2)^2}$$

goes like this:

Step 1

$$z_1 = 1, \quad z_2 = 5$$

Step 2

$$b_1 = \frac{2}{11}, \quad b_2 = \frac{2}{17}$$

$$A = \begin{bmatrix} 0.5 & 0.1667 \\ 0.1667 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0165 & 0.0036 \\ 0.0036 & 0.0014 \end{bmatrix}$$

Step 3

$$\gamma_{\text{opt}} = 0.2021$$

Step 4

$$\begin{array}{cc} z_1 & z_2 \\ \gamma_{\text{opt}}^{-1} b_1 & \gamma_{\text{opt}}^{-1} b_2 \end{array} = \begin{array}{cc} 1 & 5 \\ 0.8997 & 0.5821 \end{array}$$

$$G(s) = \frac{-1.0035s + 18.9965}{1.0035s + 18.9965}$$

Step 5

$$Q(s) = \frac{0.3021s^2 + 1.2084s + 1.2084}{s^2 + 19.0308s + 1.8931}$$

9.5 State-Space Solution (Optional)

For completeness, this section presents a state-space procedure for solving the model-matching problem. The underlying theory is beyond the scope of this book.

Step 1 Factor T_2 as the product of an all-pass factor and a minimum-phase factor:

$$T_2 = T_{2ap} T_{2mp}.$$

Step 2 Define

$$R := \frac{T_1}{T_{2ap}},$$

factor R as $R = R_1 + R_2$ with R_1 strictly proper and all poles in $\text{Res} > 0$ and $R_2 \in \mathcal{S}$, and find a minimal realization

$$R_1(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right].$$

Step 3 Solve the Lyapunov equations

$$\begin{aligned} AL_c + L_c A' &= BB', \\ A' L_o + L_o A &= C' C. \end{aligned}$$

Step 4 Find the maximum eigenvalue λ^2 of $L_c L_o$ and a corresponding eigenvector w .

Step 5 Define

$$\begin{aligned} f(s) &= \left[\begin{array}{c|c} A & w \\ \hline C & 0 \end{array} \right], \\ g(s) &= \left[\begin{array}{c|c} -A' & \lambda^{-1} L_o w \\ \hline B' & 0 \end{array} \right], \\ X &= R - \lambda \frac{f}{g}. \end{aligned}$$

Step 6 Then $\gamma_{\text{opt}} = \lambda$ and the optimal $Q = X/T_{2mp}$.

Example For the same data as in Example 4 in the preceding section, the procedure yields the following results.

Step 1

$$T_{2ap}(s) = \frac{(s-1)(s-5)}{(s+1)(s+5)}, \quad T_{2mp}(s) = \frac{(s+1)(s+5)}{(s+1)^2}$$

Step 2

$$\begin{aligned} R(s) &= \frac{(s+1)^2(s+5)}{(10s+1)(s-1)(s-5)} \\ A &= \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -\frac{6}{11} \\ \frac{30}{17} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix} \end{aligned}$$

Step 3

$$L_c = \begin{bmatrix} 0.1488 & -0.1604 \\ -0.1604 & 0.3114 \end{bmatrix}, \quad L_o = \begin{bmatrix} 0.5 & 0.1667 \\ 0.1667 & 0.1 \end{bmatrix}$$

Step 4

$$\lambda = 0.2021, \quad w = \begin{bmatrix} 1 \\ -0.7769 \end{bmatrix}$$

Step 5

$$\begin{aligned} f(s) &= \frac{0.2231s - 4.223}{(s-1)(s-5)}, \quad g(s) = \frac{-0.2231s - 4.223}{(s+1)(s+5)} \\ X(s) &= 3.021 \frac{(s+1)(s+5)}{(10s+1)(s+18.93)} \end{aligned}$$

Step 6

$$\gamma_{\text{opt}} = 0.2021, \quad Q(s) = 3.021 \frac{(s+2)^2}{(10s+1)(s+18.93)}$$

Exercises

1. Solve the model-matching problem for

$$T_1(s) = \frac{s}{s+10}, \quad T_2(s) = \frac{s-1}{s^2+s+1}.$$

2. For the data

$$\begin{bmatrix} 1 & 1-j \\ j & -1+2j \end{bmatrix}$$

form the Pick matrix and compute its eigenvalues.

3. Find the minimum γ for which there exists a function G in $\mathcal{S}_c$ such that

$$\|G\|_\infty \leq \gamma,$$

$$G(1) = 2, \quad G(2) = 10.$$

4. Solve the model-matching problem for

$$T_1(s) = \frac{1}{s+1}, \quad T_2(s) = \frac{s^2-s+1}{(s+2)^3}.$$

5. Compute a real-rational solution to the NP problem for the data

$$\begin{bmatrix} 1 & 1-j & 1+j \\ 0.1 & 0.2j & -0.2j \end{bmatrix}$$

6. Let A and B be $n \times n$ complex Hermitian matrices with $A > 0$. Prove that $A - \gamma^{-2}B \geq 0$ iff γ^2 is $\geq$ the largest eigenvalue of $A^{-1/2}BA^{-1/2}$. (This proves Lemma 6.)

Notes and References

The NP problem is named after the mathematicians R. Nevanlinna and G. Pick, whose work dates from 1916-1919. For a complete proof of Pick's theorem together with additional references, see Garnett (1981).

In the model-matching problem, zeros of T_2 on the imaginary axis lead to boundary interpolation in the NP problem (i.e., some of a_i s are on the imaginary axis). This complicates the NP problem; a reference for this case is Khargonekar and Tannenbaum (1985).

The proof of necessity in Section 9.2 is from Youla and Saito (1967), and Nevanlinna's algorithm is taken from Walsh (1969). NP theory was first used in the context of control problems by Tannenbaum (1980, 1981). The state-space procedure in Section 9.5 is from Francis (1987), which in turn is adapted from Sarason (1967) and Silverman and Bettayeb (1980).

