

Chapter 8

Advanced Loopshaping

In Chapter 7 we saw how to convert performance specifications on S and T into specifications on the loop transfer function L . For a stable, minimum-phase plant and L having at least the same relative degree as P , the controller was obtained from $C = L/P$. In this chapter we discuss extensions to this basic idea by doing loopshaping directly with C or other quantities and by considering plants with right half-plane (RHP) poles or zeros. Finally, we introduce several optimal control problems and explore in what sense loopshaping designs can be said to be optimal. Our aim is to extend and deepen our understanding of loopshaping, to provide an introduction to optimal design, and to establish some connections between the two approaches. Much of this chapter is closely related to what has traditionally been called classical control, particularly the work of Bode.

8.1 Optimal Controllers

Recall from Section 4.5 that in general the norm

$$\|(|W_1 S|^2 + |W_2 T|^2)^{1/2}\|_\infty$$

is a reasonable performance measure, a compromise norm for the robust performance problem. Throughout this chapter we consider problems where P , W_1 , and W_2 are fixed but C is variable, so it is convenient to indicate the functional dependence of this norm on C by defining

$$\psi(C) := \|(|W_1 S|^2 + |W_2 T|^2)^{1/2}\|_\infty. \quad (8.1)$$

Throughout this chapter we refer to the *optimal* C , the controller that minimizes ψ , for the purpose of comparing it to controllers obtained via other methods in order to help evaluate their effectiveness. A procedure for determining the optimal C given P and weights W_1 and W_2 is developed in Chapter 12.

We shall treat the unity-feedback loop of Figure 8.1. Suppose that we focus on the response of e and u to the two inputs d and n (r has the same effect as n), and recall that the transfer functions from d and n to e and u are given as follows:

$$\begin{bmatrix} e \\ u \end{bmatrix} = - \begin{bmatrix} PS & S \\ T & CS \end{bmatrix} \begin{bmatrix} d \\ n \end{bmatrix}.$$

If we introduce weights W_e on e , W_u on u , W_d on d , and W_n on n , then we could make our performance specification to keep the matrix

$$\begin{bmatrix} W_e & 0 \\ 0 & W_u \end{bmatrix} \begin{bmatrix} PS & S \\ T & CS \end{bmatrix} \begin{bmatrix} W_d & 0 \\ 0 & W_n \end{bmatrix} = \begin{bmatrix} W_e S \\ W_u CS \end{bmatrix} [W_d P \quad W_n]$$

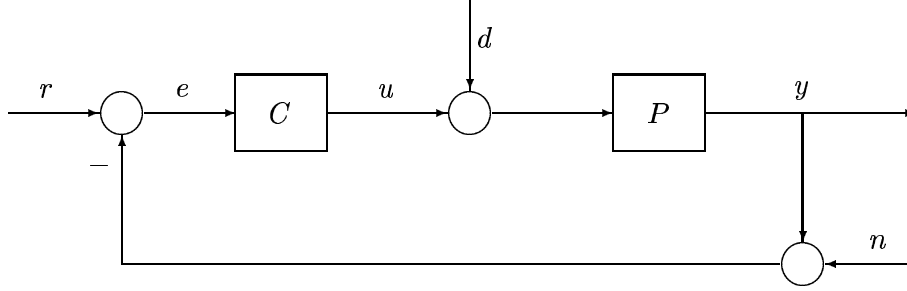


Figure 8.1: Unity-feedback loop.

small in some sense. A convenient specification is

$$\left\| (|W_e S|^2 + |W_u C S|^2)^{1/2} (|W_d P|^2 + |W_n|^2)^{1/2} \right\|_{\infty} < 1,$$

which is equivalent to

$$\left\| (|W_1 S|^2 + |W_2 T|^2)^{1/2} \right\|_{\infty} < 1,$$

where

$$|W_1| = |W_e| (|W_d P|^2 + |W_n|^2)^{1/2}, \quad |W_2| = |W_u| (|W_n|^2 |P|^{-2} + |W_d|^2)^{1/2}. \quad (8.2)$$

Thus this problem fits the type of performance specification in (8.1).

We will use this setup throughout this chapter, as it makes a useful “standard” problem for a number of reasons. First, it leads to some very interesting control problems, even when simple constant weights are used. Second, there is a rich theory available for this problem, although it will only be hinted at in this chapter. Third, it is easy to motivate problems in this framework that greatly stretch the loopshaping methods. Finally, despite its simplicity, the problem setup is easy to relate to what might arise in many practical situations.

8.2 Loopshaping with C

The loopshaping procedure developed in Chapter 7 involved converting performance specifications on S and T into specifications on the loop transfer function L , and then constructing an L to satisfy the resulting specifications and have reasonable crossover characteristics. Assuming that the plant had neither RHP poles nor zeros, and that L had at least the same relative degree as P , the controller was obtained from $C = L/P$. In this section we consider a slightly different approach that focuses more directly on C . Rather than construct L without regard to P , we could begin with a very simple controller, say $C = 1$, and compare the resulting L with the specification. It is often easy then to add simple dynamics to C to meet the specification.

In some instances it is more convenient to do loopshaping directly in terms of C rather than L . This will typically occur when the weights on S and T share substantial dynamics with P , as in (8.2). If we put a constant weight of 1 on d and n and a constant weight of 0.5 on u and e , then the weights from (8.2) become

$$W_e = W_u = 0.5, \quad W_d = W_n = 1, \quad (8.3)$$

$$|W_1| = 0.5(|P|^2 + 1)^{1/2}, \quad |W_2| = 0.5(|P|^{-2} + 1)^{1/2}. \quad (8.4)$$

With W_1 and W_2 so defined, for the performance specification to be met the loopshape will usually be very similar to P , so that the controller C will have $|C| \approx 1$. We can get some insight into why this is so by interpreting the weights in terms of the requirements they place on $|C|$ as follows.

For $L = PC$

$$|W_1 S|^2 = 0.25 \frac{|P|^2 + 1}{|L + 1|^2}, \quad |W_2 T|^2 = 0.25 \frac{|P|^{-2} + 1}{|L^{-1} + 1|^2},$$

and

$$\frac{|W_2 T|}{|W_1 S|} = |C|.$$

Assuming that $\psi(C) < 1$, at frequencies for which $|P| \gg 1$ we have that

$$|W_1| \approx 0.5|P|, \quad |W_2| \approx 0.5, \quad |W_1 S| \approx 0.5 \frac{|T|}{|C|}, \quad |W_2 T| \approx 0.5|T|,$$

and when $|P| \ll 1$,

$$|W_1| \approx 0.5, \quad |W_2| \approx 0.5 \frac{1}{|P|}, \quad |W_1 S| \approx 0.5|S|, \quad |W_2 T| \approx 0.5|CS|.$$

The crossover region can occur wherever $|P| \approx 1$. When $|P| = 1$, we have

$$|W_1| = |W_2| = \frac{\sqrt{2}}{2}.$$

Viewing this as a standard loopshaping problem, we would expect that $|L| \approx |P|$ and $|C| = 1$.

Example 1 Consider P given by

$$P(s) = \frac{0.5}{s} + \sum_{i=1}^4 \frac{0.2s}{s^2 + 2\zeta_i\omega_i s + \omega_i^2},$$

where $\omega_1 = 0.2$, $\omega_2 = 0.5$, $\omega_3 = 2$, $\omega_4 = 10$, $\zeta_i = 0.02$. The Bode magnitude plot of P is shown in Figure 8.2, and the resulting weights W_1 and W_2 on S and T are shown in Figure 8.3.

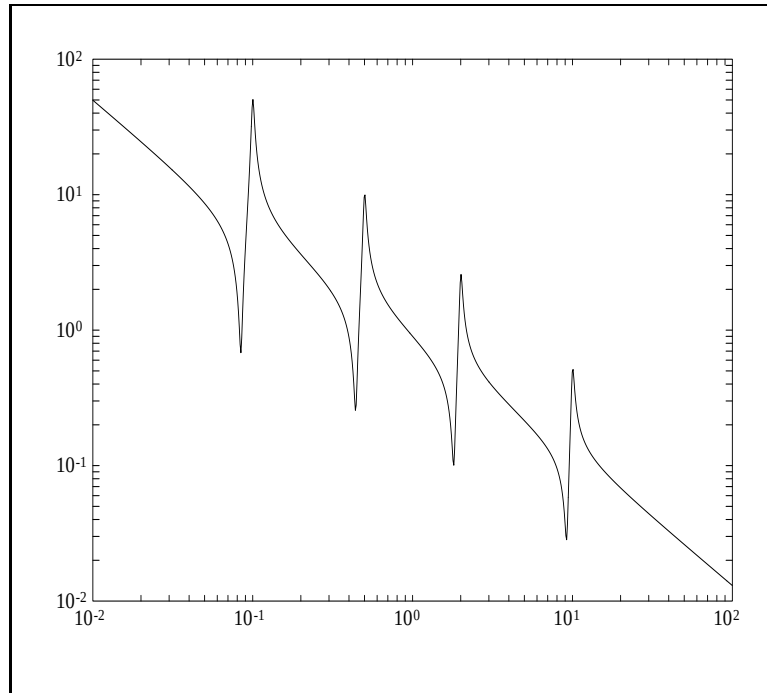
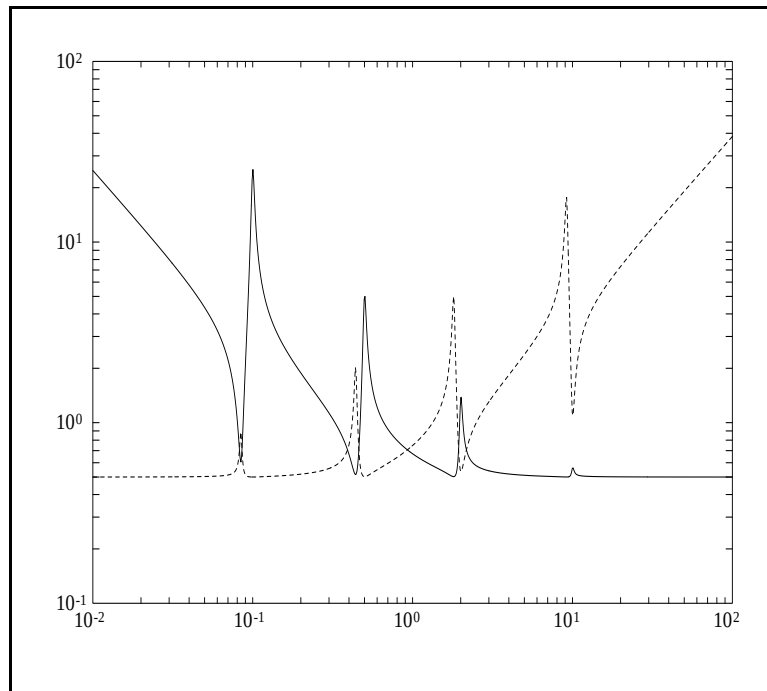
The complicated weights would appear to make this a tricky loopshaping problem, but in fact the controller $C = 1$ meets the specification. The quantity

$$(|W_1 S|^2 + |W_2 T|^2)^{1/2} \quad (8.5)$$

for $C = 1$ is plotted in Figure 8.4. The optimal ψ for this problem is approximately 0.69, so $C = 1$ is very close to optimal.

This example illustrates the point that loopshaping directly using C can often be much simpler than loopshaping with L and solving for C . While the example is somewhat contrived, similar things can happen quite naturally. In particular, this example exhibits some characteristics of plants that arise in the control of flexible structures.

Example 2 We will now use the same setup but with a slightly simpler plant P which we will motivate with a simple mechanical analog of a two-mode flexible structure (Figure 8.5). Shown is a one-dimensional rigid beam of mass M , length $2l$, and moment of inertia I connected to a

Figure 8.2: Bode plot of $|P|$.Figure 8.3: Bode plots of $|W_1|$ (solid) and $|W_2|$ (dash).

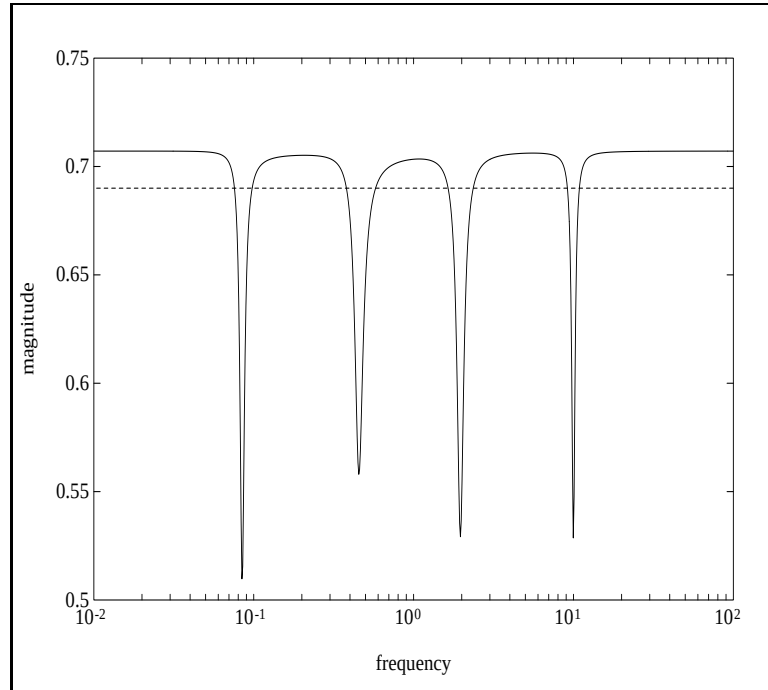


Figure 8.4: $(|W_1 S|^2 + |W_2 T|^2)^{1/2}$ for $C = 1$ (solid) and optimal C (dash).

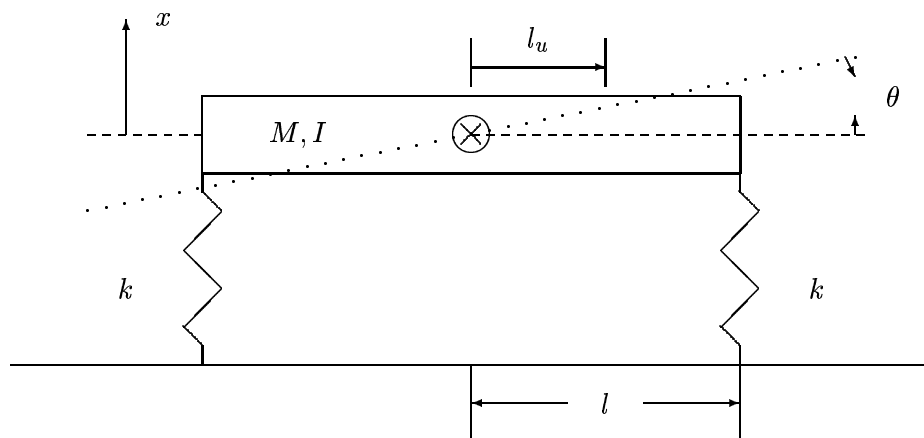


Figure 8.5: Mechanical structure, Example 2.

fixed base by two springs, each with spring constant k . We assume that the beam undergoes small vertical deflections x of the center of mass and small rotations θ about the center of mass. If we apply a vertical force u at position l_u from the center of mass, the linearized equations of motion are

$$\begin{aligned} M\ddot{x} + 2kx &= u, \\ I\ddot{\theta} + 2kl^2\theta &= l_u u. \end{aligned}$$

Taking $M = 2$, $I = 0.5$, $l = 2$, $k = 0.25$, and $l_u = 1$, we get that the transfer functions from u to x and θ are respectively

$$\frac{0.5}{s^2 + 0.5^2}, \quad \frac{2}{s^2 + 2^2}.$$

We will now measure $y := \dot{x} \pm \dot{\theta}$, the vertical velocity of the beam at the position ± 1 from the center of mass. For $+1$, the measurement is at the same location as the force input u ; this is called *collocated*. In the -1 case the measurement is at the other side of the beam, *noncollocated*. The resulting transfer function from u to y is then

$$P(s) = \frac{0.5s}{s^2 + 0.5^2} \pm \frac{2s}{s^2 + 2^2} = \frac{(2 \pm 0.5)s(1 \pm s^2)}{(s^2 + 0.5^2)(s^2 + 2^2)}. \quad (8.6)$$

A plot of $|P|$ is shown in Figure 8.6. The collocated system has zeros at $\pm j$ and the noncollocated

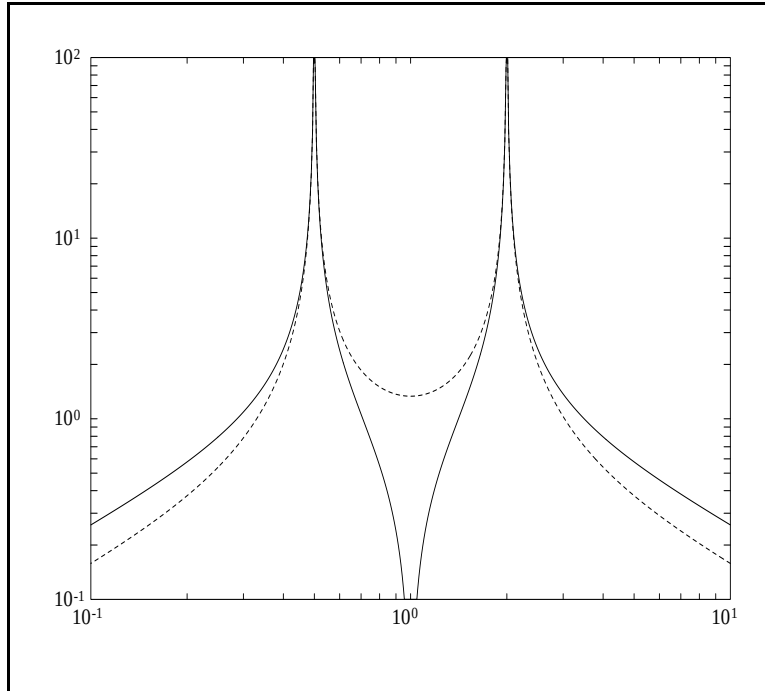


Figure 8.6: Bode plots of $|P|$ for $+$ (solid) and $-$ (dash).

system has zeros at ± 1 . As expected, we shall see that the noncollocated system is more difficult to control. For the collocated case it turns out that the optimal controller for the weights (8.3) is exactly $C = 1$, which is again much simpler than the resulting loopshape. It also turns out that this result holds for all positive values of M , I , l , l_u , and k , excluding those for which the

system is not mechanically possible. In each case, the optimal controller is $C = 1$ and the optimal $\psi = \sqrt{2}/2 \approx 0.707$. A proof of this, along with a discussion of the noncollocated case, is given in the last section of this chapter.

It is interesting to note that both examples above have multiple crossover frequencies, that is, several distinct frequencies at which $|L| = 1$. By contrast, most previous examples had only one crossover frequency. Indeed, most problems considered in classical control texts have a single crossover, and this might be considered typical. There are, however, certain application domains, such as the control of flexible structures, where multiple crossovers are common. It turns out that such systems have some interesting characteristics which may appear counterintuitive to readers unfamiliar with them. These issues will be explored more fully in the last section of this chapter.

It is not unusual for a reasonable controller to be much simpler than the resulting loopshape even when the problem setup is different from the one considered in the examples above. It often makes sense to begin the loopshaping design process with L equal to some constant times P and then add dynamic elements to try to get the right loopshape.

Example 3 As a slightly different example, consider the same setup as in Example 1 but with

$$P(s) = \frac{1}{s+1} + \frac{0.1s}{s^2 + 2\zeta\omega_1 s + \omega_1^2},$$

where $\zeta = 0.02$ and $\omega_1 = 0.5$. Suppose also that the weight on the error e is

$$W_e(s) = 0.5 \frac{s+0.5}{s+0.01},$$

with $W_u = 0.5$ and $W_d = W_n = 1$ as before. This gives us the performance objective as in (8.1), (8.2), and (8.4) except that now

$$|W_1| = |W_e|(|P|^2 + 1)^{1/2}, \quad |W_2| = 0.5(|P|^{-2} + 1)^{1/2}.$$

The Bode magnitude plot of P is shown in Figure 8.7 and the resulting weights W_1 and W_2 on S and T are shown in Figure 8.8.

If we compare the loopshaping constraints in Figure 8.9 with the Bode plot of P in Figure 8.7 we see immediately that $C = 1$ will not work because there is not enough gain at low frequency. Adding the simple lag compensator

$$C(s) = \frac{s+1}{s+0.01}$$

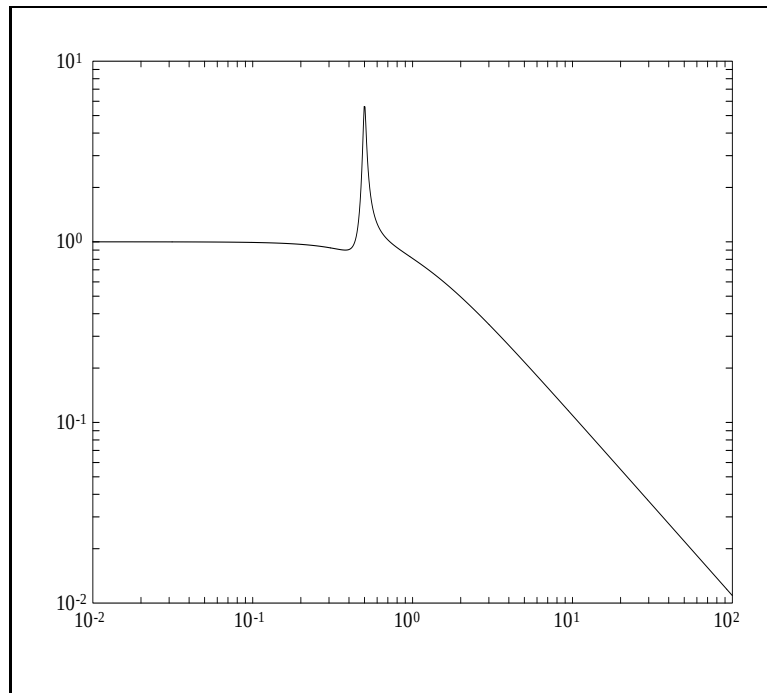
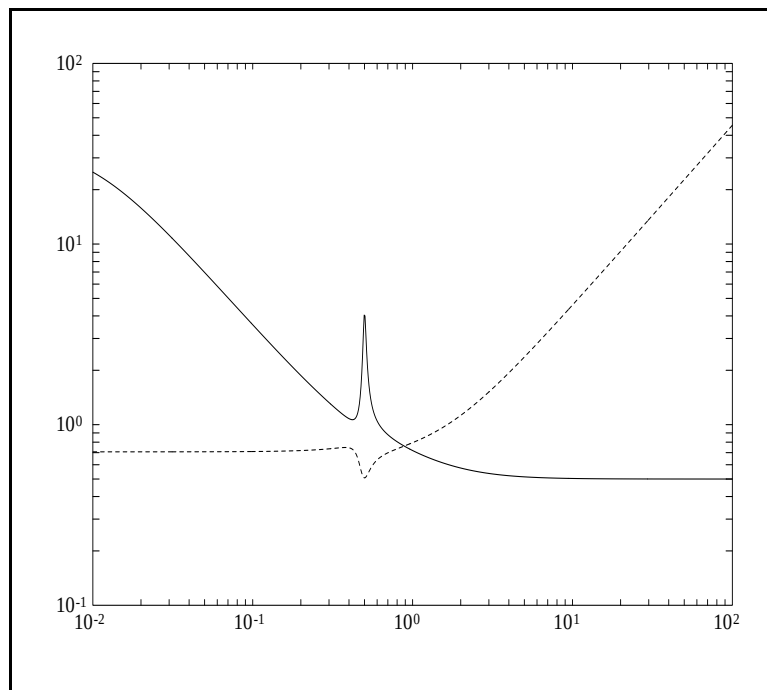
gives the loopshape shown in Figure 8.9 and $(|W_1 S|^2 + |W_2 T|^2)^{1/2}$ as shown in Figure 8.10. Figure 8.10 also shows the optimal level for this problem. The simple controller found here is very close to optimal.

8.3 Plants with RHP Poles and Zeros

Plants with RHP Zeros

Suppose that we begin a loopshaping problem with a plant P_0 , controller C_0 , and loop transfer function $L_0 = P_0 C_0$ with neither RHP poles nor zeros and a single crossover. Now consider the effect of adding a single RHP zero at $s = z$ to the plant by multiplying by the all-pass function

$$P_z(s) = \frac{z-s}{z+s},$$

Figure 8.7: Bode plot of $|P|$.Figure 8.8: Bode plots of $|W_1|$ (solid) and $|W_2|$ (dash).

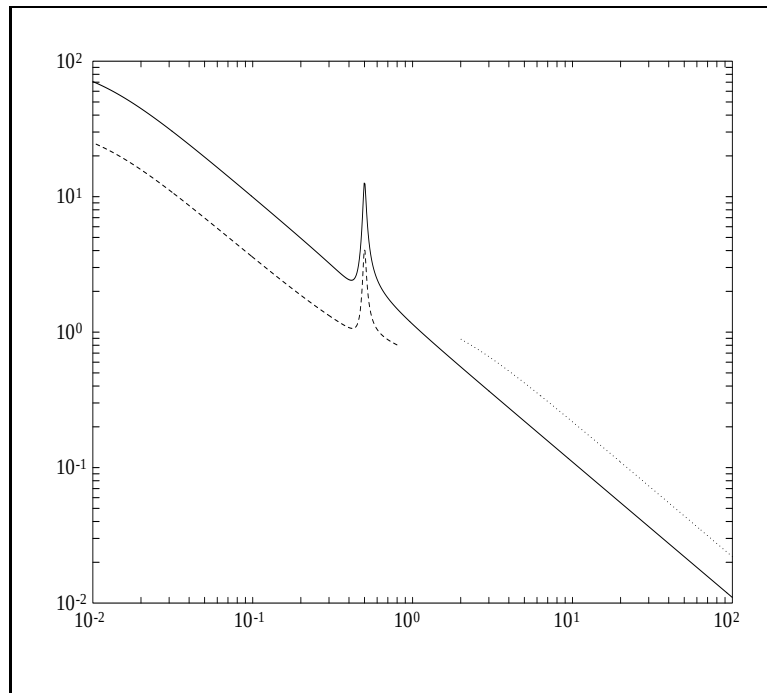


Figure 8.9: Loopshaping constraints and $|L|$ (solid) for $C = (s + 1)/(s + 0.01)$.

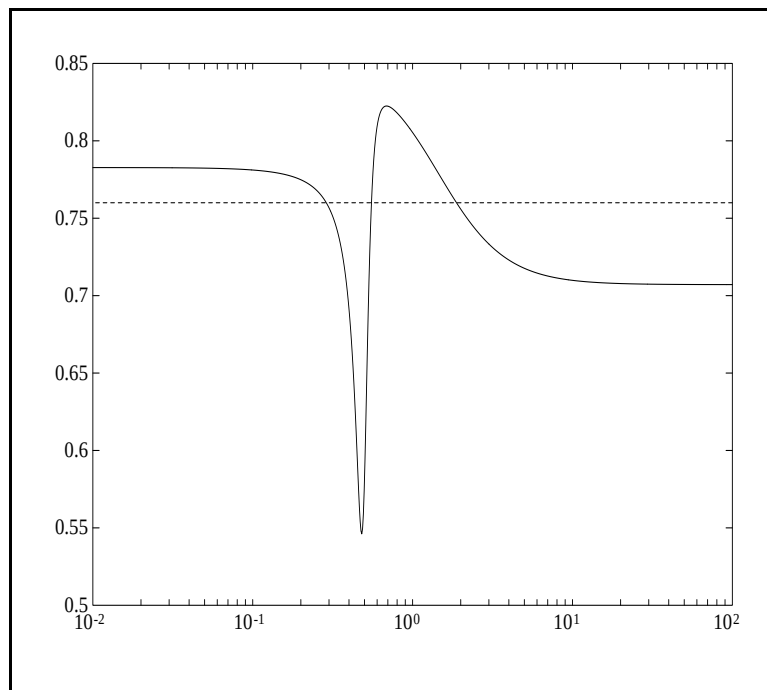


Figure 8.10: $(|W_1 S|^2 + |W_2 T|^2)^{1/2}$ for $C = (s + 1)/(s + 0.01)$ (solid) and optimal C (dash).

where the sign of P_z is chosen so that $P_z(0) = 1$. This also adds the same factor to the loop transfer function, which becomes

$$L(s) = L_0(s) \frac{z - s}{z + s}.$$

From the results of Chapter 6 we would expect problems unless z is larger than the crossover frequency: Recall that

$$\|W_1 S\|_\infty \geq |W_1(z)|,$$

so that $|W_1|$ must not be large at z .

We may also consider the effect of the RHP zero in terms of the gain-phase relations from Section 7.2. The phase at crossover will be the original phase of L_0 plus the phase of P_z :

$$\begin{aligned} \angle L(j\omega) &= \angle L_0(j\omega) + \angle P_z(j\omega), \\ \angle P_z(j\omega) &= 2\angle(-j\omega + z). \end{aligned}$$

The phase from P_z is negative and can be substantial as ω approaches z , with $\angle P_z(jz) = -\pi/2$. Again, we see that if z approaches the crossover frequency, the additional phase will degrade the closed-loop system performance or even lead to instability. Thus RHP zeros make the problem worse for systems with one crossover. As an example, let the initial loopshape be $L_0(s) = 1/s$ with L as above. The crossover frequency is $\omega = 1$ and the closed-loop system is stable if $z > 1$ and unstable if $z \leq 1$.

If P_z has a complex pair of RHP zeros, then

$$P_z(s) = \frac{\omega_z^2 - 2\zeta_z \omega_z s + s^2}{\omega_z^2 + 2\zeta_z \omega_z s + s^2}$$

and

$$\angle P_z(j\omega) = 2\angle(-2\zeta_z \omega_z j\omega + \omega_z^2 - \omega^2).$$

Observe that for lightly damped zeros where ζ_z is small, the phase changes abruptly near $\omega = \omega_z$. Otherwise, the same remarks apply here as in the case of one RHP zero. The simplest strategy to adopt in doing loopshaping designs for systems with RHP zeros is to proceed as usual while keeping track of the extra phase.

It would seem from this discussion that RHP zeros are always undesirable, that we would always avoid them if possible, and that we would never deliberately introduce them in our controllers. It is clearly true that all other things being equal, there is no advantage in having RHP zeros in the plant, because we could always add them in the controller if they were desirable. The issue of using RHP zeros in the controller is more subtle. Basically, they are clearly undesirable when there is only one crossover, but may be useful when there are multiple crossovers. This issue will be considered more fully in later sections on optimality.

Plants with RHP Poles

The problems created by RHP poles are similar to those created by RHP zeros, but there are important differences. Suppose that we begin again with a loopshaping problem with a plant P_0 , controller C_0 , and loop transfer function $L_0 = P_0 C_0$ with neither RHP poles nor zeros and a single crossover. Now consider the effect of adding a single RHP pole at $s = p$ to the plant by multiplying it by the all-pass function

$$P_p(s) = \frac{s + p}{s - p}.$$

This also adds the same factor to the loop transfer function L . The sign of P_p is chosen so that $P_p(0) = -1$. For p small, this gives the right number of encirclements for L to give closed-loop stability. Now we expect that we will have problems unless p is *smaller* than the crossover frequency: Recall that

$$\|W_2 T\|_\infty \geq |W_2(p)|,$$

so that $|W_2|$ must not be large at p .

We may also consider the effect of the RHP pole in terms of the gain-phase relations. The phase at crossover will be the original phase of L_0 plus the phase of P_p :

$$\begin{aligned}\angle L(j\omega) &= \angle L_0(j\omega) + \angle P_p(j\omega), \\ \angle P_p(j\omega) &= 2\angle(j\omega + p).\end{aligned}$$

We may illustrate this with the same example. Let the initial loopshape be $L_0(s) = 1/s$ with L as above. The crossover frequency is $\omega = 1$ and the closed-loop system is stable if $p < 1$ and unstable if $p \geq 1$. Similar effects hold when there is more than one RHP pole.

All things being equal, we would prefer to avoid RHP poles in our plants. Even though they can be stabilized by feedback, there is a price to pay in terms of closed-loop performance. Recall that there are times when we must add RHP poles in our compensator just to stabilize the system. Are there other times when we would want to add RHP poles? Clearly no if we have only one crossover, but if there are multiple crossovers it might be advantageous. This will be discussed in a later section on optimality.

Including RHP Poles and Zeros in Uncertainty Description

A somewhat more formal strategy for handling RHP poles and zeros is to include them in the uncertainty description as follows. Suppose that we have a RHP zero at z . Then we can factor P as

$$P(s) = P_0(s) \frac{z-s}{z+s} = P_0(s) \left(1 + \frac{-2s}{z+s}\right)$$

and cover this with a multiplicative perturbation to get

$$P = P_0 (1 + W_z \Delta), \quad W_z(s) = \frac{2s}{z+s}.$$

A somewhat tighter cover is given by

$$\begin{aligned}P(s) &= P_0(s) \left(\frac{z}{z+s} - \frac{s}{z+s} \right) \\ &= P_0(s) \frac{z}{z+s} [1 + W_z(s) \Delta(s)], \\ W_z(s) &= \frac{s}{z}.\end{aligned}$$

Robust stability for uncertainty in this form would involve a test on $\|W_z T\|_\infty$.

Why would we want to do this? If we put all the RHP zeros into the plant uncertainty, then we can use any design technique for plants with no RHP zeros, provided that we account for the extra uncertainty. Basically, one makes sure that T is small enough where necessary. The way we have covered the RHP zero above makes the model have little uncertainty at low frequencies and considerable uncertainty at high frequencies. The transition occurs at frequencies near z . It is also

easy to cover the all-pass part with an uncertainty that is large at low frequencies and small at high frequencies.

We can model RHP poles similarly. Care must be taken, however, because RHP poles don't naturally go into multiplicative uncertainty. Suppose that we have a RHP pole at p . Then we can factor P as

$$P(s) = P_0(s) \frac{s+p}{s-p} = P_0(s) \left(1 + \frac{-2p}{s+p}\right)^{-1}$$

and cover this with a perturbation to get

$$P = P_0 \frac{1}{1 + W_p \Delta}, \quad W_p(s) = \frac{2p}{s+p}.$$

This introduces an additional weight on S . A somewhat tighter cover is given by

$$\begin{aligned} P(s) &= P_0(s) \left(\frac{s}{s+p} - \frac{p}{s+p} \right)^{-1} \\ &= P_0(s) \frac{s+p}{s} \frac{1}{1 + W_p(s) \Delta(s)}, \\ W_p(s) &= \frac{p}{s}. \end{aligned}$$

How conservative is this covering method? For most problems with a single crossover and for which the performance objectives are achievable, this approach will work well. Basically, we need to make sure that S is small enough where there are RHP poles and T is small enough where there are RHP zeros. There are more complicated problems, particularly those involving multiple crossovers, where the impact of RHP poles and zeros might be quite different. This issue will be considered again in the final section of this chapter.

Examples

We will now consider several examples that illustrate loopshaping for systems with RHP zeros and poles.

Example 1 Consider

$$P(s) = P_0(s) \frac{z-s}{z+s}, \quad P_0(s) = \frac{1}{s}.$$

As in Section 8.2,

$$|W_1| = 0.5(|P|^2 + 1)^{1/2}, \quad |W_2| = 0.5(|P|^{-2} + 1)^{1/2}, \quad (8.7)$$

or

$$W_1(s) = 0.5 \left(1 + \frac{1}{s}\right), \quad W_2(s) = 0.5(s + 1). \quad (8.8)$$

The obvious controller for P_0 is $C = 1$, which is also optimal with $|W_1 S| = |W_2 T| = 0.5$ and $\psi(C) = \sqrt{2}/2$. This simple controller will work fine for $z \gg 1$, but deteriorates as z approaches 1 and does not stabilize for $z \leq 1$ because of the additional phase lag caused by the all-pass factor. Recall that for any controller

$$\|W_1 S\|_\infty \geq |W_1(z)| = 0.5 \left(1 + \frac{1}{z}\right),$$

but it is possible to improve substantially on $C = 1$ as z gets close to 1.

We will now focus on $z = 2$. For $C = 1$ the loopshape constraints and the loopshape $L = CP$ are shown in Figure 8.11. The closed-loop performance is shown in Figure 8.12. Note that $\psi(C) > 2$, with both $|W_1S|$ and $|W_2T|$ exceeding the specification at low and high frequencies, with a large peak in the middle. This is due to too much phase lag in L at crossover ($\omega = 1$). Recall from the gain-phase formula in Section 7.2 that we may improve the phase at crossover by reducing the low-frequency gain, reducing the crossover frequency, and increasing the high-frequency gain. A simple controller that does this is of the form

$$C(s) = \frac{k(s+2)}{(s+2k^2)}, \quad k > 1. \quad (8.9)$$

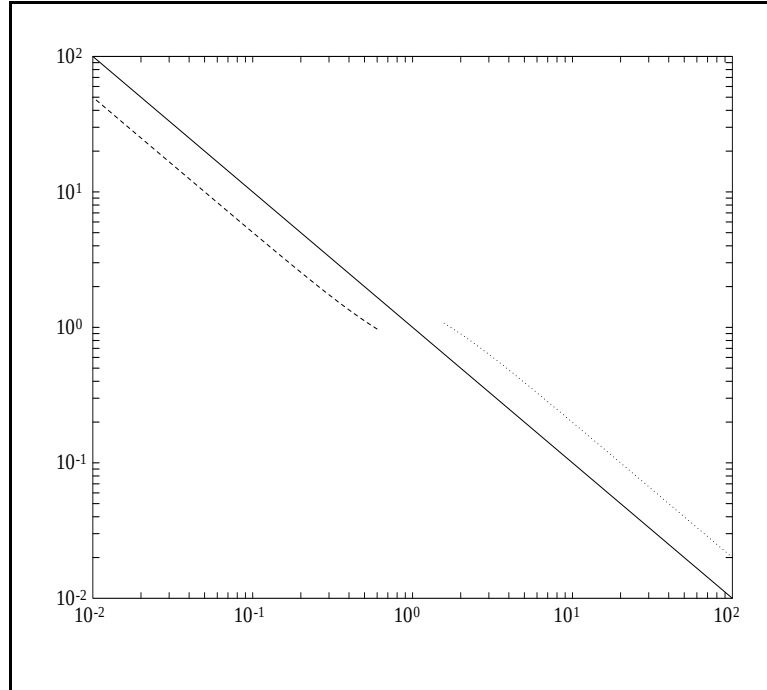


Figure 8.11: Loopshaping constraints (dash and dot) and $|L|$ (solid) for $C = 1$.

It turns out that the optimal C has $k = 1.78$, which yields an optimal $\psi(C) = 1.02$. A value of k very close to optimal is also easily arrived at by a little trial and error. The loopshape for the optimal C is shown in Figure 8.13 and the closed-loop performance is shown in Figure 8.14. The controller reduces the low-frequency gain and the crossover frequency and raises the high-frequency gain of L , as expected.

Next we will try an alternative design by covering the RHP zero with uncertainty and then loopshaping with the resulting minimum-phase nominal plant. For general z write

$$P(s) = \frac{1}{s} \left(\frac{z}{z+s} - \frac{s}{z+s} \right) = P_0(s) [1 + W_z(s)\Delta(s)],$$

where

$$P_0(s) = \frac{1}{s} \frac{z}{z+s}, \quad W_z(s) = \frac{s}{z}. \quad (8.10)$$

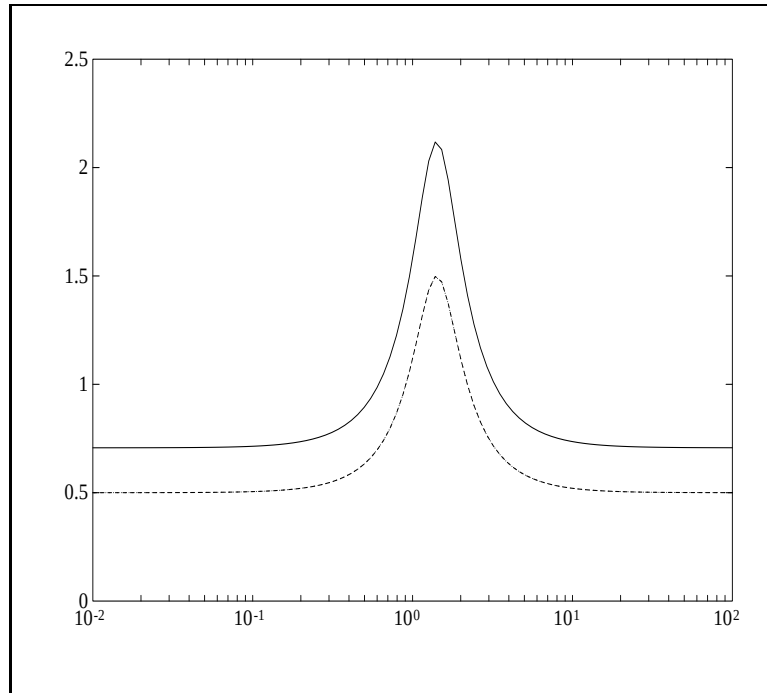


Figure 8.12: $(|W_1 S|^2 + |W_2 T|^2)^{1/2}$ (solid), $|W_1 S|$ and $|W_2 T|$ (both dash) for $C = 1$.

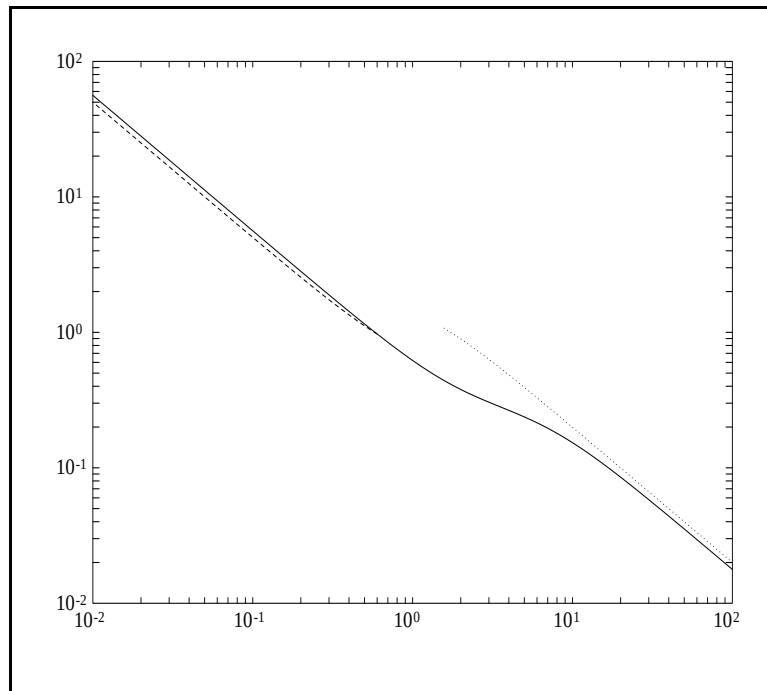


Figure 8.13: Loopshaping constraints (dash and dot) and $|L|$ (solid) for optimal C .

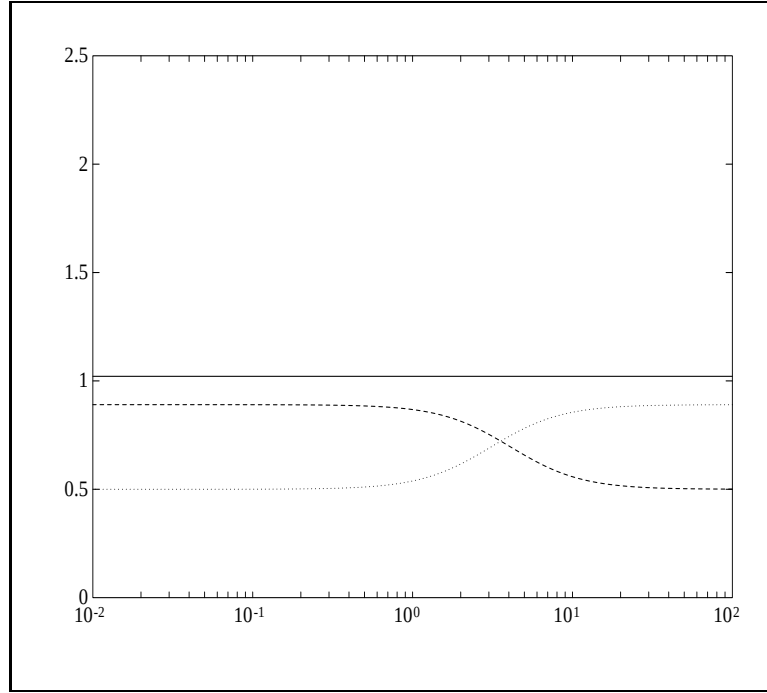


Figure 8.14: $(|W_1S|^2 + |W_2T|^2)^{1/2}$ (solid), $|W_1S|$ (dash), and $|W_2T|$ (dot), optimal C .

We now have two weights on T : W_2 and W_z . We need to find a weight that covers both of these. A reasonable approximation is

$$|W_{\text{tot}}| = |W_2| + |W_z| + |W_2||W_z|.$$

It is routine to show that

$$\left| \frac{(1 + \Delta W_z)W_2T}{1 + \Delta W_zT} \right| \leq \frac{|W_2T|(1 + |W_z|)}{1 - |W_zT|}$$

and

$$\frac{|W_2T|(1 + |W_z|)}{1 - |W_zT|} < 1 \quad \text{iff} \quad |W_{\text{tot}}T| < 1,$$

so using W_{tot} is a safe, but possibly conservative approximation. As an alternative way of thinking about this in terms of uncertainty descriptions, note that

$$\{(1 + |W_2|\Delta_1)(1 + |W_z|\Delta_2) : |\Delta_1| \leq 1, |\Delta_2| \leq 1\} \subset \{1 + |W_{\text{tot}}|\Delta : |\Delta| \leq 1\}.$$

If $z \gg 1$, this additional “uncertainty” will be negligible and the controller $C = 1$ will be fine.

Again, for the case $z = 2$ the plot for $|W_{\text{tot}}|$ is shown in Figure 8.15, together with plots of $|W_2|$ and $|W_z|$. To help us understand how conservative we have been by these approximations, we can compare

$$(|W_1S|^2 + |W_2T|^2)^{1/2} \tag{8.11}$$

for P with RHP zero at $s = 2$ with

$$[|W_1S|^2 + (1 + |W_z|)^2|W_2T|^2]^{1/2} \frac{1}{1 - |W_zT|} \tag{8.12}$$

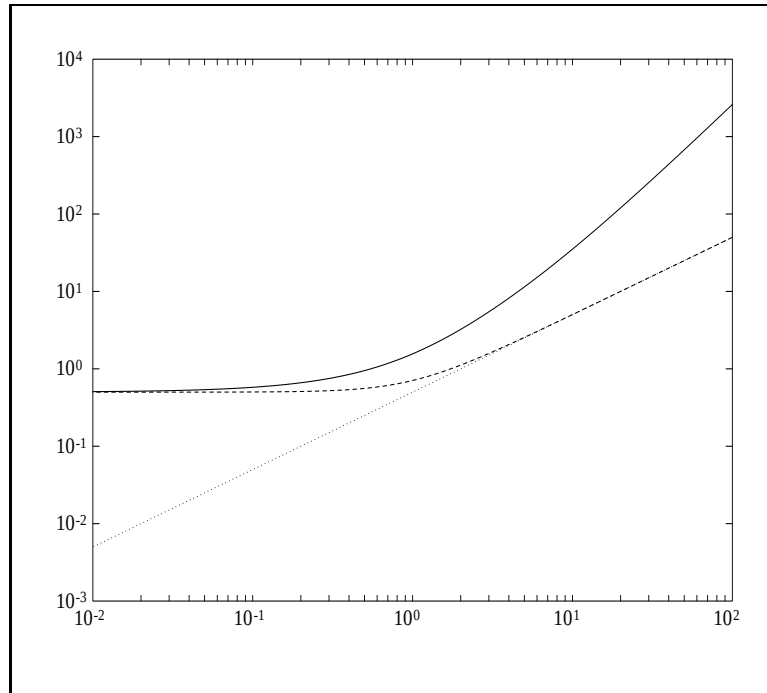


Figure 8.15: $|W_2|$ (dash), $|W_z|$ (dot), and $|W_{\text{tot}}|$ (solid).

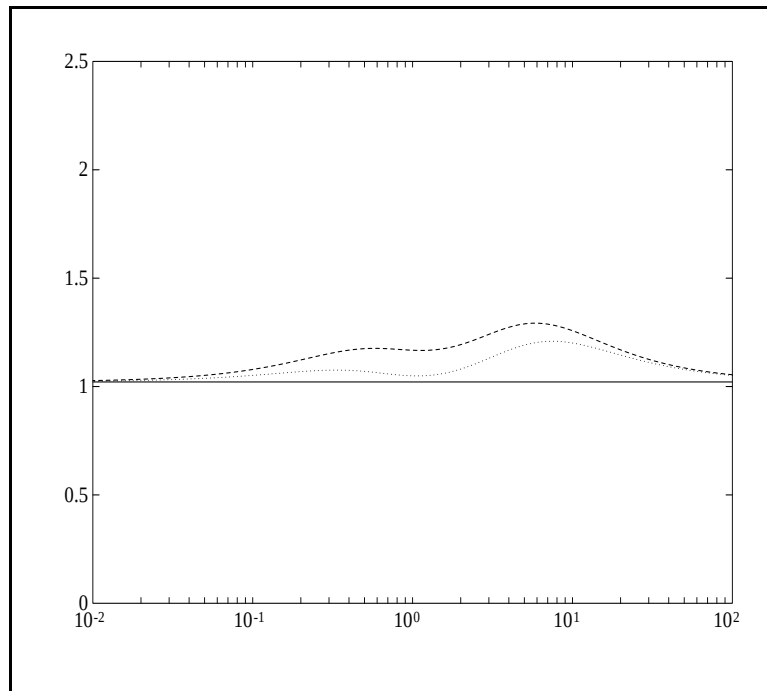


Figure 8.16: Plots of (8.11) (solid), (8.12) (dash), and (8.13) (dot).

and

$$(|W_1 S|^2 + |W_{\text{tot}} T|^2)^{1/2}, \quad (8.13)$$

where in (8.12) and (8.13) we use the plant P_0 from (8.10). By construction, we have that (8.11) $\leq$ (8.12), but (8.13) is not necessarily an upper bound for (8.11). All three quantities are plotted in Figure 8.16.

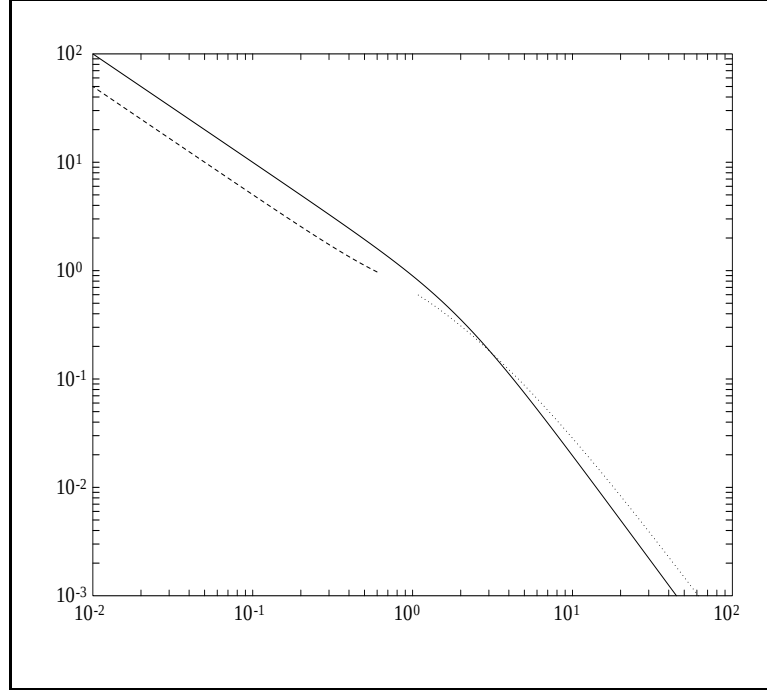


Figure 8.17: Loopshaping constraints (dash and dot) and $|L|$ (solid), $C = 1$.

For $P = P_0$ and $C = 1$, the loopshape $L = CP$ is shown in Figure 8.17, along with the loopshaping constraints. As before, there is too much phase lag in the crossover region, which is verified in Figure 8.18. As before, we may improve the phase at crossover by reducing the low-frequency gain, reducing the crossover frequency, and increasing the high-frequency gain. The optimal controller should once again be adequate, as is verified in Figures 8.19 and 8.20.

This example illustrates how the simple loopshaping ideas from this chapter and the preceding one can be extended to handle plants with RHP zeros.

Example 2 Now consider the problem of stabilizing the plant

$$P(s) = \frac{\alpha s - 1}{\alpha - s}, \quad \alpha \in (0, 1). \quad (8.14)$$

It is easily checked that a constant controller $C = k$ stabilizes this system iff $k \in (\alpha, 1/\alpha)$. We may compare this with the conclusions we would arrive at if we were to cover the RHP zero and pole with uncertainty about a nominally stable, minimum-phase plant. For example, we may write

$$\frac{\alpha s - 1}{\alpha - s} = \frac{1}{s} \left(\frac{1 + \alpha s \Delta_1(s)}{1 + \alpha/s \Delta_2(s)} \right). \quad (8.15)$$

We can pose this in the standard form of $\psi(C)$ in (8.1) with plant $P(s) = 1/s$ and weights $W_1(s) = \alpha/s$ and $W_2(s) = \alpha s$. If we now consider loopshaping for this problem, we will meet this

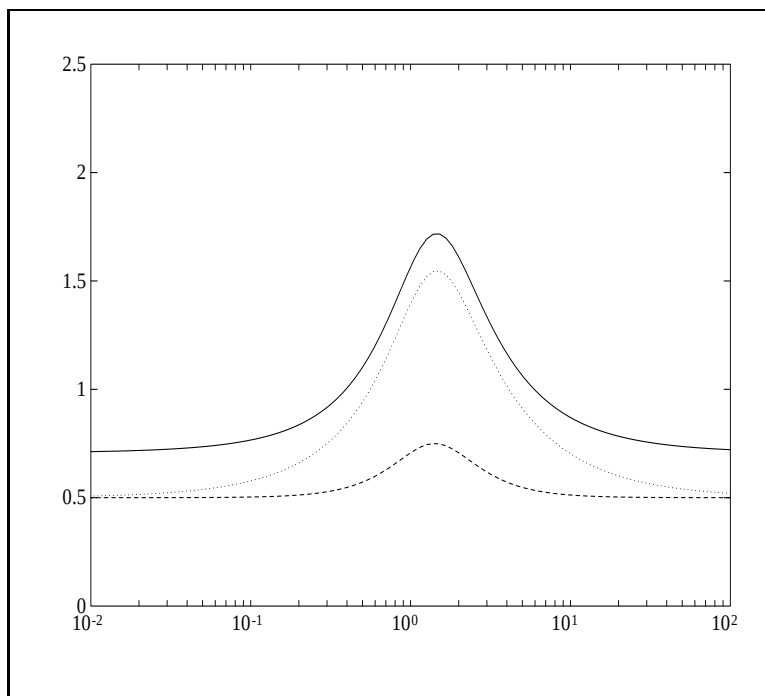


Figure 8.18: $(|W_1 S|^2 + |W_{\text{tot}} T|^2)^{1/2}$ (solid), $|W_1 S|$ (dash), and $|W_{\text{tot}} T|$ (dot), $C = 1$.

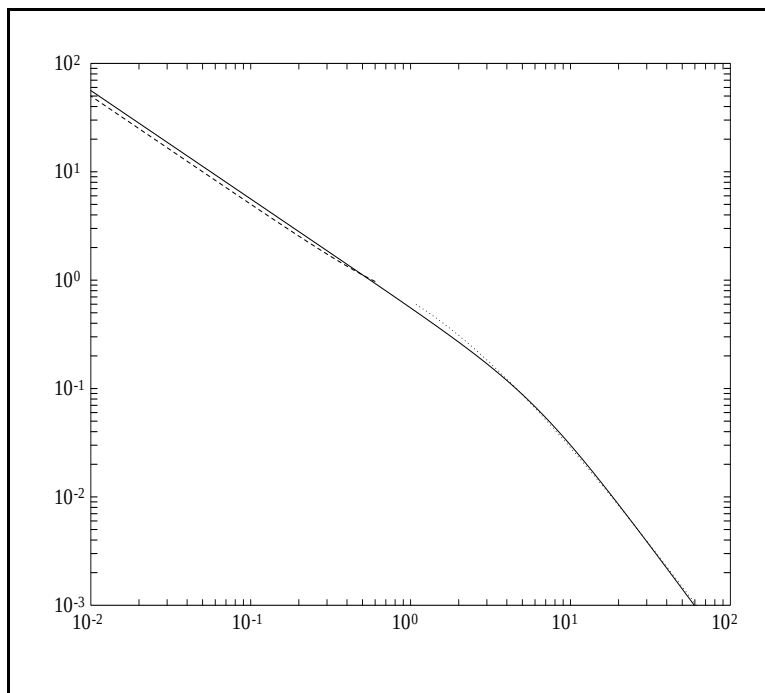


Figure 8.19: Loopshaping constraints (dash and dot) and $|L|$ (solid) for optimal C .

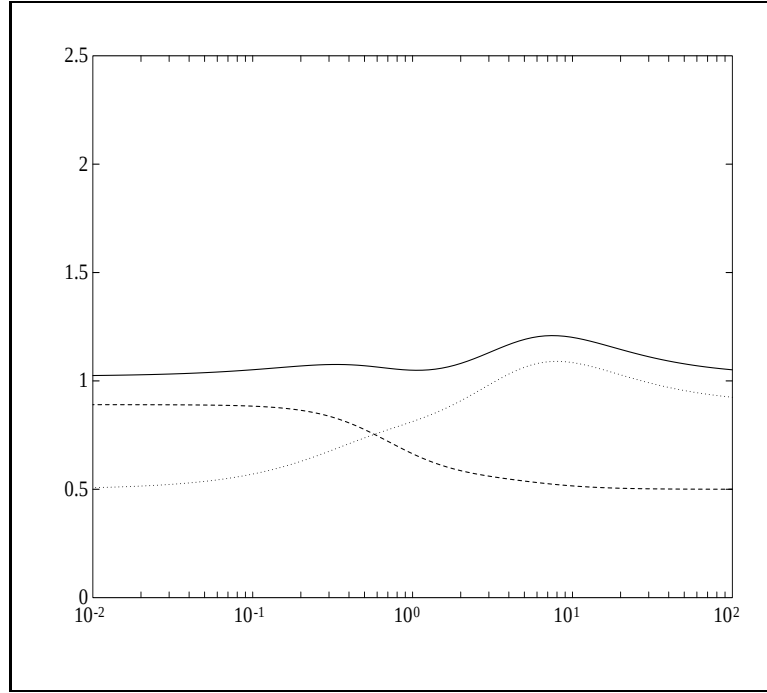


Figure 8.20: $(|W_1S|^2 + |W_{\text{tot}}T|^2)^{1/2}$ (solid), $|W_1S|$ (dash), and $|W_{\text{tot}}T|$ (dot), optimal C .

specification with a constant controller $C = k$ roughly if $k \in (\alpha, 1/\alpha)$, as before. Thus for this problem with both a RHP pole and zero, the approximation of covering the RHP pole and zero with uncertainty produces very little conservatism.

8.4 Shaping S , T , or Q

It is possible to do designs directly in terms of S and T rather than translate the specifications into constraints on the loopshape. This is especially useful when P is either stable or minimum-phase, or both. Then the stability constraints on S and T are particularly simple. For example, if there are only RHP zeros, these must appear in T , but T is otherwise unconstrained. The performance objective due to weights on T can be directly handled by the choice of T , but S is not as easy. Since $S = 1 - T$, we can make S small by making T close to 1. Although this can sometimes be a bit awkward, it is often the case that by looking at T and S directly in addition to L , we can arrive at a design more quickly.

Another alternative to loopshaping is to use Q , appearing in the parametrization of all stabilizing controllers. To summarize this, recall (Section 5.1) that if we have a stable P , we can parametrize the set of all stabilizing controllers as

$$C = \frac{Q}{1 - PQ},$$

where Q is any stable transfer function. In terms of this free parameter Q , we have that $S = 1 - PQ$ and $T = PQ$. As Q approaches $1/P$, then S approaches 0 and C approaches ∞ . Thus for minimum-phase P , we can make S arbitrarily small, as expected. For non-minimum-phase P , recall that we can factor $P = P_{ap}P_{mp}$ where P_{ap} is all-pass and P_{mp} is minimum-phase. We can approximately invert the minimum-phase part by letting $Q = F/P_{mp}$, where F is a low-pass filter so that Q

is proper. We can then shape the low-pass F to trade off between S and T . This approach is essentially shaping T since $T = FP_{ap}$.

Example Take

$$P(s) = \frac{1}{s} \left(\frac{2-s}{2+s} \right). \quad (8.16)$$

Consider directly shaping T and look at tradeoffs between S and T and the limitations imposed by the RHP zero at $s = 2$. For internal stability we must have $T(2) = 0$ and $T(0) = 1$, so we can parametrize a family of allowable T s with

$$\begin{aligned} T(s) &= \left(\frac{2-s}{2+s} \right) \frac{1}{1+\tau s}, \\ S(s) &= 1 - T(s) \\ &= \frac{s}{s+2} \left(\frac{\tau s + 2\tau + 2}{\tau s + 1} \right). \end{aligned}$$

This gives

$$\begin{aligned} L(s) &= \frac{T(s)}{S(s)} \\ &= \frac{2-s}{s(\tau s + 2\tau + 2)}, \\ C(s) &= \frac{s+2}{\tau s + 2\tau + 2}. \end{aligned}$$

For this parametrization, $1/\tau$ is roughly the closed-loop bandwidth, so we are pushing the bandwidth up by making τ small. This is shown in Figures 8.21 and 8.22, where S , T , and L are plotted for $\tau = 0.01, 0.1, 1, 10$, and 100 . There are negligible changes in S , T , and L at low frequency as τ is decreased from 0.1 to 0.01 , although the high-frequency characteristics change substantially. This illustrates another way the limits are imposed by RHP zeros.

If we take limits as $\tau \rightarrow 0$, we get

$$T \rightarrow \frac{2-s}{2+s}, \quad S \rightarrow \frac{2s}{s+2}, \quad L \rightarrow \frac{2-s}{2s}, \quad C \rightarrow \frac{2+s}{2}. \quad (8.17)$$

Thus even at the expense of an infinite bandwidth controller, we cannot get disturbance rejection much above $\omega = 1$. Recall that if we take the weight

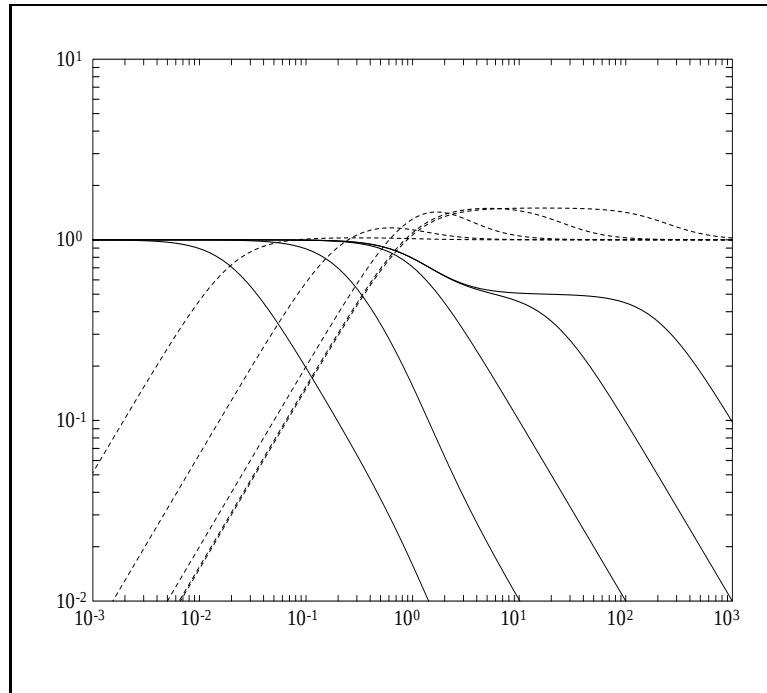
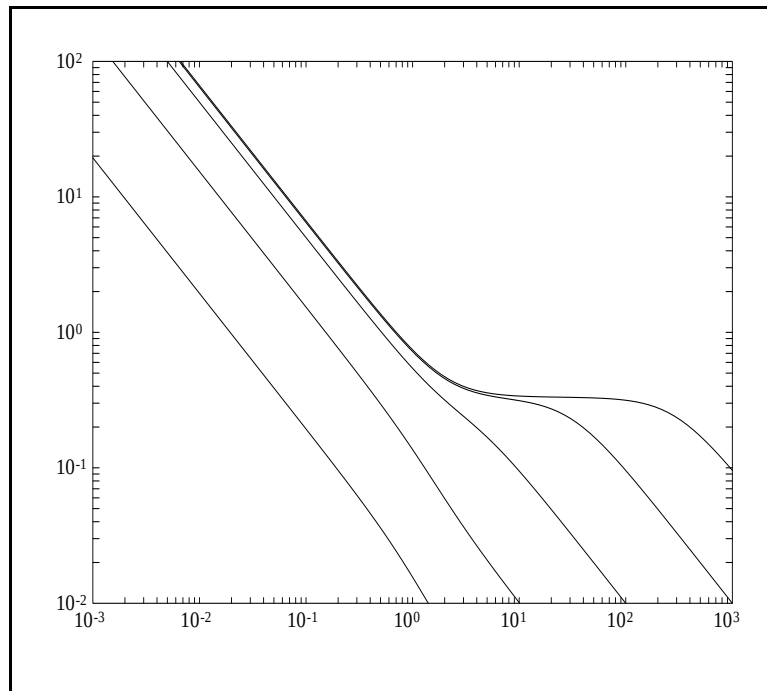
$$W_1(s) = \frac{s+2}{2s},$$

then for any stabilizing controller

$$\|W_1 S\|_\infty \geq \left. \frac{s+2}{2s} \right|_{s=2} = 1,$$

so the S in (8.17) cannot be uniformly improved on. For very low bandwidths (i.e., for $\tau \gg 1$ and $\omega \ll 1$), the RHP zero has negligible impact as

$$T \approx \frac{1}{\tau s + 1}, \quad S \approx \frac{\tau s}{\tau s + 1}, \quad L \approx \frac{1}{\tau s}, \quad C \approx \frac{1}{\tau}.$$

Figure 8.21: $|T|$ (solid) and $|S|$ (dash) for $0.01 \leq \tau \leq 100$.Figure 8.22: $|L|$ for $0.01 \leq \tau \leq 100$.

This example illustrates how we may easily explore the tradeoffs between S and T for P with RHP zeros by putting the zeros in T and then exploring a one-parameter family of T and hence S . Similar tradeoffs may be explored for plants with RHP poles by parametrizing a family of S . For plants with both RHP poles and zeros, it is necessary to use the more complicated parametrization developed in Chapter 5.

8.5 Further Notions of Optimality

Each of the design methods presented so far—shaping L , S , T , or Q —gives the designer something handy to manipulate to obtain a controller. In each case, the resulting controller will have to be examined in the closed-loop system to see if it is satisfactory—every controller results in some tradeoff between S and T , and the designer must decide if that tradeoff makes sense for a particular problem. However, some controllers are intrinsically poor. In particular, we would like to avoid design techniques that yield controllers that can be *uniformly* improved upon, that is, where both $|S|$ and $|T|$ can be reduced at every frequency. Such controllers clearly do a poor job on the tradeoff between S and T .

In this section we define several notions of optimality which we will use to help understand the notion of an intrinsically poor controller. In what follows, all controllers are assumed to be internally stabilizing and all weights are assumed to be stable and not identically zero; C_o is a fixed controller, and S_o and T_o its corresponding sensitivity and complementary sensitivity functions; C is some generic, variable controller and S and T its corresponding functions.

The first type of optimality we will consider is Pareto optimality. A controller C_o is *Pareto (Par.) optimal* if there is no C such that at every frequency $|S| < |S_o|$ and $|T| < |T_o|$; equivalently, for every C , at some frequency either $|S| \geq |S_o|$ or $|T| \geq |T_o|$. A controller C_o is *strongly Pareto (str. Par.) optimal* if there is no $C \neq C_o$ such that at every frequency $|S| \leq |S_o|$ and $|T| \leq |T_o|$; equivalently, for every $C \neq C_o$, at some frequency either $|S| > |S_o|$ or $|T| > |T_o|$. The class of strongly Pareto optimal controllers is quite large: some of them give good S and T tradeoffs, some do not. But controllers that are not strongly Pareto optimal, that is, are outside this class, are evidently poor because they can be improved uniformly.

Our primary objective in this section and the next is to show that loopshaping generally yields strongly Pareto optimal controllers. Unfortunately, we must take a circuitous route to establish this, introducing several intermediate notions of optimality. There are added benefits, however, in that these additional notions of optimality have some independent interest, and we shall see that optimal controllers, the subject of Chapter 12, are also Pareto optimal.

We have argued that the norm

$$\|(|W_1 S|^2 + |W_2 T|^2)^{1/2}\|_\infty$$

is a reasonable performance measure, a compromise norm for the robust performance problem. Recall that for fixed P with C variable, we defined

$$\psi(C) := \|(|W_1 S|^2 + |W_2 T|^2)^{1/2}\|_\infty.$$

Using $\psi(C)$, we can define two additional notions of optimality, together with strengthened versions involving uniqueness that are analogous to strongly Pareto optimal.

1. For given weights W_1 and W_2 , C_o is *optimal* if $\psi(C_o) \leq \psi(C)$ for every C , and *uniquely optimal* if $\psi(C_o) < \psi(C)$ for every $C \neq C_o$.

2. C_o is *potentially* (pot.) *optimal* if there exist weights W_1, W_2 such that C_o is optimal for these weights and *potentially uniquely* (pot. uni.) *optimal* if there exist weights W_1, W_2 such that C_o is uniquely optimal for these weights.

These notions, along with Pareto optimal, are related according to the Venn diagram shown in Figure 8.23, which is easily verified. Potentially uniquely optimal is the strongest of the four notions.

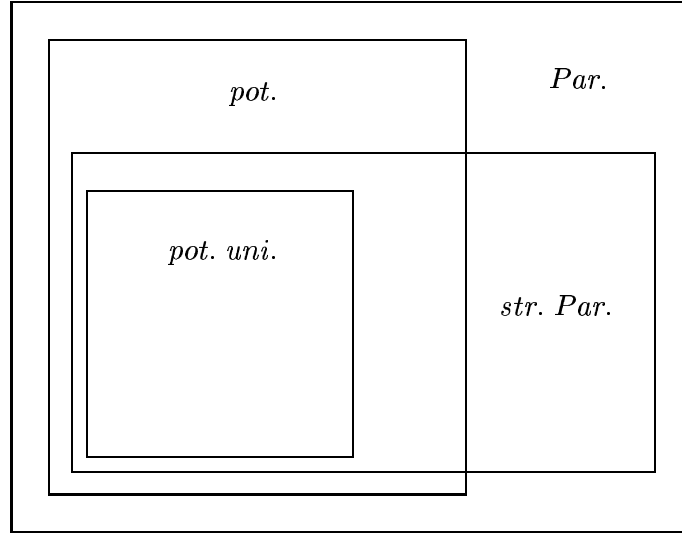


Figure 8.23: Notions of optimality.

Self-Optimality

In this subsection an even stronger type of optimality is introduced whose significance is due to three features:

1. It is easy to characterize.
2. Loopshaping generally produces controllers with this type of optimality.
3. It implies all the notions above, and in particular, strong Pareto optimality.

Fix a controller C_o , define $L_o := PC_o$, and define weights as follows: W_1^{-1} is the minimum-phase factor of S_o and W_2^{-1} is the minimum-phase factor of T_o . Thus the weights are defined in terms of the controller itself. Observe that W_2^{-1} may be strictly proper, and that both W_1^{-1} and W_2^{-1} may have zeros on the imaginary axis. Thus the weights W_1 and W_2 may not be bounded on the imaginary axis. Since $W_1 S_o$ equals the all-pass factor of S_o , it has constant magnitude 1 on the imaginary axis. Similarly for $W_2 T_o$. Thus

$$|W_1 S_o|^2 + |W_2 T_o|^2 = 2, \quad \forall \omega.$$

Given W_1 and W_2 so defined, C_o is called *self-optimal* if it is optimal with respect to these weights. Similarly, it is *uniquely self-optimal* if it is uniquely optimal with respect to these weights.

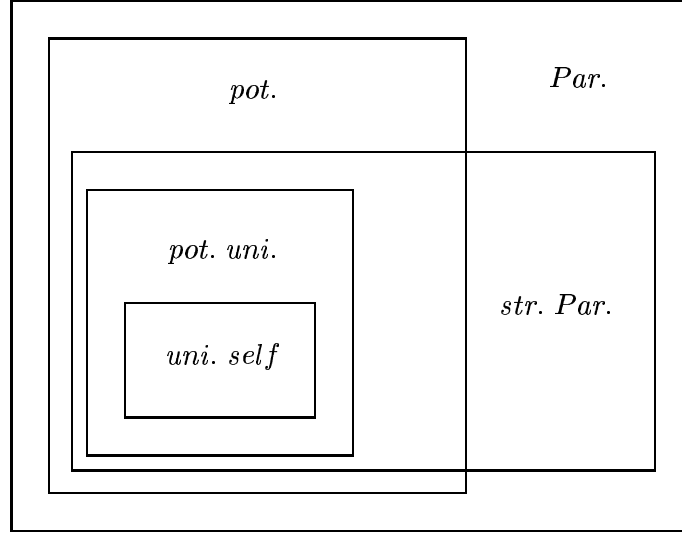


Figure 8.24: Notions of optimality.

It will be shown below that for almost all controllers, a self-optimal controller is uniquely self-optimal. The Venn diagram, including uniquely self-optimal is shown in Figure 8.24.

It is convenient to introduce the following notation. For a transfer function G having no poles on the imaginary axis, let

$$\begin{aligned} \#p(G) &:= \text{number of open RHP poles of } G, \\ \#z(G) &:= \text{number of open RHP zeros of } G, \\ \#i(G) &:= \text{number of imaginary axis zeros of } G. \end{aligned}$$

The main result is as follows.

Theorem 1 *If $\#i(L_o - 1) = 0$, C_o is self-optimal iff C_o is uniquely self-optimal iff*

$$\#z(L_o - 1) > \#p(C_o) + \#z(C_o). \quad (8.18)$$

We are restricting attention to the case $\#i(L_o - 1) = 0$ primarily for technical reasons, as it simplifies the development with a negligible loss of generality. Note that $\#i(L_o - 1) = 0$ iff $L_o(j\omega) \neq 1, \forall \omega$. This would hold for almost all controllers, because if $\#i(L_o - 1) \neq 0$, then a small change, say by a constant gain, in L_o would make $\#i(L_o - 1) = 0$. It can easily be shown that if $\#i(L_o - 1) \geq 1$, then C_o is always self-optimal, but characterizing uniquely self-optimal is more difficult.

Observe from the principle of the argument that in terms of the Nyquist plot of L_o ,

$$\begin{aligned} \#z(L_o - 1) &= \#p(L_o - 1) + (\text{no. clockwise enc. of } +1) \\ &= \#p(L_o) + (\text{no. clockwise enc. of } +1) \\ &= \#p(P) + \#p(C_o) + (\text{no. clockwise enc. of } +1). \end{aligned}$$

So condition (8.18) is equivalent to

$$\#p(P) + (\text{no. clockwise enc. of } +1) > \#z(C_o).$$

A typical loopshaping controller for a stable plant will have $\#z(C_o) = 0$ and

$$(\text{no. clockwise enc. of } +1) > 0,$$

and will thus be self-optimal.

Example For a simple illustration, suppose that

$$P(s) = \frac{s+1}{100s+1}.$$

The controller $C_o(s) = 10$ yields

$$L_o(s) = 10 \frac{s+1}{100s+1}.$$

The Nyquist plot of L_o has one clockwise encirclement of $+1$, so C_o is self-optimal. So is $C_o(s) = K$ for any K in $(1, 100)$. In particular, such controllers cannot be improved upon uniformly over all frequencies.

It can be shown using the phase formula (Section 7.2) that any controller that is minimum-phase and for which L_o has a single gain crossover is self-optimal.

Proof of Theorem 1 Define

$$\Gamma_o(j\omega) := |W_1(j\omega)S_o(j\omega)|^2 + |W_2(j\omega)T_o(j\omega)|^2,$$

so that

$$\psi(C_o)^2 = \sup_{\omega} \Gamma_o(j\omega).$$

As we have already seen, $\Gamma_o(j\omega) = 2$. Similarly, for another controller C , define

$$\Gamma(j\omega) := |W_1(j\omega)S(j\omega)|^2 + |W_2(j\omega)T(j\omega)|^2,$$

so that

$$\psi(C)^2 = \sup_{\omega} \Gamma(j\omega).$$

Thus C_o is self-optimal

$$\begin{aligned} &\iff (\forall C) \psi(C_o) \leq \psi(C) \\ &\iff (\forall C)(\exists \omega) 2 \leq \Gamma(j\omega) \\ &\iff \text{it is not true that } (\exists C)(\forall \omega) 2 > \Gamma(j\omega) \end{aligned}$$

and C_o is uniquely self-optimal

$$\iff \text{it is not true that } (\exists C \neq C_o)(\forall \omega) 2 \geq \Gamma(j\omega).$$

So the theorem statement is equivalent to

$$\begin{aligned} (\exists C)(\forall \omega) 2 > \Gamma(j\omega) &\iff (\exists C \neq C_o)(\forall \omega) 2 \geq \Gamma(j\omega) \\ &\iff \#z(L_o - 1) \leq \#p(C_o) + \#z(C_o). \end{aligned}$$

It is convenient to turn this statement into one in terms of S_o rather than C_o . As we saw in Chapter 6, the constraints placed on S_o by the requirement that C_o achieve internal stability are

$$\begin{aligned} S_o &\in \mathcal{S}, \\ S_o &= 0 \text{ at RHP poles of } P, \\ S_o &= 1 \text{ at RHP zeros of } P, \end{aligned}$$

with appropriate multiplicity. Then S satisfies these constraints too iff it has the form

$$S = S_o + AY,$$

where A is the all-pass factor formed from the RHP poles and zeros of P , and Y is an arbitrary element of $\mathcal{S}$. It is convenient first to assume that P and C_o are biproper and have no poles or zeros on the imaginary axis, so that neither S_o nor T_o have imaginary axis zeros, and the weights are stable and biproper. Then (8.19) is equivalent to

$$\begin{aligned} (\exists Y \in \mathcal{S})(\forall \omega) \quad 2 &> |W_1(S_o + AY)|^2 + |W_2(T_o - AY)|^2 \iff \\ (\exists Y \in \mathcal{S}, Y \neq 0)(\forall \omega) \quad 2 &\geq |W_1(S_o + AY)|^2 + |W_2(T_o - AY)|^2 \iff \\ \#z(L_o - 1) &\leq \#p(C_o) + \#z(C_o) \end{aligned} \quad (8.19)$$

($j\omega$ has been dropped to simplify the notation). Recall that

$$|W_1| = \frac{1}{|S_o|}, \quad |W_2| = \frac{1}{|T_o|},$$

so

$$\begin{aligned} |W_1(S_o + AY)|^2 + |W_2(T_o - AY)|^2 &= \left|1 + \frac{AY}{S_o}\right|^2 + \left|1 - \frac{AY}{T_o}\right|^2 \\ &= 2 + 2\operatorname{Re}\left(\left(\frac{1}{S_o} - \frac{1}{T_o}\right)AY\right) \end{aligned} \quad (8.20)$$

$$+ \left(\frac{1}{|S_o|^2} + \frac{1}{|T_o|^2}\right)|AY|^2. \quad (8.21)$$

Thus (8.19) is equivalent to the condition

$$(\exists Y \in \mathcal{S})(\forall \omega) \quad 0 > 2\operatorname{Re}(XY) + W|Y|^2 \quad (8.22)$$

iff

$$(\exists Y \in \mathcal{S}, Y \neq 0)(\forall \omega) \quad 0 \geq 2\operatorname{Re}(XY) + W|Y|^2 \quad (8.23)$$

iff

$$\#z(L_o - 1) \leq \#p(C_o) + \#z(C_o), \quad (8.24)$$

where

$$X := \left(\frac{1}{S_o} - \frac{1}{T_o}\right)A$$

is biproper with no imaginary poles or zeros and

$$W := \left(\frac{1}{|S_o|^2} + \frac{1}{|T_o|^2}\right)|A|^2 > 0$$

is bounded for all ω . That (8.22) implies (8.23) is immediate by inspection. We will complete the proof by first showing that (8.24) is equivalent to

$$\#z(X) \leq \#p(X) \quad (8.25)$$

and then that (8.23) $\implies$ (8.25) and (8.25) $\implies$ (8.22).

To see that (8.24) $\iff$ (8.25) holds, note that

$$\begin{aligned} X &= \left(L_o + 1 - \frac{L_o + 1}{L_o} \right) A = \frac{(L_o + 1)(L_o - 1)}{L_o} A \\ &= \frac{[\text{num}(PC_o) + \text{den}(PC_o)][\text{num}(PC_o) - \text{den}(PC_o)]}{\text{num}(PC_o)\text{den}(PC_o)} A. \end{aligned}$$

Now the polynomial $\text{num}(PC_o) + \text{den}(PC_o)$ has all its zeros in the left half-plane, by internal stability; and the numerator of A cancels the zeros in $\text{num}(PC_o)\text{den}(PC_o)$ coming from the RHP poles and zeros of P . Thus

$$\#z(X) = \#z[\text{num}(PC_o) - \text{den}(PC_o)] = \#z(L_o - 1)$$

and

$$\#p(X) = \#p(C_o) + \#z(C_o).$$

For (8.23) $\implies$ (8.25), note that if the Nyquist plot of XY lies in the closed left half-plane, then in particular it does not encircle the origin. Since XY is not identically zero, by the principle of the argument

$$\#z(XY) = \#p(XY).$$

But $\#z(X) \leq \#z(XY)$ and $\#p(XY) \leq \#p(X)$.

For (8.25) $\implies$ (8.22), note that (8.22) is equivalent to

$$(\exists Y \in \mathcal{S})(\forall \omega) \ 0 > 2\text{Re}(XY),$$

since we can scale Y so that the quadratic term $W|Y|^2$ is negligible. We will construct a $Y \in \mathcal{S}$ such that $\text{Re}(XY) < 0$, $\forall \omega$. If $\#z(X) \leq \#p(X)$, we can write $X = X_1 X_2$, where X_1 has only RHP poles and zeros with $\#z(X) = \#z(X_1) = \#p(X_1)$, and X_2 has $\#z(X_2) = 0$ and $\#p(X_2) = \#p(X) - \#p(X_1)$. Thus both X_1 and X_2 are biproper. If

$$Y(s) = \frac{X_1(-s)}{X_2(s)},$$

then $Y \in \mathcal{S}$ and is biproper and

$$X(j\omega)Y(j\omega) = X_1(j\omega)X_1(-j\omega) = |X_1(j\omega)|^2 > 0, \quad \forall \omega.$$

To complete the proof, we must drop the restriction that L_o is biproper and has no imaginary axis poles or zeros. This means that S_o and T_o may have imaginary zeros, and thus W_1 and W_2 may have imaginary poles. In order for (8.21) to be bounded, we must have AY share the imaginary zeros of S_o and T_o , including those at ∞ .

The simplest way to proceed is to assume that A is no longer all-pass but has in addition the imaginary zeros of S_o and T_o and enough LHP poles so that A has the appropriate behavior at ∞ . The location of the LHP poles is unimportant. Then

$$S = S_o + AY$$

still parametrizes all S that can arise from internally stabilizing controllers and produce a finite Γ . The proof can now proceed exactly as before. Observe that X is still biproper and has no imaginary poles or zeros, because of the construction of A . ■

Example and Implications

In the preceding subsection it was remarked that loopshaping produces self-optimal controllers. In particular, for stable plants and loopshapes with a single crossover, where $\#z(L_o - 1) = (\# \text{ clockwise enc. of } +1) = 1$, then the controller is self-optimal iff it has no RHP poles or zeros. For $\#z(L_o - 1) > 1$, controllers with no RHP poles or zeros are also always self-optimal. This helps make clear the conventional wisdom about why RHP controller poles and zeros are undesirable. Similar insight comes from the results on performance limitations due to RHP zeros and the integral gain-phase relations from earlier chapters.

Are there ever times when it would make sense to introduce RHP controller poles or zeros deliberately? Theorem 1 leaves open this possibility for situations where $\#z(L_o - 1) > 1$, which can occur when there are multiple crossovers.

Example To illustrate this we will reconsider the simple beam, Example 2 from Section 8.2, with P given in (8.6) and with weights from (8.3). Recall that for the noncollocated case the plant has a RHP zero at $s = 1$. Consider the controller

$$C_-(s) = 2.4 \left(\frac{s-1}{s+1} \right) \left(\frac{s^2 - s + 1}{(s+0.1)(s+10)} \right). \quad (8.26)$$

A Bode plot of C_- is shown in Figure 8.25. The resulting closed-loop performance for C_- is shown in Figure 8.26, along with the performance for the optimal controller. Observe that C_- is nearly optimal. (It was actually obtained by rounding coefficients in the optimal controller to convenient values.) Also, the optimal performance of $\psi \approx 1.26$ is poorer than the optimal performance of $\psi = \sqrt{2}/2 \approx 0.707$ for the collocated case.

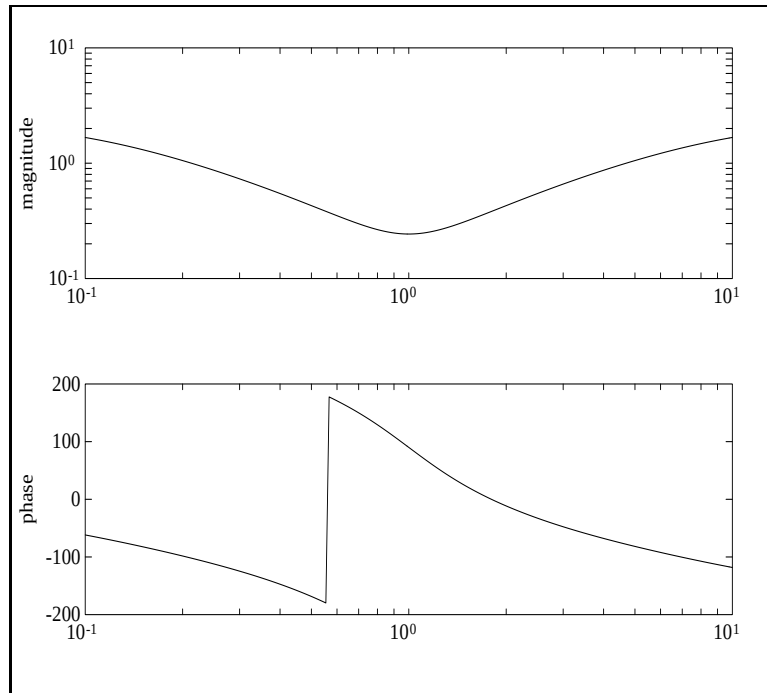
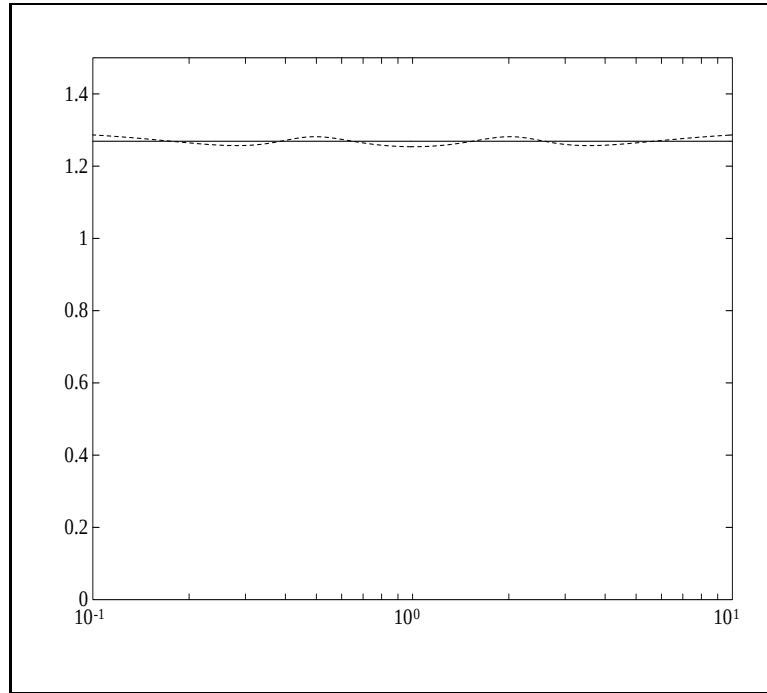


Figure 8.25: Bode plots of $|C_-|$ (upper) and $\angle C_-$ (lower).

The controller C_- has no RHP poles and three RHP zeros, so that the loop transfer function PC_- has a total of four RHP zeros. Since $\#z(PC_- - 1) = 4$, we have from Theorem 1 that this

Figure 8.26: ψ for optimal C (solid) and C_- (dash).

controller is also self-optimal. It is possible to give a loopshaping interpretation of this controller, which is not much different than an all-pass. Intuitively, a good controller for this problem must have the appropriate phase at the two resonant frequencies ($\omega = 0.5, 2$) for stability, and from the arguments in Section 8.2 its gain should not deviate too greatly from 1. Thus a controller which is nearly all-pass would seem appropriate.

The ideas on dealing with RHP zeros presented in Section 8.3 are not terribly useful in this example. Since there are necessarily crossover frequencies both above and below the frequency $\omega = 1$ where the zero occurs, it is not clear how to cover this with uncertainty. The controller C_- actually places an additional three RHP zeros near this frequency. While a loopshaping methodology could potentially still apply to this example, the actual details of this controller are hardly obvious. The loopshaping methods developed in this book always involve some trial and error, but this example seems to require a great deal. This would also be true for many other problems of this complexity with RHP zeros at frequencies between multiple crossover frequencies.

This example illustrates how RHP controller zeros can be effective for problems with multiple crossovers. What about poles? Theorem 1 appears to be symmetric in the controller RHP poles and zeros, but this is somewhat misleading since RHP poles have an impact on the need for encirclements of -1 and hence on the number $\#z(PC - 1)$. Nevertheless, in the problem considered in this section, with perhaps slightly different weights, there will typically be multiple crossovers and $\#z(PC - 1) \geq 4$, so C can have up to a total of three RHP poles and zeros and still be self-optimal. Thus we would expect that relatively small changes in the weights could lead to controllers with a variety of RHP poles and zeros. Recall also that certain plants can only be stabilized with unstable controllers, so any problem with such a plant would necessarily have controllers with RHP poles. On the other hand, the vast majority of feedback control problems in engineering applications have a single crossover, and controller RHP poles and zeros would clearly be undesirable.

Now we return to the collocated version of this problem and use the characterization of self-

optimality from the preceding subsection to prove that the controller $C = 1$ is optimal for all physically possible values of the parameters M , I , l , l_u , and k . This is a remarkable robust performance result and does not hold in the more general noncollocated case. It illustrates why collocated control problems are so popular among researchers in control of flexible structures. We have not focused on this type of parametric uncertainty in this book, because the available results tend not to be very general. For example, the optimality of $C = 1$ for this problem depends not only on the special structure of P but also on the special structure of the performance specification. Nevertheless, the methods developed in this book can be useful in studying specific problems involving parametric uncertainty, as illustrated by this example.

To prove the optimality of $C = 1$, note that for all values of the parameters

$$P(s) = \frac{s}{Ms^2 + 2k} + \frac{l_u^2 s}{Is^2 + 2kl^2} = \frac{s[(I + Ml_u^2)s^2 + 2k(1 + l^2)]}{(Ms^2 + 2k)(Is^2 + 2kl^2)}$$

has all its poles and zeros on the imaginary axis, with poles at

$$s = \pm j\sqrt{\frac{2k}{M}}, \pm j\sqrt{\frac{2kl^2}{I}}$$

and zeros at

$$s = 0, \pm j\sqrt{\frac{2k(1 + l^2)}{(I + Ml_u^2)}}.$$

To be physically meaningful, all the parameters must be positive, and the geometry of the problem restricts the moment of inertia such that

$$0 < I < Ml^2,$$

so we have that

$$0 < \sqrt{\frac{2k}{M}} < \sqrt{\frac{2k(1 + l^2)}{(I + Ml_u^2)}} < \sqrt{\frac{2kl^2}{I}}.$$

Thus the zeros and poles alternate on the imaginary axis and

$$\operatorname{Re}P(j\omega) = 0, \quad \forall \omega.$$

It is clear from the Nyquist diagram of P that

$$\#z(P + 1) = 0 \quad \text{and} \quad \#z(P - 1) = 4,$$

so the controller $C = 1$ is both stabilizing and self-optimal for all parameter values. Recall that the weights W_1 and W_2 that are used to define self-optimality are those that make $|W_1 S|$ and $|W_2 T|$ all-pass. Clearly, we could use weights that are a constant multiple of these as well. Thus to use self-optimality to show that $C = 1$ is optimal for

$$\psi(C) := \|(|W_1 S|^2 + |W_2 T|^2)^{1/2}\|_\infty$$

with

$$|W_1| = 0.5(|P|^2 + 1)^{1/2}, \quad |W_2| = 0.5(|P|^{-2} + 1)^{1/2},$$

we must verify that $|W_1 S|$ and $|W_2 T|$ are all-pass. But since $\operatorname{Re}P(j\omega) = 0$, $\forall \omega$, we have that

$$|P + 1|^2 = |P|^2 + 1, \quad \forall s = j\omega,$$

and thus

$$|W_1 S|^2 = 0.25 \frac{|P|^2 + 1}{|P + 1|^2} = 0.25, \quad |W_2 T|^2 = 0.25 \frac{|P|^{-2} + 1}{|P^{-1} + 1|^2} = 0.25,$$

as desired. This completes the proof.

What are the implications of this example and the results on self-optimality for loopshaping? Each of the design methods presented so far—shaping L , S , T , or Q —gives the designer something handy to manipulate to obtain a controller. In each case, the resulting controller will have to be examined in the closed-loop system to see if it is satisfactory—every controller results in some tradeoff between S and T , and the designer must decide if that tradeoff makes sense for a particular problem. If a reasonable tradeoff between S and T can be found by any method, and the resulting controller is self-optimal, then there is no other controller that can uniformly improve on the tradeoff—any other controller just achieves some other tradeoff. What makes self-optimality so useful is that it is very easily checked, and the loopshaping methods introduced so far generally yield self-optimal controllers.

Unfortunately, the designer may find that for some problems loopshaping does not easily lead to a reasonable tradeoff between competing performance objectives. It is for these problems that directly solving for an optimal controller is most useful. Optimal controllers are the subject of the remainder of this book.

Exercises

1. Verify the Venn diagrams in Section 8.5.
2. Prove that if $\#i(L_o - 1) \geq 1$, then C_o is self-optimal.
3. Develop notions of self-optimality that involving S and T separately, and prove results analogous to Theorem 1.

Notes and References

The problem setup in Section 8.1 is studied in much greater depth in McFarlane and Glover (1990), who develop an interesting theory of control design using a combination of loopshaping and optimality. For a recent treatment of the control of flexible structures, see Joshi (1989). Controller design by shaping Q , known in the process control literature as internal model control (IMC), is developed in detail in Morari and Zafriou (1989). Section 8.5 is based on Lenz et al. (1988).

