

Chapter 5

Stabilization

In this chapter we study the unity-feedback system with block diagram shown in Figure 5.1. Here

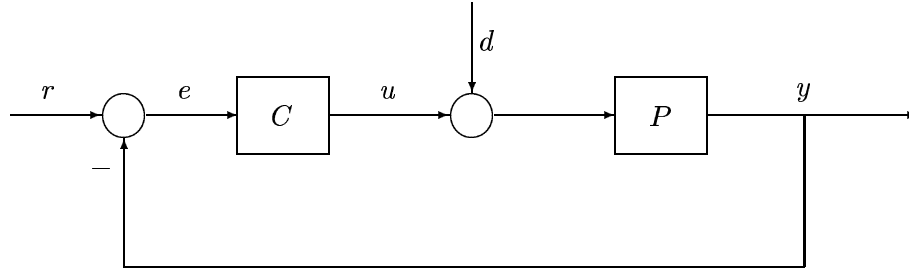


Figure 5.1: Unity-feedback system.

P is strictly proper and C is proper.

Most synthesis problems can be formulated in this way: Given P , design C so that the feedback system (1) is internally stable, and (2) acquires some additional desired property; for example, the output y asymptotically tracks a step input r . The method of solution is to parametrize all C s for which (1) is true, and then to see if there exists a parameter for which (2) holds. In this chapter such a parametrization is derived and then applied to two problems: achieving asymptotic performance specs and internal stabilization by a stable controller.

5.1 Controller Parametrization: Stable Plant

In this section we assume that P is already stable, and we parametrize all C s for which the feedback system is internally stable. Introduce the symbol $\mathcal{S}$ for the family of all stable, proper, real-rational functions. Notice that $\mathcal{S}$ is closed under addition and multiplication: If $F, G \in \mathcal{S}$, then $F + G, FG \in \mathcal{S}$. Also, $1 \in \mathcal{S}$. (Thus $\mathcal{S}$ is a commutative ring with identity.)

Theorem 1 *Assume that $P \in \mathcal{S}$. The set of all C s for which the feedback system is internally stable equals*

$$\left\{ \frac{Q}{1 - PQ} : Q \in \mathcal{S} \right\}.$$

Proof ($\subset$) Suppose that C achieves internal stability. Let Q denote the transfer function from r to u , that is,

$$Q := \frac{C}{1 + PC}.$$

Then $Q \in \mathcal{S}$ and

$$C = \frac{Q}{1 - PQ}.$$

($\supset$) Conversely, suppose that $Q \in \mathcal{S}$ and define

$$C := \frac{Q}{1 - PQ}. \quad (5.1)$$

According to the definition in Section 3.2, the feedback system is internally stable iff the nine transfer functions

$$\frac{1}{1 + PC} \begin{bmatrix} 1 & -P & -1 \\ C & 1 & -C \\ PC & P & 1 \end{bmatrix}$$

all are stable and proper. After substitution from (5.1) and clearing of fractions, this matrix becomes

$$\begin{bmatrix} 1 - PQ & -P(1 - PQ) & -(1 - PQ) \\ Q & 1 - PQ & -Q \\ PQ & P(1 - PQ) & 1 - PQ \end{bmatrix}.$$

Clearly, these nine entries belong to $\mathcal{S}$. ■

Note that all nine transfer functions above are affine functions of the free parameter Q ; that is, each is of the form $T_1 + T_2 Q$ for some T_1, T_2 in $\mathcal{S}$. In particular the sensitivity and complementary sensitivity functions are

$$\begin{aligned} S &= 1 - PQ, \\ T &= PQ. \end{aligned}$$

Let us look at a simple application. Suppose that we want to find a C so that the feedback system is internally stable and y asymptotically tracks a step r (when $d = 0$). Parametrize C as in the theorem. Then y asymptotically tracks a step iff the transfer function from r to e (i.e., S) has a zero at $s = 0$, that is,

$$P(0)Q(0) = 1.$$

This equation has a solution Q in $\mathcal{S}$ iff $P(0) \neq 0$. Conclusion: The problem has a solution iff $P(0) \neq 0$; when this holds, the set of all solutions is

$$\left\{ C = \frac{Q}{1 - PQ} : Q \in \mathcal{S}, Q(0) = \frac{1}{P(0)} \right\}.$$

Observe that Q inverts P at dc. Also, you can check that a controller of the latter form has a pole at $s = 0$, as it must by Theorem 3 of Chapter 3.

Example For the plant

$$P(s) = \frac{1}{(s + 1)(s + 2)}$$

suppose that it is desired to find an internally stabilizing controller so that y asymptotically tracks a ramp r . Parametrize C as in the theorem. The transfer function S from r to e must have (at least) two zeros at $s = 0$, where r has two poles. Let us take

$$Q(s) = \frac{as + b}{s + 1}.$$

This belongs to $\mathcal{S}$ and has two variables, a and b , for the assignment of the two zeros of S . We have

$$\begin{aligned} S(s) &= 1 - \frac{as + b}{(s + 1)^2(s + 2)} \\ &= \frac{s^3 + 4s^2 + (5 - a)s + (2 - b)}{(s + 1)^2(s + 2)}, \end{aligned}$$

so we should take $a = 5, b = 2$. This gives

$$\begin{aligned} Q(s) &= \frac{5s + 2}{s + 1}, \\ C(s) &= \frac{(5s + 2)(s + 1)(s + 2)}{s^2(s + 4)}. \end{aligned}$$

The controller is internally stabilizing and has two poles at $s = 0$.

5.2 Coprime Factorization

Now suppose that P is not stable and we want to find an internally stabilizing C . We might try as follows. Write P as the ratio of coprime polynomials,

$$P = \frac{N}{M}.$$

By Euclid's algorithm (reviewed below) we can get two other polynomials X, Y satisfying the equation

$$NX + MY = 1.$$

Remembering Theorem 3.1 (the feedback system is internally stable iff the characteristic polynomial has no zeros in $\text{Re } s \geq 0$), we might try to make the left-hand side equal to the characteristic polynomial by setting

$$C = \frac{X}{Y}.$$

The trouble is that Y may be 0; even if not, this C may not be proper.

Example 1 For $P(s) = 1/s$, we can take $N(s) = 1, M(s) = s$. One solution to the equation $NX + MY = 1$ is $X(s) = 1, Y(s) = 0$, for which X/Y is undefined. Another solution is $X(s) = -s + 1, Y(s) = 1$, for which X/Y is not proper.

The remedy is to arrange that N, M, X, Y are all elements of $\mathcal{S}$ instead of polynomials. Two functions N and M in $\mathcal{S}$ are *coprime* if there exist two other functions X and Y also in $\mathcal{S}$ and satisfying the equation

$$NX + MY = 1.$$

Notice that for this equation to hold, N and M can have no common zeros in $\text{Re } s \geq 0$ nor at the point $s = \infty$ —if there were such a point s_0 , there would follow

$$0 = N(s_0)X(s_0) + M(s_0)Y(s_0) \neq 1.$$

It can be proved that this condition is also sufficient for coprimeness.

Let G be a real-rational transfer function. A representation of the form

$$G = \frac{N}{M}, \quad N, M \in \mathcal{S},$$

where N and M are coprime, is called a *coprime factorization* of G over $\mathcal{S}$. The purpose of this section is to present a method for the construction of four functions in $\mathcal{S}$ satisfying the two equations

$$G = \frac{N}{M}, \quad NX + MY = 1.$$

The construction of N and M is easy.

Example 2 Take $G(s) = 1/(s - 1)$. To write $G = N/M$ with N and M in $\mathcal{S}$, simply divide the numerator and denominator polynomials, 1 and $s - 1$, by a common polynomial with no zeros in $\text{Re } s \geq 0$, say $(s + 1)^k$:

$$\frac{1}{s - 1} = \frac{N(s)}{M(s)}, \quad N(s) = \frac{1}{(s + 1)^k}, \quad M(s) = \frac{s - 1}{(s + 1)^k}.$$

If the integer k is greater than 1, then N and M are not coprime—they have a common zero at $s = \infty$. So

$$N(s) = \frac{1}{s + 1}, \quad M(s) = \frac{s - 1}{s + 1}$$

suffice.

More generally, to get N and M we could divide the numerator and denominator polynomials of G by $(s + 1)^k$, where k equals the maximum of their degrees. What is not so easy is to get the other two functions, X and Y , and this is why we need Euclid's algorithm.

Euclid's algorithm computes the greatest common divisor of two given polynomials, say $n(\lambda)$ and $m(\lambda)$. When n and m are coprime, the algorithm can be used to compute polynomials $x(\lambda)$, $y(\lambda)$ satisfying

$$nx + my = 1.$$

Procedure A: Euclid's Algorithm

Input: polynomials n, m

Initialize: If it is not true that $\text{degree}(n) \geq \text{degree}(m)$, interchange n and m .

Step 1 Divide m into n to get quotient q_1 and remainder r_1 :

$$n = mq_1 + r_1,$$

$$\text{degree } r_1 < \text{degree } m.$$

Step 2 Divide r_1 into m to get quotient q_2 and remainder r_2 :

$$m = r_1 q_2 + r_2,$$

$$\text{degree } r_2 < \text{degree } r_1.$$

Step 3 Divide r_2 into r_1 :

$$r_1 = r_2 q_3 + r_3,$$

$$\text{degree } r_3 < \text{degree } r_2.$$

Continue.

Stop at Step k when r_k is a nonzero constant.

Then x, y are obtained as illustrated by the following example for $k = 3$. The equations are

$$n = m q_1 + r_1,$$

$$m = r_1 q_2 + r_2,$$

$$r_1 = r_2 q_3 + r_3,$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 \\ q_2 & 1 & 0 \\ -1 & q_3 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & -q_1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} n \\ m \end{bmatrix}.$$

Solve for r_3 by, say, Gaussian elimination:

$$r_3 = (1 + q_2 q_3)n + [-q_3 - q_1(1 + q_2 q_3)]m.$$

Set

$$\begin{aligned} x &= \frac{1}{r_3}(1 + q_2 q_3), \\ y &= \frac{1}{r_3}[-q_3 - q_1(1 + q_2 q_3)]. \end{aligned}$$

Example 3 The algorithm for $n(\lambda) = \lambda^2$, $m(\lambda) = 6\lambda^2 - 5\lambda + 1$ goes like this:

$$\begin{aligned} q_1(\lambda) &= \frac{1}{6}, \\ r_1(\lambda) &= \frac{5}{6}\lambda - \frac{1}{6}, \\ q_2(\lambda) &= \frac{36}{5}\lambda - \frac{114}{25}, \\ r_2(\lambda) &= \frac{6}{25}. \end{aligned}$$

Since r_2 is a nonzero constant, we stop after Step 2. Then the equations are

$$n = m q_1 + r_1,$$

$$m = r_1 q_2 + r_2,$$

yielding

$$r_2 = (1 + q_1 q_2)m - q_2 n.$$

So we should take

$$x = -\frac{q_2}{r_2}, \quad y = \frac{1 + q_1 q_2}{r_2},$$

that is,

$$x(\lambda) = -30\lambda + 19, \quad y(\lambda) = 5\lambda + 1.$$

Next is a procedure for doing a coprime factorization of G . The main idea is to transform variables, $s \rightarrow \lambda$, so that polynomials in λ yield functions in $\mathcal{S}$.

Procedure B

Input: G

Step 1 If G is stable, set $N = G$, $M = 1$, $X = 0$, $Y = 1$, and stop; else, continue.

Step 2 Transform $G(s)$ to $\tilde{G}(\lambda)$ under the mapping $s = (1 - \lambda)/\lambda$. Write $\tilde{G}$ as a ratio of coprime polynomials:

$$\tilde{G}(\lambda) = \frac{n(\lambda)}{m(\lambda)}.$$

Step 3 Using Euclid's algorithm, find polynomials $x(\lambda)$, $y(\lambda)$ such that

$$nx + my = 1.$$

Step 4 Transform $n(\lambda)$, $m(\lambda)$, $x(\lambda)$, $y(\lambda)$ to $N(s)$, $M(s)$, $X(s)$, $Y(s)$ under the mapping $\lambda = 1/(s + 1)$.

The mapping used in this procedure is not unique; the only requirement is that polynomials n , and so on, map to N , and so on, in $\mathcal{S}$.

Example 4 For

$$G(s) = \frac{1}{(s - 1)(s - 2)}$$

the algorithm gives

$$\begin{aligned} \tilde{G}(\lambda) &= \frac{\lambda^2}{6\lambda^2 - 5\lambda + 1}, \\ n(\lambda) &= \lambda^2, \\ m(\lambda) &= 6\lambda^2 - 5\lambda + 1, \\ x(\lambda) &= -30\lambda + 19, \\ y(\lambda) &= 5\lambda + 1 \quad (\text{from Example 3}), \\ N(s) &= \frac{1}{(s + 1)^2}, \end{aligned}$$

$$\begin{aligned}
M(s) &= \frac{(s-1)(s-2)}{(s+1)^2}, \\
X(s) &= \frac{19s-11}{s+1}, \\
Y(s) &= \frac{s+6}{s+1}.
\end{aligned}$$

5.3 Coprime Factorization by State-Space Methods (Optional)

This optional section presents a state-space procedure for computing a coprime factorization over $\mathcal{S}$ of a proper G . This procedure is more efficient than the polynomial method in the preceding section.

We start with a new data structure. Suppose that A, B, C, D are real matrices of dimensions

$$n \times n, \quad n \times 1, \quad 1 \times n, \quad 1 \times 1.$$

The transfer function going along with this quartet is

$$D + C(sI - A)^{-1}B.$$

Note that the constant D equals the value of the transfer function at $s = \infty$; if the transfer function is strictly proper, then $D = 0$. It is convenient to write

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

instead of

$$D + C(sI - A)^{-1}B.$$

Beginning with a realization of G ,

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right],$$

the goal is to get state-space realizations for four functions N, M, X, Y , all in $\mathcal{S}$, such that

$$G = \frac{N}{M}, \quad NX + MY = 1.$$

First, we look at how to get N and M . If the input and output of G are denoted u and y , respectively, then the state model of G is

$$\dot{x} = Ax + Bu, \tag{5.2}$$

$$y = Cx + Du. \tag{5.3}$$

Choose a real matrix F , $1 \times n$, such that $A + BF$ is stable (i.e., all eigenvalues in $\text{Res} < 0$). Now define the signal $v := u - Fx$. Then from (5.2) and (5.3) we get

$$\dot{x} = (A + BF)x + Bv,$$

$$u = Fx + v,$$

$$y = (C + DF)x + Dv.$$

Evidently from these equations, the transfer function from v to u is

$$M(s) := \left[\begin{array}{c|c} A + BF & B \\ \hline F & 1 \end{array} \right], \quad (5.4)$$

and that from v to y is

$$N(s) := \left[\begin{array}{c|c} A + BF & B \\ \hline C + DF & D \end{array} \right]. \quad (5.5)$$

Therefore,

$$u = Mv, \quad y = Nv,$$

so that $y = NM^{-1}u$, that is, $G = N/M$. Clearly, N and M are proper, and they are stable because $A + BF$ is. Thus $N, M \in \mathcal{S}$. Suggestion: Test the formulas above for the simplest case, $G(s) = 1/s$ ($A = 0, B = 1, C = 1, D = 0$).

The theory behind the formulas for X and Y is beyond the scope of this book. The procedure is to choose a real matrix H , $n \times 1$, so that $A + HC$ is stable, and then set

$$X(s) := \left[\begin{array}{c|c} A + HC & H \\ \hline F & 0 \end{array} \right], \quad (5.6)$$

$$Y(s) := \left[\begin{array}{c|c} A + HC & -B - HD \\ \hline F & 1 \end{array} \right]. \quad (5.7)$$

In summary, the procedure to do a coprime factorization of G is this:

Step 1 Get a realization (A, B, C, D) of G .

Step 2 Compute matrices F and H so that $A + BF$ and $A + HC$ are stable.

Step 3 Using formulas (5.4) to (5.7), compute the four functions N, M, X, Y .

5.4 Controller Parametrization: General Plant

The transfer function P is no longer assumed to be stable. Let $P = N/M$ be a coprime factorization over $\mathcal{S}$ and let X, Y be two functions in $\mathcal{S}$ satisfying the equation

$$NX + MY = 1. \quad (5.8)$$

Theorem 2 *The set of all C s for which the feedback system is internally stable equals*

$$\left\{ \frac{X + MQ}{Y - NQ} : Q \in \mathcal{S} \right\}.$$

It is useful to note that Theorem 2 reduces to Theorem 1 when $P \in \mathcal{S}$. To see this, recall from Section 5.2 (Step 1 of Procedure B) that we can take

$$N = P, \quad M = 1, \quad X = 0, \quad Y = 1$$

when $P \in \mathcal{S}$. Then

$$\frac{X + MQ}{Y - NQ} = \frac{Q}{1 - PQ}.$$

The proof of Theorem 2 requires a preliminary result.

Lemma 1 *Let $C = N_C/M_C$ be a coprime factorization over $\mathcal{S}$. Then the feedback system is internally stable iff*

$$(NN_C + MM_C)^{-1} \in \mathcal{S}.$$

The proof of this lemma is almost identical to the proof of Theorem 3.1, and so is omitted.

Proof of Theorem 2 ($\supset$) Suppose that $Q \in \mathcal{S}$ and

$$C := \frac{X + MQ}{Y - NQ}.$$

To show that the feedback system is internally stable, define

$$N_C := X + MQ, \quad M_C := Y - NQ.$$

Then from the equation

$$NX + MY = 1$$

it follows that

$$NN_C + MM_C = 1.$$

Therefore, $C = N_C/M_C$ is a coprime factorization, and from Lemma 1 the feedback system is internally stable.

($\subset$) Conversely, let C be any controller achieving internal stability. We must find a Q in $\mathcal{S}$ such that

$$C = \frac{X + MQ}{Y - NQ}.$$

Let $C = N_C/M_C$ be a coprime factorization over $\mathcal{S}$ and define

$$V := (NN_C + MM_C)^{-1}$$

so that

$$NN_C V + MM_C V = 1. \tag{5.9}$$

By Lemma 1, $V \in \mathcal{S}$. Let Q be the solution of

$$M_C V = Y - NQ. \tag{5.10}$$

Substitute (5.10) into (5.9) to get

$$NN_C V + M(Y - NQ) = 1. \tag{5.11}$$

Also, add and subtract NMQ in (5.8) to give

$$N(X + MQ) + M(Y - NQ) = 1. \tag{5.12}$$

Comparing (5.11) and (5.12), we see that

$$N_C V = X + MQ. \tag{5.13}$$

Now (5.10) and (5.13) give

$$C = \frac{N_C V}{M_C V} = \frac{X + MQ}{Y - NQ}.$$

It remains to show that $Q \in \mathcal{S}$. Multiply (5.10) by X and (5.13) by Y , then subtract and switch sides:

$$(NX + MY)Q = YN_C V - XM_C V.$$

But the left-hand side equals Q by (5.8), while the right-hand side belongs to $\mathcal{S}$. So we are done. ■

Theorem 2 gives an automatic way to stabilize a plant.

Example Let

$$P(s) = \frac{1}{(s-1)(s-2)}.$$

Apply Procedure B to get

$$\begin{aligned} N(s) &= \frac{1}{(s+1)^2}, \\ M(s) &= \frac{(s-1)(s-2)}{(s+1)^2}, \\ X(s) &= \frac{19s-11}{s+1}, \\ Y(s) &= \frac{s+6}{s+1}. \end{aligned}$$

According to the theorem, the controller

$$C(s) = \frac{X(s)}{Y(s)} = \frac{19s-11}{s+6}$$

achieves internal stability.

As before, when P was stable, all closed-loop transfer functions are affine functions of Q if C is parametrized as in the theorem statement. For example, the sensitivity and complementary sensitivity functions are

$$\begin{aligned} S &= M(Y - NQ), \\ T &= N(X + MQ). \end{aligned}$$

Finally, it is sometimes useful to note that Lemma 1 suggests another way to solve the equation $NX + MY = 1$ given coprime N and M . First, find a controller C achieving internal stability for $P = N/M$ —this might be easier than solving for X and Y . Next, write a coprime factorization of C : $C = N_C/M_C$. Then Lemma 1 says that

$$V := NN_C + MM_C$$

is invertible in $\mathcal{S}$. Finally, set $X = N_C V^{-1}$ and $Y = M_C V^{-1}$.

5.5 Asymptotic Properties

How to find a C to achieve internal stability and asymptotic properties simultaneously is perhaps best shown by an example.

Let

$$P(s) = \frac{1}{(s-1)(s-2)}.$$

The problem is to find a proper C so that

1. The feedback system is internally stable.
2. The final value of y equals 1 when r is a unit step and $d = 0$.
3. The final value of y equals zero when d is a sinusoid of 10 rad/s and $r = 0$.

The first step is to parametrize all stabilizing C s. Suitable N , M , X , Y are given in the example of the preceding section. From Theorem 2 C must have the form

$$C = \frac{X + MQ}{Y - NQ} \quad (5.14)$$

for some Q in $\mathcal{S}$ in order to satisfy (1). For such C the transfer function from r to y equals $N(X + MQ)$. By the final-value theorem (2) holds iff

$$N(0)[X(0) + M(0)Q(0)] = 1. \quad (5.15)$$

Similarly, the transfer function from d to y equals $N(Y - NQ)$, so (3) holds iff

$$N(10j)[Y(10j) - N(10j)Q(10j)] = 0. \quad (5.16)$$

So the problem reduces to the purely algebraic one of finding a function Q in $\mathcal{S}$ satisfying (5.15) and (5.16), which reduce to

$$\begin{aligned} Q(0) &= 6, \\ Q(10j) &= (6 + 10j)(1 + 10j) = -94 + 70j. \end{aligned} \quad (5.17)$$

This last equation is really two real equations:

$$\operatorname{Re} Q(10j) = -94, \quad (5.18)$$

$$\operatorname{Im} Q(10j) = 70. \quad (5.19)$$

So we must find a function Q in $\mathcal{S}$ satisfying (5.17), (5.18), and (5.19).

A method that will certainly work is to let Q be a polynomial in $(s + 1)^{-1}$ with enough variable coefficients. This guarantees that $Q \in \mathcal{S}$. Since we need to satisfy three equations, we should allow three coefficients. So take Q in the form

$$Q(s) = x_1 + x_2 \frac{1}{s + 1} + x_3 \frac{1}{(s + 1)^2}.$$

The three equations (5.17)-(5.19) lead to one of the form $Ax = b$, where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Solve for x . In this case the solution is

$$x_1 = -79, \quad x_2 = -723, \quad x_3 = 808.$$

This gives

$$Q(s) = \frac{-79s^2 - 881s + 6}{(s + 1)^2}.$$

Finally, we get C from (5.14):

$$C(s) = \frac{-60s^4 - 598s^3 + 2515s^2 - 1794s + 1}{s(s^2 + 100)(s + 9)}.$$

In summary, the procedure consists of four steps:

1. Parametrize all internally stabilizing controllers.
2. Reduce the asymptotic specs to interpolation constraints on the parameter.
3. Find (if possible) a parameter to satisfy these constraints.
4. Back-substitute to get the controller.

5.6 Strong and Simultaneous Stabilization

Practicing control engineers are reluctant to use unstable controllers, especially if the plant itself is stable. System integrity is the motivation: For example, if a sensor or actuator fails, or is deliberately turned off during start-up or shutdown, and the feedback loop opens, overall stability is maintained if both plant and controller individually are stable. If the plant itself is unstable, the argument against using an unstable controller is less compelling. However, knowledge of when a plant is or is not stabilizable with a stable controller is useful for another problem, namely, simultaneous stabilization, meaning stabilization of several plants by the same controller.

The issue of simultaneous stabilization arises when a plant is subject to a discrete change, such as when a component burns out. Simultaneous stabilization of two plants can also be viewed as an example of a problem involving highly structured uncertainty. A set of plants with exactly two elements is the most extreme example of highly structured uncertainty, standing at the opposite end of the spectrum from the unstructured disk-like uncertainty, which is the type of uncertainty focused on in this book.

Say that a plant is *strongly stabilizable* if internal stabilization can be achieved with C itself a stable transfer function. We start with an example of a plant that is not strongly stabilizable.

Example 1 Consider the plant transfer function

$$P(s) = \frac{s-1}{s(s-2)}.$$

Every C achieving internal stability is itself unstable. To prove this, start with a coprime factorization of P :

$$\begin{aligned} N(s) &= \frac{s-1}{(s+1)^2}, \\ M(s) &= \frac{s(s-2)}{(s+1)^2}, \\ X(s) &= \frac{14s-1}{s+1}, \\ Y(s) &= \frac{s-9}{s+1}. \end{aligned}$$

According to Theorem 2, all stabilizing controllers have the form

$$C = \frac{X + MQ}{Y - NQ}$$

for some Q in $\mathcal{S}$. Since $X + MQ$ and $Y - NQ$ too are coprime—they satisfy the equation

$$N(X + MQ) + M(Y - NQ) = 1$$

—they have no common zero in $\text{Res} \geq 0$. So to show that all such C s are unstable, it suffices to show that $Y - NQ$ has a zero in $\text{Res} \geq 0$ for every Q in $\mathcal{S}$. Now

$$N(1) = 0, \quad N(\infty) = 0,$$

so for every Q in $\mathcal{S}$

$$\begin{aligned} (Y - NQ)(1) &= Y(1) \\ &= -4, \\ (Y - NQ)(\infty) &= Y(\infty) \\ &= 1. \end{aligned}$$

Notice that the two numbers on the right-hand side have opposite sign. Thus as s moves along the positive real axis, the function $(Y - NQ)(s)$ changes sign. By continuity, it must equal zero at some such point, that is, $Y - NQ$ has a real zero somewhere on the positive real axis.

The poles and zeros of P must share a certain property in order for P to be strongly stabilizable. In the following theorem, the point at $s = \infty$ is included among the real zeros of P .

Theorem 3 *P is strongly stabilizable iff it has an even number of real poles between every pair of real zeros in $\text{Res} \geq 0$.*

To illustrate, continue with the example above. The zeros, including the point at infinity, are at $s = 1, \infty$. Between this pair is a single pole, at $s = 2$. This plant therefore fails the test.

As another example, consider

$$P(s) = \frac{(s-1)^2(s^2-s+1)}{(s-2)^2(s+1)^3}.$$

On the positive real axis, including ∞ , P has three zeros, two at $s = 1$ and one at $s = \infty$. It has two other zeros in $\text{Res} \geq 0$, which, not being real, are irrelevant. In counting poles between pairs of zeros we only have to consider distinct zeros (there are no poles between coincident zeros). Between zeros at $s = 1, \infty$ lie two poles, at $s = 2$. So this P is strongly stabilizable.

Proof of Theorem 3, Necessity The proof is just as in Example 1. Assume that the pole-zero test fails. To show that every stabilizing controller is unstable, start with a coprime factorization of P ,

$$P = \frac{N}{M}, \quad NX + MY = 1,$$

and some stabilizing controller,

$$C = \frac{X + MQ}{Y - NQ}, \quad Q \in \mathcal{S}.$$

It suffices to show that $Y - NQ$ has a zero in $\text{Res} \geq 0$.

By assumption, there is some pair of real zeros of N in $\text{Res} \geq 0$, at $s = \sigma_1, \sigma_2$, say, with an odd number of zeros of M in between. It follows that $M(\sigma_1)$ and $M(\sigma_2)$ have opposite sign; then so do $Y(\sigma_1)$ and $Y(\sigma_2)$, since $MY = 1$ at the right half-plane zeros of N . Hence the function $Y - NQ$ has a real zero somewhere between $s = \sigma_1$ and $s = \sigma_2$. ■

The proof of sufficiency is first illustrated by means of an example.

Example 2 Take the plant transfer function

$$P(s) = \frac{s-1}{(s-2)^2}.$$

This has two poles, at $s = 2$, between the two zeros at $s = 1, \infty$, so P is strongly stabilizable. To get a stable, stabilizing C , we should get a Q in $\mathcal{S}$ such that the inverse of $U := Y - NQ$ belongs to $\mathcal{S}$. Equivalently, we should get a U in $\mathcal{S}$ such that $U^{-1} \in \mathcal{S}$ and $U = Y$ at the two zeros of N , namely, $s = 1, \infty$. For this P we have

$$N(s) = \frac{s-1}{(s+1)^2}, \quad M(s) = \frac{(s-2)^2}{(s+1)^2}.$$

Now

$$Y(1) = \frac{1}{M(1)} = 4, \quad Y(\infty) = \frac{1}{M(\infty)} = 1.$$

So the problem reduces to constructing a U in $\mathcal{S}$ such that

$$U^{-1} \in \mathcal{S}, \quad U(1) = 4, \quad U(\infty) = 1.$$

The latter problem can be solved in two steps. First, get a U_1 in $\mathcal{S}$ such that

$$U_1^{-1} \in \mathcal{S}, \quad U_1(1) = 4.$$

The easiest choice is the constant $U_1(s) = 4$. Now we look for U of the form

$$U = (1 + aF)^l U_1,$$

where a is a constant, l an integer, and $F \in \mathcal{S}$. To guarantee that $U(1) = U_1(1)$ we should arrange that $F(1) = 0$, for example,

$$F(s) = \frac{s-1}{s+1}.$$

Then for $U(\infty) = 1$ we need $(1+a)^l 4 = 1$, that is,

$$a = \left(\frac{1}{4}\right)^{1/l} - 1, \tag{5.20}$$

and for $U^{-1} \in \mathcal{S}$ it suffices to have $\|aF\|_\infty < 1$ (i.e., $|a| < 1/\|F\|_\infty = 1$). So suitable l and a can be obtained by first choosing l large enough that

$$\left| \left(\frac{1}{4}\right)^{1/l} - 1 \right| < 1$$

and then getting a from (5.20), for example, $l = 1, a = -3/4$. This gives

$$U(s) = \left(1 - \frac{3}{4} \frac{s-1}{s+1}\right)^4 = \frac{s+7}{s+1}.$$

Finally, U, M, N uniquely determine C , as follows:

$$U = Y - NQ \implies Q = \frac{Y - U}{N} \implies C = \frac{X + MQ}{Y - NQ} = \frac{1 - MU}{NU}.$$

For this example we get $C(s) = 27/(s + 7)$. Notice that we did not actually have to construct X and Y .

Now for the constructive procedure that proves sufficiency in Theorem 3. The general procedure is fairly involved, so a simplifying assumption will be made that P 's unstable poles and zeros (including ∞) are all real and distinct. (Of course, Theorem 3 holds without this assumption.)

Procedure

Step 0 Write $P = N/M$ with N, M coprime. Arrange the non-negative real zeros of N as follows:

$$0 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_m = \infty.$$

Define $r_i := 1/M(\sigma_i)$, $i = 1, \dots, m$. Then P is strongly stabilizable iff $r_1, \dots, r_m$ all have the same sign. If this is true, continue.

Step 1 Set $U_1(s) = r_1$.

Continue. Assume that U_k has been constructed to satisfy

$$U_k, U_k^{-1} \in \mathcal{S}, \quad U_k(\sigma_i) = r_i, \quad i = 1, \dots, k.$$

Step $k + 1$ Choose F in $\mathcal{S}$ to have zeros at $s = \sigma_1, \dots, \sigma_k$. Choose $l \geq 1$ and a so that

$$[1 + aF(\sigma_{k+1})]^l U_k(\sigma_{k+1}) = r_{k+1},$$

$$|a| < \frac{1}{\|F\|_\infty}.$$

Set $U_{k+1} = (1 + aF)^l U_k$.

Continue to Step m .

Step $m + 1$ Set $U = U_m$ and $C = (1 - MU)/(NU)$.

Now we return to the problem of simultaneous stabilization and see that it can be reduced to one of strong stabilization. Two plants P_1 and P_2 are *simultaneously stabilizable* if internal stability is achievable for both by a common controller. Bring in coprime factorizations:

$$P_i = \frac{N_i}{M_i}, \quad N_i X_i + M_i Y_i = 1, \quad i = 1, 2$$

and define

$$N = N_2 M_1 - N_1 M_2, \quad M = N_2 X_1 + M_2 Y_1, \quad P = \frac{N}{M}.$$

For example, if P_1 is already stable, we may take

$$N_1 = P_1, \quad M_1 = 1, \quad X_1 = 0, \quad Y_1 = 1,$$

in which case

$$N = N_2 - P_1 M_2, \quad M = M_2,$$

so $P = P_2 - P_1$.

Theorem 4 P_1 and P_2 are simultaneously stabilizable iff P is strongly stabilizable.

Proof The controllers stabilizing P_i are

$$\frac{X_i + M_i Q_i}{Y_i - N_i Q_i}, \quad Q_i \in \mathcal{S}.$$

Thus P_1 and P_2 are simultaneously stabilizable iff there exist Q_1, Q_2 in $\mathcal{S}$ such that

$$\frac{X_1 + M_1 Q_1}{Y_1 - N_1 Q_1} = \frac{X_2 + M_2 Q_2}{Y_2 - N_2 Q_2}.$$

Since these two fractions both have coprime factors, this equation holds iff there exists U in $\mathcal{S}$ such that

$$\begin{aligned} U^{-1} &\in \mathcal{S}, \\ X_1 + M_1 Q_1 &= U(X_2 + M_2 Q_2), \\ Y_1 - N_1 Q_1 &= U(Y_2 - N_2 Q_2). \end{aligned}$$

To simplify the bookkeeping, write these last two equations in matrix form:

$$\begin{bmatrix} 1 & Q_1 \end{bmatrix} \begin{bmatrix} X_1 & Y_1 \\ M_1 & -N_1 \end{bmatrix} = U \begin{bmatrix} 1 & Q_2 \end{bmatrix} \begin{bmatrix} X_2 & Y_2 \\ M_2 & -N_2 \end{bmatrix}.$$

Postmultiply this equation by

$$\begin{bmatrix} N_2 & Y_2 \\ M_2 & -X_2 \end{bmatrix}$$

to get

$$\begin{bmatrix} 1 & Q_1 \end{bmatrix} \begin{bmatrix} X_1 & Y_1 \\ M_1 & -N_1 \end{bmatrix} \begin{bmatrix} N_2 & Y_2 \\ M_2 & -X_2 \end{bmatrix} = U \begin{bmatrix} 1 & Q_2 \end{bmatrix}.$$

Now the first column of the matrix

$$\begin{bmatrix} X_1 & Y_1 \\ M_1 & -N_1 \end{bmatrix} \begin{bmatrix} N_2 & Y_2 \\ M_2 & -X_2 \end{bmatrix}$$

is

$$\begin{bmatrix} M \\ N \end{bmatrix};$$

denote the second column by

$$\begin{bmatrix} X \\ Y \end{bmatrix}.$$

Then the previous matrix equation is equivalent to the two equations

$$\begin{aligned} M + N Q_1 &= U, \\ X + Y Q_1 &= U Q_2. \end{aligned}$$

To recap, P_1 and P_2 are simultaneously stabilizable iff there exist Q_1, Q_2, U in $\mathcal{S}$ such that

$$\begin{aligned} U^{-1} &\in \mathcal{S}, \\ M + N Q_1 &= U, \\ X + Y Q_1 &= U Q_2. \end{aligned}$$

Clearly, this is equivalent to the condition, there exist Q_1, U in $\mathcal{S}$ such that

$$\begin{aligned} U^{-1} &\in \mathcal{S}, \\ M + NQ_1 &= U \end{aligned}$$

[because we can get Q_2 via $(X + YQ_1)/U$]. But it can be checked that N and M are coprime. Thus from Lemma 1 the previous condition is equivalent to, P can be stabilized by some stable controller, namely, Q_1 . ■

Example 3 Consider

$$P_1(s) = \frac{1}{s+1}, \quad P_2(s) = \frac{as+b}{(s+1)(s-1)},$$

where a and b are real constants with $a \neq 1$. Since P_1 is stable, we have

$$P(s) = P_2(s) - P_1(s) = -\frac{(1-a)s - (1+b)}{(s+1)(s-1)}.$$

This has zeros at

$$s = \frac{1+b}{1-a}, \infty$$

and a simple unstable pole at $s = 1$. So P is strongly stabilizable, or P_1 and P_2 are simultaneously stabilizable, iff either the zero $(1+b)/(1-a)$ is negative or it lies to the right of the unstable pole, that is,

$$\text{either } \frac{1+b}{1-a} < 0 \quad \text{or} \quad \frac{1+b}{1-a} > 1.$$

Simultaneous stabilization for more than two plants is still an unsolved problem.

5.7 Cart-Pendulum Example

An interesting stabilization problem is afforded by the cart-pendulum example, a common toy control system. The setup is shown in Figure 5.2. The system consists of a cart of mass M that slides in one dimension x on a horizontal surface, with a ball of mass m at the end of a rigid massless pendulum of length l . The cart and ball are treated as point masses, with the pivot at the center of the cart. There is assumed to be no friction and no air resistance. Shown as inputs are a horizontal force u on the cart and a force d on the ball perpendicular to the pendulum. The other signals shown are the angle θ and the position of the ball $y = x + l \sin \theta$.

Elementary dynamics yields the following equations of motion:

$$(M+m)\ddot{x} + ml(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) = u + d \cos \theta, \quad (5.21)$$

$$m(\ddot{x} \cos \theta + l\ddot{\theta} - g \sin \theta) = d. \quad (5.22)$$

These are nonlinear equations which can be linearized about an equilibrium position, of which there are two: $(x, \theta) = (0, 0)$ and $(x, \theta) = (0, \pi)$ (i.e., the pendulum either up or down).

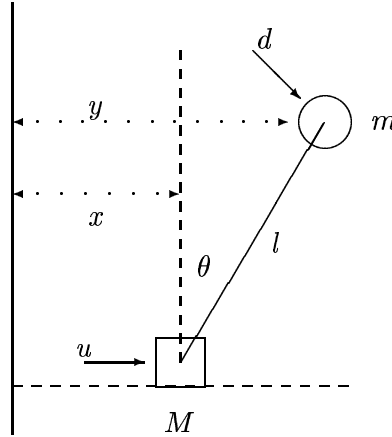


Figure 5.2: Cart-pendulum example.

Linearization About Pendulum Up

The two linearized equations are

$$\begin{aligned} (M + m)\ddot{x} + ml\ddot{\theta} &= u + d, \\ \ddot{x} + l\ddot{\theta} - g\theta &= \frac{1}{m}d. \end{aligned}$$

Take Laplace transforms to get

$$\begin{bmatrix} (M + m)s^2 & mls^2 \\ s^2 & ls^2 - g \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} \hat{u} + \hat{d} \\ \frac{1}{m}\hat{d} \end{bmatrix}.$$

Thus

$$\begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix} = \frac{1}{D(s)} \begin{bmatrix} ls^2 - g & -g \\ -s^2 & \frac{M}{m}s^2 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{d} \end{bmatrix},$$

where

$$D(s) = s^2[Ms^2 - (M + m)g].$$

Finally,

$$\hat{y} = \hat{x} + l\hat{\theta} = \frac{1}{D(s)} \begin{bmatrix} -g & \frac{M}{m}ls^2g \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{d} \end{bmatrix}.$$

In particular, the transfer functions from u to x and y are, respectively,

$$\frac{ls^2 - g}{D(s)}, \quad \frac{-g}{D(s)}.$$

These are both unstable, having right half-plane poles at

$$s = 0, \quad 0, \quad \sqrt{\frac{(M + m)g}{Ml}}.$$

Also, the transfer function from u to x has a right half-plane zero at $s = \sqrt{g/l}$.

Linearization About Pendulum Down

Replacing θ by $\pi + \theta$ in equations (5.21) and (5.22) and linearizing, we get

$$\begin{aligned} (M + m)\ddot{x} - ml\ddot{\theta} &= u - d, \\ -\ddot{x} + l\ddot{\theta} + g\theta &= \frac{1}{m}d, \end{aligned}$$

so

$$\begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix} = \frac{1}{D(s)} \begin{bmatrix} ls^2 + g & -g \\ s^2 & \frac{M}{m}s^2 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{d} \end{bmatrix},$$

and

$$\hat{y} = \hat{x} - l\hat{\theta} = \frac{1}{D(s)} \begin{bmatrix} g & -\frac{M}{m}ls^2g \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{d} \end{bmatrix},$$

where

$$D(s) = s^2[Ms^2 + (M + m)g].$$

The transfer functions from u to x and y are now, respectively,

$$\frac{ls^2 + g}{D(s)}, \quad \frac{g}{D(s)}.$$

Let us look at the problem of stabilizing the u -to- x transfer function with the pendulum in the up position. The transfer function is

$$\frac{ls^2 - g}{s^2[Ms^2 - (M + m)g]}.$$

Since this has an unstable pole, namely,

$$\sqrt{\frac{(M + m)g}{Ml}},$$

between two real zeros at $s = \sqrt{g/l}, \infty$, from the preceding section this transfer function is not strongly stabilizable. Having no finite zeros, the u -to- y transfer function is, however.

Is the cart-pendulum stabilizable if we measure x and control d ? First of all, what does this mean? The cart-pendulum as configured is really a multivariable system: It has two inputs, u and d , and two outputs, x and θ (y is a linear combination of these two). So really there are four loops we could close: from x to u and d and from θ to u and d . Let us contemplate closing just from x to d . We would like all closed-loop transfer functions to be stable, for example, u -to- θ , x -to- θ , and so on. Is this possible?

(This analysis applies only to the linearized system. Since there are poles on the imaginary axis, the stability of the linear system does not determine even the local stability of the nonlinear system.)

We shall return to this example in the next chapter.

Exercises

1. Compute a coprime factorization over $\mathcal{S}$ of

$$G(s) = \frac{s^3}{s^2 - s + 1}.$$

2. For

$$P(s) = \frac{3}{s - 4}$$

compute a controller C so that the feedback system is internally stable and the tracking error e goes to 0 when r is a ramp and $d = 0$.

3. For

$$P(s) = \frac{1}{s(s^2 + 0.2s + 1)}$$

find an internally stabilizing C so that the final value of $r - y$ equals zero when r is a unit ramp and d is a sinusoid of frequency 2 rad/s.

4. Suppose that $P(s) = 1/s$ and $C = Q/(1 - PQ)$, where Q is a proper real-rational function. Characterize those functions Q for which the feedback system is internally stable.
5. Suppose that N, M are coprime functions in $\mathcal{S}$. Prove that if $NM^{-1} \in \mathcal{S}$, then $M^{-1} \in \mathcal{S}$. Is this true without the coprimeness assumption?
6. The problem is to find an internally stabilizing C so that e tends to zero asymptotically when r is a step and $d = 0$. When is the problem solvable? Characterize all solutions. (Do not assume P is stable.)

7. Let

$$P(s) = \frac{s}{(s - 1)(10s + 1)}.$$

Find a C to achieve internal stability. What are the closed-loop poles? What is the dc gain from d to y ?

8. For formulas (5.4) to (5.7), verify that $NX + MY = 1$.
9. Consider the feedback system with plant P and controller C . Assume internal stability. Consider a coprime factorization of P over $\mathcal{S}$, $P = N/M$. Suppose that P is perturbed to

$$P = \frac{N + \Delta_1}{M + \Delta_2},$$

where

$$\begin{aligned} \Delta_1, \Delta_2 &\in \mathcal{S}, \\ \|\Delta_1\|_\infty, \|\Delta_2\|_\infty &\leq \gamma. \end{aligned}$$

Find a bound on γ for robust stability.

10. Compute a stable, stabilizing controller for

$$P(s) = \frac{(s - 1)(s^2 + s + 1)}{(s - 2)(s - 3)(s + 1)^2}.$$

11. Study simultaneous stabilization of the cart-pendulum system in the up and down configurations.

Notes and References

The idea behind Theorem 1 is due to Newton et al. (1957). They observed that while the transfer function from r to y is nonlinear in C , it is linear in Q , the transfer function from r to u . So they proposed to design Q to achieve desired performance and then obtain C by back-substitution. Theorem 1 itself is due to Zames (1981). The controller parametrization in Theorem 1 is used in the field of process control, where it is called *internal model control* [because the controller $C = Q/(1 - PQ)$ contains a model of the plant P] (Morari and Zafriou, 1989).

The original form of Theorem 2 is due independently to Youla et al. (1976) and Kucera (1979). Its present form is due to Desoer et al. (1980), who saw the advantage of doing coprime factorization over $\mathcal{S}$ instead of the ring of polynomials. This idea in turn is due to Vidyasagar (1972). State-space formulas for coprime factorization were first derived by Khargonekar and Sontag (1982). The formulas in Section 5.3 are from Nett et al. 1984. Section 5.5 is adapted from Francis and Vidyasagar (1983). The algebraic point of view has been explored in detail by Desoer and co-workers (e.g., Desoer and Gustafson, 1984) and by Vidyasagar (1985). The notion of strong stabilization and Theorem 3 are due to Youla et al. (1974). Simultaneous stabilization was first treated by Sacks and Murray (1982). The simple proofs of Theorems 3 and 4 given here are borrowed from Vidyasagar (1985). The controller parametrization of Theorem 2 has been exploited in a CAD method for controller design (Boyd et al., 1988).

