

Chapter 12

Design for Robust Performance

This chapter presents a mathematical technique for designing a controller to achieve robust performance. Chapter 7 proposed loopshaping as a graphical method when P and P^{-1} are stable. Without these assumptions loopshaping is very awkward and the methodical procedure in this chapter can be used.

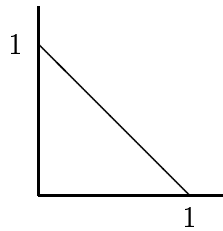
12.1 The Modified Problem

As defined in Section 4.3, the robust performance problem is to design a proper controller C so that the feedback system for the nominal plant is internally stable and the inequality

$$\| |W_1 S| + |W_2 T| \|_\infty < 1 \quad (12.1)$$

holds. Also as mentioned in Chapter 7, the exact problem as just stated remains unsolved. So we look for a nearby problem that is solvable.

We seek to replace inequality (12.1) with a tractable one. Fix a frequency and define $x := |W_1 S|$ and $y := |W_2 T|$. The region in the (x, y) -plane where $x + y < 1$ is the right-angle triangle shown here:



By imagining the circle with center 0, radius $1/\sqrt{2}$ you can see that

$$x^2 + y^2 < 1/2 \Rightarrow x + y < 1.$$

Thus a sufficient condition for (12.1) is

$$\| |W_1 S|^2 + |W_2 T|^2 \|_\infty < \frac{1}{2}. \quad (12.2)$$

In this way we arrive at the *modified robust performance problem*: Find a proper, internally stabilizing C so that (12.2) holds. This problem is a compromise; it is not exactly the problem we

would like to solve, but it is nearby and has the desirable feature of being solvable without too much difficulty.

We will solve the modified problem under the following assumptions:

1. P is strictly proper and has neither poles nor zeros on the imaginary axis.
2. W_1 and W_2 are stable and proper.
3. W_1 and W_2 have no common zeros on the imaginary axis.

Some of these assumptions could be relaxed, but with an increase in messiness.

We'll need an additional tool, covered in the next section.

12.2 Spectral Factorization

For a rational function $F(s)$ with real coefficients, let $\overline{F}$ denote the function $F(-s)$. Thus

$$\overline{F}(j\omega) = F(-j\omega) = \overline{F(j\omega)}.$$

We saw in Lemma 6.2 that, provided $F \in \mathcal{S}$, it has a factorization of the form $F = F_{\text{ap}}F_{\text{mp}}$. All the right half-plane zeros go into the all-pass factor, and the ones in the left half-plane or on the imaginary axis go into the minimum-phase factor. The all-pass factor has the property $\overline{F_{\text{ap}}}(s)F_{\text{ap}}(s) = 1$.

There is a related factorization if F has the property $\overline{F} = F$ and no zeros or poles on the imaginary axis. This means that the zeros of F form a pattern that is symmetrical with respect to both the real and imaginary axes. To see this, simply note that if z is a zero, so is $-z$ because $F(s) = F(-s)$. It follows that for every zero z in the right half-plane, the numerator of F is divisible by

$$(z - s)(z + s).$$

Similarly for the poles. Hence we can write F in terms of its gain, poles, and zeros like this:

$$F(s) = cF_1(s), \quad F_1(s) = \frac{\prod(z_i - s)(z_i + s)}{\prod(p_i - s)(p_i + s)}.$$

Note that $\{z_i\}$ and $\{p_i\}$ are the right half-plane zeros and poles. Note also that $F_1(0) > 0$. From F_1 form a function G by selecting only the factors corresponding to zeros and poles in $\text{Re } s < 0$, that is,

$$G(s) := \frac{\prod(z_i + s)}{\prod(p_i + s)}.$$

Then G and G^{-1} are stable. The zeros and poles of F_1 in the right half-plane are absorbed into the function $\overline{G}$ and we have the factorization

$$F = \overline{G}cG \text{ with } G, G^{-1} \text{ stable.}$$

Finally, if $c > 0$, we define F_{sf} as

$$F_{sf}(s) := \sqrt{c} \frac{\prod(z_i + s)}{\prod(p_i + s)}$$

to get $F = \overline{F_{sf}}F_{sf}$. Since $F_1(0) > 0$, $c > 0$ iff $F(0) > 0$.

The function F_{sf} —which is like a square root of $F(s)$ —therefore has the properties

$$F = \overline{F_{sf}} F_{sf} \text{ with } F_{sf}, F_{sf}^{-1} \text{ stable}$$

and is called a *spectral factor* of F . This factorization of F is called a *spectral factorization*.

An alternative characterization of the condition $\overline{F} = F$ is this: $\overline{F} = F$ iff F is a rational function in s^2 .

The above is summarized as follows.

Lemma 1 *If the real-rational function F has the properties $\overline{F} = F$, no zeros or poles on the imaginary axis, and $F(0) > 0$, then it has a spectral factorization.*

Example 1 The function

$$F(s) = \frac{1}{1-s^2}$$

can be factored as

$$F(s) = \frac{1}{1-s} \frac{1}{1+s},$$

so a suitable spectral factor is $F_{sf}(s) = 1/(1+s)$. So is $-1/(1+s)$.

Example 2 Consider the function

$$F(s) = 10 \frac{4-s^2}{25+6s^2+s^4}.$$

Its poles are $\pm 1 \pm 2j$ and its zeros are ± 2 . The function G is therefore

$$G(s) = \frac{2+s}{(1+2j+s)(1-2j+s)} = \frac{2+s}{5+2s+s^2}.$$

So a spectral factor is

$$F_{sf}(s) = \sqrt{10} \frac{2+s}{5+2s+s^2}.$$

Example 3 The function

$$F(s) = \frac{2s^2-1}{s^4-s^2+1}$$

has no spectral factorization because $F(0) < 0$.

12.3 Solution of the Modified Problem

The solution of the modified problem involves transforming it into the model-matching problem of Chapter 9. Since this transformation is a little involved, it is worth describing it first on a high level, as follows.

1. We use the by now familiar factorization of all internally stabilizing controllers from Theorem 5.2. Factorize P :

$$P = \frac{N}{M}, \quad NX + MY = 1, \quad N, M, X, Y \in \mathcal{S}.$$

The formula for C is

$$C = \frac{X + MQ}{Y - NQ}, \quad Q \in \mathcal{S}.$$

In terms of Q we have

$$S = M(Y - NQ), \quad T = N(X + MQ).$$

So the modified problem reduces to this: find Q in $\mathcal{S}$ such that

$$\| |W_1 M(Y - NQ)|^2 + |W_2 N(X + MQ)|^2 \|_\infty < \frac{1}{2}. \quad (12.3)$$

Let's simplify notation by defining

$$\begin{aligned} R_1 &:= W_1 M Y, & S_1 &:= W_2 N X, \\ R_2 &:= W_1 M N, & S_2 &:= -W_2 M N. \end{aligned}$$

so that (12.3) becomes

$$\| |R_1 - R_2 Q|^2 + |S_1 - S_2 Q|^2 \|_\infty < \frac{1}{2}. \quad (12.4)$$

2. Inequality (12.4) involves the sum of two squares in Q . To get closer to the model-matching problem we shall transform (12.4) so that only one square in Q appears, that is, transform it into

$$\| |U_1 - U_2 Q|^2 + U_3 \|_\infty < \frac{1}{2} \quad (12.5)$$

for suitable functions U_i , $i = 1, 2, 3$. The first two, U_1 and U_2 , will belong to $\mathcal{S}$, while U_3 will be real-rational with the property $\overline{U_3} = U_3$.

3. In what follows we shall drop $j\omega$ and also introduce U_4 , a spectral factor of $\frac{1}{2} - U_3$:

$$\begin{aligned} (12.5) &\Leftrightarrow |U_1 - U_2 Q|^2 + U_3 < \frac{1}{2}, \quad \forall \omega \\ &\Leftrightarrow |U_1 - U_2 Q|^2 < \frac{1}{2} - U_3, \quad \forall \omega \\ &\Leftrightarrow |U_1 - U_2 Q|^2 < |U_4|^2, \quad \forall \omega \\ &\Leftrightarrow |U_4^{-1} U_1 - U_4^{-1} U_2 Q|^2 < 1, \quad \forall \omega \\ &\Leftrightarrow \|U_4^{-1} U_1 - U_4^{-1} U_2 Q\|_\infty < 1. \end{aligned}$$

So the final model-matching problem is to find a Q in $\mathcal{S}$ satisfying

$$\|U_4^{-1} U_1 - U_4^{-1} U_2 Q\|_\infty < 1.$$

Now for the details. For equivalence of (12.4) and (12.5) it suffices to have the following equation hold:

$$(\overline{R_1} - \overline{R_2 Q})(R_1 - R_2 Q) + (\overline{S_1} - \overline{S_2 Q})(S_1 - S_2 Q) = (\overline{U_1} - \overline{U_2 Q})(U_1 - U_2 Q) + U_3.$$

Multiply all factors out and collect terms. The result is an identity in Q provided that the following three equations hold:

$$\overline{R_2}R_2 + \overline{S_2}S_2 = \overline{U_2}U_2, \quad (12.6)$$

$$\overline{R_2}R_1 + \overline{S_2}S_1 = \overline{U_2}U_1, \quad (12.7)$$

$$\overline{R_1}R_1 + \overline{S_1}S_1 = \overline{U_1}U_1 + U_3. \quad (12.8)$$

The idea is to get U_1 and U_2 satisfying (12.6) and (12.7), and then to get U_3 from (12.8). In fact, we can solve for U_3 right away:

$$U_3 = \frac{\overline{W_1}W_1\overline{W_2}W_2}{\overline{W_1}W_1 + \overline{W_2}W_2}. \quad (12.9)$$

Proof From (12.7) we have

$$U_1 = \frac{\overline{R_2}R_1 + \overline{S_2}S_1}{\overline{U_2}}$$

and hence

$$\overline{U_1}U_1 = \frac{(\overline{R_1}R_2 + \overline{S_1}S_2)(\overline{R_1}\overline{R_2} + \overline{S_1}\overline{S_2})}{\overline{U_2}U_2}.$$

Substituting from (12.6) in the denominator gives

$$\overline{U_1}U_1 = \frac{(\overline{R_1}R_2 + \overline{S_1}S_2)(\overline{R_1}\overline{R_2} + \overline{S_1}\overline{S_2})}{\overline{R_2}R_2 + \overline{S_2}S_2}.$$

Using this in (12.8) gives

$$U_3 = \overline{R_1}R_1 + \overline{S_1}S_1 - \frac{(\overline{R_1}R_2 + \overline{S_1}S_2)(\overline{R_1}\overline{R_2} + \overline{S_1}\overline{S_2})}{\overline{R_2}R_2 + \overline{S_2}S_2}.$$

Finally, substitute into this the definitions of R_i and S_i and simplify to get (12.9). ■

Now we turn to the solution of (12.6) and (12.7). We want solutions U_1 and U_2 in $\mathcal{S}$. To see what is involved, let us look at an example.

Example 1 Suppose that

$$\begin{aligned} R_1(s) &= \frac{1}{2+s}, & S_1(s) &= 0, \\ R_2(s) &= \frac{1}{1+s}, & S_2(s) &= 1. \end{aligned}$$

Then equations (12.6) and (12.7) are

$$\frac{(\sqrt{2}-s)(\sqrt{2}+s)}{(1-s)(1+s)} = \overline{U_2}(s)U_2(s), \quad (12.10)$$

$$\frac{1}{(1-s)(2+s)} = \overline{U_2}(s)U_1(s). \quad (12.11)$$

To satisfy (12.10), an obvious choice is the spectral factor

$$U_2(s) = \frac{\sqrt{2}+s}{1+s}.$$

But then (12.11) has the unique solution

$$U_1(s) = \frac{1}{(2+s)(\sqrt{2}-s)},$$

which is unsatisfactory, not being in $\mathcal{S}$. So as a solution of (12.10) let us take

$$U_2(s) = \frac{\sqrt{2}+s}{1+s}V(s),$$

where V is an all-pass function, as yet unknown. Again, the solution of (12.11) is

$$U_1(s) = \frac{1}{(2+s)(\sqrt{2}-s)}V(s).$$

So to get U_1 in $\mathcal{S}$ we should let V have a zero at $s = \sqrt{2}$, for example,

$$V(s) = \frac{\sqrt{2}-s}{\sqrt{2}+s}.$$

The following procedure yields functions U_1 and U_2 in $\mathcal{S}$ satisfying (12.6) and (12.7).

Procedure A

Input: R_1, R_2, S_1, S_2

Step 1 Set $F := \overline{R_2}R_2 + \overline{S_2}S_2$. Comment: F has no zeros or poles on the imaginary axis, $\overline{F} = F$, and $F(0) > 0$.

Step 2 Compute a spectral factor F_{sf} of F .

Step 3 Choose an all-pass function V such that

$$\frac{\overline{R_2}R_1 + \overline{S_2}S_1}{\overline{F_{sf}}}V \in \mathcal{S}.$$

Step 4 Set

$$U_1 := \frac{\overline{R_2}R_1 + \overline{S_2}S_1}{\overline{F_{sf}}}V, \quad U_2 := F_{sf}V.$$

Finally, the following procedure solves the modified robust performance problem.

Procedure B

Input: P, W_1, W_2

Step 1 Compute

$$U_3 = \frac{\overline{W_1}W_1\overline{W_2}W_2}{\overline{W_1}W_1 + \overline{W_2}W_2}.$$

Check if $\|U_3\|_\infty < \frac{1}{2}$, that is,

$$\left\| \frac{|W_1W_2|^2}{|W_1|^2 + |W_2|^2} \right\|_\infty < \frac{1}{2}.$$

If so, continue. If not, the problem is not solvable; stop.

Step 2 Coprime factorization of P :

$$P = \frac{N}{M}, \quad NX + MY = 1.$$

Step 3 Set

$$\begin{aligned} R_1 &:= W_1MY, & S_1 &:= W_2NX, \\ R_2 &:= W_1MN, & S_2 &:= -W_2MN. \end{aligned}$$

Step 4 Apply Procedure A to get U_1 and U_2 .

Step 5 Compute a spectral factor U_4 of $\frac{1}{2} - U_3$.

Step 6 Set $T_1 := U_4^{-1}U_1$ and $T_2 := U_4^{-1}U_2$. Comment: $T_1, T_2 \in \mathcal{S}$ and T_2 has no zeros on the imaginary axis.

Step 7 Compute γ_{opt} , the minimum model-matching error. If $\gamma_{\text{opt}} < 1$, continue. If $\gamma_{\text{opt}} \geq 1$, the modified robust performance problem is not solvable; stop.

Step 8 Compute Q , the solution to the model-matching problem. If Q is not proper, roll it off at a sufficiently high frequency while maintaining $\|T_1 - T_2Q\|_\infty < 1$.

Step 9 Set

$$C = \frac{X + MQ}{Y - NQ}.$$

The purpose of the next example is solely to illustrate the procedures above. It should be emphasized that the plant is very simple and the design could more easily be done using loopshaping.

Example 2 Take

$$P(s) = \frac{1}{s+1}, \quad W_1(s) = \frac{a}{s+1}, \quad W_2(s) = \frac{0.02s}{0.01s+1}.$$

The positive constant a is left unspecified at this point. In fact we will find the largest a for which the modified robust performance problem is solvable.

Step 1

$$U_3(s) = -\frac{0.0004a^2s^2}{a^2 - (0.0001a^2 + 0.0004)s^2 + 0.0004s^4}$$

The ∞ -norm of U_3 is computed for selected values of a :

a	Norm
50	0.4444
52	0.4601
54	0.4757
56	0.4912
58	0.5064
60	0.5217

By interpolation, the supremum value of a for which $\|U_3\|_\infty < 1/2$ is about 57.2.

Step 2 $N = P, M = 1, X = 0, Y = 1$

Step 3

$$R_1(s) = \frac{a}{s+1}, \quad R_2(s) = \frac{a}{(s+1)^2}, \quad S_1(s) = 0, \quad S_2(s) = -\frac{0.02s}{(s+1)(0.01s+1)}$$

Step 4 Procedure A:

Step 1

$$F(s) = \frac{a^2 - (0.0001a^2 + 0.0004)s^2 + 0.0004s^4}{(1-s)^2(1+s)^2(1-0.01s)(1+0.01s)}$$

Step 2

$$F_{sf}(s) = \frac{a + bs + 0.02s^2}{(1+s)^2(1+0.01s)}$$

$$b := (0.0001a^2 + 0.04a + 0.0004)^{1/2}$$

Step 3

$$V(s) = \frac{a - bs + 0.02s^2}{a + bs + 0.02s^2}$$

Step 4

$$U_1(s) = a^2 \frac{1 - 0.01s}{(1+s)(a + bs + 0.02s^2)}, \quad U_2(s) = \frac{a - bs + 0.02s^2}{(1+s)^2(1+0.01s)}$$

Step 5

$$U_4(s) = \frac{a + cs + 0.02s^2}{\sqrt{2}(a + bs + 0.02s^2)}$$

$$c := (-0.0007a^2 + 0.04a + 0.0004)^{1/2}$$

Step 6

$$T_1(s) = \sqrt{2}a^2 \frac{1 - 0.01s}{(1+s)(a + cs + 0.02s^2)}$$

$$T_2(s) = \sqrt{2} \frac{(a + bs + 0.02s^2)(a - bs + 0.02s^2)}{(a + cs + 0.02s^2)(1+s)^2(1+0.01s)}$$

Step 7 γ_{opt} is computed for selected values of a :

a	γ_{opt}
36	0.9381
37	0.9560
38	0.9742
39	0.9928
40	1.0118

So $a = 39$ is feasible, while $a = 40$ is not. Let's proceed with $a = 36$.

Step 8 The solution to the model-matching problem is

$$Q_{im}(s) = \frac{0.3317s^4 + 55.19s^3 + 2838s^2 + 64215s + 61432}{s^3 + 97.42s^2 + 3978s + 62585}.$$

This must be rolled off: Set

$$Q(s) = Q_{im}(s) \frac{1}{\tau s + 1}.$$

The value $\tau = 0.0009$ yields $\|T_1 - T_2 Q\|_\infty = 0.9996$, and is therefore satisfactory.

Step 9 $C = Q/(1 - PQ)$.

The resulting Bode plots are displayed in Figure 12.1.

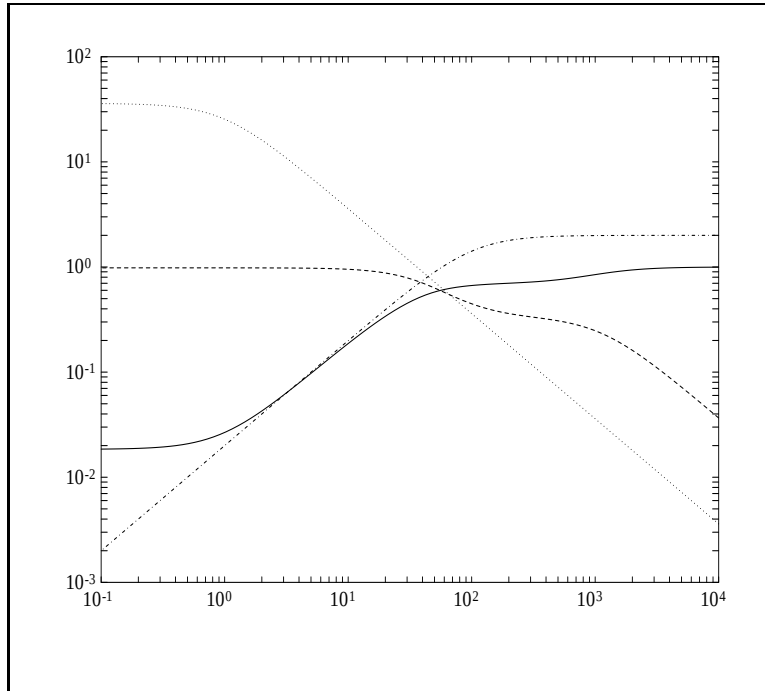


Figure 12.1: Bode plot of $|S|$ (solid), $|T|$ (dash), $|W_1|$ (dot), and $|W_2|$ (dot-dash).

12.4 Design Example: Flexible Beam Continued

In the preceding section we presented a procedure for designing a controller to achieve a tradeoff between S and T . The procedure has some appeal pedagogically, being elementary and based in

the frequency domain. However, it must be admitted that it would not be suitable even for a medium-sized problem because computations with rational functions are clumsy and numerically sensitive. Modern software for control system design is based on state-space methods, the theory behind which is beyond the scope of this book. One such software package is MATLAB with the μ -Tools Toolbox. In this section we continue the flexible beam example begun in Section 10.3. The actual computations were performed using MATLAB, so only the results are reported. Our main purpose in including this example is to discuss further the issue of weights selection, in particular, the choice of weights for time-domain specs.

Recall that the transfer function for the plant is

$$P(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{s(5s^3 + 3.5682s^2 + 139.5021s + 0.0929)}.$$

The plant input is a voltage to a power amplifier and the output is the tip deflection of the beam. The performance specs in Section 10.3 were in terms of the unit step response:

$$\begin{aligned} \text{settling time} &\approx 8 \text{ s}, \\ \text{overshoot} &\leq 10\%. \end{aligned}$$

Using the performance weight

$$W_1(s) = \frac{0.9}{1.0210} \frac{s^2 + 1.2s + 1}{(s + 0.001)(s + 1.2)(0.001s + 1)},$$

we designed a controller achieving these two specs.

In this section we place an additional constraint, namely, an amplitude constraint on the plant input:

$$|u(t)| \leq 0.5, \quad \forall t. \quad (12.12)$$

This reflects the fact that the power amplifier will saturate if its input is outside this range. The Section 10.3 design violates this spec, the signal $u(t)$ exhibiting a very large surge over the time interval $[0, 0.03]$. No frequency-domain design procedure can treat a specification like (12.12) precisely: An amplitude bound in the time domain does not translate over precisely to anything tractable in the frequency domain. So we have to be content with a trial-and-error procedure.

Intuitively, we may have to relax the step-response spec in order to achieve (12.12); this means decreasing the gain of W_1 . So let us take

$$W_1(s) = a \frac{s^2 + 1.2s + 1}{(s + 0.001)(s + 1.2)(0.001s + 1)},$$

where a is a design parameter.

The transfer function from reference input r to plant input u is CS (not T). So to achieve (12.12) it makes sense to introduce a new weight, W_3 , and associated modified criterion,

$$\| |W_1 S|^2 + |W_3 CS|^2 \|_\infty < 1. \quad (12.13)$$

Although this is different from the criterion in Section 12.3, it can be handled in exactly the same way—the changes are obvious; for example, whereas in Section 12.3 we had

$$W_2 T = W_2 N(X + MQ),$$

we now have

$$W_3CS = W_3M(X + MQ).$$

The observation above, that $u(t)$ exhibits an initial large surge, suggests penalizing high-frequency control actuation by taking W_3 to be a high-pass filter, of the form

$$W_3(s) = b \frac{s + 0.01c}{s + c}.$$

The constant c was taken to be $c = 1/0.03 = 33.3$ rad/s, corresponding to the surge interval $[0, 0.03]$.

We are left with the two parameters a and b in the weights: The larger the value of a , the better the step response; the larger the value of b , the larger the penalty on control. These parameters were determined by the following iterative procedure:

1. Set $a = 0.9/1.0210$, the value for the previous design.
2. Decrease b from some starting value until (12.13) is achievable by some controller. Obtain such a controller and do a step-response simulation. Check if the specs are satisfied.
3. If necessary, decrease a and repeat Step 2.

The values $a = 0.8$, $b = 1$ were obtained, and the corresponding controller has numerator

$$\begin{aligned} &1.424s^7 + 9.076 \times 10^2s^6 + 3.141 \times 10^4s^5 + 1.117 \times 10^5s^4 + 9.073 \times 10^5s^3 \\ &+ 1.961 \times 10^6s^2 + 1.306 \times 10^3s + 0.01406 \end{aligned}$$

and denominator

$$\begin{aligned} &s^8 + 1.013 \times 10^3s^7 + 1.326 \times 10^4s^6 + 1.129 \times 10^5s^5 + 6.326 \times 10^5s^4 \\ &+ 2.348 \times 10^6s^3 + 4.940 \times 10^6s^2 + 3.440 \times 10^6s + 3.435 \times 10^3. \end{aligned}$$

The order of the controller (8) equals the sum of the orders of P , W_1 , and W_3 . Figure 12.2 shows the resulting Bode plots and Figure 12.3 the step responses. The specs have been met.

It is interesting to compare the step response of y in Figure 12.3 with the step response in Figure 10.3, where the control input was unconstrained: The former exhibits a pronounced initial time lag, evidently the price paid for constrained control.

Exercises

1. This exercise seeks to illustrate why minimization of the performance measure

$$\| |W_1S| + |W_2T| \|_\infty \tag{12.14}$$

is harder than minimization of

$$\| |W_1S|^2 + |W_2T|^2 \|_\infty. \tag{12.15}$$

Consider the space $\mathbb{R}^2$ of 2-vectors $x = (x_1, x_2)$. There are many possible norms on this space; let us focus on

$$\|x\|_1 := |x_1| + |x_2| \text{ and } \|x\|_2 := (|x_1|^2 + |x_2|^2)^{1/2},$$

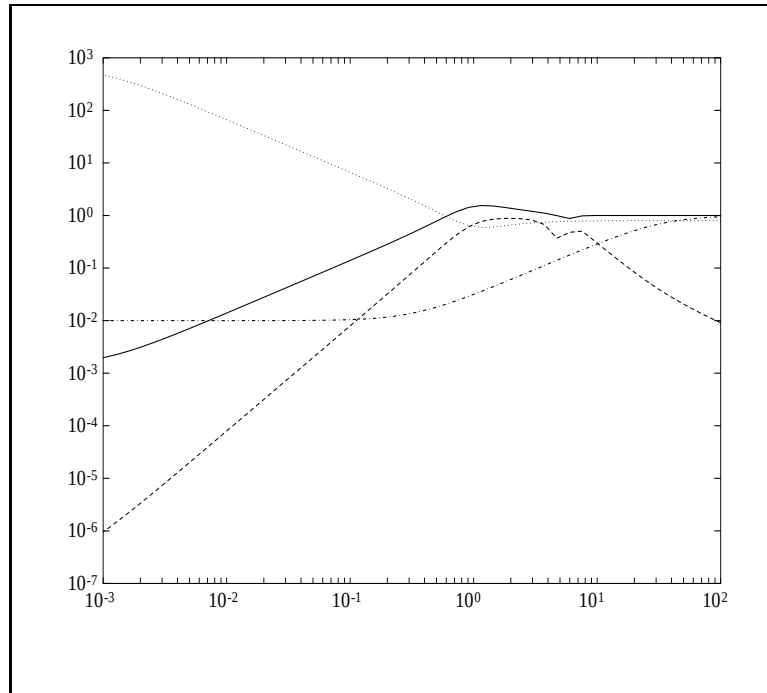


Figure 12.2: Bode plot of $|S|$ (solid), $|CS|$ (dash), $|W_1|$ (dot), and $|W_3|$ (dot-dash).

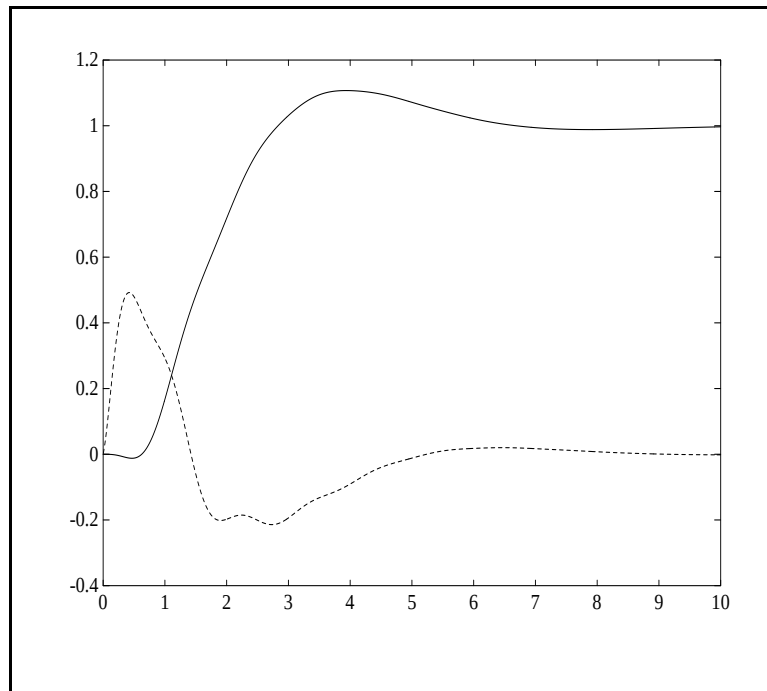


Figure 12.3: Step response of y (solid) and u (dash).

which are analogous to (12.14) and (12.15), respectively. Note that $\|\cdot\|_2$ is the usual Euclidean norm. Sketch the two unit balls,

$$\{x : \|x\|_1 \leq 1\}, \quad \{x : \|x\|_2 \leq 1\}.$$

To illustrate that $\|\cdot\|_1$ is harder to work with than $\|\cdot\|_2$, we will look at an approximation problem. Let $\mathcal{M}$ denote the subspace spanned by $(1, 2)$, that is, the straight line through this point and the origin. For $x = (4, 1)$, compute the vector y in $\mathcal{M}$ which is closest to x in that it minimizes $\|x - y\|_2$. Note that y equals the orthogonal projection of x onto $\mathcal{M}$. Now compute y to minimize $\|x - y\|_1$. Observe that y and $x - y$ are not orthogonal.

2. Prove directly that a necessary condition for solvability of the modified robust performance problem is

$$\left\| \frac{|W_1 W_2|^2}{|W_1|^2 + |W_2|^2} \right\|_\infty < \frac{1}{2}.$$

3. Show that

$$\||W_1 S| + |W_2 T|\|_\infty < 1 \Rightarrow \||W_1 S|^2 + |W_2 T|^2\|_\infty < 1.$$

4. Consider the example in Section 12.3. Let a_{sup} denote the supremum value of a for which the robust performance problem is solvable. It was shown that the value $a = 39$ is feasible for the modified robust performance problem, and hence $a_{\text{sup}} \geq 39$. Using Exercise 3, find an upper bound for a_{sup} .
5. Do a loopshape design for the example in Section 12.3, trying to make a as large as possible.

Notes and References

This chapter is based on Doyle (1983) and Francis (1983). In the literature the problem of minimizing

$$\||W_1 S|^2 + |W_2 T|^2\|_\infty$$

over all internally stabilizing controllers is called the *mixed sensitivity problem*. It was solved (for the more general case of multivariable plants) by Kwakernaak (1985) and Verma and Jonckheere (1984). The mixed sensitivity problem is a special case of a more general ∞ -norm optimization problem that is treated in Doyle (1984), Francis (1987), and Foias and Tannenbaum (1988).

