

Chapter 11

Stability Margin Optimization

In Section 4.2 we looked at several measures of stability margin (e.g., gain and phase margin). In this chapter we pose the problem of designing a controller whose sole purpose is to maximize the stability margin. The maximum obtainable stability margin is a measure of how difficult the plant is to control; for example, a plant with a right half-plane pole near a zero has a relatively small optimal stability margin, and hence is relatively difficult to control.

Three measures of stability margin are treated, namely, the ∞ -norm of a multiplicative perturbation, gain margin, and phase margin. It is shown that the problem of optimizing these stability margins can be reduced to a model-matching problem, as studied in Chapter 9. The reduction for gain and phase margins requires some conformal mappings, presented in Section 11.2.

11.1 Optimal Robust Stability

To establish our point of view, we begin with a general statement of the robust stability problem. Consider the usual unity-feedback system with plant transfer function P and controller transfer function C . It is hypothesized that P is not fixed but belongs to some set $\mathcal{P}$. The *robust stability problem* is to find, if one exists, a controller C that achieves internal stability for every P in $\mathcal{P}$. We would like to know two things: conditions on $\mathcal{P}$ for C to exist and a procedure to construct such C . In this generality the robust stability problem remains unsolved.

We concentrate in this section on the special case where $\mathcal{P}$ consists of multiplicative perturbations of a nominal plant P . Following Section 4.1, let $\mathcal{P}$ be the family of plants of the form $(1 + \Delta W_2)P$, where

1. P and $(1 + \Delta W_2)P$ have the same number of poles in $\text{Res} \geq 0$.
2. $\|\Delta\|_\infty \leq \epsilon$. (In Section 4.1 we took $\epsilon = 1$.)

Here W_2 is a fixed stable proper weighting function and $\epsilon > 0$. We write $\mathcal{P}_\epsilon$ to show the explicit dependence on ϵ . Let ϵ_{sup} denote the least upper bound on ϵ such that some C stabilizes every plant in $\mathcal{P}_\epsilon$. So ϵ_{sup} is the maximum stability margin for this model of uncertainty.

The key result in Section 4.2 (Theorem 4.1) was that a controller C achieves internal stability for every plant in $\mathcal{P}_\epsilon$ iff it achieves internal stability for P and $\|W_2 T\|_\infty < 1/\epsilon$, where T is the nominal complementary sensitivity function,

$$T = \frac{PC}{1 + PC}. \quad (11.1)$$

Define

$$\gamma_{\inf} := \inf_C \|W_2 T\|_{\infty}, \quad (11.2)$$

the infimum being over all internally stabilizing controllers. Then $\epsilon_{\sup} = 1/\gamma_{\inf}$.

Proof If $\epsilon < \epsilon_{\sup}$, there exists a C internally stabilizing all of $\mathcal{P}_{\epsilon}$, and therefore

$$\|W_2 T\|_{\infty} < \frac{1}{\epsilon}.$$

This implies that $\gamma_{\inf} < 1/\epsilon$. Since ϵ could have been arbitrarily close to $\epsilon_{\sup}$, it must be that $\gamma_{\inf} \leq 1/\epsilon_{\sup}$. The reverse inequality is proved in a similar way. ■

We would like to compute $\epsilon_{\sup}$, equivalently, $\gamma_{\inf}$. Computing $\gamma_{\inf}$ reduces to a model-matching problem in the following way. As in Section 5.2, do a coprime factorization of P :

$$P = \frac{N}{M}, \quad NX + MY = 1.$$

By Theorem 5.2 the formula

$$C = \frac{X + MQ}{Y - NQ}, \quad Q \in \mathcal{S}$$

expresses all controllers achieving internal stability for the nominal plant P . Substitution of these formulas into (11.1) gives

$$T = N(X + MQ),$$

so that

$$W_2 T = W_2 N(X + MQ),$$

and hence

$$\gamma_{\inf} = \inf_{Q \in \mathcal{S}} \|W_2 N(X + MQ)\|_{\infty}. \quad (11.3)$$

Equation (11.3) suggests the model-matching problem of Section 9.1:

$$\gamma_{\text{opt}} = \min_{Q_{\text{im}} \text{ stable}} \|T_1 - T_2 Q_{\text{im}}\|_{\infty}. \quad (11.4)$$

Evidently, $T_1 = W_2 N X$, $T_2 = -W_2 N M$. So that T_2 has no zeros on the imaginary axis (an assumption in Section 9.1), we will *assume* that P has neither poles nor zeros on the imaginary axis and W_2 has no zeros on the imaginary axis. The difference between the two problems is that Q must be stable and proper in (11.3), but Q_{im} need only be stable in (11.4); the infimum in (11.3) is not achieved, while the minimum in (11.4) is achieved. Nevertheless, $\gamma_{\inf}$ equals γ_{opt} , the minimum model-matching error, a consequence of the fact that N , hence $W_2 N$, is strictly proper (Exercise 7).

The optimization problem defined by (11.3) is very much like the performance design problem in Section 10.2. The procedure there can be easily adapted to give the following one for computing $\epsilon_{\sup}$ together with a controller C which, for any $\epsilon < \epsilon_{\sup}$, achieves internal stability for all plants in $\mathcal{P}_{\epsilon}$.

Procedure

Input: P, W_2

Step 1 Do a coprime factorization of P : Find four functions in $\mathcal{S}$ satisfying the equations

$$P = \frac{N}{M}, \quad NX + MY = 1.$$

Step 2 Solve the model-matching problem for $T_1 = W_2NX$, $T_2 = -W_2NM$. Let Q_{im} denote its solution and let γ_{opt} denote the minimum model-matching error. Then $\epsilon_{\text{sup}} = 1/\gamma_{\text{opt}}$.

Step 3 Let ϵ be an arbitrary number $< \epsilon_{\text{sup}}$. Set

$$J(s) := \frac{1}{(\tau s + 1)^k},$$

where k is just large enough that $Q_{\text{im}}J$ is proper and τ is just small enough that

$$\|W_2N(X + MQ_{\text{im}}J)\|_{\infty} < \frac{1}{\epsilon}.$$

Step 4 Set $Q = Q_{\text{im}}J$.

Step 5 Set $C = (X + MQ)/(Y - NQ)$.

Example Consider the plant

$$P(s) = \frac{s-1}{(s+1)(s-p)}, \quad 0 < p \neq 1$$

with an unstable pole at $s = p$ and a zero at $s = 1$. We might anticipate some difficulty if $p \approx 1$. Suppose that the uncertainty weight is the high-pass function

$$W_2(s) = \frac{s+0.1}{s+1}.$$

The procedure above goes like this:

Step 1

$$\begin{aligned} N(s) &= \frac{s-1}{(s+1)^2} \\ M(s) &= \frac{s-p}{s+1} \\ X(s) &= \frac{(p+1)^2}{p-1} \\ Y(s) &= \frac{s-(p+3)/(p-1)}{s+1} \end{aligned}$$

Step 2 Factor N as $N = N_{\text{ap}}N_{\text{mp}}$ with

$$N_{\text{ap}}(s) = \frac{s-1}{s+1}, \quad N_{\text{mp}}(s) = \frac{1}{s+1}.$$

Then

$$\|W_2N(X + MQ)\|_\infty = \|W_2N_{\text{mp}}(X + MQ)\|_\infty,$$

so an equivalent model-matching problem has $T_1 = W_2N_{\text{mp}}X$, $T_2 = -W_2N_{\text{mp}}M$, that is,

$$T_1(s) = \frac{(p+1)^2(s+0.1)}{(p-1)(s+1)^2}, \quad T_2(s) = -\frac{(s+0.1)(s-p)}{(s+1)^3}.$$

Since T_2 has only one zero in the right half-plane, namely, at $s = p$, the minimum model-matching error is (see Section 9.1)

$$\gamma_{\text{opt}} = |T_1(p)| = \left| \frac{p+0.1}{p-1} \right|,$$

so

$$\epsilon_{\text{sup}} = \left| \frac{p-1}{p+0.1} \right|.$$

The graph of ϵ_{sup} versus p decreases monotonically as p approaches 1 from above or below. Thus less and less uncertainty can be tolerated as p approaches 1.

To proceed, let's take a particular value for p , say $p = 0.5$, for which $\epsilon_{\text{sup}} = 0.8333$. The solution of the model-matching problem, Q_{im} , satisfies

$$T_1 - T_2Q_{\text{im}} = T_1(p),$$

which yields

$$Q_{\text{im}}(s) = -1.2 \frac{(s+1)(s-1.25)}{s+0.1}.$$

Step 3 Set $\epsilon = 0.8$ (arbitrary) and

$$J(s) = \frac{1}{\tau s + 1}.$$

The value $\tau = 0.01$ gives

$$\|W_2N(X + MQ_{\text{im}}J)\|_\infty = 1.2396 < \frac{1}{\epsilon} = 1.25.$$

Step 4

$$Q(s) = -1.2 \frac{(s+1)(s-1.25)}{(s+0.1)(0.01s+1)}$$

Step 5

$$C(s) = -\frac{(s+1)(124.5s^2 + 240.45s + 120)}{s^3 + 227.1s^2 + 440.7s + 220}$$

11.2 Conformal Mapping

For our treatment of optimal gain and phase margins, we need some preliminaries on conformal mapping.

Let $\mathcal{D}$ denote the open unit disk. Also, let $\mathcal{H}_+$ denote the open right half-plane. A well-known technique in signals and systems is to map $\mathcal{H}_+$ onto $\mathcal{D}$. One such mapping is

$$s \mapsto \frac{1-s}{1+s}.$$

This is a *conformal mapping*; that is, it is analytic in $\mathcal{H}_+$ and its inverse,

$$z \mapsto \frac{1-z}{1+z},$$

is analytic in $\mathcal{D}$. Because such a mapping exists, $\mathcal{H}_+$ and $\mathcal{D}$ are said to be *conformally equivalent*.

What proper subsets of $\mathbb{C}$ are conformally equivalent to $\mathcal{D}$? It turns out that the subsets need only be open and simply connected (i.e., have no holes)—for example, an annulus is not simply connected. This fact is known as the Riemann mapping theorem.

Example 1 Let $\mathcal{G}_1$ be the complement of the negative real axis,

$$\{s \in \mathbb{C} : s \text{ is real and } \leq 0\}.$$

It is easy to see that $\mathcal{G}_1$ is open and simply connected. We construct a conformal mapping $\phi_1 : \mathcal{G}_1 \rightarrow \mathcal{D}$ as the composition of two mappings:

$$\psi_1 : \mathcal{G}_1 \rightarrow \mathcal{H}_+, \quad \psi_1(s) = \sqrt{s}$$

and

$$\psi_2 : \mathcal{H}_+ \rightarrow \mathcal{D}, \quad \psi_2(s) = \frac{1-s}{1+s}.$$

Then

$$\phi_1(s) := \psi_2(\psi_1(s)) = \frac{1-\sqrt{s}}{1+\sqrt{s}}.$$

It is worth pointing out that conformal mappings are not unique, so the function ϕ_1 is just one possibility.

Example 2 Let a be a positive real number and let $\mathcal{G}_2$ be the complement of the horizontal ray from $-a$ left, that is, the complement of

$$\{s \in \mathbb{C} : s \text{ is real and } \leq -a\}.$$

The function

$$s \mapsto 1 + sa^{-1}$$

maps $\mathcal{G}_2$ conformally onto $\mathcal{G}_1$. Composing it with ϕ_1 gives

$$\phi_2(s) := \frac{1 - \sqrt{1 + sa^{-1}}}{1 + \sqrt{1 + sa^{-1}}},$$

a conformal mapping from $\mathcal{G}_2$ onto $\mathcal{D}$, taking 0 to 0.

Example 3 Let $a = a_1 + ja_2$ be a complex number in the first quadrant (i.e., $a_1, a_2 > 0$). Consider the two rays from a vertically up and from $\bar{a}$ vertically down; their union is the set

$$\{a + j\omega : \omega \geq 0\} \cup \{\bar{a} - j\omega : \omega \geq 0\}.$$

Let $\mathcal{G}_3$ denote the complement of this set. We construct a conformal mapping ϕ_3 from $\mathcal{G}_3$ onto $\mathcal{D}$ in several steps. First, the function $\psi_1(s) := s - a_1$ translates the rays to the imaginary axis. Second, $\psi_2(s) := js$ rotates them onto the real axis. Thus, $\psi_2 \circ \psi_1$ maps $\mathcal{G}_3$ conformally onto the complement of the set

$$\{s : s \in \mathbb{R}, |s| \geq a_2\}.$$

Third,

$$\psi_3(s) := \sqrt{\frac{1 - s/a_2}{1 + s/a_2}}$$

maps the latter set conformally onto $\mathcal{H}_+$. Now let's pause and see where 0 is mapped under $\psi_3 \circ \psi_2 \circ \psi_1$:

$$0 \mapsto -a_1 \mapsto -ja_1 \mapsto c,$$

where

$$c := \sqrt{\frac{1 + ja_1/a_2}{1 - ja_1/a_2}}.$$

Finally,

$$\psi_4(s) := \frac{s - c}{s + \bar{c}}$$

maps $\mathcal{H}_+$ onto $\mathcal{D}$. So a suitable ϕ_3 is

$$\phi_3 := \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1.$$

It too takes 0 to 0.

The conformal mappings in Examples 2 and 3 both take 0 to 0, a property that will be needed in the applications to follow. This property makes them unique up to rotation. More precisely, suppose that $\mathcal{G}$ is an open, simply connected subset of $\mathbb{C}$, and ϕ_1 and ϕ_2 are two conformal mappings from $\mathcal{G}$ onto $\mathcal{D}$ taking 0 to 0; then there exists an angle α such that

$$\phi_1 = e^{j\alpha} \phi_2$$

(i.e., ϕ_1 is a rotation of ϕ_2). Consequently, ϕ_1 and ϕ_2 have equal magnitudes at points of evaluation in $\mathcal{G}$; that is, for z in $\mathcal{G}$, $|\phi_i(z)|$ does not depend on the particular conformal mapping.

11.3 Gain Margin Optimization

This section continues with the robust stability problem, but now

$$\mathcal{P} = \{kP : 1 \leq k \leq k_1\}.$$

Here P is the nominal plant transfer function and k is a real gain that is uncertain and may lie anywhere in the interval $[1, k_1]$. (The family

$$\mathcal{P} = \{kP : k_0 \leq k \leq k_1\}, \quad 0 < k_0 < k_1$$

is only superficially more general; it can be reduced to the case $k_0 = 1$ by scaling.) We ask the question: How large can k_1 be but yet there exists a controller achieving internal stability for every plant in $\mathcal{P}$? Let $k_{\sup}$ denote the supremum such k_1 . We'll get a formula for $k_{\sup}$ under the simplifying *assumption* that P has neither poles nor zeros on the imaginary axis.

It turns out that $k_{\sup}$ is closely related to the infimum norm of the (unweighted) complementary sensitivity function [see (11.2)],

$$\gamma_{\inf} := \inf_C \|T\|_{\infty}.$$

Of course, as in Section 11.1 this can be converted to a model-matching problem, an analysis of which will show that

1. $\gamma_{\inf} = 0$ if P is stable.
2. $\gamma_{\inf} = 1$ if P is unstable but minimum phase.
3. $\gamma_{\inf} > 1$ if P is unstable and non-minimum phase.

Theorem 1 *If P is stable or minimum phase, then $k_{\sup} = \infty$. Otherwise,*

$$k_{\sup} = \left(\frac{\gamma_{\inf} + 1}{\gamma_{\inf} - 1} \right)^2.$$

Partial Proof A complete proof is fairly long, so parts will be omitted.

That $k_{\sup} = \infty$ when P is stable is trivial: Take the zero controller, $C = 0$. So we proceed under the assumption that P is unstable.

It suffices to show that there exists a C stabilizing kP for all $1 \leq k \leq k_1$ iff

$$k_1 < \left(\frac{\gamma_{\inf} + 1}{\gamma_{\inf} - 1} \right)^2$$

or equivalently iff

$$\gamma_{\inf} < \frac{\sqrt{k_1} + 1}{\sqrt{k_1} - 1}. \quad (11.5)$$

We will prove necessity, in three steps. Assume that such C exists.

Step 1 Since C stabilizes the nominal plant, P , it must have the form

$$C = \frac{X + MQ}{Y - NQ}$$

for some Q in $\mathcal{S}$. Fix $1 < k \leq k_1$ and invoke Lemma 5.1: Since C stabilizes kP ,

$$M(Y - NQ) + kN(X + MQ) \text{ is invertible in } \mathcal{S}.$$

This leads in turn to the following chain:

$$\begin{aligned} & MY + kNX + (k-1)MNQ \text{ is invertible in } \mathcal{S} \\ \Rightarrow & (1 - NX) + kNX + (k-1)MNQ \text{ is invertible in } \mathcal{S} \\ \Rightarrow & 1 + (k-1)N(X + MQ) \text{ is invertible in } \mathcal{S} \\ \Rightarrow & \frac{1}{k-1} + N(X + MQ) \text{ is invertible in } \mathcal{S}. \end{aligned}$$

Let $\overline{\mathcal{H}_+}$ denote the closed right half-plane together with the point at infinity. We conclude that the function $N(X + MQ)$ maps $\overline{\mathcal{H}_+}$ into the complement of the single point $-1/(k-1)$. Since this holds for all $1 < k \leq k_1$, $N(X + MQ)$ maps $\overline{\mathcal{H}_+}$ into the complement of the set

$$\{s \in \mathbb{C} : s \text{ is real and } \leq -a\}, \quad a := \frac{1}{k_1 - 1}.$$

Let $\mathcal{G}$ denote this complement.

Note that $N(X + MQ)$ equals T , the complementary sensitivity function corresponding to P and C . It follows that T satisfies certain interpolation conditions [see equation (6.2)]. Let $\{p_i\}$ and $\{z_i\}$ denote the right half-plane poles and zeros of P ; for simplicity, these are assumed to be distinct. Then

$$T(p_i) = 1, \quad T(z_i) = 0.$$

Step 2 As in Example 2 of the preceding section, bring in a conformal mapping from $\mathcal{G}$ onto the open unit disk, $\mathcal{D}$:

$$\phi(s) := \frac{1 - \sqrt{1 + sa^{-1}}}{1 + \sqrt{1 + sa^{-1}}}.$$

Define the function $G = \phi \circ T$. From the properties of T we get that G maps $\overline{\mathcal{H}_+}$ into $\mathcal{D}$ and

$$G(p_i) = \phi(1), \quad G(z_i) = \phi(0) = 0.$$

Step 3 The final step is to scale G : Define

$$H := \frac{1}{\phi(1)} G.$$

Then H maps $\overline{\mathcal{H}_+}$ into the disk of radius $1/|\phi(1)|$ [i.e., $\|H\|_\infty < 1/|\phi(1)|$] and

$$H(p_i) = 1, \quad H(z_i) = 0.$$

The function H is analytic in $\mathcal{H}_+$, but not necessarily real-rational because ϕ does not preserve rationality. Nevertheless, it can be proved that there exists a function, say K , which is real-rational and has the foregoing properties of H .

We can conclude that there exists a function K in $\mathcal{S}$ having the properties

$$\|K\|_\infty < \frac{1}{|\phi(1)|}, \tag{11.6}$$

$$K(p_i) = 1, \quad K(z_i) = 0.$$

These properties in turn imply that K is a complementary sensitivity function for some controller; that is, $K = N(X + MQ_1)$ for some Q_1 in $\mathcal{S}$. But then from (11.6)

$$\gamma_{\inf} < \frac{1}{|\phi(1)|}.$$

It remains to compute that

$$\phi(1) = \frac{1 - \sqrt{k_1}}{1 + \sqrt{k_1}}.$$

This proves (11.5).

Notice that k_{sup} equals the value of k_1 satisfying the equation

$$\gamma_{\text{inf}} = \frac{1}{|\phi(1)|}. \blacksquare$$

The following procedure gives a way to compute a controller that achieves an upper gain margin of k_1 , any number less than k_{sup} . The idea is to reverse the argument in the proof above. At a high level, the procedure is as follows:

1. Solve the model-matching problem corresponding to $\inf \|T\|_{\infty}$; let the solution be Q_{im} .
2. Get a suitable roll-off function J .
3. Let K be the complementary sensitivity function corresponding to the controller parameter $Q_{\text{im}}J$.
4. Set $G = \phi(1)K$.
5. Solve $G = \phi \circ T$ for T , which will be the resulting complementary sensitivity function.
6. Solve $T = N(X + MQ)$ for Q .
7. Set $C = (X + MQ)/(Y - NQ)$.

Procedure: P Unstable, Non-minimum Phase, No Imaginary Axis Poles or Zeros

Input: P

Step 1 Do a coprime factorization of P :

$$P = \frac{N}{M}, \quad NX + MY = 1.$$

Step 2 Solve the model-matching problem for $T_1 = NX$, $T_2 = -NM$. Let Q_{im} denote its solution and let γ_{opt} denote the minimum model-matching error. Then

$$k_{\text{sup}} = \left(\frac{\gamma_{\text{opt}} + 1}{\gamma_{\text{opt}} - 1} \right)^2.$$

Step 3 Let k_1 be arbitrary with $1 < k_1 < k_{\text{sup}}$. Set

$$J(s) := \frac{1}{(\tau s + 1)^k},$$

where k is just large enough that $Q_{\text{im}}J$ is proper and τ is just small enough that

$$\|N(X + MQ_{\text{im}}J)\|_{\infty} < \frac{\sqrt{k_1} + 1}{\sqrt{k_1} - 1}.$$

Step 4 Set

$$\begin{aligned} K &= N(X + MQ_{\text{im}}J), \\ G &= \frac{1 - \sqrt{k_1}}{1 + \sqrt{k_1}}K, \\ T &= \frac{1}{k_1 - 1} \left[\left(\frac{1 - G}{1 + G} \right)^2 - 1 \right], \\ Q &= \frac{T - NX}{NM}. \end{aligned}$$

Step 5 Set $C = (X + MQ)/(Y - NQ)$.

Example Consider the plant

$$P(s) = \frac{s - 1}{(s + 1)(s - p)}, \quad 0 < p \neq 1,$$

which was studied in Section 11.1. The procedure above goes like this:

Step 1

$$\begin{aligned} N(s) &= \frac{s - 1}{(s + 1)^2} \\ M(s) &= \frac{s - p}{s + 1} \\ X(s) &= \frac{(p + 1)^2}{p - 1} \\ Y(s) &= \frac{s - (p + 3)/(p - 1)}{s + 1} \end{aligned}$$

Step 2 Factor N as $N = N_{\text{ap}}N_{\text{mp}}$ with

$$N_{\text{ap}}(s) = \frac{s - 1}{s + 1}, \quad N_{\text{mp}}(s) = \frac{1}{s + 1}.$$

Then

$$\|N(X + MQ)\|_{\infty} = \|N_{\text{mp}}(X + MQ)\|_{\infty},$$

so an equivalent model-matching problem has $T_1 = N_{\text{mp}}X$, $T_2 = -N_{\text{mp}}M$, that is,

$$T_1(s) = \frac{(p + 1)^2}{(p - 1)(s + 1)}, \quad T_2(s) = -\frac{s - p}{(s + 1)^2}.$$

Thus

$$\gamma_{\text{opt}} = |T_1(p)| = \left| \frac{p + 1}{p - 1} \right|$$

and

$$k_{\text{sup}} = \left(\frac{p + 1 + |p - 1|}{p + 1 - |p - 1|} \right)^2 = \begin{cases} p^2, & p \geq 1 \\ p^{-2}, & p < 1. \end{cases}$$

As with ϵ_{sup} , this function has its minimum at $p = 1$.

To proceed, let's take $p = 2$ (arbitrary), for which $k_{\text{sup}} = 4$. The solution of the model-matching problem, Q_{im} , satisfies

$$T_1 - T_2 Q_{\text{im}} = T_1(p),$$

which yields

$$Q_{\text{im}}(s) = 3(s + 1).$$

Step 3 Set $k_1 = 3.5$ (arbitrary) and

$$J(s) = \frac{1}{\tau s + 1}.$$

The value $\tau = 0.01$ gives

$$\|N(X + M Q_{\text{im}} J)\|_{\infty} = 3.0827 < \frac{\sqrt{k_1} + 1}{\sqrt{k_1} - 1} = 3.2967.$$

Step 4

$$\begin{aligned} K(s) &= 3 \frac{(s-1)(1.03s+1)}{(s+1)^2(0.01s+1)} \\ G(s) &= -0.9100 \frac{(s-1)(1.03s+1)}{(s+1)^2(0.01s+1)} \\ T(s) &= \frac{0.0375s^5 + 3.8231s^4 + 7.3882s^3 - 0.1831s^2 - 7.4257s - 3.64}{0.0003s^6 + 0.0041s^5 + 0.1190s^4 + 0.9378s^3 + 11.1662s^2 + 19.4563s + 9.1204} \\ Q(s) &= \frac{0.0352s^4 + 3.9357s^3 + 22.1648s^2 + 59.8368s + 41.5724}{0.0003s^4 + 0.0036s^3 + 0.1116s^2 + 0.7175s + 9.6670} \end{aligned}$$

11.4 Phase Margin Optimization

Now the family of plants to be stabilized is

$$\mathcal{P} = \left\{ e^{-j\theta} P : -\theta_1 \leq \theta \leq \theta_1 \right\},$$

where P is the nominal plant transfer function and θ is an uncertain phase lying anywhere in the interval $[-\theta_1, \theta_1]$; θ_1 is an angle in $(0, \pi]$. Let θ_{sup} denote the supremum θ_1 for which there exists a stabilizing controller. Under the *assumption* that P has neither poles nor zeros on the imaginary axis, a formula can be derived for θ_{sup} in terms of $\gamma_{\text{inf}} := \inf \|T\|_{\infty}$ just as in the preceding section.

Theorem 2 *If P is stable or minimum phase, then $\theta_{\text{sup}} = \pi$. Otherwise,*

$$\theta_{\text{sup}} = 2 \sin^{-1} \frac{1}{\gamma_{\text{inf}}}.$$

Proof The proof of Theorem 1 can be adapted with a few alterations by starting with $k = e^{-j\theta}$. Again, the case that P is stable is trivial.

First, in Step 1 the appropriate set $\mathcal{G}$ is the complement of

$$\left\{ -\frac{1}{k-1} : k = e^{-j\theta}, -\theta_1 \leq \theta \leq \theta_1 \right\}.$$

The latter set is the union of the vertical rays from

$$a = a_1 + ja_2 := \frac{1}{2} + j \frac{\sin \theta_1}{1 - \cos \theta_1}$$

up and from $\bar{a}$ down. Second, in Step 2, from Example 3 of the preceding section a conformal mapping from $\mathcal{G}$ onto $\mathcal{D}$ is

$$\phi = \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1,$$

where

$$\begin{aligned} \psi_1(s) &= s - \frac{1}{2}, \\ \psi_2(s) &= js, \\ \psi_3(s) &= \sqrt{\frac{1 - s/a_2}{1 + s/a_2}}, \\ \psi_4(s) &= \frac{s - c}{s + \bar{c}}, \\ c &= \sqrt{\frac{1 + j/(2a_2)}{1 - j/(2a_2)}}. \end{aligned}$$

Finally, as in the last sentence in the proof of Theorem 1, θ_{sup} equals the value of θ_1 satisfying the equation

$$\gamma_{\text{inf}} = \frac{1}{|\phi(1)|}.$$

So it remains to show that

$$|\phi(1)| = \sin \frac{\theta_1}{2}.$$

Elementary computations give

$$\phi(1) = -\frac{\text{Im}c}{\bar{c}}, \quad |\phi(1)| = \sin \frac{\theta_1}{2}. \blacksquare$$

Example Consider once again the plant

$$P(s) = \frac{s-1}{(s+1)(s-p)}, \quad 0 < p \neq 1.$$

We found in the preceding section that

$$\gamma_{\text{inf}} = \left| \frac{p+1}{p-1} \right|.$$

Therefore,

$$\theta_{\text{sup}} = 2 \sin^{-1} \left| \frac{p-1}{p+1} \right|.$$

Exercises

1. Compute ϵ_{sup} for

$$P(s) = \frac{s-1}{(s-2)(s-3)}, \quad W_2(s) = \frac{s+1}{s+100}.$$

Select some $\epsilon < \epsilon_{\text{sup}}$ and compute a robust controller.

2. Take

$$P(s) = \frac{1}{s-1}, \quad W_2(s) = \frac{as}{0.01s+1},$$

with a a positive number. Compute the least upper bound on a for which robust stability is achievable. Pick some a less than this upper bound and design a robust controller.

3. With $\gamma_{\text{inf}} = \inf \|T\|_{\infty}$ and P having neither poles nor zeros on the imaginary axis, prove

$\gamma_{\text{inf}} = 0$ if P is stable,

$\gamma_{\text{inf}} = 1$ if P is unstable but minimum phase,

$\gamma_{\text{inf}} > 1$ if P is unstable and non-minimum phase.

4. Compute the maximum gain margin k_{sup} for

$$P(s) = \frac{s-1}{(s-2)(s-3)}.$$

Select some $k_1 < k_{\text{sup}}$ and compute a robust controller.

5. Repeat the Exercise 4 but for phase margin.

6. Recall from Section 4.2 that $1/\|S\|_{\infty}$ equals the distance from the critical point -1 to the nearest point on the Nyquist plot of PC , and in this way qualifies as a stability margin. The smaller $\|S\|_{\infty}$, the larger the stability margin. Compute $\inf \|S\|_{\infty}$ for the plant in Exercise 4.

7. Using the equation

$$W_2N(X + MQJ) = W_2N(X + MQ)J + W_2NX(1 - J),$$

prove that γ_{inf} in (11.3) and γ_{opt} in (11.4) are equal.

Notes and References

The problem in Section 11.1 was first solved by Kimura (1984). The gain margin problem was first solved by Tannenbaum (1980, 1981) using Nevanlinna-Pick theory. Khargonekar and Tannenbaum (1985) showed the mathematical equivalence of the problems of gain margin optimization, sensitivity minimization, and robust stabilization. Yan and Anderson (1990) considered a problem that mixes performance and gain margin.

