

Chapter 10

Design for Performance

The performance criterion $\|W_1 S\|_\infty < 1$ was introduced in Section 3.4. The associated design problem is to find a proper C for which the feedback system is internally stable and $\|W_1 S\|_\infty < 1$. When does such a C exist and how can it be computed? These questions are easy when the inverse of the plant transfer function is stable. When the inverse is unstable, the questions are more interesting. The solutions presented in this chapter use model-matching theory.

10.1 P^{-1} Stable

We assume in this section that P has no zeros in $\text{Re } s \geq 0$, or in other words, P^{-1} is stable. The weighting function W_1 is assumed to be stable and strictly proper. The latter condition is not too serious a loss of generality. We will see that under these conditions it is always possible, indeed quite easy, to design a proper C which is internally stabilizing and makes $\|W_1 S\|_\infty < 1$.

Let k be a positive integer and τ a positive real number, and consider the transfer function

$$J(s) := \frac{1}{(\tau s + 1)^k}.$$

Sketch the Bode plot of J : The magnitude starts out at 1, is relatively flat out to the corner frequency $\omega = 1/\tau$, and then rolls off to $-\infty$ with slope $-k$; the phase starts out at 0, is relatively flat up to, say, $\omega = 0.1/\tau$, and then rolls off to $-k\pi/2$ radians. So for low frequency, $J(j\omega) \approx 1$. This function has the useful property that it approximates 1 beside a strictly proper function.

Lemma 1 *If G is stable and strictly proper, then*

$$\lim_{\tau \rightarrow 0} \|G(1 - J)\|_\infty = 0.$$

Proof Let $\epsilon > 0$ and $\omega_1 > 0$. By the argument above regarding the Bode plot of J , if τ is sufficiently small, then the Nyquist plot of J lies in the disk with center 1, radius ϵ for $\omega \leq \omega_1$, and in the disk with center 0, radius 1 for $\omega > \omega_1$. Now $\|G(1 - J)\|_\infty$ equals the maximum of

$$\max_{\omega \leq \omega_1} |G(j\omega)[1 - J(j\omega)]|$$

and

$$\max_{\omega > \omega_1} |G(j\omega)[1 - J(j\omega)]|.$$

The first of these is bounded above by $\epsilon\|G\|_\infty$, and the second by

$$\|1 - J\|_\infty \max_{\omega > \omega_1} |G(j\omega)|.$$

Since

$$\|1 - J\|_\infty \leq \|1\|_\infty + \|J\|_\infty = 2,$$

we have

$$\|G(1 - J)\|_\infty \leq \max \left\{ \epsilon\|G\|_\infty, 2 \max_{\omega > \omega_1} |G(j\omega)| \right\}.$$

This holds for τ sufficiently small. But the right-hand side can be made arbitrarily small by suitable choice of ϵ and ω_1 because

$$\lim_{\omega_1 \rightarrow \infty} \max_{\omega > \omega_1} |G(j\omega)| = |G(j\infty)| = 0.$$

We conclude that for every $\delta > 0$, if τ is small enough, then

$$\|G(1 - J)\|_\infty \leq \delta.$$

This is the desired conclusion. ■

We'll develop the design procedure first with the additional assumption that P is stable. By Theorem 5.1 the set of all internally stabilizing C s is parametrized by the formula

$$C = \frac{Q}{1 - PQ}, \quad Q \in \mathcal{S}.$$

Then W_1S is given in terms of Q by

$$W_1S = W_1(1 - PQ).$$

To make $\|W_1S\|_\infty < 1$ we are prompted to set $Q = P^{-1}$. This is indeed stable, by assumption, but not proper, hence not in $\mathcal{S}$. So let's try $Q = P^{-1}J$ with J as above and the integer k just large enough to make $P^{-1}J$ proper (i.e., k equals the relative degree of P). Then

$$W_1S = W_1(1 - J),$$

whose ∞ -norm is < 1 for sufficiently small τ , by Lemma 1.

In summary, the design procedure is as follows.

Procedure: P and P^{-1} Stable

Input: P, W_1

Step 1 Set k = the relative degree of P .

Step 2 Choose τ so small that

$$\|W_1(1 - J)\|_\infty < 1,$$

where

$$J(s) := \frac{1}{(\tau s + 1)^k}.$$

Step 3 Set $Q = P^{-1}J$.

Step 4 Set $C = Q/(1 - PQ)$.

When P is unstable, the parametrization in Theorem 5.2 is used.

Procedure: P^{-1} Stable

Input: P, W_1

Step 1 Do a coprime factorization of P : Find four functions in $\mathcal{S}$ satisfying the equations

$$P = \frac{N}{M}, \quad NX + MY = 1.$$

Step 2 Set $k =$ the relative degree of P .

Step 3 Choose τ so small that

$$\|W_1MY(1 - J)\|_\infty < 1,$$

where

$$J(s) := \frac{1}{(\tau s + 1)^k}.$$

Step 4 Set $Q = YN^{-1}J$.

Step 5 Set $C = (X + MQ)/(Y - NQ)$.

Example Consider the unstable plant and weighting function

$$P(s) = \frac{1}{(s - 2)^2}, \quad W_1(s) = \frac{100}{s + 1}.$$

This weight has bandwidth 1 rad/s, so it might be used to get good tracking (i.e., approximately 1% tracking error, up to this frequency). The previous procedure for these data goes as follows:

Step 1 First, do a coprime factorization of P over $\mathcal{S}$:

$$\begin{aligned} N(s) &= \frac{1}{(s + 1)^2}, \\ M(s) &= \frac{(s - 2)^2}{(s + 1)^2}, \\ X(s) &= 27 \frac{s - 1}{s + 1}, \\ Y(s) &= \frac{s + 7}{s + 1}. \end{aligned}$$

Step 2 $k = 2$

Step 3 Choose τ so that the ∞ -norm of

$$\frac{100(s-2)^2(s+7)}{(s+1)^4} \left[1 - \frac{1}{(\tau s + 1)^2} \right]$$

is < 1 . The norm is computed for decreasing values of τ :

τ	∞ -Norm
10^{-1}	199.0
10^{-2}	19.97
10^{-3}	1.997
10^{-4}	0.1997

So take $\tau = 10^{-4}$.

Step 4

$$Q(s) = \frac{(s+1)(s+7)}{(10^{-4}s+1)^2}$$

Step 5

$$C(s) = 10^4 \frac{(s+1)^3}{s(s+7)(10^{-4}s+2)}$$

This section concludes with a result stated but not proved in Section 6.2. It concerns the performance problem where the weight W_1 satisfies

$$|W_1(j\omega)| = \begin{cases} \frac{1}{\epsilon}, & \omega_1 \leq \omega \leq \omega_2 \\ \frac{1}{\delta}, & \text{else.} \end{cases}$$

Thus the criterion $\|W_1 S\|_\infty < 1$ is equivalent to the conditions

$$\begin{aligned} |S(j\omega)| &< \epsilon, & \omega_1 \leq \omega \leq \omega_2 \\ |S(j\omega)| &< \delta, & \text{else.} \end{aligned} \tag{10.1}$$

Lemma 2 *If P^{-1} is stable, then for every $\epsilon > 0$ and $\delta > 1$, there exists a proper C such that the feedback system is internally stable and (10.1) holds.*¹

Proof The idea is to approximately invert P over the frequency range $[0, \omega_2]$ while rolling off fast enough at higher frequencies. From Theorem 5.2 again, the formula for all internally stabilizing proper controllers is

$$C = \frac{X + MQ}{Y - NQ}, \quad Q \in \mathcal{S}.$$

For such C

$$S = M(Y - NQ). \tag{10.2}$$

Now fix $\epsilon > 0$ and $\delta > 1$. We may as well suppose that $\epsilon < 1$. Choose $c > 0$ so small that

$$c\|MY\|_\infty < \epsilon, \tag{10.3}$$

¹The assumption on P in Lemma 2 is slightly stronger than necessary; see the statement in Section 6.2.

$$(1 + c)^2 < \delta. \quad (10.4)$$

Since P is strictly proper, so is N . This fact together with the equation

$$NX + MY = 1$$

shows that

$$M(j\infty)Y(j\infty) = 1.$$

Since $|M(j\omega)Y(j\omega)|$ is a continuous function of ω , it is possible to choose $\omega_3 \geq \omega_2$ such that

$$|M(j\omega)Y(j\omega)| \leq 1 + c, \quad \forall \omega \geq \omega_3. \quad (10.5)$$

The assumption on P implies that N^{-1} is stable (but not proper). Choose a function V in $\mathcal{S}$ with the following three properties:

1. VN^{-1} is proper.
2. $|1 - V(j\omega)| \leq c, \quad \forall \omega \leq \omega_3.$
3. $\|1 - V\|_\infty \leq 1 + c.$

The idea behind the choice of V can be explained in terms of its Nyquist plot: It should lie in the disk with center 1, radius c up to frequency ω_3 (property 2) and in the disk with center 1, radius $1 + c$ thereafter (property 3). In addition, V should roll off fast enough so that VN^{-1} is proper. It is left as an exercise to convince yourself that such a V exists—a function of the form

$$\frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)^k}$$

will work.

Finally, take Q to be

$$Q := VN^{-1}Y.$$

Substitution into (10.2) gives

$$S = MY(1 - V).$$

Thus for $\omega \leq \omega_3$

$$\begin{aligned} |S(j\omega)| &\leq c\|MY\|_\infty && \text{from property 2} \\ &< \epsilon && \text{from (10.3)} \end{aligned}$$

and for $\omega > \omega_3$

$$\begin{aligned} |S(j\omega)| &\leq (1 + c)|M(j\omega)Y(j\omega)| && \text{from property 3} \\ &\leq (1 + c)^2 && \text{from (10.5)} \\ &< \delta && \text{from (10.4). } \blacksquare \end{aligned}$$

10.2 P^{-1} Unstable

We come now to the first time in this book that we need a nonclassical method, namely, interpolation theory. To simplify matters we will assume in this section that

- P has no poles or zeros on the imaginary axis, only distinct poles and zeros in the right half-plane, and at least one zero in the right half-plane (i.e., P^{-1} is unstable).
- W_1 is stable and strictly proper.

It would be possible to relax these assumptions, but the development would be messier.

To motivate the procedure to follow, let's see roughly how the design problem of finding an internally stabilizing C so that $\|W_1 S\|_\infty < 1$ can be translated into an NP problem. The definition of S is

$$S = \frac{1}{1 + PC}.$$

For C to be internally stabilizing it is necessary and sufficient that $S \in \mathcal{S}$ and PC have no right half-plane pole-zero cancellations (Theorem 3.2). Thus, S must interpolate the value 1 at the right half-plane zeros of P and the value 0 at the right half-plane poles (see also Section 6.1); that is, S must satisfy the conditions

$$\begin{aligned} S(z) &= 1 \quad \text{for } z \text{ a zero of } P \text{ in } \text{Res} > 0, \\ S(p) &= 0 \quad \text{for } p \text{ a pole of } P \text{ in } \text{Res} > 0. \end{aligned}$$

The weighted sensitivity function $G := W_1 S$ must therefore satisfy

$$\begin{aligned} G(z) &= W_1(z) \quad \text{for } z \text{ a zero of } P \text{ in } \text{Res} > 0, \\ G(p) &= 0 \quad \text{for } p \text{ a pole of } P \text{ in } \text{Res} > 0. \end{aligned}$$

So the requirement of internal stability imposes interpolation constraints on G . The performance spec $\|W_1 S\|_\infty < 1$ translates into $\|G\|_\infty < 1$. Finally, the condition $S \in \mathcal{S}$ requires that G be analytic in the right half-plane.

One approach to the design problem might be to find a function G satisfying these conditions, then to get S , and finally to get C by back-substitution. This has a technical snag because the requirement that C be proper places an additional constraint on G not handled by our NP theory of the Chapter 9. For this reason we proceed via controller parametrization.

Bring in again a coprime factorization of P :

$$P = \frac{N}{M}, \quad NX + MY = 1.$$

The controller parametrization formula is

$$C = \frac{X + MQ}{Y - NQ}, \quad Q \in \mathcal{S},$$

and for such C the weighted sensitivity function is

$$W_1 S = W_1 M(Y - NQ).$$

The parameter Q must be both stable and proper. Our approach is first to drop the properness requirement and find a suitable parameter, say, Q_{im} , which is improper but stable, and then to get a suitable Q by rolling Q_{im} off at high frequency. The reason this works is that W_1 is strictly proper, so there is no performance requirement at high frequency. The method is outlined as follows:

ProcedureInput: P, W_1 **Step 1** Do a coprime factorization of P : Find four functions in $\mathcal{S}$ satisfying the equations

$$P = \frac{N}{M}, \quad NX + MY = 1.$$

Step 2 Find a stable function Q_{im} such that

$$\|W_1 M(Y - NQ_{\text{im}})\|_{\infty} < 1.$$

Step 3 Set

$$J(s) := \frac{1}{(\tau s + 1)^k},$$

where k is just large enough that $Q_{\text{im}}J$ is proper and τ is just small enough that

$$\|W_1 M(Y - NQ_{\text{im}}J)\|_{\infty} < 1.$$

Step 4 Set $Q = Q_{\text{im}}J$.**Step 5** Set $C = (X + MQ)/(Y - NQ)$.

That Step 3 is feasible follows from the equation

$$W_1 M(Y - NQ_{\text{im}}J) = W_1 M(Y - NQ_{\text{im}})J + W_1 MY(1 - J).$$

The first term on the right-hand side has ∞ -norm less than 1 from Step 2 and the fact that $\|J\|_{\infty} \leq 1$, while the ∞ -norm of the second term goes to 0 as τ goes to 0 by Lemma 1.

Step 2 is the model-matching problem, find a stable function Q_{im} to minimize

$$\|T_1 - T_2 Q_{\text{im}}\|_{\infty},$$

where $T_1 := W_1 MY$ and $T_2 := W_1 MN$. Step 2 is feasible iff γ_{opt} , the minimum model-matching error, is < 1 .

10.3 Design Example: Flexible Beam

This section presents an example to illustrate the procedure of the preceding section. The example is based on a real experimental setup at the University of Toronto.

The control system, depicted in Figure 10.1, has the following components: a flexible beam, a high-torque dc motor at one end of the beam, a sonar position sensor at the other end, a digital computer as the controller with analog-to-digital interface hardware, a power amplifier to drive the motor, and an antialiasing filter. The objective is to control the position of the sensed end of the beam.

A plant model was obtained as follows. The beam is pinned to the motor shaft and is free at the sensed end. First the beam itself was modeled as an ideal Euler-Bernoulli beam with no damping; this yielded a partial differential equation model, reflecting the fact that the physical model of the

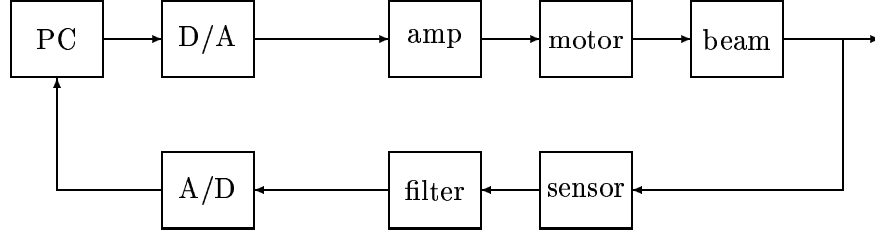


Figure 10.1: Flexible beam setup.

beam has an infinite number of modes. The model is therefore linear but infinite-dimensional. The corresponding transfer function from torque input at the motor end to tip deflection at the sensed end has the form

$$\sum_{i=0}^{\infty} \frac{c_i}{s^2 + \omega_i^2}.$$

Then damping was introduced, yielding the form

$$\sum_{i=0}^{\infty} \frac{c_i}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}.$$

The first term is c_0/s^2 and corresponds to the rigid-body slewing motion about the pinned end. The second term,

$$\frac{c_1}{s^2 + 2\zeta_1\omega_1 s + \omega_1^2},$$

corresponds to the first flexible mode. And so on. The motion was found to be adequately modeled by the first four flexible modes. Then the damping ratios and natural frequencies were determined experimentally. Finally, the amplifier, motor, and sensor were introduced into the model. The antialiasing filter was ignored for the purpose of design.

For simplicity we shall take the plant transfer function to be

$$P(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{s(5s^3 + 3.5682s^2 + 139.5021s + 0.0929)}.$$

The poles are

$$0, -0.0007, -0.3565 \pm 5.2700j.$$

The first two poles correspond to the rigid-body motion; the one at $s = -0.0007$ has been perturbed away from the origin by the back EMF in the motor. The two complex poles correspond to the first flexible mode, the damping ratio being 0.0675. The zeros are

$$-4.9081, 5.5308.$$

Because of the zero at $s = 5.5308$ the plant is non-minimum phase, reflecting the fact that the actuator (the motor) and the sensor are not located at the same point on the beam. The procedure of the preceding section requires no poles on the imaginary axis, so the model is (harmlessly) perturbed to

$$P(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{5s^4 + 3.5682s^3 + 139.5021s^2 + 0.0929s + 10^{-6}}.$$

A common way to specify desired closed-loop performance is by a step response test. For this flexible beam the spec is that a step reference input (r) should produce a plant output (y) satisfying

$$\begin{aligned}\text{settling time} &\approx 8\text{s}, \\ \text{overshoot} &\leq 10\%.\end{aligned}$$

We will accomplish this by shaping $T(s)$, the transfer function from r to y , so that it approximates a standard second-order system: The ideal $T(s)$ is

$$T_{\text{id}}(s) := \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

A settling time of 8 s requires

$$\frac{4.6}{\zeta\omega_n} \approx 8$$

and an overshoot of 10% requires

$$\exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) = 0.1.$$

The solutions are $\zeta = 0.5912$ and $\omega_n = 0.9583$. Let's round to $\zeta = 0.6$ and $\omega_n = 1$. So the ideal $T(s)$ is

$$T_{\text{id}}(s) = \frac{1}{s^2 + 1.2s + 1}.$$

Then the ideal sensitivity function is

$$S_{\text{id}}(s) := 1 - T_{\text{id}}(s) = \frac{s(s + 1.2)}{s^2 + 1.2s + 1}.$$

Now take the weighting function $W_1(s)$ to be $S_{\text{id}}(s)^{-1}$, that is,

$$W_1(s) = \frac{s^2 + 1.2s + 1}{s(s + 1.2)}.$$

The rationale for this choice is a rough argument that goes as follows. Consider Step 2 of the procedure in the preceding section; from it the function

$$F := W_1 M(Y - NQ_{\text{im}})$$

equals a constant times an all-pass function. The procedure then rolls off Q_{im} to result in the weighted sensitivity function

$$W_1 S := W_1 M(Y - NQ_{\text{im}}J).$$

So $W_1 S \approx F$ except at high frequency, that is,

$$S \approx F S_{\text{id}}.$$

Now F behaves approximately like a time delay except at high frequency (this is a property of all-pass functions). So we arrive at the rough approximation

$$S \approx (\text{time delay}) \times S_{\text{id}}.$$

Hence our design should produce

$$\text{actual step response} \approx \text{delayed ideal step response.}$$

One further adjustment is required in the problem setup: W_1 must be stable and strictly proper, so the function above is modified to

$$W_1(s) = \frac{s^2 + 1.2s + 1}{(s + 0.001)(s + 1.2)(0.001s + 1)}.$$

The procedure can now be applied.

Step 1 Since $P \in \mathcal{S}$ we take $N = P, M = 1, X = 0, Y = 1$.

Step 2 The model-matching problem is to minimize

$$\|W_1 M(Y - NQ_{\text{im}})\|_{\infty} = \|W_1(1 - PQ_{\text{im}})\|_{\infty}.$$

Since P has only one right half-plane zero, at $s = 5.5308$, we have from Section 9.1

$$\min \|W_1(1 - PQ_{\text{im}})\|_{\infty} = |W_1(5.5308)| = 1.0210.$$

Thus the spec $\|W_1 S\|_{\infty} < 1$ is not achievable for this P and W_1 . Let us scale W_1 as

$$W_1 \leftarrow \frac{0.9}{1.0210} W_1.$$

Then $|W_1(5.5308)| = 0.9$ and the optimal Q_{im} is

$$Q_{\text{im}} = \frac{W_1 - 0.9}{W_1 P},$$

that is,

$$Q_{\text{im}}(s) = \frac{s(0.0008s^5 + 0.0221s^4 + 0.1768s^3 + 0.7007s^2 + 3.8910s + 0.0026)}{s^3 + 6.1081s^2 + 6.8897s + 4.9801}.$$

Step 3 Set

$$J(s) := \frac{1}{(\tau s + 1)^3}.$$

Compute $\|W_1(1 - PQ_{\text{im}}J)\|_{\infty}$ for decreasing values of τ until the norm is < 1 :

τ	∞ -Norm
0.1	1.12
0.05	1.01
0.04	0.988

Take $\tau = 0.04$.

Step 4 $Q = Q_{\text{im}}J$

Step 5 $C = Q/(1 - PQ)$

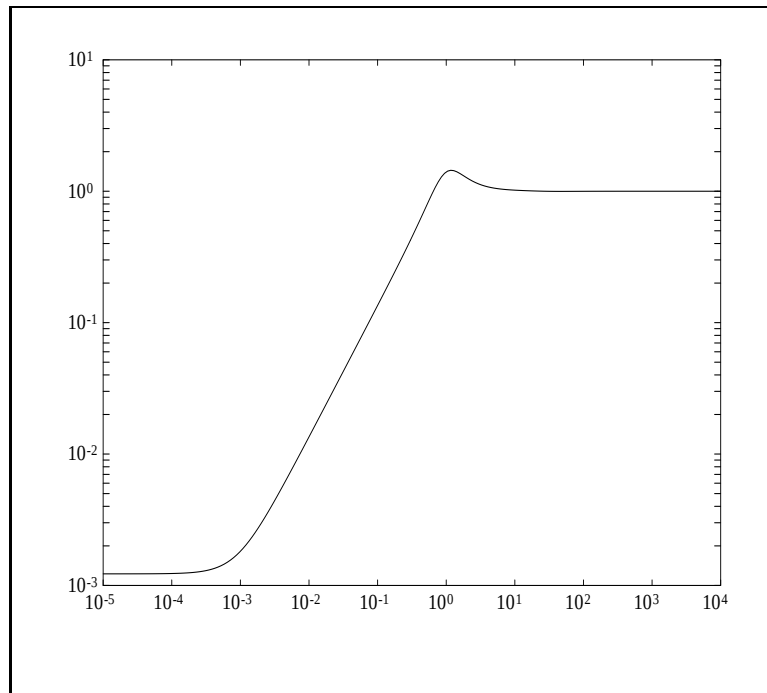
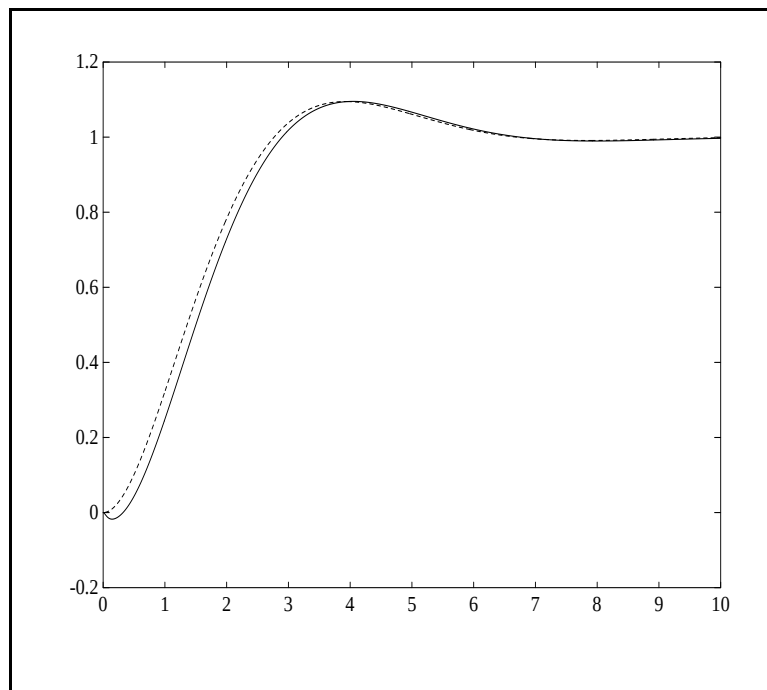
Figure 10.2: Bode plot of $|S|$.

Figure 10.3: Actual (solid) and ideal (dash) step responses.

A Bode magnitude plot of the resulting sensitivity function is shown in Figure 10.2. Figure 10.3 shows the step response of the plant output together with the ideal step response (i.e., that of T_{id}). The performance specs are met.

The design above, while achieving the step response goal, may not be satisfactory for other reasons; for example, internal signals may be too large (in fact, for this design the input to the power amplifier would saturate during the step response test). The problem is that we have placed no limits on controller gain or bandwidth. We return to this example and see how to correct this deficiency in Chapter 12.

Another disadvantage of the design above is that C is of high order. The common way to alleviate this is to approximate Q by a lower-order transfer function (it is difficult to reduce C directly because of the internal stability constraint, whereas Q only has to be stable and proper).

10.4 2-Norm Minimization

The subject of this section is minimization of the 2-norm of some designated closed-loop transfer function. As a typical case, we consider the transfer function from d to y , namely, PS . Motivation for considering the 2-norm was presented in Section 2.3. For example, if $d(t)$ is the unit impulse, then the 2-norm of $y(t)$ equals $\|PS\|_2$. Or, $\|PS\|_2$ equals the 2-norm to ∞ -norm gain from d to y .

To make the problem more flexible, bring in a weighting function W —appropriate assumptions on W will be introduced as required. The problem for consideration is: Given P and W , design an internally stabilizing C to minimize $\|WPS\|_2$. The method of solution is again to parametrize all internally stabilizing C s and then to select an optimal parameter, but now optimal for the 2-norm.

We need some preliminaries. First, let $\mathcal{S}_0$ denote the subset of $\mathcal{S}$ of all strictly proper stable transfer functions, $\mathcal{S}_0^\perp$ the set of strictly proper transfer functions analytic in $\text{Re } s \leq 0$, and $\mathcal{S}_0 + \mathcal{S}_0^\perp$ the set of all sums. Thus $\mathcal{S}_0 + \mathcal{S}_0^\perp$ consists precisely of all strictly proper transfer functions with no poles on the imaginary axis. By Lemma 2.1 the 2-norm is finite for all functions in $\mathcal{S}_0 + \mathcal{S}_0^\perp$. Furthermore, every function F in $\mathcal{S}_0 + \mathcal{S}_0^\perp$ can be uniquely expressed as

$$F = F_{\text{st}} + F_{\text{un}}, \quad F_{\text{st}} \in \mathcal{S}_0, F_{\text{un}} \in \mathcal{S}_0^\perp,$$

for example, by partial fraction expansion (F_{st} is stable, F_{un} is unstable).

Pythagoras's theorem holds in this setting.

Lemma 3 *If $F \in \mathcal{S}_0$ and $G \in \mathcal{S}_0^\perp$, then*

$$\|F + G\|_2^2 = \|F\|_2^2 + \|G\|_2^2.$$

Proof

$$\begin{aligned} \|F + G\|_2^2 &= \frac{1}{2\pi} \int |F(j\omega) + G(j\omega)|^2 d\omega \\ &= \|F\|_2^2 + \|G\|_2^2 + 2\text{Re} \left[\frac{1}{2\pi} \int \overline{F(j\omega)} G(j\omega) d\omega \right]. \end{aligned}$$

So it suffices to show the last integral equals zero. Convert it into a contour integral by closing the imaginary axis with an infinite-radius semicircle in the left half-plane:

$$\frac{1}{2\pi} \int \overline{F(j\omega)} G(j\omega) d\omega = \frac{1}{2\pi j} \oint F(-s) G(s) ds.$$

But the right-hand side equals zero by Cauchy's theorem. ■

For the usual representations

$$\begin{aligned} P &= \frac{N}{M}, \quad NX + MY = 1, \\ C &= \frac{X + MQ}{Y - NQ}, \quad Q \in \mathcal{S}, \end{aligned}$$

we have

$$WPS = WNY - WN^2Q.$$

So the problem reduces to: Obtain Q in $\mathcal{S}$ to minimize

$$\|WNY - WN^2Q\|_2.$$

Evidently, this is a 2-norm model-matching problem. It is much easier than the ∞ -norm version.

Let us now assume that $W \in \mathcal{S}$. The idea is to factor $U := WN^2$ as $U = U_{ap}U_{mp}$. So that the minimum-phase factor U_{mp} has a stable inverse (so that we can back-solve for Q), assume in addition that W and P have no zeros on the imaginary axis. Now we shall use the fact that U_{ap} has unit magnitude on the imaginary axis as follows. For fixed Q in $\mathcal{S}$ we have

$$\begin{aligned} \|WNY - WN^2Q\|_2^2 &= \|WNY - U_{ap}U_{mp}Q\|_2^2 \\ &= \|U_{ap}(U_{ap}^{-1}WNY - U_{mp}Q)\|_2^2 \\ &= \|U_{ap}^{-1}WNY - U_{mp}Q\|_2^2 \\ &= \|(U_{ap}^{-1}WNY)_{\text{un}} + (U_{ap}^{-1}WNY)_{\text{st}} - U_{mp}Q\|_2^2 \\ &= \|(U_{ap}^{-1}WNY)_{\text{un}}\|_2^2 + \|(U_{ap}^{-1}WNY)_{\text{st}} - U_{mp}Q\|_2^2. \end{aligned}$$

Lemma 3 was used in the last equality. It is now clear that the unique optimal, generally improper Q is

$$Q_{\text{im}} = U_{mp}^{-1}(U_{ap}^{-1}WNY)_{\text{st}} = (WN^2)_{mp}^{-1} [(WN^2)_{ap}^{-1}WNY]_{\text{st}}$$

and the consequent minimum value of $\|WNY - WN^2Q\|_2$ equals $\|(U_{ap}^{-1}WNY)_{\text{un}}\|_2$. This Q_{im} must be rolled off at high frequency to get a proper suboptimal Q , just as in Section 10.1; the details are routine and are therefore omitted.

Example In this example the 2-norm of the plant output y is minimized for a unit step disturbance d . Thus $W(s) = 1/s$. A modification of the derivation above is required because this W is unstable. Take

$$P(s) = \frac{1-s}{s^2+s+2}.$$

This being stable, we should take

$$C = \frac{Q}{1-PQ}, \quad Q \in \mathcal{S}.$$

Temporarily relax the requirement that Q be proper. The function whose 2-norm is to be minimized is

$$WPS = WP(1-PQ). \tag{10.6}$$

For this to have finite 2-norm we must guarantee that $P(1 - PQ)$ has a zero at $s = 0$ to cancel the pole of W ; this requires $1 - PQ$ to have a zero at $s = 0$, that is, $Q(0) = 1/P(0) = 2$. The set of all stable Q s satisfying $Q(0) = 2$ is

$$Q(s) = 2 + sQ_1(s), \quad Q_1 \text{ stable.}$$

Substitute this into (10.6) to get

$$WPS = T - UQ_1,$$

where

$$\begin{aligned} T(s) &:= W(s)P(s)[1 - 2P(s)] \\ &= \frac{(1-s)(s+3)}{(s^2+s+2)^2}, \\ U(s) &:= P(s)^2 \\ &= \frac{(1-s)^2}{(s^2+s+2)^2}. \end{aligned}$$

Then as above, the optimal improper Q_1 equals $U_{mp}^{-1}(U_{ap}^{-1}T)_{st}$, that is,

$$Q_{1im}(s) = \frac{(s^2+s+2)^2}{(s+1)^2} \left[\frac{(s+1)^2(1-s)(s+3)}{1-s(s^2+s+2)^2} \right]_{st} = \frac{s^3+2s^2+2s-1}{(s+1)^2}.$$

To get a proper Q , this should be made strictly proper, so set

$$Q_1(s) = \frac{s^3+2s^2+2s-1}{(\tau s+1)^2(s+1)^2}.$$

Then

$$Q(s) = 2 + s \frac{s^3+2s^2+2s-1}{(\tau s+1)^2(s+1)^2}.$$

As $\tau \rightarrow 0$, this Q recovers optimality of $\|WPS\|_2$.

Exercises

1. Design a controller to achieve $\|W_1S\|_\infty < 1$ for

$$\begin{aligned} P(s) &= \frac{s-1}{(s+1)^2}, \\ W_1(s) &= 0.62 \frac{s^2+1.2s+1}{(s+0.001)(s+1.2)(0.001s+1)}. \end{aligned}$$

Plot the resulting step response of the plant output.

2. Take

$$P(s) = \frac{(s-1)(s-2)}{(s+1)^3}, \quad W_1(s) = \frac{a}{100s+1}.$$

Design a controller to achieve $\|W_1S\|_\infty < 1$. You'll have to adjust the parameter a so that the spec is achievable. Sketch Bode plots of $|S|$, $|W_1|$, and $|CS|$.

3. Repeat with

$$P(s) = \frac{s-1}{(s-2)(s+1)}.$$

4. This problem looks at performance design with P minimum phase but having a pole on the imaginary axis. Take

$$P(s) = \frac{1}{s}, \quad W_1(s) = \frac{100}{s+1}.$$

- (a) Perturb P to $P(s) = 1/(s+\epsilon)$, $\epsilon > 0$. Find a controller C (internally stabilizing) so that $\|W_1 S\|_\infty < 1$. Let ϵ go to zero in the coefficients of C . Does the resulting C solve the performance design problem for the original P ?
- (b) Factor P as

$$P = P_1 P_2, \quad P_1(s) = \frac{1}{s+1}, \quad P_2(s) = \frac{s+1}{s}.$$

Solve the performance design problem for P_1 ; let C_1 be the solution. Set $C = C_1/P_2$. Does the resulting C solve the performance design problem for the original P ? If so, explain why.

5. Take

$$P(s) = \frac{s-1}{(s-2)(s+1)}.$$

Let S and T denote the sensitivity and complementary sensitivity functions. Prove that

$$\inf \|S\|_\infty = \inf \|T\|_\infty, \quad (10.7)$$

where both infima are over all proper controllers which internally stabilize the feedback system. Hint to save you some work: The result can be proved without actually computing a coprime factorization of P .

6. Take $P(s) = 1/s^2$. By the method in the proof of Lemma 2, design an internally stabilizing C such that

$$\begin{aligned} |S(j\omega)| &< 0.05, \quad \forall \omega \leq 1, \\ \|S\|_\infty &< 1.4. \end{aligned}$$

7. Prove that if G is stable and strictly proper, then

$$\lim_{\tau \rightarrow 0} \|G(1-J)\|_2 = 0.$$

8. Take

$$P(s) = \frac{10}{(s-1)(s+2)}.$$

Compute an internally stabilizing C to minimize the 2-norm of the tracking error $y-r$ for r the unit step.

Notes and References

The material in Section 10.1 is drawn from Zames (1981), Bensoussan (1984), Francis and Zames (1984), and Francis (1987). In the literature the problem of minimizing $\|W_1 S\|_\infty$ is called the *weighted sensitivity problem*. Lemma 2 is from Bensoussan (1984) for P stable and Francis (1987) for the general case. Section 10.2 is based on Zames and Francis (1983), Francis and Zames (1984), and Khargonekar and Tannenbaum (1985).

There is a way to design for performance which yields a proper Q directly, but it involves boundary NP interpolation.

The technique in Section 10.4, 2-norm minimization, is also known as *minimum variance control*: If d is a zero-mean stationary random signal with power spectral density $W(-s)W(s)$, the variance of y equals exactly $\|WPS\|_2^2$. A good treatment of this subject is given in Morari and Zafiriou (1989).