

6.241: Dynamic Systems—Fall 2003

RECITATION 7

Input-Output Stability

In this recitation, we review the ∞ -norm, 1-norm, and 2-norm on the vector spaces $\mathcal{L}_p^n(\mathbb{R})$, where $p = \infty, 1$, or 2 respectively (or ℓ_p^n for discrete time signals) discussed in lecture. The Fourier Transform is defined and a proof for Parseval's relation is presented.

$\mathcal{L}_p$ Spaces

A dynamic system could be thought of as an operator that acts on a set of signals, just as a $m \times n$ matrix is an operator that acts on the vector space $\mathbb{R}^n$. We are interested in considering the effect of our dynamic system on particular sets of signals, specifically, vector spaces whose elements are certain signals and on which we can define a notion of the size of a signal (i.e. a norm). Once we have defined our vector space and norm, we can define a notion of size or “maximum amplification” of the dynamic system, just as we had done for matrices (recall that the induced norm was a measure of maximum amplification of a matrix). The size or induced norm of the system will depend on the norm we pick, so we quickly review signal norms below.

Consider the following sets of signals:

	Continuous Time $\mathcal{L}_p^n(\mathbb{R})$	Discrete Time $\ell_p^n(\mathbb{Z})$
$p = \infty$	$\{f : \mathbb{R} \rightarrow \mathbb{R}^n \mid \sup_t \ f(t)\ _\infty < \infty\}$	$\{f : \mathbb{Z} \rightarrow \mathbb{R}^n \mid \sup_k \ f(k)\ _\infty < \infty\}$
$p = 1$	$\{f : \mathbb{R} \rightarrow \mathbb{R}^n \mid \int_{\mathbb{R}} \ f(t)\ _1 dt < \infty\}$	$\{f : \mathbb{Z} \rightarrow \mathbb{R}^n \mid \sum_{k \in \mathbb{Z}} \ f(k)\ _1 < \infty\}$
$p = 2$	$\{f : \mathbb{R} \rightarrow \mathbb{R}^n \mid \int_{\mathbb{R}} \ f(t)\ _2^2 dt < \infty\}$	$\{f : \mathbb{Z} \rightarrow \mathbb{R}^n \mid \sum_{k \in \mathbb{Z}} \ f(k)\ _2^2 < \infty\}$

Check that each of these is a vector space. That is, if $f, g \in \mathcal{L}_p$, then $f+g \in \mathcal{L}_p$ and $\alpha f \in \mathcal{L}_p$, where α is any scalar (note that this latter condition includes the condition that the zero vector must belong to the vector space; just let α equal zero). Furthermore, you can verify that the following are valid norms for each of the above spaces:

		Continuous Time $\mathcal{L}_p^n(\mathbb{R})$	Discrete Time $\ell_p^n(\mathbb{Z})$
Peak Magnitude	$p = \infty$	$\sup_t \ f(t)\ _\infty$	$\sup_k \ f(k)\ _\infty$
Action	$p = 1$	$\int_{\mathbb{R}} \ f(t)\ _1 dt$	$\sum_{k \in \mathbb{Z}} \ f(k)\ _1$
Energy	$p = 2$	$[\int_{\mathbb{R}} \ f(t)\ _2^2 dt]^{\frac{1}{2}}$	$[\sum_{k \in \mathbb{Z}} \ f(k)\ _2^2]^{\frac{1}{2}}$

Finally, we note that although the power of a signal, $P_f = \lim_{T \rightarrow \infty} (\frac{1}{2T} \int_{-T}^T f(\tau)^T f(\tau) d\tau)$ (CT) and $P_f = \lim_{N \rightarrow \infty} (\frac{1}{2N} \sum_{k=-(N-1)}^{N-1} f(k)^T f(k))$ (DT), is not a norm, because a nonzero signal can have zero power, power is a useful signal measure.

Example 1: Consider the following continuous time signal

$$w(t) = \begin{cases} 1 & \text{if } t \in [2^{2k}, 2^{2k+1}], \text{ for } k = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

By inspection, we have that $\|w\|_\infty = 1$ and $\|w\|_1 = \|w\|_2 = \infty$. Furthermore, P_w does not exist. This can be concluded as follows. First note that power is the limit of $f(T)$ as $T \rightarrow \infty$ where $f(T) = (\frac{1}{2T} \int_{-T}^T w(\tau)^T w(\tau) d\tau)$. On one hand, for $f(2^{2N+1})$,

$$f(2^{2N+1}) = \frac{1}{2 \cdot 2^{2N+1}} \sum_{k=1}^N 2^{2k+1} - 2^{2k} = \frac{1}{4} \sum_{k=1}^N 4^{k-N} \rightarrow \frac{1}{4} \frac{1}{1 - \frac{1}{4}} = \frac{1}{3} \text{ as } N \rightarrow \infty$$

And on the otherhand,

$$f(2^{2(N+1)}) = \frac{1}{2} f(2^{2N+1}) \rightarrow \frac{1}{6} \text{ as } N \rightarrow \infty$$

That is, the power of the signal does not exist, because the integral over a length $2T$ oscillates between 2 fixed values as $T \rightarrow \infty$.

Example 2: Prove that for discrete time signals, $w(k)$ (for simplicity consider scalar-values signals, i.e., $w(k) : \mathbb{Z} \rightarrow \mathbb{R}$.) that are bounded in the p norm are bounded in the $p+q$ norm for $q > 0$.

We are given that $w \in \ell_p$ and we want to show that $w \in \ell_{p+q}$. That is, we know that $\sum_k |w(k)|^p < \infty$ and we want to show that $\sum_k |w(k)|^{p+q} < \infty$. Let's split the domain of $w(k)$ into two sets, $K_0 = \{k \in \mathbb{Z} | |w(k)| < 1\}$ and $K_1 = \{k \in \mathbb{Z} | |w(k)| \geq 1\}$. So, $\sum_k |w(k)|^{p+q} = \sum_{k \in K_0} |w(k)|^{p+q} + \sum_{k \in K_1} |w(k)|^{p+q} < \infty$, from which we can conclude that $\sum_{k \in K_0} |w(k)|^{p+q} < \infty$ and $\sum_{k \in K_1} |w(k)|^{p+q} < \infty$. The latter inequality implies that signals in ℓ_p must have a finite number of elements $w(k)$ with magnitude greater than or equal to 1. This in turn implies that $\sum_{k \in K_1} |w(k)|^{p+q} < \infty$. Now recall that $\|w\|_{p+q}^{p+q} = \sum_{k \in K_0} |w(k)|^{p+q} + \sum_{k \in K_1} |w(k)|^{p+q}$, so it remains to show that $\sum_{k \in K_0} |w(k)|^{p+q} < \infty$. Recall that for $k \in K_0$, $|w(k)| < 1$. Thus, $|w(k)|^{p+q} < |w(k)|^p$. This implies that $\sum_{k \in K_0} |w(k)|^{p+q} < \sum_{k \in K_0} |w(k)|^p < \infty$, which is what we want.

Example 3: In general, for continuous time signals, the $\mathcal{L}_p$ spaces are different spaces. For example, the signal $f(t) = t^{-3/4}$ on $t \in (0, 1]$ and zero elsewhere is in $\mathcal{L}_1(\mathbb{R})$ but not in $\mathcal{L}_2(\mathbb{R})$. Conversely, the signal $f(t) = t^{-3/4}$ on $t \in [1, \infty)$ and zero elsewhere is in $\mathcal{L}_2(\mathbb{R})$ but not in $\mathcal{L}_1(\mathbb{R})$.

System Norms

Now that we have defined some measures of signals, we can define the notion of an input-output stable system H . A system H (not necessarily finite dimensional) is *p-stable* if there exists a constant $C_p \in \mathbb{R}$ such that

$$\|y\|_p \leq C_p \|u\|_p.$$

The smallest such constant C_p is called the *induced norm* of the system H . This is completely analogous to the induced-norm of a matrix. Recall that

$$\|A\|_{p-ind} = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

The induced norm of a system is

$$\|H\|_{p-ind} = \sup_{u \neq 0} \frac{\|y\|}{\|u\|} = \sup_{u \neq 0} \frac{\|H(u)\|}{\|u\|}.$$

Fourier Transform

In general, we can define the Fourier integral for vectors of signals $f(t) = [f_1(t), f_2(t), \dots, f_n(t)]^T$. However, for better insight, we shall assume in the discussion that f is a scalar signal. Extensions to signal vectors can be readily made.

Fourier Integral. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be an integrable function (i.e. $f \in \mathcal{L}_1^1$). The Fourier transform of f is the function:

$$F(j\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt.$$

This function is well-defined because we assumed that $f \in \mathcal{L}_1$

$$|F(j\omega)| = \left| \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt \right| \leq \int_{-\infty}^{+\infty} |f(t)| dt < \infty.$$

In general, we expect that $F(w)$ may not be defined in the above sense for elements of $\mathcal{L}_2$ since these do not necessarily belong to $\mathcal{L}_1$. Still, it can be shown that there is a unique extension of the fourier transform from the set of functions on which it is well-defined to $\mathcal{L}_2$. Alternatively, the Fourier integral can be appropriately defined as for functions $f \in \mathcal{L}_2$,

$$\lim_{M \rightarrow \infty} \int_{-M}^M f(t) e^{-j\omega t} dt.$$

which can be shown to exist in the sense of $\mathcal{L}_2$ and is equal to $F(j\omega)$ (see [1]).

Finally, it can be shown that the mapping $f \rightarrow F$ is bijective. That is, the inverse of the fourier transform exists and is unique. Moreover,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega.$$

Parseval's Relation. If $f(t)$ and $F(j\omega)$ are a Fourier transform pair, then, $\|F\|_2^2 = 2\pi\|f\|_2^2$, i.e.,

$$\int_{-\infty}^{+\infty} \|f(t)\|_2^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|F(j\omega)\|_2^2 d\omega.$$

Proof. Taking complex conjugates transposes of both sides of

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega,$$

we obtain,

$$(f(t))' = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (F(j\omega))' e^{-j\omega t} d\omega.$$

Now,

$$\begin{aligned} \int_{-\infty}^{+\infty} \|f(t)\|_2^2 dt &= \int_{-\infty}^{+\infty} (f(t))' f(t) dt \\ &= \int_{-\infty}^{+\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} (F(j\omega))' e^{-j\omega t} d\omega \right] f(t) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} (F(j\omega))' \left[\int_{-\infty}^{+\infty} e^{-j\omega t} f(t) dt \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} (F(j\omega))' F(j\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|F(j\omega)\|_2^2 d\omega \end{aligned}$$

Discrete Time Fourier Transform. Finally, the fourier transform for a discrete time signal, $f[n]$, is defined as

$$F(e^{j\omega}) = \sum_{-\infty}^{+\infty} f[n] e^{-j\omega n},$$

and,

$$f[n] = \frac{1}{2\pi} \int_{2\pi} F(e^{j\omega}) e^{j\omega n} d\omega.$$

Parseval's relation for DT signals is similar to that for CT signals:

$$\sum_{-\infty}^{+\infty} \|f[n]\|_2^2 = \frac{1}{2\pi} \int_{2\pi} \|F(e^{j\omega})\|_2^2 d\omega.$$

References

- [1] Adams, M., and Guillemin, V. "Measure Theory and Probability," *Birkhauser*: 1996.
- [2] Oppenheim, A., Willsky, A., and Nawab, H. "Signals and Systems," *Prentice Hall*: 1997.