

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Department of Electrical Engineering and Computer Science

**6.241: Dynamic Systems—Fall 2003**

MIDTERM EXAM SOLUTIONS

**Problem 1** (i)  $\|Ax - b\|^2 = 3x^2 - 8x + 6$ , a quadratic with imaginary roots (hence positive minimum). Differentiating,  $\frac{d}{dx}(3x^2 - 8x + 6) = 6x - 8$ . Set equal to zero and solve for  $x$ ,  $x = 4/3$ .

(ii)  $\begin{bmatrix} \Delta^* & e^* \end{bmatrix} = \operatorname{argmin}\{\| \begin{bmatrix} \Delta & e \end{bmatrix} \|_F \mid \begin{bmatrix} A & -b \end{bmatrix} + \begin{bmatrix} \Delta & e \end{bmatrix} \text{ is singular}\}; \| \begin{bmatrix} \Delta^* & e^* \end{bmatrix} \|_F = \sigma_2(\begin{bmatrix} A & -b \end{bmatrix})$ .

(iii)  $x(t) = e^{At}x_o$ , where  $e^{At} = I + At + A^2t^2/2 + A^3t^3/6$  since  $A^4, A^5, \dots$  are zero. Now,

$$At = \begin{bmatrix} 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^2t^2/2 = \begin{bmatrix} 0 & 0 & t^2/2 & 0 \\ 0 & 0 & 0 & t^2/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and } A^3t^3/6 = \begin{bmatrix} 0 & 0 & 0 & t^3/6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad \text{So,}$$

$$x(t) = \begin{bmatrix} 1 + t + t^2/2 + t^3/6 \\ 1 + t + t^2/2 \\ 1 + t \\ 1 \end{bmatrix}.$$

(iv)  $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ ,  $B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$ ,  $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$ , and  $D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$ , where  $A_1 = 1$ ,  $B_1 = 1$ ,  $C_1 = 1$ , and  $D_1 = 0$  from  $\frac{1}{s-1}$  and  $A_2 = -2$ ,  $B_2 = -2$ ,  $C_2 = 1$ , and  $D_2 = 1$  from  $\frac{s}{s+2} = 1 - \frac{2}{s+2}$ .

(v) Note that any  $(x_1, 0)$  is an equilibrium point. Since there are no isolated equilibrium points, the system is not asymptotically stable. However, it is stable i.s.L., since  $V = x_1^2 + x_2^2$  is a Lyapunov function for which  $\dot{V} = -8x_2^2(x_1^2 + x_2^2)$  is negative semidefinite.

**Problem 2** (i) The system becomes unstable when the eigenvalues of  $A + \Delta$  cross the imaginary axis, i.e.  $\det(jw_o I - (A + \Delta)) = 0$  for some  $w_o \in \mathbb{R}$ . Now, the  $\Delta$  with the smallest 2-induced norm that results in  $\det((jw_o I - A) - \Delta) = 0$  has size greater or equal to  $\sigma_{\min}(jw_o I - A)$  for a particular  $w_o$ . To find the smallest, we minimize over all  $w \in \mathbb{R}$ , so,  $\min_{\Delta}\{\|\Delta\|_2 \mid \text{system is unstable}\} = \min_{\Delta, w}\{\|\Delta\|_2 \mid jwI - (A + \Delta) \text{ is singular}\} \geq \min_w \sigma_{\min}(jwI - A)$  (\*). Now, since  $A$  is real and symmetric,  $A = U\Lambda U'$  where  $\Lambda$  is a diagonal matrix with the eigenvalues of  $A$  on the diagonal, and  $U$  is orthonormal. So,  $jwI - A = U(jwI - \Lambda)U'$ , and  $(\sigma(jwI - A))^2 = \operatorname{eig}((jwI - A)(jwI - A)') = \operatorname{eig}(U(jwI - \Lambda)(jwI - \Lambda)'U') = \operatorname{eig}(U(jwI - \Lambda)(-jwI - \Lambda)U') = \text{the diagonal elements of } w^2 I + \Lambda^2$ . So,  $\min_w \sigma_{\min}(jwI - A) = \min_w (w^2 + \lambda_{\min}^2(A))^{\frac{1}{2}} = |\lambda_{\min}(A)|$ . Finally, we note that the inequality in (\*) is actually an equality, i.e.,  $\min_{\Delta}\{\|\Delta\|_2 \mid \text{system is unstable}\} = |\lambda_{\min}(A)|$  since we can find a real  $\Delta$  for which the minimum is achieved, namely,  $\Delta = -\sigma_n(A)u_n v_n'$  where  $n$  corresponds to the index of the smallest singular value of  $A$  and  $u_n$  and  $v_n$  come from the svd of  $A$  and are real since  $A$  is symmetric and real.

(ii) First, since  $Q$  is positive definite,  $Q = F'F$ , and  $F$  is invertible. Let  $z = Fx$ , we have  $\max_x \{x'Px \mid x'Qx \leq 1\} = \max_z \{z'(F^{-1})'PF^{-1}z \mid z'z \leq 1\} = \max_z \{z'(F^{-1})'PF^{-1}z \mid z'z = 1\} = \lambda_{\max}((F^{-1})'PF^{-1}) = \lambda_{\max}(F^{-1}(F^{-1})'P) = \lambda_{\max}(Q^{-1}P)$ , where the third equality follows because  $(F^{-1})'PF^{-1}$  is symmetric and the fourth equality is due to the fact that the eigenvalues of  $(F^{-1})'PF^{-1}$  are the same as the eigenvalues of  $F^{-1}(F^{-1})'P$ :  $(F^{-1})'PF^{-1}x = \lambda x \iff F^{-1}(F^{-1})'P(F^{-1}x) = \lambda(F^{-1}x)$ . In general, we can relate the eigenvalues of  $AB$  to the eigenvalues of  $BA$  by the following claim.

Suppose  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times m}$  with  $m \geq n$ ,  $AB$  has  $m$  eigenvalues,  $BA$  has  $n$  eigenvalues, and  $m$  of the eigenvalues of  $AB$  are those of  $BA$ , including multiplicity, and the remaining eigenvalues of  $AB$  are at zero. The proof of the statement is similar the proof of  $\det(I - AB) = \det(I - BA)$ , but has a small modification, thanks to which the statement immediately follows. Consider

$$\begin{bmatrix} I & A \\ B & \lambda I \end{bmatrix} \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ B & \lambda I - BA \end{bmatrix}$$

and

$$\begin{bmatrix} I & A \\ B & \lambda I \end{bmatrix} \begin{bmatrix} \lambda I & 0 \\ -B & I \end{bmatrix} = \begin{bmatrix} \lambda I - AB & A \\ 0 & \lambda I \end{bmatrix},$$

and let  $P = \begin{bmatrix} I & A \\ B & \lambda I \end{bmatrix}$ , then  $\det(P) = \det(\lambda I - BA)$ ,  $\lambda^m \det(P) = \lambda^n \det(\lambda I - AB)$ , so  $\det(\lambda I - AB) = \lambda^{m-n} \det(\lambda I - BA)$ .

(iii) The statement is false, let  $r \geq 2$ ,  $C = \begin{bmatrix} I_r & 0 \end{bmatrix}$  and  $A = \begin{bmatrix} J_r & 0 \\ 0 & I_{n-r} \end{bmatrix}$ , where  $J_r =$

$$\begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \text{ and } J_r \in \mathbb{R}^{r \times r}.$$

**Problem 3** (a)(i)  $vw'$  is  $n \times n$  matrix that has rank 1, so,  $n - 1$  of the eigenvalues are at the origin. For stability, we require that the remaining eigenvalue is inside the unit disk. Now,  $\det(\lambda I - vw') = \lambda^{n-1} \det(\lambda - w'v) = \lambda^{n-1}(\lambda - w'v) = 0$  implies  $\lambda = w'v$ . So, the condition for stability is  $|w'v| < 1$ .

(ii)  $y_k = (w'v)^{k-1} c'vw'x_o$ , for  $k \geq 1$  and  $y_o = c'x_o$ .

(iii) First, we note that  $v(w + \delta)'$  is rank 1, so any  $\delta$  will result in  $n - 1$  eigenvalues at the origin. Furthermore, since we may assume  $v(w + \delta)'$  is real, the remaining eigenvalue,  $(w + \delta)'v$ , must be real (if a real matrix has a complex eigenvalue, then the complex conjugate is also an eigenvalue). So, we seek to find the smallest  $\delta$  such that  $(w + \delta)'v = 1$  or  $(w + \delta)'v = -1$ , i.e., find the  $\delta$  for either case and take the smaller of the two. That is,  $\min_{\delta} \{\|\delta\|_2 \mid (w + \delta)'v \in \{1, -1\}\} = \min\{\min\{\|\delta\|_2 \mid (w + \delta)'v = 1\}, \min\{\|\delta\|_2 \mid (w + \delta)'v = -1\}\}$ . Note, we need to find the solutions for two underdetermined systems of equations,  $\min\{\min\{\|\delta\|_2 \mid (w + \delta)'v = 1\}, \min\{\|\delta\|_2 \mid (w + \delta)'v = -1\}\} = \min\{\min\{\|\delta\|_2 \mid v'\delta = 1 - w'v\}, \min\{\|\delta\|_2 \mid v'\delta = -1 - w'v\}\} = \min\{(1 - w'v)(v/\|v\|^2), (-1 - w'v)(v/\|v\|^2)\} = \min\{(1 - w'v), (-1 - w'v)\}(v/\|v\|^2)$ .

(b)(i)  $w'v = 1$  so the system is not asymptotically stable.

(ii) First, note that  $v$  is a right eigenvector of  $vw'$  and  $w/(w'v)$  is a left eigenvector and  $w'v$  is the eigenvalue. Since  $c'v = 0$  and  $w'b = 0$ ,  $v$  is a right eigenvector of  $vw' + \alpha bc'$  and  $w/(w'v)$  is a left eigenvector and  $w'v$  is the eigenvalue. Similarly,  $b$  is a right eigenvector of  $bc'$  and  $c/(c'b)$  is a left eigenvector and  $c'b$  is the eigenvalue. Furthermore,  $b$  is a right eigenvector of  $vw' + \alpha bc'$  and  $c/(c'b)$  is a left eigenvector and  $c'b$  is the eigenvalue. Thus,

$$vw' + \alpha bc' = \begin{bmatrix} v & b \end{bmatrix} \begin{bmatrix} w'v & \\ & \alpha c'b \end{bmatrix} \begin{bmatrix} w'/(w'v) \\ c'/(c'b) \end{bmatrix},$$

and

$$(vw' + \alpha bc')^n = \begin{bmatrix} v & b \end{bmatrix} \begin{bmatrix} (w'v)^n & \\ & (\alpha c'b)^n \end{bmatrix} \begin{bmatrix} w'/(w'v) \\ c'/(c'b) \end{bmatrix}.$$

So,  $\Phi(k, l) = \Phi(k - l) = v(w'v)^{k-l-1}w' + b(\alpha c'b)^{k-l-1}c'$  for  $k > l$  and  $\Phi(0) = I$ .

(iii) One way is to note that  $v$  and  $b$  are eigenvectors of  $vw' + \alpha bc'$  since  $c'v = 0$  and  $w'b = 0$ , and  $v$  is an eigenvector of  $vw'$ , and  $b$  is an eigenvector of  $bc'$ . That is,  $(vw' + \alpha bc')v = \lambda_1 v$  and  $(vw' + \alpha bc')b = \lambda_2 b$ , from which we have that  $\lambda_1 = w'v = 1$  is an eigenvalue of the system no matter what  $\alpha$  we use. Equivalently, you may have calculated  $\det(\lambda I - (vw' + \alpha bc')) = \lambda^2(\lambda - 1)(\lambda - \alpha)$ , from which it is clear that 1 is an eigenvalue of the system.