

## Exercises

**Exercise 14.1 Bounded Perturbation** Recall Example 14.3. In this problem we want to improve the lower bound on  $\gamma(A)$ .

- (a) To improve the lower bound, we use the information that if  $\Delta$  is real, then poles appear in complex conjugate pair. Define

$$A_w = \begin{pmatrix} A & wI \\ -wI & A \end{pmatrix}.$$

Show that

$$\gamma(A) \geq \min_{w \in \mathbb{R}} \sigma_{\min}[A_w].$$

- (b) If you think harder about your proof above, you will be able to further improve the lower bound. In fact, it follows that

$$\gamma(A) \geq \min_{w \in \mathbb{R}} \sigma_{2n-1}[A_w]$$

where  $\sigma_{2n-1}$  is the next to last singular value. Show this result.

**Exercise 15.1 Non-causal Systems** In this chapter, we only focused on causal operators, although the results derived were more general. As an example, consider a particular CT LTI system with a bi-lateral Laplace transform:

$$G(s) = \frac{s+2}{(s-2)(s+1)}.$$

- (a) Check the *p-stability* and *causality* of the system in the following cases:

(i) the ROC (Region of Convergence) is  $R_1 = \{s \in \mathbb{C} \mid \operatorname{Re}(s) < -1\}$  where  $\operatorname{Re}(s)$  denotes the real part of  $s$ ;

(ii) the ROC is  $R_2 = \{s \in \mathbb{C} \mid -1 < \operatorname{Re}(s) < 2\}$ ;

(iii) the ROC is  $R_3 = \{s \in \mathbb{C} \mid \operatorname{Re}(s) > 2\}$ .

- (b) In the cases where the system is not *p-stable* for  $p = 2$  and  $p = \infty$ , find a bounded input that makes the output unbounded, i.e., find an input  $u \in L_p$  that produces an output  $y \notin L_p$ , for  $p = 2, \infty$ .

**Exercise 15.2** In nonlinear systems,  $p$ -stability may be satisfied in only a local region around zero. In that case, a system will be locally  $p$ -stable if:

$$\|Gu\|_p \leq C\|u\|_p, \quad \text{for all } u \text{ with } \|u\|_p \leq \delta$$

Consider the system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ z &= Cx + Du \\ y &= g(z) \end{aligned}$$

Where  $g$  is a continuous function on  $[-T, T]$ . Which of the following systems is  $p$ -stable, locally  $p$ -stable or unstable for  $p \geq 1$ :

- (a)  $g(x) = \cos x$ .
- (b)  $g(x) = \sin x$ .
- (c)  $g(x) = \text{Sat}(x)$  where

$$\text{Sat}(x) = \begin{cases} x & |x| \leq 1 \\ 1 & |x| \geq 1 \end{cases}$$

**Exercise 16.1** We have shown in class that the  $\mathcal{H}_\infty$ -norm arises from measuring the largest I/O energy-to-energy amplification. In this problem, we want to show that the  $\mathcal{H}_\infty$  norm can also be seen as determining the largest power-to-power amplification. For that, consider a CT LTI system with transfer function  $H(s)$ , input  $u \in \mathbb{R}^n$  and output  $y \in \mathbb{R}^n$ . Recall that the  $\mathcal{H}_\infty$ -norm of the system, which we simply denote by  $\|H\|_\infty$ , is given by

$$\|H\|_\infty = \sup_w \sigma_{\max} [H(jw)].$$

Remember also that the power of a continuous-time signal  $v$  is given by

$$P_v = \lim_{L \rightarrow \infty} \left( \frac{1}{2L} \int_{-L}^L v'(t)v(t)dt \right)$$

Define the set  $X = \left\{ \sum_{i=1}^N u_i e^{jw_i t} : u_i \in \mathbb{R}^n, w_i \in \mathbb{R} \right\}$ . In this problem, we want to show that  $\sup_{P_u=1} P_y = \|H\|_\infty^2$ , at least for  $u \in X$  (although the result holds more generally for any  $u$  that has a well-defined *power spectrum*).

- (a) Let  $u \in X$ . Compute the power of  $u$ .
- (b) Compute the power of the system output  $y$ .
- (c) Now show that  $\sup_{P_u=1} P_y \leq \|H\|_\infty^2$ .
- (d) Show that equality can be achieved by picking the appropriate input  $u \in X$ .

**Exercise 16.2 (a)** Consider a DT LTI system (not necessarily finite-dimensional) specified by the I/O convolution relationship  $y(k) = h * u(k)$ , where the unit sample response  $h(k)$  has  $z$ -transform  $H(z)$ . Show that if the system is  $\infty$ -stable then it is 2-stable. (Hint: Show that  $\|h * u\|_2 \leq \|h\|_1 \|u\|_2$ .) Although the converse is true for finite-dimensional LTI systems, it is not true in general, as part (b) below shows.

**(b)** First show that if a sequence is  $\ell_1$  then its  $z$ -transform evaluated on the unit circle is continuous, i.e., if  $h \in \ell_1$  then  $H(e^{j\theta})$  is continuous in  $\theta$ . Now show that the system with transfer function

$$H(z) = e^{\left(\frac{1}{z-1}\right)}$$

is 2-stable but not  $\infty$ -stable. You have to show that  $H(z)$  is analytic outside the unit circle and bounded on the circle, but that it is not continuous on the circle.