

Lectures on Dynamic Systems and Control

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Chapter 7

State-Space Models

7.1 Introduction

A central question in dealing with a causal discrete-time (DT) system with input u , output y , is the following:

Given the input at some time n , i.e. given $u[n]$, how much information do we need about past inputs, i.e. about $u[k]$ for $k < n$, in order to determine the present output, namely $y[n]$?

The same question can be asked for continuous-time (CT) systems. This question addresses the issue of *memory* in the system. Why is this a central question? Some reasons:

- The answer gives us an idea of the complexity, or number of degrees of freedom, associated with the dynamic behavior of the system. The more information we need about past inputs in order to determine the present output, the richer the variety of possible output behaviors.
- In a control application, the answer to the above question suggests the required degree of complexity of the controller, because the controller has to remember enough about the past to determine the effects of present control actions on the response of the system.
- For a computer algorithm that acts causally on a data stream, the answer to the above question suggests how much memory will be needed to run the algorithm.

We now describe the general structure of *state-space models*, for which the preceding question has an immediate and transparent answer.

7.2 General Description

For a *causal* system with m inputs $u_j(t)$ and p outputs $y_i(t)$ (hence $m + p$ manifest variables), an n th-order state-space description is one that introduces n latent variables $x_\ell(t)$ called *state variables* in order to obtain a particular form for the constraints that define the model. Letting

$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix},$$

an n th-order state-space description takes the form

$$\dot{x}(t) = f(x(t), u(t), t) \quad (\text{state evolution equations}) \quad (7.1)$$

$$y(t) = g(x(t), u(t), t) \quad (\text{instantaneous output equations}) \quad (7.2)$$

To save writing the same equations over for both continuous and discrete time, we interpret

$$\dot{x}(t) = \frac{dx(t)}{dt}, \quad t \in \mathbb{R} \text{ or } \mathbb{R}^+$$

for CT systems, and

$$\dot{x}(t) = x(t+1), \quad t \in \mathbb{Z} \text{ or } \mathbb{Z}^+$$

for DT systems. We will only consider finite-order (or finite-dimensional, or *lumped*) state-space models, although there is also a rather well developed (but much more subtle and technical) theory of infinite-order (or infinite-dimensional, or *distributed*) state-space models.

DT Models

The key feature of a state-space description is the following property, which we shall refer to as the *state property*. Given the present state vector (or “state”) and present input at time t , we can compute: (i) the present output, using (7.2); and (ii) the next state using (7.1). It is easy to see that this puts us in a position to do the same thing at time $t + 1$, and therefore to continue the process over any time interval. Extending this argument, we can make the following claim:

State Property of DT state-Space Models

Given the initial state $x(t_0)$
and input $u(t)$ for $t_0 \leq t < t_f$
(with t_0 and t_f arbitrary),
we can compute the output $y(t)$ for $t_0 \leq t < t_f$
and the state $x(t)$ for $t_0 < t \leq t_f$.

Thus, the state at any time t_0 summarizes everything about the past that is relevant to the future. Keeping in mind this fact — that the state variables are the *memory variables* (or, in more physical situations, the *energy storage* variables) of a system — often guides us quickly to good choices of state variables in any given context.

CT Models

The same state property turns out to hold in the CT case, at least for $f(\cdot)$ that are well behaved enough for the state evolution equations to have a unique solution for all inputs of interest and over the entire time axis — these will typically be the only sorts of CT systems of interest to us. A demonstration of this claim, and an elucidation of the precise conditions under which it holds, would require an excursion into the theory of differential equations beyond what is appropriate for this course. We can make this result plausible, however, by considering the Taylor series approximation

$$x(t_0 + \epsilon) \approx x(t_0) + \left(\frac{dx(t)}{dt} \right)_{t=t_0} \epsilon \quad (7.3)$$

$$= x(t_0) + f(x(t_0), u(t_0), t_0) \epsilon \quad (7.4)$$

where the second equation results from applying the state evolution equation (7.1). This suggests that we can approximately compute $x(t_0 + \epsilon)$, given $x(t_0)$ and $u(t_0)$; the error in the approximation is of order ϵ^2 , and can therefore be made smaller by making ϵ smaller. For sufficiently well behaved $f(\cdot)$, we can similarly step forwards from $t_0 + \epsilon$ to $t_0 + 2\epsilon$, and so on, eventually arriving at the final time t_f , taking on the order of ϵ^{-1} steps in the process. The accumulated error at time t_f is then of order $\epsilon^{-1} \cdot \epsilon^2 = \epsilon$, and can be made arbitrarily small by making ϵ sufficiently small. Also note that, once the state at any time is determined and the input at that time is known, then the output at that time is immediately given by (7.2), even in the CT case.

The simple-minded Taylor series approximation in (7.4) corresponds to the crudest of numerical schemes — the “forward Euler” method — for integrating a system of equations of the form (7.1). Far more sophisticated schemes exist (e.g. Runge-Kutta methods, Adams-Gear schemes for “stiff” systems that exhibit widely differing time scales, etc.), but the forward Euler scheme suffices to make plausible the fact that the state property highlighted above applies to CT systems as well as DT ones.

Example 7.1 RC Circuit

This example demonstrates a fine point in the definition of a state for CT systems. Consider an RC circuit in series with a voltage source u . Using KVL, we get the following equation describing the system:

$$-u + v_R + RC\dot{v}_C = 0.$$

It is clear that v_C defines a state for the system as we described before. Does v_R define a state? If $v_R(t_0)$ is given, and the input $u(t)$, $t_0 \leq t < t_f$ is known, then

one can compute $v_C(t_0)$ and using the state property $v_C(t_f)$ can be computed from which $v_R(t_f)$ can be computed. This says that $v_R(t)$ defines a state which contradicts our intuition since it is not an energy storage component.

There is an easy fix of this problem if we assume that all inputs are piece-wise continuous functions. In that case we define the state property as the ability to compute future values of the state from the initial value $x(t_0)$ and the input $u(t)$, $t_0 < t < t_f$. Notice the strict inequality. We leave it to you to verify that this definition rules out v_R as a state variable.

Linearity and Time-Invariance

If in the state-space description (7.1), (7.2), we have

$$f(x(t), u(t), t) = f(x(t), u(t)) \quad (7.5)$$

$$g(x(t), u(t), t) = g(x(t), u(t)) \quad (7.6)$$

then the model is *time-invariant* (in the sense defined earlier, for behavioral models). This corresponds to requiring time-invariance of the functions that specify how the state variables and inputs are combined to determine the state evolution and outputs. The results of experiments on a time-invariant system depend only on the inputs and initial state, not on *when* the experiments are performed.

If, on the other hand, the functions $f(\cdot)$ and $g(\cdot)$ in the state-space description are linear functions of the state variables and inputs, i.e. if

$$f(x(t), u(t), t) = A(t)x(t) + B(t)u(t) \quad (7.7)$$

$$g(x(t), u(t), t) = C(t)x(t) + D(t)u(t) \quad (7.8)$$

then the model is *linear*, again in the behavioral sense. The case of a *linear and periodically varying* (LPV) model is often of interest; when $A(t) = A(t+T)$, $B(t) = B(t+T)$, $C(t) = C(t+T)$, and $D(t) = D(t+T)$ for all t , the model is LPV with period T .

Of even more importance to us is the case of a model that is **linear and time-invariant** (LTI). For an LTI model, the state-space description simplifies to

$$f(x(t), u(t), t) = Ax(t) + Bu(t) \quad (7.9)$$

$$g(x(t), u(t), t) = Cx(t) + Du(t) \quad (7.10)$$

We will primarily study LTI models in this course. Note that LTI state-space models are sometimes designated as (A, B, C, D) or

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] ,$$

as these four matrices completely specify the state-space model.

System	Type
$\dot{x}(t) = tx^2(t)$	NLTV
$\dot{x}(t) = x^2(t)$	NLTI
$\dot{x}(t) = tx(t)$	LTV
$\dot{x}(t) = (\cos t)x(t)$	LPV
$\dot{x}(t) = x(t)$	LTI

Table 7.1: Some examples of linear, nonlinear, time-varying, periodically-varying, and time-invariant state-space descriptions.

Some examples of the various classes of systems listed above are given in Table 7.1. More elaborate examples follow.

One might think that the state-space formulation is restrictive since it only involves first-order derivatives. However, by appropriately choosing the state variables, higher-order dynamics can be described. The examples in this section and on homework will make this clear.

Example 7.2 (Mass-Spring System)

For the mass-spring system in Example 6.2, we derived the following system representation:

$$M\ddot{z} = -kz + u.$$

To put this in state space form, choose position and velocity as state variables:

$$\begin{aligned} x_1 &= z \\ x_2 &= \dot{z}. \end{aligned} \tag{7.11}$$

Therefore,

$$\begin{aligned} \dot{x}_1 &= \dot{z} = x_2 \\ \dot{x}_2 &= -\frac{k}{M}z + \frac{1}{M}u = -\frac{k}{M}x_1 + \frac{1}{M}u. \end{aligned}$$

The input is the force u and let the output be the position of the mass. The resulting state space description of this system is

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} x_2 \\ -\frac{k}{M}x_1 + \frac{1}{M}u \end{bmatrix} \\ y &= x_1. \end{aligned}$$

The above example suggests something that is true in general for mechanical systems: the natural state variables are the position and velocity variables (associated with potential energy and kinetic energy respectively).

Example 7.3 (Nonlinear Circuit)

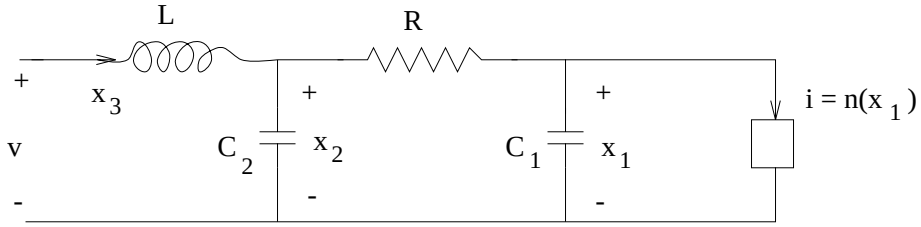


Figure 7.1: Nonlinear circuit.

We wish to put the relationships describing the above circuit's behavior in state-space form, taking the voltage v as an input, and choosing as output variables the voltage across the nonlinear element and the current through the inductor. The constituent relationship for the nonlinear admittance in the circuit diagram is $i_{nonlin} = \mathcal{N}(v_{nonlin})$, where $\mathcal{N}(\cdot)$ denotes some nonlinear function.

Let us try taking as our state variables the capacitor voltages and inductor current, because these variables represent the energy storage mechanisms in the circuit. The corresponding state-space description will express the rates of change of these variables in terms of the instantaneous values of these variables and the instantaneous value of the input voltage v . It is natural, therefore, to look for expressions for $C_1\dot{x}_1$ (the current through C_1), for $C_2\dot{x}_2$ (the current through C_2), and for $L\dot{x}_3$ (the voltage across L).

Applying KCL to the node where R , C_1 , and the nonlinear device meet, we get

$$C_1\dot{x}_1 = \frac{(x_2 - x_1)}{R} - \mathcal{N}(x_1)$$

Applying KCL to the node where R , C_2 and L meet, we find

$$C_2\dot{x}_2 = x_3 - \frac{(x_2 - x_1)}{R}$$

Finally, KVL applied to a loop containing L yields

$$L\dot{x}_3 = v - x_2$$

Now we can combine these three equations to obtain a state-space description of this system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{C_1} \left(\frac{x_2 - x_1}{R} - \mathcal{N}(x_1) \right) \\ \frac{1}{C_2} \left(x_3 - \frac{x_2 - x_1}{R} \right) \\ -\frac{1}{L}x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L}v \end{bmatrix} \quad (7.12)$$

$$y = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} . \quad (7.13)$$

Observe that the output variables are described by an instantaneous output equation of the form (7.2). This state-space description is time-invariant but nonlinear. This makes sense, because the circuit does contain a nonlinear element!

Example 7.4 (Discretization)

Assume we have a continuous-time system described in state-space form by

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

Let us now sample this system with a period of T , and approximate the derivative as a forward difference:

$$\frac{1}{T} (x((k+1)T) - x(kT)) = Ax(kT) + Bu(kT), \quad k \in \mathbb{Z}. \quad (7.14)$$

It is convenient to change our notation, writing $x[k] \equiv x(kT)$, and similarly for u and y . Our sampled equation can thereby be rewritten as

$$\begin{aligned} x[k+1] &= (I + TA)x[k] + TBu[k] \\ &= \hat{A}\mathbf{x}[k] + \hat{B}\mathbf{u}[k], \\ y[k] &= Cx[k] + Du[k]. \end{aligned} \quad (7.15)$$

which is in standard state-space form.

In many modern applications, control systems are implemented digitally. For that purpose, the control engineer must be able to analyze both discrete-time as well as continuous-time systems. In this example a crude sampling method was used to obtain a discrete-time model from a continuous-time one. We will discuss more refined discretization methods later on in this book.

It is also important to point out that there are physical phenomena that directly require or suggest discrete-time models; not all discrete-time models that one encounters in applications are discretizations of continuous-time ones.

7.3 Linearization

Much of our attention in this course will be focused on *linear* models. Linear models frequently arise as descriptions of small perturbations away from a nominal solution of the system. Consider, for example, the continuous-time (CT) state-space model

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \\ y(t) &= g(x(t), u(t), t) \end{aligned} \quad (7.16)$$

where $x(t)$ is the n -dimensional state-vector at time t , $u(t)$ is the m -dimensional vector of inputs, and $y(t)$ is the p -dimensional vector of outputs. Suppose $x_o(t)$, $u_o(t)$ and $y_o(t)$ constitute a *nominal solution* of the system, i.e. a collection of CT signals that jointly satisfy the equations in (7.16). Now let the control and initial condition be perturbed from their nominal values to $u(t) = u_o(t) + \delta u(t)$ and $x(0) = x_o(0) + \delta x(0)$ respectively, and let the state trajectory accordingly be perturbed to $x(t) = x_o(t) + \delta x(t)$. Substituting these new values into (7.16) and performing a (multivariable) Taylor series expansion to first-order terms, we find

$$\begin{aligned}\delta \dot{x}(t) &\approx \left[\frac{\partial f}{\partial x} \right]_o \delta x(t) + \left[\frac{\partial f}{\partial u} \right]_o \delta u(t) \\ \delta y(t) &\approx \left[\frac{\partial g}{\partial x} \right]_o \delta x(t) + \left[\frac{\partial g}{\partial u} \right]_o \delta u(t)\end{aligned}\tag{7.17}$$

where the $n \times n$ matrix $[\partial f / \partial x]_o$ denotes the Jacobian of $f(., ., .)$ with respect to x , i.e. a matrix whose ij -th entry is the partial derivative of the i th component of $f(., ., .)$ with respect to the j th component of x , and where the other Jacobian matrices in (7.17) are similarly defined. The subscript $_o$ indicates that the Jacobians are evaluated along the nominal trajectory, i.e. at $x(t) = x_o(t)$ and $u(t) = u_o(t)$. The linearized model (7.17) is evidently linear, of the form

$$\begin{aligned}\delta \dot{x}(t) &= A(t) \delta x(t) + B(t) \delta u(t) \\ \delta y(t) &= C(t) \delta x(t) + D(t) \delta u(t).\end{aligned}\tag{7.18}$$

When the original nonlinear model is time-invariant, the linearized model will also be time-invariant *if the nominal solution is constant* (i.e. if the nominal solution corresponds to a constant *equilibrium*); however, the linearized model *may be time varying* if the nominal solution is time varying (even if the original nonlinear model is time-invariant), and will be *periodic* — i.e., have periodically varying coefficients — if the nominal solution is periodic (as happens when the nominal solution corresponds to operation in some cyclic or *periodic steady state*).

The same development can be carried out for discrete-time (DT) systems, but we focus in this lecture on the CT case.

Example 7.5 (Linearizing a Nonlinear Circuit Model)

Consider linearizing the state-space model we obtained for the nonlinear circuit in Example 7.3. We ended up there with a nonlinear model of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{C_1} \left(\frac{x_2 - x_1}{R} - \mathcal{N}(x_1) \right) \\ \frac{1}{C_2} \left(x_3 - \frac{x_2 - x_1}{R} \right) \\ -\frac{1}{L} x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} v \end{bmatrix} .\tag{7.19}$$

For the linearization, all that happens is each x_j is replaced by δx_j , and $\mathcal{N}(x_1)$ is replaced by $[d\mathcal{N}(x_1)/dx_1]_o \delta x_1$, resulting in a linear state-space model of the form

$$\delta \dot{x}(t) = A \delta x(t) + B \delta v(t)\tag{7.20}$$

with

$$A = \begin{pmatrix} -\frac{1}{RC_1} - \frac{1}{C_1} \left[\frac{dN}{dx_1} \right]_o & \frac{1}{RC_1} & 0 \\ \frac{1}{RC_2} & -\frac{1}{RC_2} & \frac{1}{C_2} \\ 0 & \frac{1}{L} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{L} \end{pmatrix} \quad (7.21)$$

Example 7.6 (Linearizing the Inverted Pendulum)

Recall from Example 6.3 the equations that describe the dynamics of the inverted pendulum. Those equations are nonlinear due to the presence of the terms $\sin(\theta)$, $\cos(\theta)$, and $(\dot{\theta})^2$. We can linearize these equations around $\theta = 0$ and $\dot{\theta} = 0$, by assuming that $\theta(t)$ and $\dot{\theta}(t)$ remain small. Recall that for small θ

$$\begin{aligned} \sin(\theta) &\approx \theta - \frac{1}{6}\theta^3 \\ \cos(\theta) &\approx 1 - \frac{1}{2}\theta^2, \end{aligned}$$

and using the linear parts of these relations the linearized system of equations takes the form

$$\begin{aligned} \left(1 - \frac{ml}{ML}\right) \ddot{s} + \frac{ml}{M} \frac{g}{L} \theta &= \frac{1}{M} u, \\ \left(1 - \frac{ml}{ML}\right) \ddot{\theta} - \frac{g}{L} \theta &= -\frac{1}{ML} u. \end{aligned}$$

Using as state vector

$$x = \begin{bmatrix} s \\ \dot{s} \\ \theta \\ \dot{\theta} \end{bmatrix},$$

the following state-space model can be easily obtained:

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\alpha \frac{ml}{ML} g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha \frac{g}{L} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\alpha}{M} \\ 0 \\ -\frac{\alpha}{LM} \end{pmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x, \end{aligned}$$

where the constant α is given by

$$\alpha = \frac{1}{\left(1 - \frac{ml}{ML}\right)}.$$

Exercises

Exercise 7.1 Consider the nonlinear difference equation

$$y(k+n) = F[y(k+n-1), \dots, y(k), u(k+n-1), \dots, u(k), k]$$

where n is a fixed integer, and k is the time index.

- (a) Find a state-space representation of order $2n-1$ for this difference equation.
- (b) Find an n th-order state-space representation in LTI case (what is the form of F in this case?), using z-transforms for guidance (natural state variables are the coefficients of the initial-condition terms in the z-transformed version of the difference equation — try a third-order difference equation — remind of forward shift theorem from z-transforms). This part will guide the solution of (c).
- (c) Find an n th-order state-space representation for the nonlinear system in (a) for the case where $F[.]$ has the special form

$$F[.] = \sum_{i=1}^n f_i[y(k+n-i), u(k+n-i)]$$

(Hint: Note that the difference equation in part (b) has this form; use your definition of state variables in (b) to guide your choice here.)

Exercise 7.2 Consider a causal continuous-time system with input-output representation $y(t) = h * u(t)$, where $*$ denotes convolution and $h(t)$ is the impulse response of the system:

$$h(t) = 2e^{-t} - ce^{-2t} \quad \text{for } t \geq 0$$

Here c denotes a constant.

- (a) Suppose $c = 2$. Use only the input-output representation of the system to show that the variables $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$ qualify as state variables of the system at time t .
- (b) Compute the transfer function of the system, and use it to describe what may be special about the case $c = 2$.

Exercise 7.3 The input $u(t)$ and output $y(t)$ of a system are related by the equation

$$\frac{dy(t)}{dt} + a_0(t)y(t) = b_0(t)u(t) + b_1(t)\frac{du(t)}{dt}$$

Find a linear, time varying state-space representation of this system.

Exercise 7.4 Given the *periodically varying* system $x(k+1) = A(k)x(k) + B(k)u(k)$ of period N , with $A(k+N) = A(k)$ and $B(k+N) = B(k)$, define the *sampled state* $z[k]$ and the associated *extended input vector* $v[k]$ by

$$z[k] = x(kN) , \quad v[k] = \begin{pmatrix} u(kN) \\ u(kN+1) \\ \vdots \\ u(kN+N-1) \end{pmatrix}$$

Now show that $z[k+1] = Fz[k] + Gv[k]$ for *constant* matrices F and G (i.e. matrices independent of k) by determining F and G explicitly.

Exercise 7.5 Let the state space representations of two given systems be

$$x_i(k+1) = A_i x_i(k) + B_i u_i(k) , \quad y_i(k) = C_i x_i(k) , \quad i = 1, 2$$

Determine a state-space representation in the form

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

for the new system obtained when systems 1 and 2 are interconnected (a) in series, (b) in parallel, and in a feedback loop. Assume the size of the inputs and outputs of the two systems are consistent for each of the above configuration to make sense.

Exercise 7.6 Consider a pendulum comprising a mass m at the end of a light but rigid rod of length r . The angle of the pendulum from its equilibrium position is denoted by θ . Suppose a torque $u(t)$ can be applied about the axis of support of the pendulum (e.g. suppose the pendulum is attached to the axis of an electric motor, with the current through the motor being converted to torque). A simple model for this system takes the form

$$mr^2\ddot{\theta}(t) + f\dot{\theta}(t) + mgr \sin \theta(t) = u(t)$$

where the term $f\dot{\theta}$ represents a frictional torque, with f being a positive coefficient, and g is the acceleration due to gravity.

- (a) Find a state-space representation for this model. Is your state-space model linear? time invariant?
- (b) What nominal input $u_o(t)$ corresponds to the nominal motion $\theta_o(t) = \Omega t$ for all t , where Ω is some fixed constant?
- (c) Linearize your state-space model in (a) around the nominal solution in (b). Is the resulting model linear? Is it time invariant or periodically varying?

Exercise 7.7 Consider the horizontal motion of a particle of unit mass sliding under the influence of gravity on a frictionless wire. It can be shown that, if the wire is bent so that its height h is given by $h(x) = V_\alpha(x)$, then a state-space model for the motion is given by

$$\begin{aligned}\dot{x} &= z \\ \dot{z} &= -\frac{d}{dx}V_\alpha(x),\end{aligned}$$

Suppose $V_\alpha(x) = x^4 - \alpha x^2$.

- (a) Verify that the above model has $(z, x) = (0, 0)$ as equilibrium point for any α in the interval $-1 \leq \alpha \leq 1$, and it also has $(z, x) = \left(0, \pm\sqrt{\frac{\alpha}{2}}\right)$ as equilibrium points when α is in the interval $0 < \alpha \leq 1$.
- (b) Derive the linearized system at each of these equilibrium points.