

Lectures on Dynamic Systems and Control

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Chapter 23

CT Reachability, Canonical Forms

23.1 Introduction

The definition of reachability for CT systems is identical to that of DT systems. However, while in the DT case Reachability can be checked through simple matrix conditions, it is not so clear that one can derive simple matrix conditions for CT systems. It is somewhat a surprising result that the reachability condition for CT systems is exactly the same as DT systems.

23.2 The Reachability Problem in CT

We turn now to the n^{th} -order CT model

$$\dot{x}(t) = Ax(t) + Bu(t) . \quad (23.1)$$

Consider whether and how we may choose the input $u(t)$, $t \in [0, L]$, so as to move the system from $x(0) = 0$ to a desired target state $x(L) = d$ at a given time $L > 0$. If there is such an input, we say that the state d is *reachable* in time L . We shall soon show that the choice of L is not critical (unlike in the DT case, where the choice of time interval was not critical provided it was not less than n steps).

The relationship of $x(L)$ to $u(t)$ under the above conditions is given by

$$x(L) = \int_0^L e^{(L-t)A} Bu(t) dt \quad (23.2)$$

$$= \int_0^L F^T(t) u(t) dt \quad (23.3)$$

$$= \langle F(t), u(t) \rangle_L, \quad (23.4)$$

where the Gram product in (23.4) is defined by (23.3), and

$$F^T(t) = e^{(L-t)A} B. \quad (23.5)$$

The set $\mathbb{R}$ of reachable states forms a *subspace*, because

$$\left. \begin{aligned} x_a(L) &= \langle F(t), u_a(t) \rangle_L \\ \& \\ x_b(L) &= \langle F(t), u_b(t) \rangle_L \end{aligned} \right\} \implies \alpha x_a(L) + \beta x_b(L) = \langle F(t), \alpha u_a(t) + \beta u_b(t) \rangle_L \quad (23.6)$$

i.e. any linear combination of reachable states is reachable. (This assumes, of course, that there are no constraints placed on $u(t)$.) We therefore refer to $\mathbb{R}$ as the *reachable subspace*. (Strictly speaking, we should make clear that all this is for target states at time L , but as already mentioned, the choice of L turns out to be irrelevant.) If $\mathbb{R}$ is the entire state space, *i.e.* if the entire state space can be reached, then we refer to the system (23.1) as a *reachable system*.

The key characterization of $\mathbb{R}$ is the following result.

Theorem 23.1

The reachable subspace $\mathbb{R}$ is related to the *reachability Gramian* (at time L), namely $\mathcal{P}_L = \langle F(t), F(t) \rangle_L$ as follows:

$$\mathbb{R} = Ra(\mathcal{P}_L) \quad (23.7)$$

$$= Ra\left(\int_0^L F^T(t)F(t) dt\right) \quad (23.8)$$

(where (23.8) makes explicit the definition of the reachability Gramian — we leave it to you to verify that $\mathcal{P}_L$ is symmetric and positive semidefinite.)

Proof.

We first show that

$$\mathbb{R} \subset Ra(\mathcal{P}_L) \quad (23.9)$$

or equivalently, that

$$Ra^\perp(\mathcal{P}_L) \subset \mathbb{R}^\perp \quad (23.10)$$

For this, note that

$$\begin{aligned} q^T \mathcal{P}_L = 0 &\implies q^T \mathcal{P}_L q = 0 \\ &\iff \langle F(t)q, F(t)q \rangle = 0 \\ &\iff q^T F^T(t) = 0 \\ &\implies q^T x(L) = 0 \end{aligned} \quad (23.11)$$

where the last implication follows from (23.2), (23.3) and (23.4). So any vector in $Ra^\perp(\mathcal{P}_L)$ is also in $\mathbb{R}^\perp$.

Now we show that $\mathbb{R} = Ra(\mathcal{P}_L)$ by showing that any target state $d \in Ra(\mathcal{P}_L)$ can be reached. Suppose $d = \mathcal{P}_L \alpha$, and pick $u(t) = F(t)\alpha$. Then

$$\begin{aligned} x(L) &= \int_0^L F^T(t)F(t)\alpha dt \\ &= \mathcal{P}_L \alpha = d. \end{aligned} \quad (23.12)$$

A characterization that does not involve integrals or matrix exponentials is provided by the following result.

Theorem 23.2

$$Ra(\mathcal{P}_L) = Ra \left(\begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & B \end{bmatrix} \right) \quad (23.13)$$

$$= Ra(R_n), \quad (23.14)$$

where the definition of the *reachability matrix* R_n in (23.14) is clear from (23.13).

Proof.

We shall prove, equivalently, that the orthogonal complements of the above two subspaces are equal.

$$\begin{aligned} q^T \mathcal{P}_L = 0 &\implies q^T e^{A(L-t)} B = 0 \quad (\text{as in the proof of Theorem 23.1}) \\ &\implies \begin{cases} q^T B = 0 & (\text{set } t = L \text{ above}) \\ q^T AB = 0 & (\text{differentiate and set } t = L) \\ \vdots \\ q^T A^{n-1} B = 0 \end{cases} \\ &\iff q^T R_n = 0. \end{aligned}$$

Conversely, $q^T R_n = 0 \implies q^T e^{A(L-t)} B = 0$ (since, by Cayley-Hamilton, $e^{A(L-t)}$ can be written as a time-varying combination of $I, A, \dots, A^{n-1}$) $\implies q^T \mathcal{P}_L = 0$.

Corollary 23.1

The system in (23.1) is reachable iff $\text{rank } R_n = n$.

Remark 23.2.1

From Theorem 23.2, and the fact that R_n does not depend on L , note that the reachable subspace is independent of the choice of L . This is why we were not insistent on marking $d, \mathbb{R}$, etc. with something to indicate the time interval over which the target was to be reached. However, the characteristics of the control used to attain a particular target state will depend on L ; the smaller L is, the “larger” (in some sense) we expect $u(t)$ to be.

Remark 23.2.2

Theorem 23.2 shows that the condition for CT reachability is expressed in the same way — for a given (A, B) — as the condition for DT reachability. Hence all our DT results on standard forms for unreachable systems, modal tests, and so on, remain unchanged. We therefore do not repeat any of these DT results for the CT case, but count on you to explicitly note the CT versions of our earlier DT results.

Remark 23.2.3

Getting from a starting state $x(0) = s$ to a target state $x(T) = d$ requires us to find a $u(t)$ for which

$$d - e^{AT} s = \langle F(t), u(t) \rangle_T \quad (23.15)$$

For arbitrary d, s , the requisite condition is the same as that for reachability from the origin.

Remark 23.2.4

An initial state $x(0) = s$ is termed *controllable* if there is a $u(t)$ that will result in $x(T) = 0$. What (23.15) shows is that the controllable subspace is $e^{-AT} \mathbb{R}$. It follows that a CT system is controllable iff it is reachable. (In DT, this is not quite the case, as we have seen.) Because the distinction between controllability and reachability does not exist in CT, the two terms are often used interchangeably in the literature (even when talking about DT systems! — in which case what is intended is reachability).

Further Notes on the Reachability Gramian

As noted, we define the *reachability Gramian at time t* by

$$\mathcal{P}_t \triangleq \int_0^t e^{\tau A} B B^T (e^{\tau A})^T d\tau.$$

If A is stable (by which we mean that its eigenvalues are in the open left half plane), we can define the *reachability Gramian at $t = \infty$* or simply the *reachability Gramian* as follows:

$$\mathcal{P} \triangleq \lim_{t \rightarrow \infty} \mathcal{P}_t = \int_0^\infty e^{\tau A} B B^T (e^{\tau A})^T d\tau$$

Theorem 23.3 The reachability Gramian $\mathcal{P}$ satisfies the *continuous-time algebraic Lyapunov equation*:

$$A\mathcal{P} + \mathcal{P}A^T = -BB^T \quad (23.16)$$

Proof: First, note that

$$\begin{aligned} \int_0^\infty \frac{d}{dt} e^{tA} B B^T (e^{tA})^T dt &= e^{tA} B B^T (e^{tA})^T \Big|_0^\infty \\ &= -BB^T. \end{aligned}$$

But we can also write

$$\begin{aligned} \int_0^\infty \frac{d}{dt} e^{tA} B B^T (e^{tA})^T dt &= \int_0^\infty \{ A e^{tA} B B^T (e^{tA})^T + e^{tA} B B^T (e^{tA})^T A^T \} dt \\ &= A \int_0^\infty e^{tA} B B^T (e^{tA})^T dt + \left(\int_0^\infty e^{tA} B B^T (e^{tA})^T dt \right) A^T \\ &= A\mathcal{P} + \mathcal{P}A^T. \end{aligned}$$

which of course implies (23.16), and the proof is complete.

We can therefore solve for the continuous-time reachability Gramian by solving the relatively simple Lyapunov equation.

Additional Remarks on Reachability

Reachability is lost only under special conditions. For a randomly picked (or typical, or generic) pair of matrices (A, B) , we will find the system to be reachable. However, for a system assembled out of components in a structured way, *i.e.* with an A, B pair that has structured constraints, it is possible for unreachability to arise even if the entries of A, B are otherwise arbitrary (*i.e.* arbitrary except for the constraints). This is one reason for examining the notion of reachability.

It is also possible to have systems that are *nearly* unreachable, either in the sense that small perturbations of A and B will make the system unreachable (*e.g.* if the minimum singular value of $[\lambda I - A \mid B]$ for some λ is small compared to $\| [A \mid B] \|_2$), or in the sense that inordinate control effort is needed to move the state significantly away from a particular subspace. In discussing such situations, our understanding of reachability Gramians, unreachable systems, etc., provides a benchmark.

23.3 Canonical Forms for Reachable Systems

Consider an n^{th} -order reachable single-input (SI) system in CT or DT, namely

$$\text{or } \left. \begin{array}{l} \dot{x}(t) \\ x(t+1) \end{array} \right\} = Ax(t) + bu(t) \quad (23.17)$$

Its reachability matrix

$$R = [A^{n-1}b \mid A^{n-2}b \mid \cdots \mid b] \quad (23.18)$$

is $n \times n$ and invertible. Suppose we use the matrix R to carry out a similarity transformation of (23.17), so that the pair (A, b) is transformed to $(\bar{A}, \bar{b})$, with

$$AR = R\bar{A}, \quad b = R\bar{b}. \quad (23.19)$$

Now substitute (23.18) in (23.19), and use the Cayley-Hamilton theorem to write

$$A^n b = -(a_{n-1}A^{n-1}b + \cdots + a_0b) \quad (23.20)$$

where the coefficients $\{a_i\}$ are those of the characteristic polynomial of A :

$$a(\lambda) = |\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 \quad (23.21)$$

What these substitutions show is that

$$\bar{A} = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \quad (23.22)$$

The state-space description associated with the pair $(\bar{A}, \bar{b})$ will be said to be in *reachability* (or *controllability*) *canonical form*. The word “canonical” denotes “simplest”; there is also a precise technical meaning, but we bypass that. What is clear about the description $(\bar{A}, \bar{b})$ is that it has just the minimum number of coefficients needed to establish the characteristic polynomial of $\bar{A}$, and is thus as simple as one might expect to get. Note that $\bar{A}$ is in companion form, and that the reachability matrix $\bar{R}$ of the pair in (23.22) is just I .

We have established that any n^{th} -order SI reachable system can be transformed to the reachability canonical form determined by its characteristic polynomial. In particular, the SI system corresponding to the pair $(\bar{A}, \bar{b})$ below,

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (23.23)$$

is easily verified to have characteristic polynomial $a(\lambda)$ as given in (23.21), and to be reachable (its reachability matrix $\tilde{R}$ is lower triangular, with 1's on the diagonal, and is therefore nonsingular).

Hence (23.23) can be similarity transformed to its reachability canonical form, which is evidently (23.22) again. We shall refer to the form (23.23) as the *controller canonical form*.

Putting together the above results, we see that

$$\text{and } \left. \begin{array}{l} AR = R\tilde{A}, \quad b = R\tilde{b} \\ \tilde{A}\tilde{R} = \tilde{R}\tilde{A}, \quad \tilde{b} = \tilde{R}\tilde{b} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \text{and} \quad A(R\tilde{R}^{-1}) = (R\tilde{R}^{-1})\tilde{A} \\ b = (R\tilde{R}^{-1})\tilde{b} \end{array} \right., \quad (23.24)$$

i.e. the original SI reachable system (23.17) can also be transformed to controller canonical form, using the matrix $R\tilde{R}^{-1}$. Just as it was convenient to have a standard form for unreachable systems (see Lecture 14) in order to study problems associated with such systems, it will also turn out to be useful to have canonical forms for reachable systems. The controller canonical form, in particular, will permit an easy analysis of state feedback in SI systems (next lecture).

In the case of multi-input (MI) reachable systems, canonical forms can again be developed using transformation matrices derived from the reachability matrix R . Now, however, R has nm columns (for an m -input system), so there are many ways to select n independent columns from R . One way of selecting these columns is by proceeding from right to left in R , keeping columns that are independent of ones that are already selected, and discarding the rest. The resulting transformed system will then be in what can be termed the MI reachability canonical form. An alternative procedure is to pick the rightmost nonzero column b_1 in R , then Ab_1 , A^2b_1 , etc., until we reach a column that depends on previously selected columns; now pick the rightmost column b_2 that is independent of previously selected ones, followed by Ab_2 , A^2b_2 , etc.; continue until n independent columns have been picked. This leads to a transformation that produces the MI controller canonical form. We shall not pursue the details of these.

Exercises

Exercise 23.1 Consider the single-input LTI system $\dot{x}(t) = Ax(t) + bu(t)$ with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We want to reach the target state $x_f = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ from the origin in 1 second.

- (a) Find $e^{At}b$.
- (b) Find the reachability Gramian G over an interval of length 1. Show that $x_f \in \mathcal{Ra}(G)$, i.e. that $x_f = G\alpha$ for some α , and find α .
- (c) Use your results from (a) and (b) to help you find an input $u(t)$ such that

$$\int_0^1 e^{A(1-t)}bu(t) dt = x_f$$

(i.e. an input that will take you from the origin at time 0 to the target state at time 1). Express the “energy” of this input, namely

$$\int_0^1 u^2(t)dt$$

in terms of G and α , and evaluate the result. How does this input compare with the minimum-energy input required to reach x_f from the origin in 1 second?

- (d) If we choose a different target state, the energy of the input constructed by the above procedure will in general be different. Find a (possibly different) target state x_f with $\|x_f\|_2 = 1$ such that the energy of the input constructed by the above procedure is the *maximum* possible.

Exercise 23.2 Given the asymptotically stable LTI model $\dot{x}(t) = Ax(t) + Bu(t)$, show that the reachability Gramian G corresponding to the interval $[0, \infty]$ is the unique solution of the Lyapunov-type equation (see Problem 3 of Homework 6)

$$AG + GA' = -BB'$$

Exercise 23.3 An LTI model of the form $\dot{x}(t) = Ax(t) + Bu(t)$, with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

describes the perturbations in radial and tangential positions and velocities for a satellite orbiting at nominally constant angular velocity ω (so ω is a constant positive parameter in the above model). The first input component $u_1(t)$ is the radial thrust, and the second component $u_2(t)$ is the tangential thrust.

- (a) Is the system asymptotically stable?
- (b) Show that the system is reachable.
- (c) Is the system reachable if the radial thruster fails?
- (d) Is the system reachable if the tangential thruster fails?

Exercise 23.4 Consider the perturbed Single-Input dynamic system:

$$\dot{x} = Ax + (b + \delta)u,$$

where $\delta \in \mathbb{R}^n$ is a perturbation vector. Assume that the nominal system (A, b) is reachable.

- (a) Find the smallest $\|\delta\|_2$ so that the system is not reachable. This gives a robustness measure to the reachability of a system.
- (b) To improve the robustness of reachability, an engineer suggested to apply a control input that consists of a feedback component; i.e.,

$$u = f^T x + v,$$

where $f \in \mathbb{R}^n$ and v is the external signal. She/He argued that for a special choice of f you need a larger δ (than part 1) to make the system not reachable. Do you agree with her/him? Prove or disprove this claim. (If you think it is true, it suffices to find one f that does the job. If you think it is not true, prove your claim).

Exercise 23.5 Let a rocket car of unit mass be subjected to only the force of the rocket thrust. Suppose that the car is initially at position $x_1(0) = 0$ with velocity $\dot{x}(0) = x_2 = 1 \text{ m/sec}$. The equation of motion is given by:

$$\ddot{x} = -u$$

- (a) Write down the solution of the above differential equation for any input and the given initial conditions.
- (b) Is the system reachable? Compute $\mathcal{P}_T$ (the Reachability Gramian at time T)
- (c) Suppose we wish to bring the car to rest at $x = 0$ after time T sec. Find the control input of minimum energy that achieves this objective, i.e.,

$$\min \|u\|_2 \text{ such that } x_1(T) = 0, x_2(T) = 0.$$

Verify your choice. How does your answer relate to the Reachability Gramian $\mathcal{P}_T$.

- (d) Compute the optimal u for $T = .001$ sec. Comment on its characteristics.