

Lectures on Dynamic Systems and Control

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Chapter 20

Stability Robustness

20.1 Introduction

Last chapter showed how the Nyquist stability criterion provides conditions for the stability robustness of a SISO system. It is possible to provide an extension of those conditions by generalizing the Nyquist criterion for MIMO systems. This, however, turns out to be unnecessary and a direct derivation is possible through the small gain theorem, which will be presented in this chapter.

20.2 Additive Representation of Uncertainty

It is commonly the case that the nominal plant model is quite accurate for low frequencies but deteriorates in the high-frequency range, because of parasitics, nonlinearities and/or time-varying effects that become significant at higher frequencies. These high-frequency effects may have been left unmodeled because the effort required for system identification was not justified by the level of performance that was being sought, or they may be well-understood effects that were omitted from the nominal model because they were awkward and unwieldy to carry along during control design. This problem, namely the deterioration of nominal models at higher frequencies, is mitigated to some extent by the fact that almost all physical systems have strictly proper transfer functions, so that the system gain begins to roll off at high frequency.

In the above situation, with a nominal plant model given by the proper rational matrix $P_0(s)$, the actual plant represented by $P(s)$, and the difference $P(s) - P_0(s)$ assumed to be stable, we may be able to characterize the model uncertainty via a bound of the form

$$\sigma_{max} [P(j\omega) - P_0(j\omega)] \leq \ell_a(\omega) \quad (20.1)$$

where

$$\ell_a(\omega) = \begin{cases} \text{“Small”} & ; \quad |\omega| < \omega_c \\ \text{“Bounded”} & ; \quad |\omega| > \omega_c \end{cases} \quad (20.2)$$

This says that the response of the actual plant lies in a “band” of uncertainty around that of the nominal plant. Notice that no phase information about the modeling error is incorporated into this description. For this reason, it may lead to conservative results.

The preceding description suggests the following simple *additive* characterization of the uncertainty set:

$$\Omega = \{P(s) \mid P(s) = P_0(s) + W(s)\Delta(s)\} \quad (20.3)$$

where Δ is an arbitrary *stable* transfer matrix satisfying the norm condition

$$\|\Delta\|_\infty = \sup_{\omega} \sigma_{\max}(\Delta(j\omega)) \leq 1 \quad (20.4)$$

and the *stable* proper rational (matrix or scalar) weighting term $W(s)$ is used to represent any information we have on how the accuracy of the nominal plant model varies as a function of frequency. Figure 20.1 shows the additive representation of uncertainty in the context of a standard servo loop, with K denoting the compensator.

When the modeling uncertainty increases with frequency, it makes sense to use a weighting function $W(j\omega)$ that looks like a high-pass filter: small magnitude at low frequencies, increasing but bounded at higher frequencies. In the case of a matrix weight, a variation on the use of the additive term $W\Delta$ is to use a term of the form $W_1\Delta W_2$; we leave you to examine how the analysis in this lecture will change if such a two-sided weighting is used.

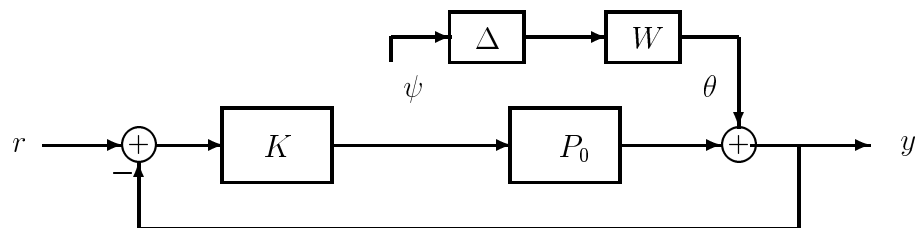


Figure 20.1: Representation of the actual plant in a servo loop via an additive perturbation of the nominal plant.

Caution: The above formulation of an additive model perturbation should *not* be interpreted as saying that the actual or perturbed plant is the *parallel combination* of the nominal system $P_0(s)$ and a system with transfer matrix $W(s)\Delta(s)$. Rather, the actual plant should be considered as being a *minimal realization* of the transfer function $P(s)$, which happens to be written in the additive form $P_0(s) + W(s)\Delta(s)$.

Some features of the above uncertainty set are worth noting:

- The *unstable* poles of all plants in the set are precisely those of the nominal model. Thus, our modeling and identification efforts are assumed to be careful enough to accurately capture the unstable poles of the system.
- The set includes models of arbitrarily large order. Thus, if the uncertainties of major concern to us were *parametric uncertainties*, i.e. uncertainties in the values of the

parameters of a particular (e.g. state-space) model, then the above uncertainty set would greatly overestimate the set of plants of interest to us.

The control design methods that we shall develop will produce controllers that are guaranteed to work for *every member* of the plant uncertainty set. Stated slightly differently, our methods will treat the system as though *every* model in the uncertainty set is a possible representation of the plant. To the extent that not all members of the set are possible plant models, our methods will be conservative.

20.3 Multiplicative Representation of Uncertainty

Another simple means of representing uncertainty that has some nice analytical properties is the *multiplicative perturbation*, which can be written in the form

$$\Omega = \{P \mid P = P_0(I + W\Delta), \|\Delta\|_\infty \leq 1\}. \quad (20.5)$$

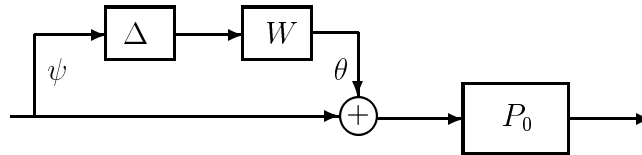


Figure 20.2: Representation of uncertainty as multiplicative perturbation at the plant input.

An alternative to this input-side representation of the uncertainty is the following output-side representation:

$$\Omega = \{P \mid P = (I + W\Delta)P_0, \|\Delta\|_\infty \leq 1\}. \quad (20.6)$$

In both the multiplicative cases above, W and Δ are stable. As with the additive representation, models of arbitrarily large order are included in the above sets. Still other variations may be imagined; in the case of matrix weights, for instance, the term $W\Delta$ can be replaced by $W_1\Delta W_2$.

The caution mentioned in connection with the additive perturbation bears repeating here: the above multiplicative characterizations should *not* be interpreted as saying that the actual plant is the *cascade combination* of the nominal system P_0 and a system $I + W\Delta$. Rather, the actual plant should be considered as being a *minimal realization* of the transfer function $P(s)$, which happens to be written in the multiplicative form.

Any unstable poles of P are poles of the nominal plant, but not necessarily the other way, because unstable poles of P_0 may be cancelled by zeros of $I + W\Delta$. In other words, the actual plant is allowed to have fewer unstable poles than the nominal plant, but all its unstable poles are confined to the same locations as in the nominal model. In view of the caution in the previous paragraph, such cancellations do *not* correspond to unstable hidden modes, and are therefore not of concern.

20.4 More General Representation of Uncertainty

Consider a nominal interconnected system obtained by interconnecting various (reachable and observable) nominal subsystems. In general, our representation of the uncertainty regarding any nominal subsystem model such as P_0 involves taking the signal ψ at the input or output of the nominal subsystem, feeding it through an “uncertainty block” with transfer function $W\Delta$ or $W_1\Delta W_2$, where each factor is stable and $\|\Delta\|_\infty \leq 1$, and then adding the output θ of this uncertainty block to either the input or output of the nominal subsystem. The one additive and two multiplicative representations described earlier are special cases of this construction, but the construction actually yields a total of three additional possibilities with a given uncertainty block. Specifically, if the uncertainty block is $W\Delta$, we get the following additional *feedback representations* of uncertainty:

- $P = P_0(I - W\Delta P_0)^{-1}$;
- $P = P_0(I - W\Delta)^{-1}$;
- $P = (I - W\Delta)^{-1}P_0$.

A useful feature of the three uncertainty representations itemized above is that the unstable poles of the actual plant P are not constrained to be (a subset of) those of the nominal plant P_0 .

Note that in all six representations of the perturbed or actual system, the signals ψ and θ become *internal* to the actual subsystem model. This is because it is the combination of P_0 with the uncertainty model that constitutes the representation of the actual model P , and the actual model is only accessed at its (overall) input and output.

In summary, then, perturbations of the above form can be used to represent many types of uncertainty, for example: high-frequency unmodeled dynamics, unmodeled delays, unmodeled sensor and/or actuator dynamics, small nonlinearities, parametric variations.

20.5 A Linear Fractional Description

We start with a given a nominal plant model P_0 , and a feedback controller K that stabilizes P_0 . The robust stability question is then: under what conditions will the controller stabilize *all* $P \in \Omega$? More generally, we assume we have an interconnected system that is nominally internally stable, by which we mean that the transfer function from an input added in at any subsystem input to the output observed at any subsystem output is always stable in the nominal system. The robust stability question is then: under what conditions will the interconnected system remain internally stable for all possible perturbed models.

If the plant uncertainty is specified (additively, multiplicatively, or using a feedback representation) via an uncertainty block of the form $W\Delta$, where W and Δ are stable, then the actual (closed-loop) system can be mapped into the very simple feedback configuration

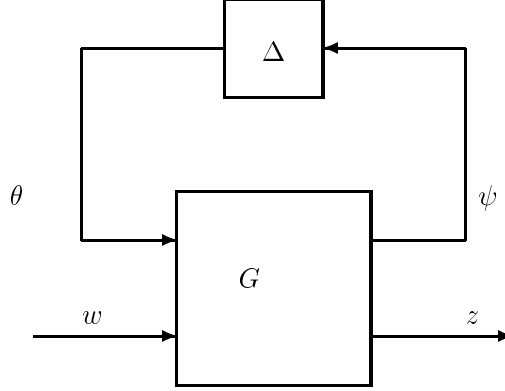


Figure 20.3: Standard model for uncertainty.

shown in Figure 20.3. (The generalization to an uncertainty block of the form $W_1\Delta W_2$ is trivial, and omitted here to avoid additional notation.)

As in the previous subsection, the signals ψ and θ respectively denote the input and output of the uncertainty block. The input w is added in at some arbitrary *accessible point* of the interconnected system, and z denotes an output taken from an arbitrary accessible point. An accessible point in our terminology is simply some subsystem input or output in the *actual* or perturbed system; the input ψ and output θ of the uncertainty block would *not* qualify as accessible points.

If we remove the perturbation block Δ in Fig. 20.3, we are left with the nominal closed-loop system, which is stable by hypothesis (since the compensator K has been chosen to stabilize the nominal plant and is lumped in G). Stability of the nominal system implies that the transfer functions relating the outputs ψ and z of the nominal system to the inputs θ and w are all stable. Thus, in the transfer function representation

$$\begin{pmatrix} \Psi(s) \\ Z(s) \end{pmatrix} = \begin{pmatrix} M(s) & N(s) \\ J(s) & L(s) \end{pmatrix} \begin{pmatrix} \Theta(s) \\ W(s) \end{pmatrix} \quad (20.7)$$

each of the transfer matrices M , N , J , and L is stable.

Now incorporating the constraint imposed by the perturbation, namely

$$\Theta = (\Delta) \Psi \quad (20.8)$$

and solving for the transfer function relating z to w in the *perturbed* system, we obtain

$$G_{wz}(s) = L + J\Delta(I - M\Delta)^{-1}N. \quad (20.9)$$

Note that M is the transfer function “seen” by the perturbation Δ , from the input θ that it imposes on the rest of the system, to the output ψ that it measures from the rest of the system. Recalling that w and z denoted arbitrary inputs and outputs at the accessible points of the actual closed-loop system, we see that internal stability of the actual (i.e. perturbed) closed-loop system requires the above transfer function be stable for all allowed Δ .

20.6 The Small-Gain Theorem

Since every term in G_{wz} other than $(I - M\Delta)^{-1}$ is known to be stable, we shall have stability of G_{wz} , and hence guaranteed stability of the actual closed-loop system, if $(I - M\Delta)^{-1}$ is stable for all allowed Δ . In what follows, we will arrive at a condition — the *small-gain condition* — that guarantees the stability of $(I - M\Delta)^{-1}$. It can also be shown (see Appendix) that if this condition is violated, then there is a stable Δ with $\|\Delta\|_\infty \leq 1$ such that $(I - M\Delta)^{-1}$ and $\Delta(I - M\Delta)^{-1}$ are unstable, and G_{wz} is unstable for some choice of z and w .

Theorem 20.1 (“Unstructured” Small-Gain Theorem) *Define the set of stable perturbation matrices $\Delta \triangleq \{\Delta \mid \|\Delta\|_\infty \leq 1\}$. If M is stable, then $(I - M\Delta)^{-1}$ and $\Delta(I - M\Delta)^{-1}$ are stable for each Δ in Δ if and only if $\|M\|_\infty < 1$.*

Proof. The proof of necessity (see Appendix) is by construction of an allowed Δ that causes $(I - M\Delta)^{-1}$ and $\Delta(I - M\Delta)^{-1}$ to be unstable if $\|M\|_\infty \geq 1$, and ensures that G_{wz} is unstable.

For here, we focus on the proof of sufficiency. We need to show that if $\|M\|_\infty < 1$ then $(I - M\Delta)^{-1}$ has no poles in the closed right half-plane for any $\Delta \in \Delta$, or equivalently that $I - M\Delta$ has no zeros there. For arbitrary $x \neq 0$ and any s_+ in the closed right half-plane (CRHP), and using the fact that both M and Δ are well-defined throughout the CRHP, we can deduce that

$$\begin{aligned} \|[I - M(s_+)\Delta(s_+)]x\|_2 &\geq \|x\|_2 - \|M(s_+)\Delta(s_+)x\|_2 \\ &\geq \|x\|_2 - \sigma_{\max}[M(s_+)\Delta(s_+)]\|x\|_2 \\ &\geq \|x\|_2 - \|M\|_\infty \|\Delta\|_\infty \|x\|_2 \\ &> 0 \end{aligned} \tag{20.10}$$

The first inequality above is a simple application of the triangle inequality. The third inequality above results from the *Maximum Modulus Theorem* of complex analysis, which says that the largest magnitude of a complex function over a region of the complex plane is found on the boundary of the region, if the function is analytic inside and on the boundary of the region. In our case, both $q'M'Mq$ and $q'\Delta'\Delta q$ are stable, and therefore analytic, in the CRHP, for unit vectors q ; hence their largest values over the CRHP are found on the imaginary axis. The final inequality in the above set is a consequence of the hypotheses of the theorem, and establishes that $I - M\Delta$ is nonsingular — and therefore has no zeros — in the CRHP.

20.7 Stability Robustness Analysis

Next, we present a few examples to illustrate the use of the small-gain theorem in stability robustness analysis.

Example 20.1 (Additive Perturbation)

For the configuration in Figure 20.1, it is easily seen that

$$M = -K(I + P_0K)^{-1}W = -(I + KP_0)^{-1}KW$$

Example 20.2 (Multiplicative Perturbation)

A multiplicative perturbation of the form of Figure 20.2 can be inserted into the closed-loop system at either the plant input or output. The procedure is then identical to Example 20.1, except that M becomes a different function. Again it is easily verified that for a multiplicative perturbation at the plant input,

$$M = -(I + KP_0)^{-1}KP_0W, \quad (20.11)$$

while a perturbation at the output yields

$$M = -(I + P_0K)^{-1}P_0KW. \quad (20.12)$$

What the above examples show is that stability robustness requires ensuring the weighted versions of certain familiar transfer functions have $\mathcal{H}_\infty$ norms that are less than 1. For instance, with a multiplicative perturbation at the output as in the last example, what we require for stability robustness is $\|TW\|_\infty < 1$, where T is the complementary sensitivity function associated with the nominal closed-loop system. This condition evidently has the same flavor as the conditions we discussed earlier in connection with nominal performance of the closed-loop system.

The small-gain theorem fails to take advantage of any special structure that there might be in the uncertainty set Δ , and can therefore be very conservative. As examples of the kinds of situations that arise, consider the following two examples.

Example 20.3

Suppose we have a system that is best represented by the model of Figure 20.4. When this system is reduced to the standard form, Δ will have a block-diagonal

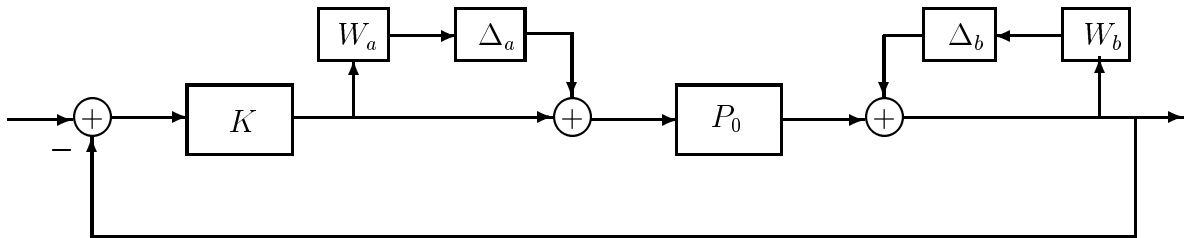


Figure 20.4: Plant with multiple uncertainties.

structure, since the two perturbations enter at different points in the system:

$$\Delta = \begin{bmatrix} \Delta_a & 0 \\ 0 & \Delta_b \end{bmatrix} \quad (20.13)$$

Thus, there is some added information about the plant uncertainty that cannot be captured by the unstructured small-gain theorem, and in general, even if $\|M\|_\infty \geq 1$ for the M that corresponds to the Δ above, there may be no admissible perturbation that will result in unstable $(I - M\Delta)^{-1}$.

Example 20.4

Suppose that in addition to norm bounds on the uncertainty, we know that the phase of the perturbation remains in the sector $[-30^\circ, 30^\circ]$. Again, even if $\|M\|_\infty \geq 1$ for the M that corresponds to the Δ for this system, there may be no admissible perturbation that will result in unstable $(I - M\Delta)^{-1}$.

In both of the preceding two examples, the unstructured small-gain theorem gives conservative results.

Relating Stability Robustness to the (SISO) Nyquist Criterion

Suppose we have a SISO nominal plant with a multiplicative perturbation, and a nominally stabilizing controller K . Then $P = P_0(1 + W\Delta)$, and the compensated open-loop transfer function is

$$PK = P_0K + P_0KW\Delta. \quad (20.14)$$

Since P_0 , K , and W are known and $|\Delta| \leq 1$ with arbitrary phase, we may deduce from (20.14) that the “real” Nyquist plot at any given frequency ω_0 is contained in a region delimited by a circle centered at $P_0(j\omega_0)K(j\omega_0)$, with radius $|P_0KW(j\omega_0)|$. This is illustrated in Figure 20.5(a). Clearly, if the circle of uncertainty ever includes -1 , there is the possibility that the “real” Nyquist plot has an extra encirclement, and hence is unstable. We may relate this to the robust stability problem as follows. From Example 20.2, the SISO system is robustly stable by the small gain theorem if

$$\left| \frac{P_0K}{1 + P_0K} W \right| < 1, \quad \forall \omega. \quad (20.15)$$

Equivalently,

$$|P_0KW| < |1 + P_0K|. \quad (20.16)$$

The right-hand side of (20.16) is the magnitude of a translation of the Nyquist plot of the nominal loop transfer function. In Figure 20.5(b), because of the translation, encirclement of zero will destabilize the system. Clearly, this cannot happen if (20.16) is satisfied. This makes the relationship of robust stability to the SISO Nyquist criterion clear.

Performance as Stability Robustness

Suppose that, for some plant model P , we wish to design a feedback controller that not only stabilizes the plant (first order of priority!), but also provides some performance benefits, such as improved output regulation in the presence of disturbances. Given that something is known

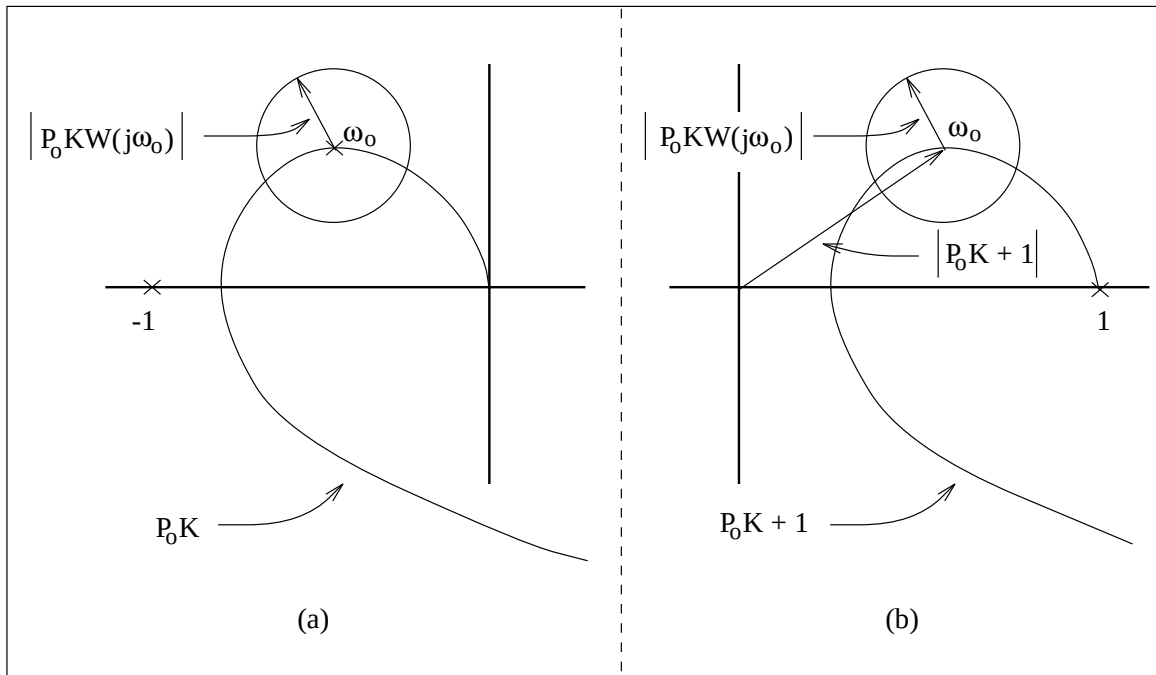


Figure 20.5: Relation of Nyquist criterion and robust stability.

about the frequency spectrum of such disturbances, the system model might look like Figure 20.6, where $\|\xi\|_2 < 1$, and the modeling filter W can be constructed to capture frequency characteristics of the disturbance. Calculating the transfer function of this loop from ξ to y , we have that $y = (I + PK)^{-1}W\xi$. We assume that the performance specification will be met if $\|(I + PK)^{-1}W\|_\infty < 1$, which does not restrict the problem, since W can always be scaled to reflect the actual magnitude of the disturbance or performance specification. This formulation looks analogous to a robust stability problem, and indeed, it can be verified that the small-gain theorem applied to the system of Figure 20.7 captures the identical constraint on the system transfer function. By mapping this system into the standard form of Figure 20.3, we find that $M = (I + PK)^{-1}W$, which is exactly the M that is needed if the small-gain condition is to yield the desired condition.

Finally, plant uncertainty has to be brought into the picture simultaneously with the

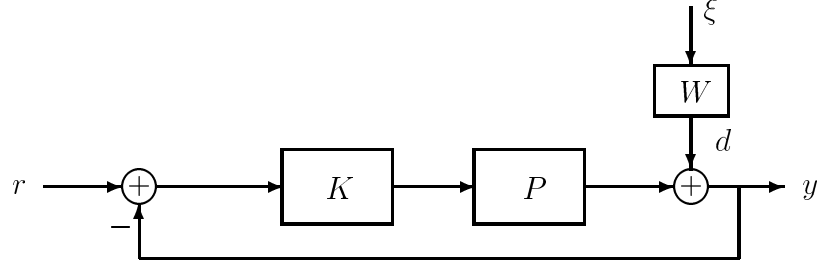


Figure 20.6: Plant with disturbance.

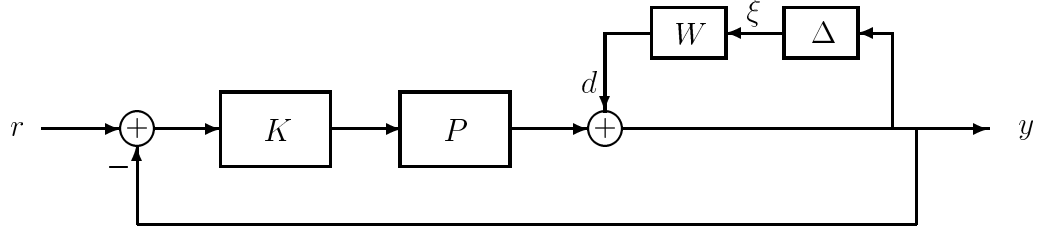


Figure 20.7: Mapping performance specifications into a stability problem.

performance constraints. This is necessary to formulate the *performance robustness* problem. It should be evident that this will lead to situations with block-diagonal Δ , as was obtained in the context of the last example in the previous subsection. The treatment of this case will require the notion of structured singular values, as we shall see in the next lecture.

Appendix

Necessity of the small gain condition for robust stability can be proved by showing that if $\sigma_{max}[M(j\omega_0)] > 1$ for some ω_0 , we can construct a Δ of norm less than one, such that the resulting closed-loop map G_{zv} is unstable. This is done as follows. Take the singular value decomposition of $M(j\omega_0)$,

$$M(j\omega_0) = U\Sigma V' = U \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} V'. \quad (20.17)$$

Since $\sigma_{max}[M(j\omega_0)] > 1$, $\sigma_1 > 1$. Then $\Delta(j\omega_0)$ can be constructed as:

$$\Delta(j\omega_0) = V \begin{bmatrix} 1/\sigma_1 & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{bmatrix} U' \quad (20.18)$$

Clearly, $\sigma_{max}\Delta(j\omega_0) < 1$. We then have

$$\begin{aligned}
(I - M\Delta)^{-1}(j\omega_0) &= I - U \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} V'V \begin{bmatrix} 1/\sigma_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} U' \\
&= U \left[I - \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \right] U' \\
&= U \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} U'
\end{aligned} \tag{20.19}$$

which is singular. Only one problem remains, which is that $\Delta(s)$ must be legitimate as the transfer function of a *stable system*, evaluating to the proper value at $s = j\omega_0$, and having its maximum singular value over all ω bounded below 1. The value of the destabilizing perturbation at ω_0 is given by

$$\Delta_0(j\omega_0) = \frac{1}{\sigma_{max}(M(j\omega_0))} v_1 u_1'$$

Write the vectors v_1 and u_1' as

$$v_1 = \begin{bmatrix} \pm|a_1|e^{j\theta_1} \\ \pm|a_2|e^{j\theta_2} \\ \vdots \\ \pm|a_n|e^{j\theta_n} \end{bmatrix}, \quad u_1' = \begin{bmatrix} \pm|b_1|e^{j\phi_1} & \pm|b_2|e^{j\phi_2} & \cdots & \pm|b_n|e^{j\phi_n} \end{bmatrix}, \tag{20.20}$$

where θ_i and ϕ_i belong to the interval $[0, \pi)$. Note that we used $\pm$ in the representation of the vectors v_1 and u_1' so that we can restrict the angles θ_i and ϕ_i to the interval $[0, \pi)$. Now we can choose the nonnegative constants $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\beta_1, \beta_2, \dots, \beta_n$ such that the phase of the function $\frac{s-\alpha_i}{s+\alpha_i}$ at $s = j\omega_0$ is θ_i , and the phase of the function $\frac{s-\beta_i}{s+\beta_i}$ at $s = j\omega_0$ is ϕ_i . Now the destabilizing $\Delta(s)$ is given by

$$\Delta(s) = \frac{1}{\sigma_{max}(M(j\omega_0))} g(s) h^T(s) \tag{20.21}$$

where

$$g(s) = \begin{bmatrix} \pm|a_1| \frac{s-\alpha_1}{s+\alpha_1} \\ \pm|a_2| \frac{s-\alpha_2}{s+\alpha_2} \\ \vdots \\ \pm|a_n| \frac{s-\alpha_n}{s+\alpha_n} \end{bmatrix}, \quad h(s) = \begin{bmatrix} \pm|b_1| \frac{s-\beta_1}{s+\beta_1} \\ \pm|b_2| \frac{s-\beta_2}{s+\beta_2} \\ \vdots \\ \pm|b_n| \frac{s-\beta_n}{s+\beta_n} \end{bmatrix}. \tag{20.22}$$

Exercises

Exercise 20.1 Consider a plant described by the transfer function matrix

$$P_\alpha(s) = \begin{pmatrix} \frac{\alpha}{s-1} & \frac{1}{s-1} \\ \frac{2s-1}{s(s-1)} & \frac{1}{s-1} \end{pmatrix}$$

where α is a real but uncertain parameter, confined to the range $[0.5, 1.5]$. We wish to design a feedback compensator $K(s)$ for robust stability of a standard servo loop around the plant.

- (a) We would like to find a value of α , say $\tilde{\alpha}$, and a scalar, stable, proper rational $W(s)$ such that the set of possible plants $P_\alpha(s)$ is contained within the “uncertainty set”

$$P_{\tilde{\alpha}}(s)[I + W(s)\Delta(s)]$$

where $\Delta(s)$ ranges over the set of stable, proper rational matrices with $\|\Delta\|_\infty \leq 1$. Try and find (no assurances that this is possible!) a suitable $\tilde{\alpha}$ and $W(s)$, perhaps by keeping in mind that what we really want to do is guarantee

$$\sigma_{max}\{P_{\tilde{\alpha}}^{-1}(j\omega)[P_\alpha(j\omega) - P_{\tilde{\alpha}}(j\omega)]\} \leq |W(j\omega)|$$

What specific choice of $\Delta(s)$ yields the plant $P_1(s)$ (i.e. the plant with $\alpha = 1$) ?

- (b) Repeat part (a), but now working with the uncertainty set

$$P_{\tilde{\alpha}}(s)[I + W_1(s)\Delta(s)W_2(s)]$$

where $W_1(s)$ and $W_2(s)$ are column and row vectors respectively, and $\Delta(s)$ is scalar. Plot the upper bound on

$$\sigma_{max}\{P_{\tilde{\alpha}}^{-1}(j\omega)[P_\alpha(j\omega) - P_{\tilde{\alpha}}(j\omega)]\}$$

that you obtain in this case.

- (c) For each of the cases above, write down a sufficient condition for robust stability of the closed-loop system, stated in terms of a norm condition involving the nominal complementary sensitivity function $T = (I + KP_{\tilde{\alpha}})^{-1}KP_{\tilde{\alpha}}$ and W — or, in part (b), W_1 and W_2 .

Exercise 20.2 It turns out that the small gain theorem holds for nonlinear systems as well. Consider a feedback configuration with a stable system M in the forward loop and a stable, unknown perturbation in the feedback loop. Assume that the configuration is well-posed. Verify that the closed loop system is stable if $\|M\|\|\Delta\| < 1$. Here the norm is the gain of the system over *any* p-norm. (This result is also true for both DT and CT systems; the same proof holds).

Exercise 20.3 The design of a controller should take into consideration quantization effects. Let us assume that the only variable in the closed loop which is subject to quantization is the output of the plant. Two very simple schemes are proposed:

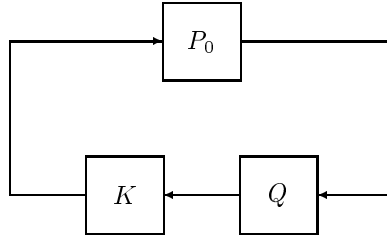


Figure 20.8: Quantization in the Closed Loop.

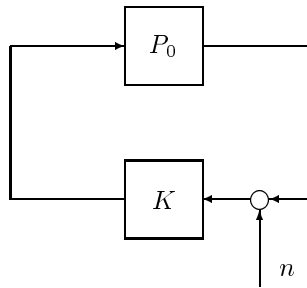


Figure 20.9: Quantization Modeled as Bounded Noise.

1. Assume that the output is passed through a quantization operator Q defined as:

$$Q(x) = a \lfloor \frac{|x|}{.5 + a} \rfloor \text{sgn}(x), \quad a > 0$$

where $\lfloor r \rfloor$ denotes the largest integer smaller than r . The output of this operator feeds into the controller as in Figure 20.8. Derive a sufficient condition that guarantees stability in the presence of Q .

2. Assume that the input of the controller is corrupted with an unknown but bounded signal, with a small bound as in Figure 20.9. Argue that the controller should be designed so that it does not amplify this disturbance at its input.

Compare the two schemes, i.e., do they yield the same result? Is there a difference?