Lectures on Dynamic Systems and Control

Mohammed Dahleh Munther A. Dahleh George Verghese
Department of Electrical Engineering and Computer Science
Massachuasetts Institute of Technology[1]

Chapter 17

Interconnected Systems and Feedback: Well-Posedness, Stability, and Performance

17.1 Introduction

Feedback control is a powerful approach to obtaining systems that are stable and that meet performance specifications, despite system disturbances and model uncertainties. To understand the fundamentals of feedback design, we will study system interconnections and some associated notions such as well-posedness and external stability. Unless otherwise noted, our standing *assumption* for the rest of the course — and a natural assumption in the control setting — will be that *all our models* for physical systems *have outputs that depend* **causally** *on their inputs.*

17.2 System Interconnections

Interconnections are very common in control systems. The system or process that is to be controlled — commonly referred to as the **plant** — may itself be the result of interconnecting various sorts of subsystems in series, in parallel, and in feedback. In addition, the plant is interfaced with sensors, actuators and the control system. Our model for the overall system represents all of these components in some idealized or nominal form, and will also include components introduced to represent uncertainties in, or neglected aspects of the nominal description.

We will start with the simplest feedback inteconnection of a plant with a controller, where the outputs from the plant are fed into a controller whose own outputs are in turn fed back as inputs to the plant. A diagram of this prototype feedback control configuration is shown in Figure 17.1.

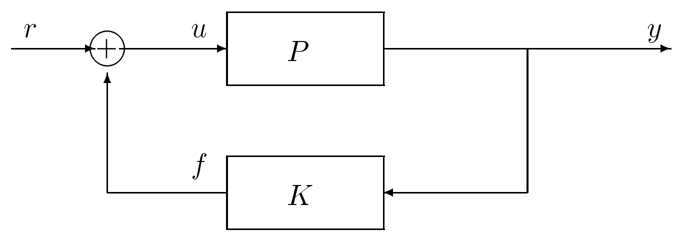


Figure 17.1: Block diagram of the prototype feedback control configuration.

The plant P and controller K could in general be nonlinear, time-varying, and infinite-dimensional, but we shall restrict attention almost entirely to **interconnections of finite-order LTI components**, whether described in state-space form or simply via their input-output transfer functions. Recall that the transfer functions of such finite-order state-space models are *proper rationals*, and are in fact *strictly proper* if there is no direct feedthrough from input to output. We shall use the notation of CT systems in the development that follows, although everything applies equally to DT systems.

The plant and controller should evidently have compatible input/output dimensions; if not, then they cannot be tied together in a feedback loop. For example, if $P(s)$ is the $p \times m$ transfer function matrix of the (nominal LTI model of the) plant in Figure 17.1, then the transfer function $K(s)$ of the (LTI) controller should be an $m \times p$ matrix.

All sorts of other feedback configurations exist; two alternatives can be found in Figures 17.2 and 17.3. For our purposes in this chapter, the differences among these various configurations are not important.

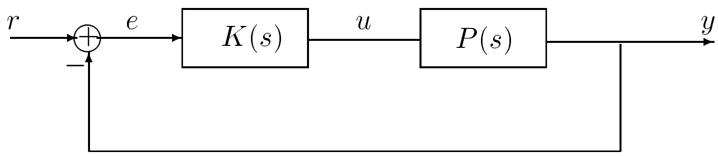


Figure 17.2: A ("servo") feedback configuration where the tracking error between the command r and output y is directly applied to the controller.

Our discussion for now will focus on the arrangement shown in Figure 17.4, which is an elaboration of Figure 17.1 that represents some additional signals of interest. Interpretations for the various (vector) signals depicted in the preceding figures are normally as follows:

- u — control inputs to plant

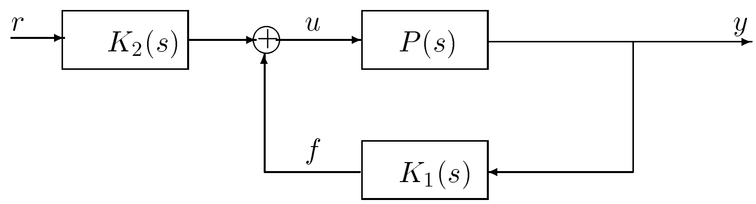


Figure 17.3: A two-parameter-compensator feedback scheme.

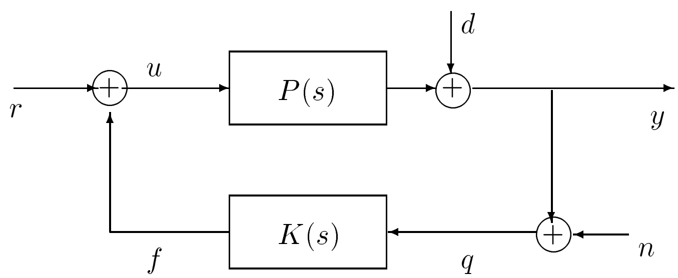


Figure 17.4: Including plant disturbances d and measurement noise n.

- y — measured outputs of plant
- d — plant disturbances, represented as acting at the output
- n — noise in the output measurements used by the feedback controller
- r — reference or command inputs
- e — tracking error $r - y$.
- f — output of feedback compensator

Transfer Functions

We now show how to obtain the transfer functions of the mappings relating the various signals found in Figure 17.4; the transform argument, s, is omitted for notational simplicity. We also depart temporarily from our convention of denoting transforms by capitals, and mark the transforms of all signals by lower case, saving upper case for transfer function matrices (i.e. transforms of impulse responses); this distinction will help the eye make its way through the expressions below, and should cause no confusion if it is kept in mind that *all quantities below are transforms.* To begin by relating the plant output to the various input signals, we can write

$$
\begin{aligned}
y &= Pu + d \\
&= P[r + K(y+n)] + d \\
(I - PK)y &= Pr + PKn + d \\
y &= (I-PK)^{-1}Pr + (I-PK)^{-1}PKn + (I-PK)^{-1}d
\end{aligned}
$$

Similarly, the control input to the plant can be written as

$$
\begin{aligned}
u &= r + K(y+n) \\
&= r + K(Pu + d + n) \\
(I - KP)u &= r + Kn + Kd \\
u &= (I-KP)^{-1}r + (I-KP)^{-1}Kn + (I-KP)^{-1}Kd
\end{aligned}
$$

The map $u \longrightarrow f$ (with the feedback loop open and $r = 0$, $n = 0$, $d = 0$) is given by $L = KP$, and is called the *loop transfer function.*

The map $d \longrightarrow y$ (with $n = 0$, $r = 0$) is given by $S_o = (I - PK)^{-1}$ and is called the *output sensitivity function.*

The map $n \longrightarrow y$ (with $d = 0$, $r = 0$) is given by $T = (I - PK)^{-1}PK$ and is called the *complementary sensitivity function.*

The map $r \longrightarrow u$ (with $d = 0$, $n = 0$) is given by $S_i = (I - KP)^{-1}$ and is called the *input sensitivity function.*

The map $r \longrightarrow y$ ($d = 0$, $n = 0$) is given by $(I - PK)^{-1}P$ is called the *system response function* .

The map $d \longrightarrow u$ (with $n = 0$, $r = 0$) is given by $(I - KP)^{-1}K$.

Note that the transfer function $(I - KP)^{-1}K$ can also be written as $K(I - PK)^{-1}$, as may be proved by rearranging the following identity:

$$(I - KP)K = K(I - PK) \quad ,$$

Similarly the transfer function $(I - PK)^{-1}P$ can be written as $P(I - KP)^{-1}$.

Note also that the output sensitivity and input sensitivity functions are different, because, except for the case when P and K are both single-input, single-output (SISO), we have

$$(I - KP)^{-1} \neq (I - PK)^{-1}.$$

17.3 Well-Posedness

We will restrict attention to the feedback structure in Figure 17.5. Our assumption is that H_1 and H_2 have some underlying state-space descriptions with inputs u_1, u_2 and outputs y_1, y_2, so their transfer functions $H_1(s)$ and $H_2(s)$ are *proper*, i.e. $H_1(\infty)$, $H_2(\infty)$ are finite. It is possible (and in fact typical for models of physical systems, since their response falls off to zero as one goes higher in frequency) that the transfer function is in fact *strictly* proper.

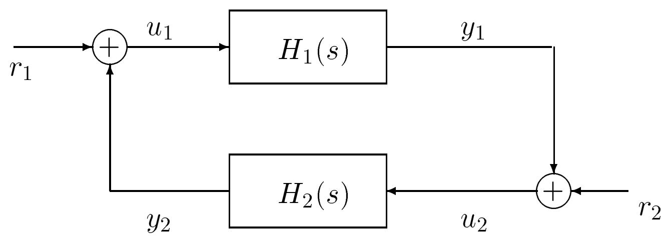


Figure 17.5: Feedback Interconnection.

The closed-loop system in Figure 17.5 can now be described in state-space form by writing down state-space descriptions for $H_1(s)$ (with input u_1 and output y_1) and $H_2(s)$ (with input u_2 and output y_2), and combining them according to the interconnection constraints represented in Figure 17.5. Suppose our state-space models for H_1 and H_2 are

$$H_1 \sim \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \quad H_2 \sim \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$$

with respective state vectors, inputs, and outputs (x_1, u_1, y_1) and (x_2, u_2, y_2), so

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1 u_1 \\ y_1 &= C_1 x_1 + D_1 u_1 \\ \dot{x}_2 &= A_2 x_2 + B_2 u_2 \\ y_2 &= C_2 x_2 + D_2 u_2 \; . \end{aligned} \tag{17.1}$$

Note that $D_1 = H_1(\infty)$ and $D_2 = H_2(\infty)$. The interconnection constraints are embodied in the following set of equations:

$$\begin{aligned} u_1 &= r_1 + y_2 = r_1 + C_2 x_2 + D_2 u_2 \\ u_2 &= r_2 + y_1 = r_2 + C_1 x_1 + D_1 u_1, \end{aligned}$$

which can be rewritten compactly as

$$\begin{bmatrix} I & -D_2 \\ -D_1 & I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & C_2 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} . \tag{17.2}$$

We shall label the interconnected system **well-posed** if the internal signals of the feedback loop, namely u_1 and u_2, are *uniquely* defined for *every* choice of the system state variables x_1, x_2 and external inputs r_1, r_2. (Note that the other internal signals, y_1 and y_2, will be uniquely defined under these conditions if and only if u_1 and u_2 are, so we just focus on the latter pair.) It is evident from (17.2) that the condition for this is the invertibility of the matrix

$$\begin{bmatrix} I & -D_2 \\ -D_1 & I \end{bmatrix}. \tag{17.3}$$

This matrix is invertible if and only if

$$I - D_1 D_2 \text{ or equivalently } I - D_2 D_1 \text{ is invertible.} \tag{17.4}$$

This result follows from the fact that if X, Y, W, and Z are matrices of compatible dimensions, and X is invertible then

$$\det \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \det(X)\det(W - ZX^{-1}Y) \tag{17.5}$$

A *sufficient* condition for (17.4) to hold is that either H_1 or H_2 (or both) be *strictly* proper; that is, either $D_1 = 0$ or $D_2 = 0$.

The significance of well-posedness is that once we have solved (17.2) to determine u_1 and u_2 in terms of x_1, x_2, r_1 and r_2, we can eliminate u_1 and u_2 from (17.1) and arrive at a state-space description of the closed-loop system, with state vector

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We leave you to write down this description explicitly. Without well-posedness, u_1 and u_2 would not be well-defined for arbitrary x_1, x_2, r_1 and r_2, which would in turn mean that there could not be a well-defined state-space representation of the closed-loop system.

The condition in (17.4) is equivalent to requiring that

$$\Big(I - H_1(s)H_2(s)\Big)^{-1} \text{ or equivalently } \Big(I - H_2(s)H_1(s)\Big)^{-1} \text{ exists and is proper.} \tag{17.6}$$

Example 17.1 Consider a discrete-time system with $H_1(z) = 1$ and $H_2(z) = 1 - z^{-1}$ in (the DT version of) Figure 17.5. In this case $(1 - H_1(\infty)H_2(\infty)) = 1 - 1 = 0$, and thus the system is **ill-posed**. Note that the transfer function from r_1 to y_1 for this system is

$$(1 - H_1H_2)^{-1}H_1 = (1 - 1 + z^{-1})^{-1} = z$$

which is not proper — it actually corresponds to the noncausal input-output relation

$$y_1(k) = r_1(k+1) \quad ,$$

which cannot be modeled by a state-space description.

Example 17.2 Again consider Figure 17.4, with $H_1(s) = \frac{s+1}{s+2}$ and $H_2(s) = \frac{s+2}{s+1}$. The expression $(1 - H_1(\infty)H_2(\infty)) = 0$, which implies that the interconnection is **ill-posed**. In this case notice that,

$$\begin{aligned}(1 - H_1(s)H_2(s)) &= 1 - 1 \\ &= 0 \quad \forall\ s \in \mathbb{C} \quad !\end{aligned}$$

Since the inverse of $(1 - H_1H_2)$ does not exist, the transfer functions relating external signals to internal signals cannot be written down.

17.4 External Stability

The inputs in Figure 17.5 are related to the signals y_1, and y_2 as follows:

$$\begin{aligned}y_1 &= H_1(y_2 + r_1) \\ y_2 &= H_2(y_1 + r_2),\end{aligned}$$

which can be written as

$$\begin{bmatrix} I & -H_1 \\ -H_2 & I \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \tag{17.7}$$

We assume that the interconnection in Figure 17.5 is *well-posed*. Let the map $\mathcal{T}(H_1, H_2)$ be defined as follows:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathcal{T}(H_1, H_2)\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}.$$

From the relations 17.7 the form of the map $\mathcal{T}(H_1, H_2)$ is given by

$$\mathcal{T}(H_1, H_2) = \begin{bmatrix} (I - H_1H_2)^{-1}H_1 & (I - H_1H_2)^{-1}H_1H_2 \\ (I - H_2H_1)^{-1}H_2H_1 & (I - H_2H_1)^{-1}H_2 \end{bmatrix}$$

We term the interconnected system **externally** p**-stable** if the map $\mathcal{T}(H_1, H_2)$ is p-stable. In our finite-order LTI case, what this requires is precisely that the poles of all the entries of the rational matrix

$$\mathcal{T}(H_1, H_2) = \begin{bmatrix} (I - H_1H_2)^{-1}H_1 & (I - H_1H_2)^{-1}H_1H_2 \\ (I - H_2H_1)^{-1}H_2H_1 & (I - H_2H_1)^{-1}H_2 \end{bmatrix}$$

be in the open left half of the complex plane.

External stability guarantees that bounded inputs r_1, and r_2 will produce bounded responses y_1, y_2, u_1, and u_2. External stability is guaranteed by asymptotic stability (or **internal stability**) of the state-space description obtained through the process described in our discussion of well-posedness. However, as noted in earlier chapters, it is possible to have external stability of the interconnection without asymptotic stability of the state-space description (because of hidden unstable modes in the system — an issue that will be discussed much more in later chapters). On the other hand, external stability is stronger than input/output stability of the mapping $(I - H_1H_2)^{-1}H_1$ between r_1 and y_1, because this mapping only involves a subset of the exposed or external variables of the interconnection.

Example 17.3 Assume we have the configuration in Figure 17.5, with $H_1 = \frac{s-1}{s+1}$ and $H_2 = -\frac{1}{s-1}$. The transfer function relating r_1 to y_1 is

$$\begin{aligned}\frac{H_1}{1 - H_1H_2} &= \frac{s-1}{s+1}\left(1 + \frac{1}{s+1}\right)^{-1} \\ &= \left(\frac{s-1}{s+1}\right)\left(\frac{s+1}{s+2}\right) \\ &= \frac{s-1}{s+2} \quad .\end{aligned}$$

Since the only pole of this transfer function is at $s = -2$, the input/output relation between r_1 and y_1 is stable. However, consider the transfer function from r_2 to u_1, which is

$$\begin{aligned}\frac{H_2}{1 - H_1H_2} &= \frac{1}{s-1}\left(\frac{1}{1 + \frac{1}{s+1}}\right) \\ &= \frac{s+1}{(s-1)(s+2)} \quad .\end{aligned}$$

This transfer function is unstable, which implies that the closed-loop system is externally unstable.

17.5 A More General Description

There are at least two reasons for going to a more general system description than those shown up to now. First, our assessment of the performance of the system may involve variables that are not among the measured/fed-back output signals of the plant. Second, the disturbances affecting the system may enter in more general ways than indicated previously. We do still want our system representation to separate out the controller portions of the system (the K's or K_1, K_2 of the earlier figures), as these are the portions that we will be designing. In this section we will introduce a general plant description that organizes the different types of inputs and outputs, and their interaction with a controller. A block diagram for a general plant description is shown in Figure 17.6.

The different signals in Figure 17.6 can be classified as follows.

- Inputs:
  1. Control input vector u, which contains the actuator signals driving the plant and generated by a controller.

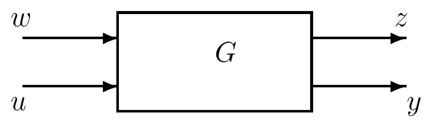


Figure 17.6: General plant description.

2. Exogeneous input vector w, which contains all other external signals, such as references and disturbances.

- Outputs:

1. Measured output vector y, which contains the signals that are available to the controller. These are based on the outputs of the sensor devices, and form the input to the controller.
2. Regulated output vector z, which contains the signals that are important for the specific application. The regulated outputs usually include the actuator signals, the tracking error signals, and the state variables that must be manipulated.

Let the transfer function matrix

$$G = \begin{bmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{bmatrix},$$

have the state-space realization

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

Example 17.4 Consider the unity feedback system in Figure 17.7, where P is a SISO plant, K is a scalar controller, y' is the output, u is the control input, v is a reference signal, and d is an external disturbance that is "shaped" by the filter H before it is injected into the measured output. The controller is driven by the difference $e = v - y'$ (the "tracking error"). The signals v and d can be taken to constitute the exogeneous input, so

$$w = \begin{bmatrix} v \\ d \end{bmatrix}.$$

In such a configuration we typically want to keep the tracking error e small, and to put a cost on the control action. We can therefore take the regulated output z to be

$$z = \begin{bmatrix} e \\ u \end{bmatrix}.$$

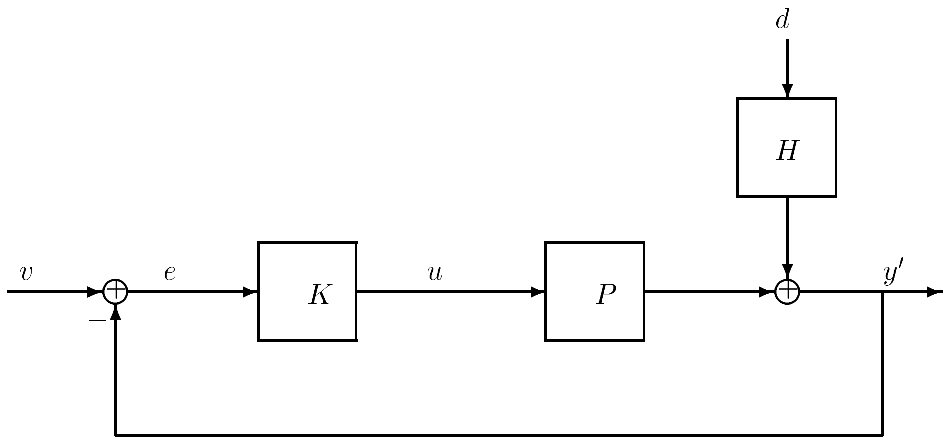


Figure 17.7: Example of a unity feedback system.

The input to the controller is e, therefore we set the measured output y to be equal to e. With these choices, the generalized plant transfer function G, which relates z and y to w and u, can be obtained from

$$\begin{aligned} z &= \begin{bmatrix} -Pu - Hd + v \\ u \end{bmatrix} = \begin{bmatrix} -P \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 & -H \\ 0 & 0 \end{bmatrix} w \\ y &= -Pu + \begin{bmatrix} 1 & -H \end{bmatrix} w. \end{aligned}$$

Let us suppose that $P = \frac{1}{s-1}$ and $H = \frac{1}{s+1}$. Then a state-space realization of G is easily obtained:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} w + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ z &= \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} w + 0u\ . \end{aligned}$$

If we close the loop, the general plant/controller structure takes the form shown in Figure 17.8.

The plant transfer matrix G is a 2×2 block matrix mapping the inputs w, u to the outputs z, y, where the part of the plant that interacts directly with the controller is just G_{yu}. The map (or transfer function) of interest in performance specifications is the map from w to z, denoted by Φ, and easily seen to be given by the following expression:

$$\Phi = G_{zw} + G_{zu}(I - KG_{yu})^{-1}KG_{yw} \tag{17.8}$$

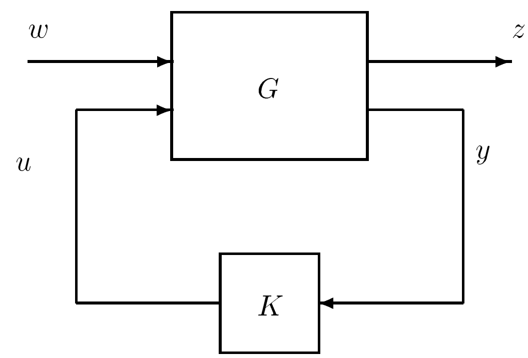


Figure 17.8: A general feedback configuration.

In this new settup we would like to determine under what conditions the closed-loop system in Figure 17.9 is **well-posed** and **externally stable**. For these purposes we inject signals r and v as shown in Figure 17.9, which is similar to what we did in the previous sections. Note that by defining the signals

$$r_1 = \begin{pmatrix} w \\ r \end{pmatrix} \quad r_2 = \begin{pmatrix} 0 \\ v \end{pmatrix}$$
$$y_1 = \begin{pmatrix} z \\ y \end{pmatrix} \quad y_2 = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

this structure is equivalent to the structure in Figure 17.5. This is illustrated in Figure 17.10, with

$$\begin{aligned} H_1 &= \begin{bmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{bmatrix} \\ H_2 &= \begin{bmatrix} 0 \\ I \end{bmatrix} K \begin{bmatrix} 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \end{aligned}$$

This interconnection is **well-posed** if and only if

$$\left(I - \begin{pmatrix} G_{zw}(\infty) & G_{zu}(\infty) \\ G_{yw}(\infty) & G_{yu}(\infty) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & K(\infty) \end{pmatrix} \right)$$

is invertible. This is the same as requiring that

$$(I - K(s)G_{yu}(s))^{-1} \text{ or equivalently } (I - G_{yu}(s)K(s))^{-1} \text{ exists and is proper}$$

The inputs in Figure 17.9 are related to the signals z, u and y as follows:

$$\begin{bmatrix} I & -G_{zu} & 0 \\ 0 & I & -K \\ 0 & -G_{yu} & I \end{bmatrix} \begin{bmatrix} z \\ u \\ y \end{bmatrix} = \begin{bmatrix} G_{zw} & 0 & 0 \\ 0 & I & K \\ G_{yw} & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ r \\ v \end{bmatrix} \tag{17.9}$$

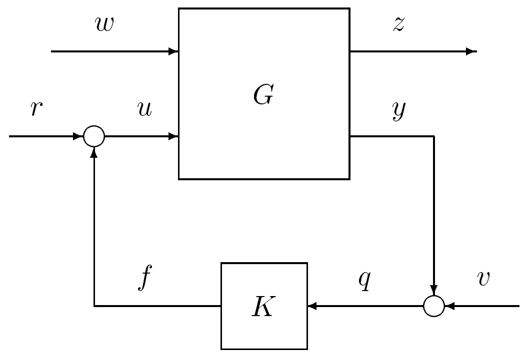


Figure 17.9: A more general feedback configuration.

Let the map $\mathcal{T}(P, K)$ be defined as follows:

$$\begin{pmatrix} z \\ u \\ y \end{pmatrix} = \mathcal{T}(P, K) \begin{pmatrix} w \\ r \\ v \end{pmatrix}$$

The interconnected system is **externally p-stable** if the map from r_1, r_2 to y_1, y_2 is p-stable, see Figure 17.10. This is equivalent to requiring that the map $\mathcal{T}(P, K)$ is p-stable.

17.6 Obtaining Stability and Performance: A Preview

In the lectures ahead we will be concerned with developing analysis and synthesis tools for studying stability and performance in the presence of plant uncertainty and system disturbances.

Stabilization

Stabilization is the first requirement in control design — without stability, one has nothing! There are two relevant notions of stability:

(a) nominal stability (stability in the absence of modeling errors), and

(b) robust stability (stability in the presence of some modeling errors).

In the previous sections, we have shown that stability analysis of an interconnected feedback system requires checking the stability of the closed-loop operator, $\mathcal{T}(P, K)$. In the case where

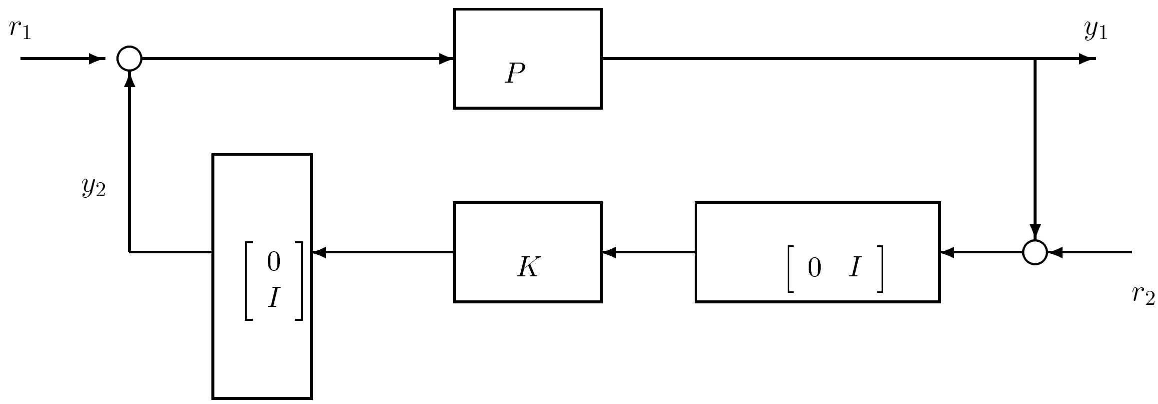


Figure 17.10: A more general feedback configuration.

modeling errors are present, such a check has to be done for every possible perturbation of the system. Efficient methods for performing this check for specified classes of modeling errors are necessary.

Meeting Performance Specifications

Performance specifications (once stability has been ensured) include disturbance rejection, command following (*i.e.*, tracking), and noise rejection. Again, we consider two notions of performance:

(a) nominal performance (performance in the absence of modeling errors), and

(b) robust performance (performance in the presence of modeling errors).

Many of the performance specifications that one may want to impose on a feedback system can be classified under the following two types of specifications:

1. Disturbance Rejection. This corresponds to minimizing the effect of the exogenous inputs w on the regulated variables z in the general 2-input 2-output description, when the exogenous inputs are only partially known. To address this problem, it is necessary to provide a model for the exogenous variables. One possibility is to assume that w has finite energy but is otherwise unknown. If we desire to minimize the energy in the z produced by this w, we can pose the performance task as involving the minimization of

$$\sup_{w \neq 0} \frac{\|\Phi w\|_2}{\|w\|_2}$$

where Φ is the map relating w to z. This is just the square root of the energy-energy gain, and is measured by the $\mathcal{H}_\infty$-norm of Φ.

Alternatively, if w is assumed to have finite peak magnitude, and we are interesed in the peak magnitude of the regulated output z, then the measure of performance is given by the peak-peak gain of the system, which is measured by the $\ell_1/\mathcal{L}_1$-norm of Φ. Other alternatives such as power-power amplification can be considered.

A rather different approach, and one that is quite powerful in the linear setting, is to model w as a stochastic process (e.g, white noise process). By measuring the variance of z, we obtain a peformance measure on Φ.

2. Fixed-Input Specifications. These specifications are based on a specific command or nominal trajectory. One can, for instance, specify a template in the time-domain within which the output is required to remain for a given class of inputs. Familiar specifications such as overshoot, undershoot, and settling time for a step input fall in this category.

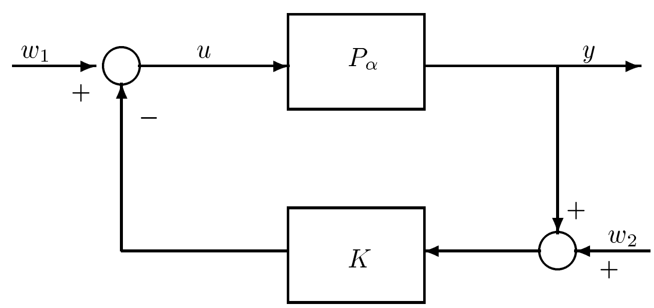
Finally, conditions for checking whether a system meets a given performance measure in the presence of prescribed modeling errors have to be developed. These topics will be revisited later on in this course.

Exercises

Exercise 17.1 Let $P(s) = e^{-2s} - 1$ be connected in a unity feedback configuration. Is this system well-posed?

Exercise 17.2 Assume that P_α and K in the diagram are given by:

$$P_\alpha(s) = \begin{pmatrix} \frac{s}{s+1} & \frac{-\alpha}{s+1} \\ \frac{1}{(s+1)} & \frac{1}{s+1} \end{pmatrix}, \quad \alpha \in \mathbb{R}, \qquad K(s) = \begin{pmatrix} \frac{s+1}{s(s+5)} & 0 \\ -\frac{s+1}{s(s+5)} & \frac{s+1}{s+5} \end{pmatrix}.$$



1. Is the closed loop system stable for all $\alpha > 0$?
2. Is the closed loop system stable for $\alpha = 0$?

Exercise 17.3 Consider the standard servo loop, with

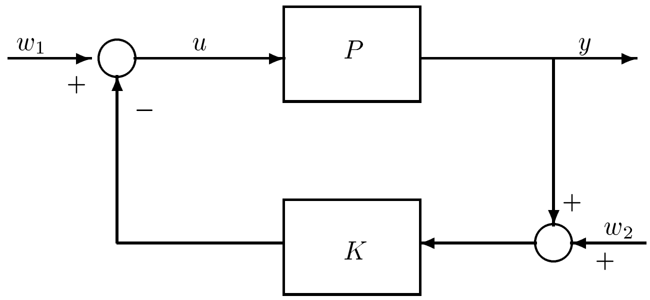
$$P(s) = \frac{1}{10s+1}, \quad K(s) = k$$

but with no measurement noise. Find the least positive gain such that the following are *all* true:

- The feedback system is internally stable.
- With no disturbance at the plant output ($d(t) \equiv 0$), and with a unit step on the command signal $r(t)$, the error $e(t) = r(t) - y(t)$ settles to $|e(\infty)| \leq 0.1$.
- Show that the $\mathcal{L}_2$ to $\mathcal{L}_\infty$ induced norm of a SISO system is given by $\mathcal{H}_2$ norm of the system.
- With zero command ($r(t) \equiv 0$), $\|y\|_\infty \leq 0.1$ for all $d(t)$ such $\|d\|_2 \leq 1$. [ADD NEW Problem]

Exercise 17.4 Parametrization of Stabilizing Controllers

Consider the diagram shown below where P is a given stable plant. We will show a simple way of parametrizing all stabilizing controllers for this plant. The plant as well as the controllers are finite dimensional.



1. Show that the feedback controller

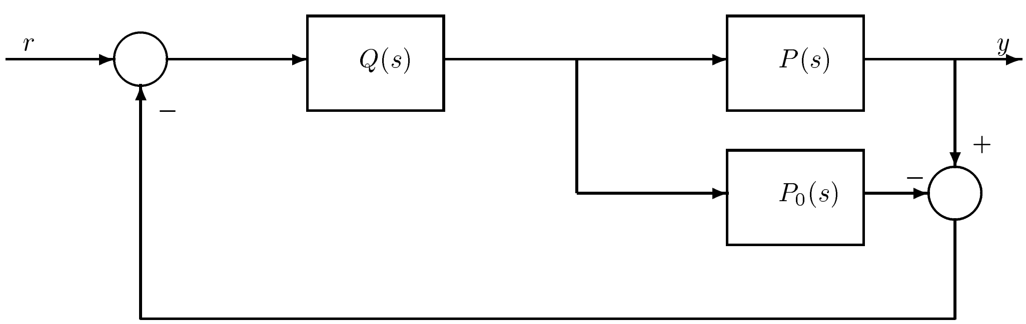
$$K = Q(I - PQ)^{-1} = (I - QP)^{-1}Q$$

   for any stable rational Q is a stabilizing controller for the closed loop system.

2. Show that every stabilizing controller is given by $K = Q(I - PQ)^{-1}$ for some stable Q. (Hint: Express Q in terms of P and K).

3. Suppose P is SISO, w_1 is a step, and $w_2 = 0$. What conditions does Q have to satisfy for the steady state value of u to be zero. Is it always possible to satisfy this condition?

Exercise 17.5 Consider the block diagram shown in the figure below.

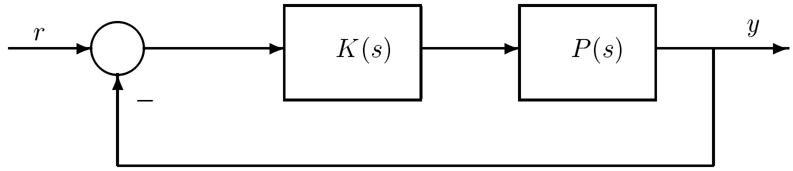


(a) Suppose $P(s) = \dfrac{2}{s-1}$, $P_0(s) = \dfrac{1}{s-1}$ and $Q = 2$. Calculate the transfer function from r to y.

(b) Is the above system internally stable?

(c) Now suppose that $P(s) = P_0(s) = H(s)$ for some $H(s)$. Under what conditions on $H(s)$ is the system internally stable for any *stable* (but otherwise arbitrary) $Q(s)$?

Exercise 17.6 Consider the system shown in the figure below.



The plant transfer function is known to be given by:

$$P(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{1}{s+1} \\ 0 & \frac{s+1}{s+2} \end{bmatrix}$$

A control engineer designed the controller $K(s)$ such that the closed-loop transfer function from r to y is:

$$H(s) = \begin{bmatrix} \frac{1}{s+4} & 0 \\ 0 & \frac{1}{s+4} \end{bmatrix}$$

(a) Compute $K(s)$.

(b) Compute the poles and zeros (with associated input zero directions) of $P(s)$ and $K(s)$.

(c) Are there pole/zero cancellations between $P(s)$ and $K(s)$?

(d) Is the system internally stable? Verify your answer.

Exercise 17.7 An engineer wanted to estimate the peak-to-peak gain of a closed loop system h (the input-output map). The controller was designed so that the system tracks a step input in the steady state. The designer simulated the step response of the system and computed the amount of overshoot (e_1) and undershoot (e_2) of the response. He/She immediately concluded that

$$\|h\|_1 \geq 1 + 2e_1 + 2e_2.$$

Is this a correct conclusion? Verify.