

Lectures on Dynamic Systems and Control

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Chapter 16

System Norms

16.1 Introduction

As in the matrix case, the measure on a system should be induced from the space of signals it operates on. Thus, the size of a system is best measured by the maximum amplification it exerts on a set of signals with unit norm. Let us denote by $\|H\|_{pi}$ the induced norm of the system G as it operates on $\mathcal{L}_p$ (ℓ_p), i.e.,

$$\|H\|_{pi} = \sup_{u \neq 0} \frac{\|Hu\|_p}{\|u\|_p}. \quad (16.1)$$

Observe that a system G is p -stable if and only if $\|H\|_{pi}$ is finite. In particular, the induced norm is the smallest constant C such that $\|y\|_p \leq C\|u\|_p$. In this chapter, we calculate the induced norms over finite-amplitude signals and over finite-energy signals.

16.2 $\mathcal{L}_\infty$ -Induced Norm

A continuous-time LTI system may be characterized by its impulse response *matrix*, $\mathcal{H}(\cdot)$, whose (i, j) th entry $h_{ij}(\cdot)$ is the impulse response from the j th input to the i th output. In other words the input-output relation is given by

$$y(t) = \int \mathcal{H}(t - \tau)u(\tau)d\tau.$$

Theorem 16.1 The $\mathcal{L}_\infty$ -induced norm of a CT LTI system with m inputs, p outputs, and impulse response matrix $\mathcal{H}(t)$ is given by

$$\|H\|_{i\infty} = \max_{1 \leq i \leq p} \sum_{j=1}^m \int |h_{ij}(t)| dt.$$

Proof: If u is an input signal that satisfies $\|u\|_\infty < \infty$, i.e. a bounded signal, then we have

$$y(t) = \int \mathcal{H}(t - \tau)u(\tau)d\tau,$$

and

$$\begin{aligned} \max_{1 \leq i \leq p} |y_i(t)| &= \max_i \left| \int \sum_{j=1}^m h_{ij}(t - \tau)u_j(\tau) d\tau \right| \\ &\leq \left[\max_i \int \sum_j |h_{ij}(t - \tau)| d\tau \right] \max_j \sup_t |u_j(t)|. \end{aligned}$$

It follows that

$$\|y\|_\infty = \sup_t \max_i |y_i(t)| \leq \left[\max_i \sum_j \int |h_{ij}(t)| dt \right] \|u\|_\infty < \infty.$$

To show that the above upper bound can be achieved, we show that for any small number $\epsilon > 0$ we can exhibit an input whose peak magnitude is 1 and that results in an output whose peak magnitude is larger than $\|\mathcal{H}\|_1 - \epsilon$. We will do this for the case when $p = m = 1$ where the impulse response is $h(t)$, for notational simplicity. Since $h(t)$ is absolutely integrable, for any given $\epsilon > 0$ there exists a time T such that

$$\int_{-T}^T |h(t)| dt \geq \|h\|_1 - \epsilon.$$

Now choose the input

$$u_T(t) = \begin{cases} \text{sgn}(h(-t)) & -T \leq t \leq T \\ 0 & |t| > T \end{cases},$$

we get

$$\begin{aligned} \sup_t |y(t)| &\geq |y(0)| = \left| \int_{-T}^T h(0 - \tau)u_T(\tau)d\tau \right| \\ &= \int_{-T}^T |h(0 - \tau)|d\tau \\ &\geq (\|h\|_1 - \epsilon) \sup_t |u_T(t)| \\ &= \|h\|_1 - \epsilon \end{aligned}$$

This induced norm is called the $\mathcal{L}_1$ -norm of $\mathcal{H}(t)$. In the scalar case, this number is just the $\mathcal{L}_1$ -norm of $h(\cdot)$, regarded as a signal.

ℓ_∞ -Induced Norm

The discrete-time case is quite similar to continuous-time where we start with a pulse response *matrix*, $\mathcal{H}(\cdot)$, whose (i, j) th entry $h_{ij}(\cdot)$ is the pulse response from the j th input to the i th output. The input-output relation is given by

$$y(t) = \sum_{\tau} \mathcal{H}(t - \tau) u(\tau).$$

Theorem 16.2 The ℓ_∞ -induced norm of a DT LTI system with m inputs, p outputs, and pulse response matrix $\mathcal{H}(t)$ is:

$$\max_{1 \leq i \leq p} \sum_{j=1}^m \sum_t |h_{ij}(t)|.$$

This induced norm is called the ℓ_1 norm of the system H .

16.3 $\mathcal{L}_2$ -Induced Norm

A CT LTI system is, in general, characterized by the impulse response matrix function $\mathcal{H}(\cdot)$. Here, we will also assume that the system is causal and its Laplace transform exists in some right half plane in the complex plane, and is denoted by $H(s)$. The input-output relation is then given by

$$Y(s) = H(s)U(s).$$

It is evident that if $H(s)$ has a pole in the RHP, then it is not $\mathcal{L}_2$ -stable (verify!). To calculate $\mathcal{L}_2$ -induced norm, we will assume that $H(s)$ is analytic in the open RHP (no poles in the open RHP).

Theorem 16.3 The $\mathcal{L}_2$ -induced norm of a causal, CT LTI system $H(s)$ which is analytic in the open RHP is

$$\sup_w \sigma_{\max} [H(jw)].$$

Proof: Recall Parseval's equality,

$$\|y\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y'(jw)Y(jw)dw.$$

Using this equality we can get the following bound on $\|y\|_2$:

$$\|y\|_2^2 \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_{\max}^2 [H(jw)] U'(jw)U(jw)dw \leq \sup_w \sigma_{\max}^2 [H(jw)] \|u\|_2^2.$$

To show that this bound is tight, we need to show that if $\sigma_{\max} [H(jw_0)] \geq c$ then there exists an input with 2-norm equal to one, that produces an output with 2-norm arbitrarily close

to c . If H is a SISO system, then the construction of such an input is straightforward; pick $u(t) = \frac{1}{T}e^{-jw_0t}$, $0 \leq t \leq T$. Then, as T approaches infinity, the $\|y\|_2$ approaches $H(jw_0)$. In the general MIMO case, this signal is multiplied by the singular vector associated with the largest singular value of $H(jw_0)$. The details of this are left as an exercise.

ℓ_2 -Induced Norm

A DT LTI causal system with a unit sample response $\mathcal{H}$ has a Z - Transform, $H(z)$, which is analytic outside some disc in the complex plain. Here we assume that $H(z)$ is analytic outside the unit disc.

Theorem 16.4 The ℓ_2 -induced norm of a causal, CT LTI system $H(z)$ which is analytic outside the open unit disc is

$$\sup_w \sigma_{\max} [H(e^{jw})] .$$

The quantity $\sup_w \sigma_{\max} [H(jw)]$ is known as the $\mathcal{H}_\infty$ -norm of the system. It measures the maximum energy amplification of finite energy inputs. It turns out (See Exercise 16.1) that this measures the maximum amplification over signals with finite power. In the SISO case, the $\mathcal{H}_\infty$ -norm corresponds to the peak value of the Bode plot.

Computation of $\|H\|_\infty$ for Continuous-time Systems

One method of computing $\|H\|_\infty$ for a CT system would be to sample the function $\sigma_{\max} [H(j\omega)]$ over a number of points in $\omega \in [0, \infty)$. In principle, if one samples finely enough, the location of the peak singular value can be found; however, what constitutes a sufficiently fine sampling of $\sigma_{\max} [H(j\omega)]$ in the frequency domain? For example, a high order system may have a frequency response which moves around considerably, with a large number of peaks and valleys. An alternative approach is to use a state space technique, which we will now discuss.

Theorem 16.5 Let $H(s)$ be a transfer function derived from a stable linear time-invariant system with description $(A, B, C, 0)$ (assume $D = 0$). Define

$$M_\gamma = \begin{bmatrix} A & \frac{1}{\gamma}BB^T \\ -\frac{1}{\gamma}C^TC & -A^T \end{bmatrix} .$$

Then, the $\|H\|_\infty < \gamma$ if and only if M_γ has no purely imaginary eigenvalues.

Proof: First, observe that

$$\begin{aligned} \|H\|_\infty < \gamma &\iff I - \frac{1}{\gamma^2}H'(j\omega)H(j\omega) \quad \text{is invertible for all } \omega \in \mathbb{R} \\ &\iff \left[I - \frac{1}{\gamma^2}H^T(-s)H(s) \right]^{-1} \quad \text{has no poles on the imaginary axis} \end{aligned}$$

The first equivalence follows directly from the definition of the $\mathcal{H}_\infty$ norm. The converse relies on the strict properness of H , i.e., if $\|H\|_\infty \geq \gamma$ then there exists at least one frequency that hits this bound. The second equivalence is straightforward.

Now, we show that M_γ is the A matrix of $(I - \frac{1}{\gamma^2}H^T(-s)H(s))^{-1}$. This is done by constructing a realization of the system shown in Figure 16.1.

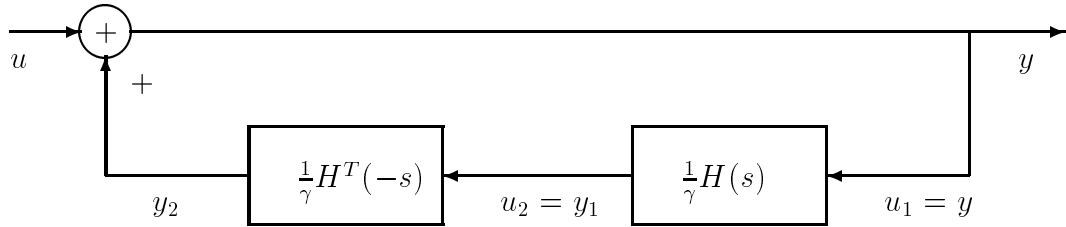


Figure 16.1: Construction of $\left[I - \frac{1}{\gamma^2}H^T(-s)H(s)\right]^{-1}$.

Notice that for this system, the transfer function mapping u to y is $\left[I - \frac{1}{\gamma^2}H^T(-s)H(s)\right]^{-1}$. Since $H(s) \sim (A, B, C, 0)$, it follows that $H^T(-s) \sim (-A^T, -C^T, B^T, 0)$. The composite system can be described as

$$\begin{aligned} \dot{x}_1 &= Ax_1 + By \\ y_1 &= \frac{1}{\gamma}Cx_1 \\ \dot{x}_2 &= -A^Tx_2 - C^Ty_1 \\ y_2 &= \frac{1}{\gamma}B^Tx_2 \\ y &= u + y_2. \end{aligned}$$

By eliminating the internal signals y_1 and y_2 we get the following realization of the closed-loop system

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} A & \frac{1}{\gamma}BB^T \\ -\frac{1}{\gamma}C^TC & -A^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & \frac{1}{\gamma}B^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + I u \end{aligned}$$

which proves the claim.

This theorem allows us to calculate $\|H\|_\infty$ to any arbitrary precision via the following bisection procedure:

1. Start with a guess for a lower bound to $\|H\|_\infty$, and denote it by γ_l .

2. Guess an upper bound to $\|H\|_\infty$, γ_u , and verify that it is an upper bound by computing the eigenvalues of M_{γ_u} and verifying that none are purely imaginary.
3. If $\gamma_u - \gamma_l$ is within the required accuracy, stop.
4. Define $\gamma = \frac{1}{2}(\gamma_l + \gamma_u)$, and compute the eigenvalues of M_γ . If there are purely imaginary eigenvalues then set $\gamma_l = \gamma$, otherwise set $\gamma_u = \gamma$.
5. Go to step 3.

16.4 The $\mathcal{H}_2$ Norm: Energy of the Response to an Impulse

Suppose we have a stable continuous-time system with a transfer function $H(s) = C(sI - A)^{-1}B$, and impulse response $\mathcal{H}(t) = Ce^{tA}B$, then we define the $\mathcal{H}_2$ norm as

$$\begin{aligned}\|H\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}(H(j\omega)^* H(j\omega)) d\omega \\ &= \|h\|_2^2 \\ &= \int_0^{\infty} \text{trace}(\mathcal{H}(t)^T \mathcal{H}(t)) dt.\end{aligned}$$

The computation of the $\mathcal{H}_2$ norm can be performed by solving a Lyapunov equation. Observe that

$$\begin{aligned}\|h\|_2^2 &= \int_0^{\infty} \text{trace}(B^T e^{tA^T} C^T C e^{tA} B) dt \\ &= \text{trace}(B^T Q B)\end{aligned}$$

where Q is referred to as the observability gramian which can be computed by solving the following Lyapunov equation

$$A^T Q + Q A + C^T C = 0.$$

Equivalently we can write

$$\begin{aligned}\|h\|_2^2 &= \int_0^{\infty} \text{trace}(C e^{tA} B B^T e^{tA^T} C^T) dt \\ &= \text{trace}(C P C^T)\end{aligned}$$

where P is referred to as the controllability gramian which can be computed by solving the following Lyapunov equation

$$A P + P A^T + B B^T = 0.$$

Finally, the $\mathcal{H}_2$ norm for DT LTI systems is defined as

$$\begin{aligned}\|H\|_2^2 &= \frac{1}{2\pi} \int_0^{2\pi} \text{trace}(H(e^{i\theta})^* H(e^{i\theta})) d\theta \\ &= \|h\|_2^2 \\ &= \sum_{t=0}^{\infty} \text{trace}(\mathcal{H}(t)^T \mathcal{H}(t)).\end{aligned}$$

This norm can also be computed via solving DT Lyapunov equations.

16.5 Submultiplicative Property

As in the matrix case, induced norms satisfy the submultiplicative property. For any 2 systems G_1, G_2 with compatible dimension such that $G_1 G_2$ is defined, it follows that

$$\|G_1 G_2\|_{pi} \leq \|G_1\|_{pi} \|G_2\|_{pi}.$$

For a BIBO stable LTI system, it follows then that both the ℓ_1 and the $\mathcal{H}_\infty$ norms satisfy this property, however, the $\mathcal{H}_2$ does not.

Example 16.1 Norm Computations

Let's consider the following two-mass two-spring system G :

$$\begin{aligned} m_1 \ddot{y}_1 + c_1 \dot{y}_1 + k_1 y_1 + c_2 (\dot{y}_1 - \dot{y}_2) + k_2 (y_1 - y_2) &= u_1 \\ m_2 \ddot{y}_2 + c_2 (\dot{y}_2 - \dot{y}_1) + k_2 (y_2 - y_1) &= u_2. \end{aligned}$$

Let $x_1 = y_1$, $x_2 = \dot{y}_1$, $x_3 = y_2$, and $x_4 = \dot{y}_2$. Then the equations above can be written in an LTI state space form as follows:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

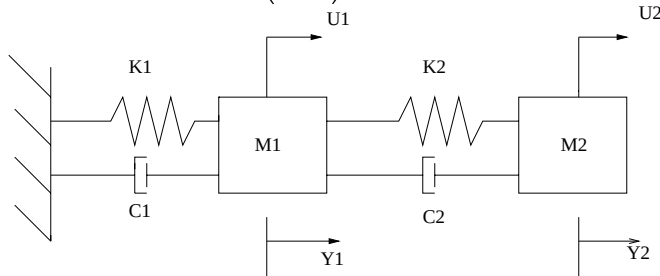


Figure 16.2: 2-mass 2-spring system

$$\begin{aligned}\dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x}.\end{aligned}$$

With $m_1 = m_2 = 1$, $c_1 = c_2 = 1$, and $k_1 = k_2 = 1$, we have the following A and B matrices:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let's first compute $\mathcal{L}_1$ norm of this system. For that purpose, we need the impulse response of the system, $\mathcal{H}(t)$, where

$$y(t) = \int \mathcal{H}(t - \tau)u(\tau)d\tau.$$

The impulse response matrix is given by $\mathcal{H}(t) = Ce^{tA}B$. To compute this matrix, we select a matrix M that puts A in a modal form. It follows that:

$$\mathcal{H}(t) = Ce^{tA}B = CM^{-1}e^{(R(t))}MB,$$

where

$$e^{Rt} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ -a_{12} & a_{11} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & -a_{34} & a_{33} \end{pmatrix}$$

with

$$\begin{aligned}a_{11} &= e^{\sigma_1 t} \cos(\omega_1 t) \\ a_{12} &= e^{\sigma_1 t} \sin(\omega_1 t) \\ a_{33} &= e^{\sigma_2 t} \cos(\omega_2 t) \\ a_{34} &= e^{\sigma_2 t} \sin(\omega_2 t)\end{aligned}$$

where $\sigma_1 = -1$, $\omega_1 = 2$, and $\sigma_2 = -.5$, $\omega_2 = \frac{\sqrt{3}}{2}$ are the real and imaginary parts of the first and second complex conjugate pairs of eigenvalues. Thus ℓ_1 norm of the system G can be computed as follows:

$$\begin{aligned}\|h\|_1 &= \max_i \sum_j \int |h_{ij}(t)|dt \\ &= \max_i \left(\frac{\sum_j \int |h_{1j}(t)|dt}{\sum_j \int |h_{2j}(t)|dt} \right) \\ &= \max \begin{pmatrix} 0.4415 \\ 0.8833 \end{pmatrix} \\ &= 0.8833\end{aligned}$$

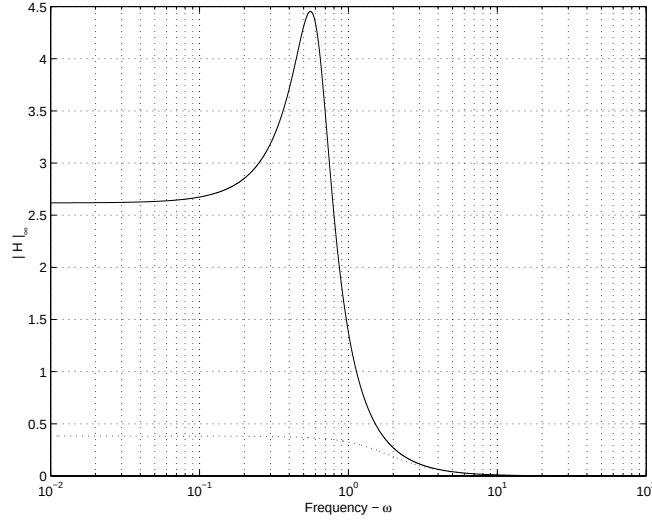


Figure 16.3: Singular values of the system - Solid: σ_{max} , Dashed: σ_{min}

Thus ℓ_1 norm of this system is 0.8833.

In order to compute $\mathcal{H}_2$ norm of the system, we solve the following Laypunov equation

$$AP + P^T A + BB^T = 0$$

which yields

$$P = \begin{pmatrix} 1 & 0 & -1.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 1.5 & 0 & 2.5 & 0 \\ 0 & 0.5 & 0 & 1 \end{pmatrix}.$$

From this, it follows that

$$\|H\|_2^2 = \text{trace}(CPC^T) = \text{trace} \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 1.5 & 0 & 2.5 & 0 \\ 0 & 0.5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^T \right) = 3.5.$$

Therefore

$$\|H\|_2 = \sqrt{3.5} = 1.8708.$$

We will compute the $\|H\|_\infty$ for a CT system by sampling the function $\sigma_{max}[H(j\omega)]$ over a number of points in $\omega \in [0, \infty)$. The plot of $\sigma_{max}[H(j\omega)]$ as a function of ω is plotted below. It can be seen directly from the plot that the maximum value of $\sigma_{max}[H(j\omega)]$ is about 4.45 at $\omega = 0.56$.

Alternatively, using the iterative procedure outlined in steps 1-5 above, we can construct the matrices M_γ for various γ values to obtain $\|H\|_\infty$. As shown in the table below, $\|H\|_\infty$ is approximately 4.4531, which matches with the approximation obtained by sampling $\sigma_{max}(H(jw))$.

γ_l	γ_u	γ	Eigenvalues of M_γ
3	5	4	$\pm 1.3067 \pm 0.9479i, \pm 0.6473i, \pm 0.4461i$
4	5	4.5	$\pm 1.3072 \pm 0.9486i, \pm 0.0287 \pm 0.5566i$
4	4.5	4.25	$\pm 1.3070 \pm 0.9483i, \pm 0.6160i, \pm 0.4885i$
4.25	4.5	4.375	$\pm 1.3071 \pm 0.9484i, \pm 0.5932i, \pm 0.5159i$
4.375	4.5	4.4375	$\pm 1.3072 \pm 0.9485i, \pm 0.5731i, \pm 0.5381i$
4.4375	4.5	4.4688	$\pm 1.3072 \pm 0.9485i, \pm 0.0164 \pm 0.5561i$
4.4375	4.4688	4.4531	$\pm 1.3072 \pm 0.9485i, \pm 0.5601i, \pm 0.5517i$

Exercises

Exercise 16.1 We have shown in class that the $\mathcal{H}_\infty$ -norm arises from measuring the largest I/O energy-to-energy amplification. In this problem, we want to show that the $\mathcal{H}_\infty$ norm can also be seen as determining the largest power-to-power amplification. For that, consider a CT LTI system with transfer function $H(s)$, input $u \in \mathbb{R}^n$ and output $y \in \mathbb{R}^n$. Recall that the $\mathcal{H}_\infty$ -norm of the system, which we simply denote by $\|H\|_\infty$, is given by

$$\|H\|_\infty = \sup_w \sigma_{max} [H(jw)] .$$

Remember also that the power of a continuous-time signal v is given by

$$P_v = \lim_{L \rightarrow \infty} \left(\frac{1}{2L} \int_{-L}^L v'(t)v(t)dt \right)$$

Define the set $X = \left\{ \sum_{i=1}^N u_i e^{jw_i t} : u_i \in \mathbb{R}^n, w_i \in \mathbb{R} \right\}$. In this problem, we want to show that $\sup_{P_u=1} P_y = \|H\|_\infty^2$, at least for $u \in X$ (although the result holds more generally for any u that has a well-defined *power spectrum*).

- (a) Let $u \in X$. Compute the power of u .
- (b) Compute the power of the system output y .
- (c) Now show that $\sup_{P_u=1} P_y \leq \|H\|_\infty^2$.
- (d) Show that equality can be achieved by picking the appropriate input $u \in X$.

Exercise 16.2 (a) Consider a DT LTI system (not necessarily finite-dimensional) specified by the I/O convolution relationship $y(k) = h * u(k)$, where the unit sample response $h(k)$ has z -transform $H(z)$. Show that if the system is ∞ -stable then it is 2-stable. (Hint: Show that $\|h * u\|_2 \leq \|h\|_1 \|u\|_2$.) Although the converse is true for finite-dimensional LTI systems, it is not true in general, as part (b) below shows.

- (b) First show that if a sequence is ℓ_1 then its z -transform evaluated on the unit circle is continuous, i.e., if $h \in \ell_1$ then $H(e^{j\theta})$ is continuous in θ . Now show that the system with transfer function

$$H(z) = e^{\left(\frac{1}{z-1}\right)}$$

is 2-stable but not ∞ -stable. You have to show that $H(z)$ is analytic outside the unit circle and bounded on the circle, but that it is not continuous on the circle.

Exercise 16.3 The table below shows the induced norms of an LTI system with respect to several input and output norms. Some entries have already been shown in this chapter or in the exercises. Verify the rest of the entries.

Output\Input	ℓ_∞	ℓ_2	Power
ℓ_∞	ℓ_1	$\mathcal{H}_2$	∞
ℓ_2	∞	$\mathcal{H}_\infty$	∞
Power	$\leq \mathcal{H}_\infty$	0	$\mathcal{H}_\infty$

Table 16.1: Comparison of induced norms.