Lyapunov exponents for the buck converter

Some numerical and analytical results

Carles Batlle, Imma Massana, Alicia Miralles

{carles,imma,almirall}@mat.upc.es.

Department of Applied Mathematics IV, EUPVG and EPSCA

Technical University of Catalonia

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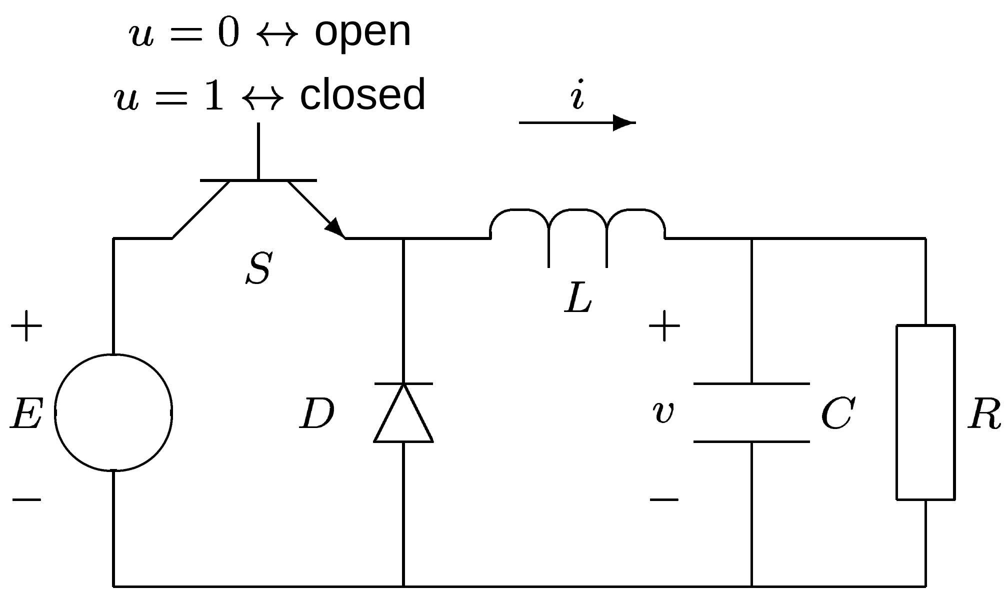
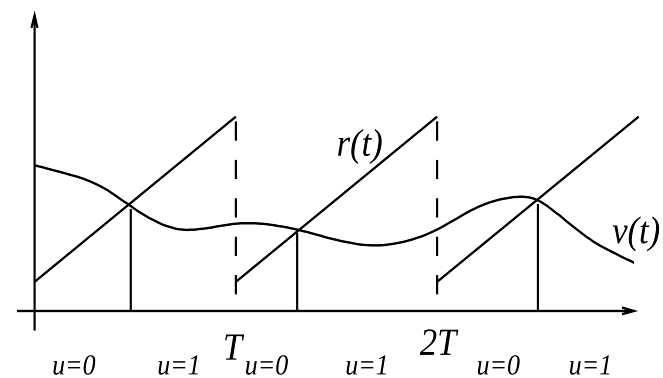
The buck converter





In adimensional variables ($v \sim x$, $i \sim y$)

$$\begin{aligned} \dot{x} &= -x + y \\ \dot{y} &= -\gamma x + \gamma V \theta(r(t) - x) \end{aligned}$$

The QR decomposition (I)

- System in $\mathbb{R}^n$ and its tangent map with respect to a trajectory $x_0(t)$, $z = x - x_0(t)$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x,t), \quad \frac{\mathrm{d}z}{\mathrm{d}t} = \left.\frac{\partial F}{\partial x}\right|_{x_0(t)} z \equiv DF(t)\cdot z$$

- Solution: $z(t) = M(t)z(0)$ where $M(t) = \mathbb{T}e^{\int_0^t DF(s)\mathrm{d}s}$
- SVD decomposition: $M(t) = U(t)\cdot A(t)\cdot V^T(t)$, with $U(t), V(t)$ orthogonal, $A(t)$ diagonal.
- Infinite-time Lyapunov exponents: eigenvalues λ_i of

$$\Lambda = \lim_{t\to+\infty}\left(M^T(t)M(t)\right)^{\frac{1}{2t}}$$

The QR decomposition (II)

- QR decomposition: $M(t) = Q(t) \cdot R(t)$, where
  - $Q(t)$ is orthogonal
  - $R(t)$ is upper-triangular and with positive diagonal elements $\Delta_i(t)$
- It can be shown that

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \log \Delta_i(t)$$

- The QR formulation is optimal: it uses the minimum number of variables.

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The QR decomposition (III)

- From $\dot{M} = DF \cdot M$ it follows

$$Q^T\dot{Q} + \dot{R}R^{-1} = Q^T \cdot DF \cdot Q \equiv S.$$

- Q can be parameterized with $n(n-1)/2$ angles θ_{ij}.
- One gets the separable equations

$$\dot{\Delta}_i = S_{ii}(\theta)\Delta_i, \quad i = 1, \ldots, n$$

plus the nonlinear equations

$$\dot{\theta}_{ij} = g_{ij}(\theta), \quad i > j$$

QR for the buck converter (I)

- If $(x_0(t), y_0(t))$ is a reference trajectory

$$DF = \begin{pmatrix} -1 & 1 \\ -\gamma(1 + V\delta(r(t) - x_0(t))) & 0 \end{pmatrix}$$

- Since $n = 2$, there's only one angle $\theta_{12}(t) \equiv \alpha(t)$ and

$$Q^T\dot{Q} = \begin{pmatrix} 0 & \dot{\alpha} \\ -\dot{\alpha} & 0 \end{pmatrix} \qquad R(t) = \begin{pmatrix} \Delta_1(t) & r_{12}(t) \\ 0 & \Delta_2(t) \end{pmatrix}$$

QR for the buck converter (II)

- The equations to be solved are

$$\begin{aligned}
\frac{\dot{\Delta}_1}{\Delta_1} &= -\cos^2\alpha + (\gamma - 1)\sin\alpha\cos\alpha + \gamma V\delta(r(t) - x_0(t))\sin\alpha\cos\alpha \\
\frac{\dot{\Delta}_2}{\Delta_2} &= -\sin^2\alpha + (1 - \gamma)\cos\alpha\sin\alpha - \gamma V\delta(r(t) - x_0(t))\cos\alpha\sin\alpha \\
\dot{\alpha} &= -\sin^2\alpha - \cos\alpha\sin\alpha - \gamma(1 + V\delta(r(t) - x_0(t)))\cos^2\alpha
\end{aligned}$$

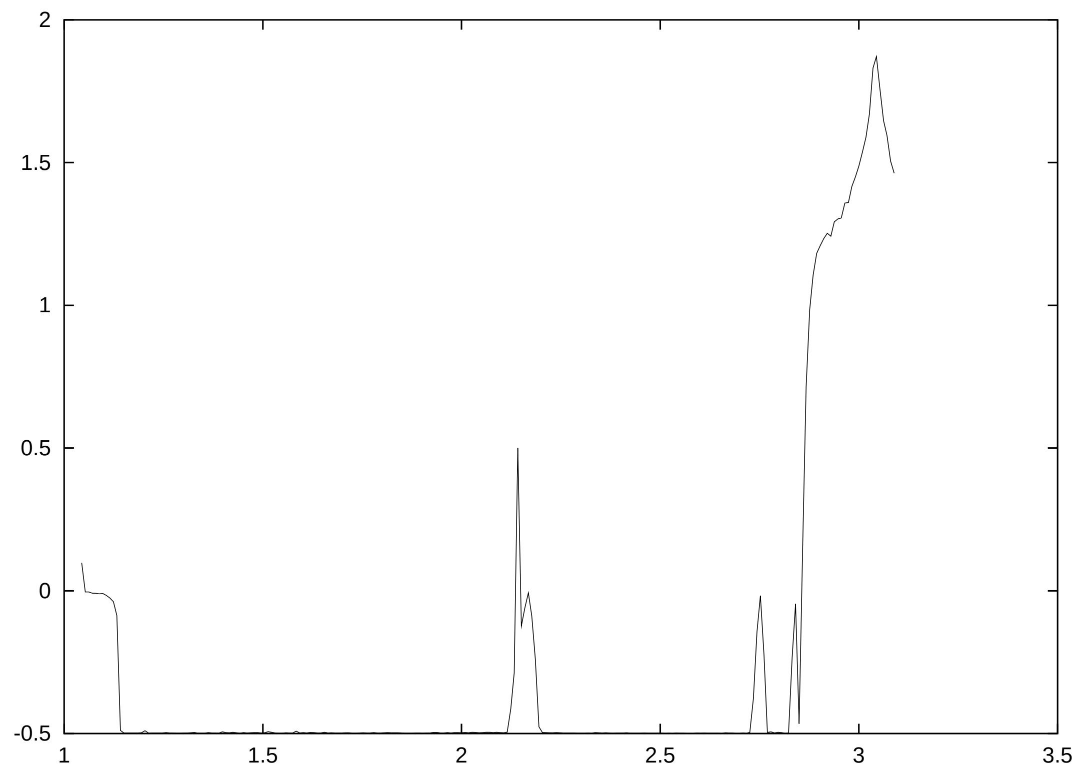
- With $\Delta_{1,2}(t) = \exp\lambda_{1,2}(t)$, one has

$$\frac{\lambda_1(t) + \lambda_2(t)}{t} = -1$$

- If $\lambda_1(t) \equiv \lambda(t)$, $\lambda(t)/t$ will asymptotically give the largest Lyapunov exponent (LLE).

LLE for the buck converter (I)

- Numerical integration, step function approximated by an $\arctan$ with high coefficient



- Gives the LLE of the dominant attracting set.

LLE for the buck converter (II)

- For a T-periodic orbit switching at $t = t_c$

$$\delta(r(t) - x_0(t)) = \frac{1}{|\dot{r}(t_c) - \dot{x}_0(t_c)|}\delta(t - t_c)$$

- If $\nu = \gamma V / |\dot{r}(t_c) - \dot{x}_0(t_c)|$ one gets

$$\begin{aligned} \dot{\alpha} &= \sin^2\alpha + \sin\alpha\cos\alpha + (\gamma + \nu\,\delta(t - t_c))\,\cos^2\alpha \\ \dot{\lambda} &= -\cos^2\alpha + (\gamma + \nu\,\delta(t - t_c) - 1)\sin\alpha\,\cos\alpha, \end{aligned}$$

- Exact integration + discontinuities at $t = t_0$:

$$\begin{aligned} \tan\alpha(t_c^+) - \tan\alpha(t_c^-) &= \nu \\ \lambda(t_c^+) - \lambda(t_c^-) &= \frac{1}{2}\log\frac{1 + \tan^2(\alpha(t_c^+))}{1 + \tan^2\alpha((t_c^-))} \end{aligned}$$

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LLE for the buck converter (III)

- Recurrence relation for $\alpha_n = \alpha(nT)$ and $\lambda_n = \lambda(nT)$:

$$
\begin{aligned}
r_{n+1} &= \frac{A+B+r_n}{1-AB-Ar_n} \\
\lambda_{n+1} &= \frac{1}{2}\log\frac{\left(\tan^2\alpha_{n+1}+1\right)\left(\tan^2\alpha_n+\tan\alpha_n+\gamma\right)}{\left(\tan^2\alpha_n+1\right)\left(\tan^2\alpha_{n+1}+\tan\alpha_{n+1}+\gamma\right)} \\
&+ \log\frac{\left((M_n+\nu)^2+M_n+\nu+\gamma\right)}{\left(M_n^2+M_n+\gamma\right)} - \frac{1}{2}T + \lambda_n
\end{aligned}
$$

where $\mu = \sqrt{\gamma - 1/4}$, $A = \tan\mu T$, $B = \nu/\mu$, $M_n = \mu r_n - 1/2$ and

$$
r_n = \tan\left(\mu t_c + \arctan\frac{\tan\alpha_n + 1/2}{\mu}\right).
$$

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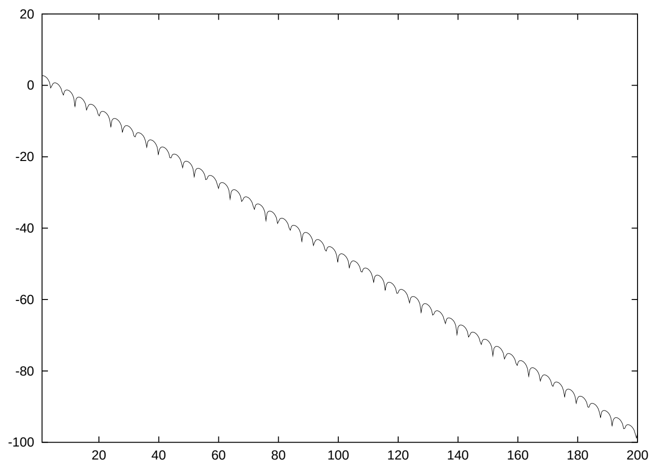
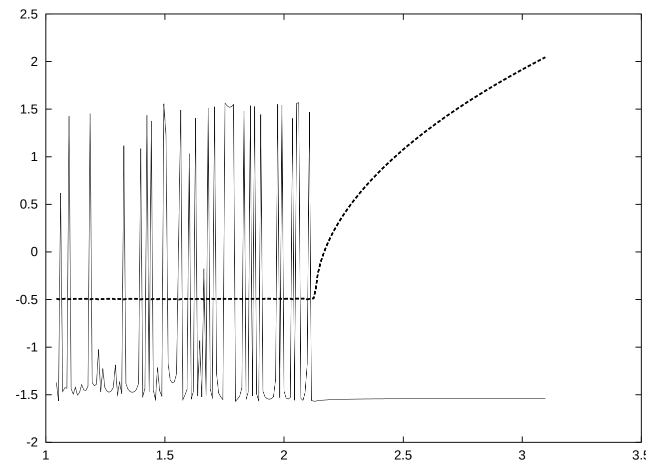
LLE for the buck converter (IV)

- Numerical iteration for λ_n (left) and asymptotic values of α_n and λ_n (right) in terms of V:

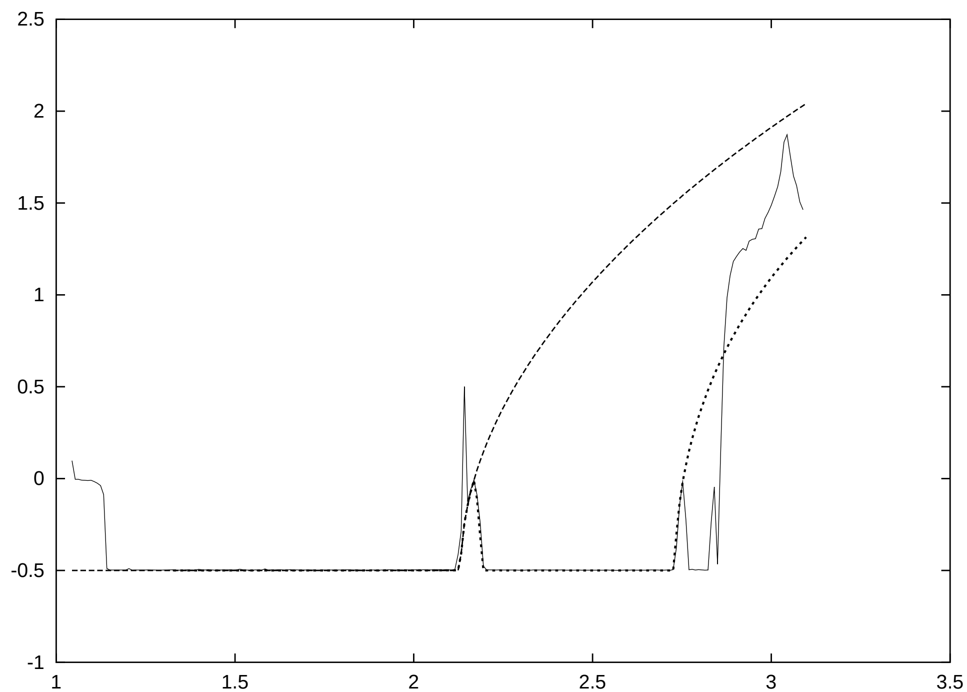




- $\lambda(t)$: linear + bounded oscillation
- When (α_n) does not converge, $\lambda = -1/2$.

Analytical results

- The above results have been proved analytically and extended to higher period orbits.
- Dominant attractor (numerical) + T-periodic (analytic) + $2T$-periodic (analytic)



Summary and things to do

- QR equations written for the buck converter
- Solved numerically for any value of V and analytically for periodic orbits.
- Analytical results reproduce numerical one when the attractor is periodic.
- *Use results for periodic orbits to compute analytical approximations of LLE in chaotic regime.*
- *Extend to higher order (corrections to linear LLE)*

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