

Lecture notes on KdV hierarchies and pseudodifferential operators

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Abstract

We review the construction of the KdV-type hierarchies of equations using the pseudodifferential operator formalism of Gelfand and Dickey and present the corresponding bihamiltonian structure.

Keywords: KdV hierarchy, Integrable systems, Bihamiltonian formalism

1 Introduction

Completely integrable systems of partial differential equations have been studied for many years.[1] The best known example is the original Korteweg-de Vries equation

$$4\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} \quad (1)$$

This equation exhibits an infinite number of nontrivial conserved quantities and has analytic solutions in spite of being nonlinear. Later, it was noticed that this was only a particular equation in an infinite set of integrable systems, a process which culminated in the work of Drinfeld and Sokolov [2], where a classification scheme based in a Lax-type formalism was constructed. Recently [3][4], integrable systems have been shown to be related to Conformal Field Theory (CFT) and this has sparked a new interest in this field.

Here we review the KdV-type hierarchies of equations using the pseudodifferential operator (PDO) formalism of Gelfand and Dickey. We emphasize the bihamiltonian structure which seems to be the fundamental mark of integrable systems. Our presentation closely follows [5].

In Section 2 we present the hierarchy of KdV equations in terms of the PDO formalism and construct the infinite set of first integrals and the soliton solutions. In Section 3 we give the algebraic scheme of an abstract hamiltonian formalism which is employed then in Section 4 to derive the bihamiltonian structure of the KdV hierarchy. We show how the second hamiltonian structure of the KdV equation gives rise to a classical representation of the Virasoro algebra. Some lengthy demonstrations are given in the appendixes.

2 KdV hierarchies

We are interested in systems of nonlinear partial differential equations which have analytic solutions and an infinite set of conserved quantities. The linear differential operator

$$L = \partial^n + u_{n-2}\partial^{n-2} + \dots + u_1\partial + u_0, \quad (2)$$

where $u_i = u_i(x)$ and $\partial = \frac{d}{dx}$, will prove to be relevant. First we need some technology.

2.1 Pseudodifferential operators

Let us consider the algebra of polynomials in u_0, u_1, \dots, u_{n-2} , and their derivatives of arbitrary order:

$$\left\{ u_i^{(j)} \right\}_{i=0, \dots, n-2}^{j=0, \dots, +\infty},$$

with real or complex coefficients. In this algebra we define a derivation obeying

$$\begin{aligned} \partial(fg) &= (\partial f)g + f(\partial g), \\ \partial u_i^{(j)} &= u_i^{(j+1)}, \end{aligned}$$

and a differential algebra, \mathcal{A} , is obtained.

Besides ∂ , other derivations ξ may be defined. We only need to know the action of ξ on the generators of the algebra. Let

$$\xi u_i^{(j)} = a_{i,j}.$$

Then, for any $f \in \mathcal{A}$,

$$\xi f = \sum_{i,k} a_{i,k} \frac{\partial f}{\partial u_i^{(k)}}.$$

Derivations commuting with ∂ will play an important role. For them we have

$$a_{i,k+1} = \xi u_i^{(k+1)} = \xi \partial u_i^{(k)} = \partial \xi u_i^{(k)} = \partial a_{i,k},$$

so $a_{i,k} = \partial^k a_i \equiv a_i^{(k)}$, with $a_{i,0} \equiv a_i$. Thus, if $[\xi, \partial] = 0$,

$$\xi f = \sum_{i=0}^{n-2} \sum_{k=0}^{+\infty} a_i^{(k)} \frac{\partial f}{\partial u_i^{(k)}}.$$

Let $a = (a_0, \dots, a_{n-2})$ design a set of $n-1$ elements of \mathcal{A} and

$$\partial_a \equiv \sum_{i=0}^{n-2} \sum_{k=0}^{+\infty} a_i^{(k)} \frac{\partial}{\partial u_i^{(k)}}. \quad (3)$$

It is easy to see that this is a derivation in \mathcal{A} . The set of all operators of this form is a Lie algebra with the Lie bracket

$$[\partial_a, \partial_b] = \partial_{\partial_a b - \partial_b a}, \quad (4)$$

Notice that $\partial_a b$ is again a set of $n-1$ elements of \mathcal{A} , with

$$(\partial_a b)_i = \sum_{j=0}^{n-2} \sum_{k=0}^{+\infty} a_j^{(k)} \frac{\partial b_i}{\partial u_j^{(k)}},$$

and expression (4) makes sense. Sometimes we will refer to ∂_a as a *vector field*, although no manifold is present. This Lie algebra will be a fundamental ingredient in the construction of the hamiltonian formalism. As a particular case, $u' = (u'_0, \dots, u'_{n-2})$ and

$$\partial_{u'} = \sum_{i=0}^{n-2} \sum_{k=0}^{+\infty} u_i^{(k+1)} \frac{\partial}{\partial u_i^{(k)}} = \partial.$$

We have the following

Proposition 2.1 *The derivation ∂_a defined by (3) commutes with ∂ .*

PROOF: First let us remark that this proposition has nothing to do with $[\partial_a, \partial_{u'}] = 0$, which is also true, where the bracket is the Lie bracket defined in (4). What we have to show is that, for any $f \in \mathcal{A}$,

$$(\partial_a \partial - \partial \partial_a) f = 0.$$

In fact, due to the derivation properties, it suffices to take as f any generator of the algebra, that is, $f = u_i^{(j)}$. Then

$$\partial \partial_a u_i^{(j)} = \sum_{k=0}^{n-2} \sum_{l=0}^{+\infty} \partial \left(a_k^{(l)} \frac{\partial u_i^{(j)}}{\partial u_k^{(l)}} \right) = \partial(a_i^{(j)}) = a_i^{(j+1)},$$

and, on the other hand,

$$\partial_a \partial u_i^{(j)} = \sum_{k=0}^{n-2} \sum_{l=0}^{+\infty} a_k^{(l)} \frac{\partial u_i^{(j+1)}}{\partial u_k^{(l)}} = a_i^{(j+1)}. \blacksquare$$

Our next step will be the definition of an integration. As long as we discuss the construction of the KdV-hierarchies, the class of functions $u_i(x)$ is not important and so, given $f \in \mathcal{A}$,

the definition of $\int f$ only requires two properties: linearity and $\int f' = 0$. Hence an integral is an homomorphism of a linear space \mathcal{A} onto another linear space \mathcal{L} such that $\partial\mathcal{A} \rightarrow 0$. We define

$$\tilde{\mathcal{A}} = \mathcal{A}/\partial\mathcal{A}.$$

We call $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ the *formal integral* and denote it by

$$\tilde{f} = \int f.$$

Any explicit integral associates an element of \mathcal{L} to each class in $\tilde{\mathcal{A}}$, and hence can be lead through our formal integral. In other words, the formal integral of $f \in \mathcal{A}$ is the equivalence class of f modulo exact derivatives.

Proposition 2.2 *The vector fields ∂_a act on the integrals according to*

$$\partial_a \tilde{f} = \int \partial_a f.$$

PROOF: We only have to show that the result does not depend on the representative chosen. If $f, g \in \tilde{f}$, we have $f - g = \partial h$ for some $h \in \mathcal{A}$. Using proposition 2.1 we get

$$\partial_a f - \partial_a g = \partial_a(f - g) = \partial_a \partial h = \partial \partial_a h$$

and so $\partial_a f$ and $\partial_a g$ belong to the same class of equivalence. ◻

Proposition 2.3 *The following relation holds*

$$\partial_a \tilde{f} = \int \sum_{i=0}^{n-2} \frac{\delta f}{\delta u_i} a_i,$$

where

$$\frac{\delta f}{\delta u_i} = \sum_{k=0}^{+\infty} (-\partial)^k \frac{\partial f}{\partial u_i^{(k)}}$$

is the standard variational derivative.

PROOF:

$$\partial_a \tilde{f} = \int \partial_a f = \int \sum_{i=0}^{n-2} \sum_{k=0}^{+\infty} a_i^{(k)} \frac{\partial f}{\partial u_i^{(k)}} = \int \sum_{i=0}^{n-2} \left(\sum_{k=0}^{+\infty} (-\partial)^k \frac{\partial f}{\partial u_i^{(k)}} \right) a_i$$

where we have used k times the property $\int f g' = - \int f' g$. ◻

Let us consider the formal series

$$X = \sum_{i=-\infty}^m X_i \partial^i, \tag{5}$$

with $X_i \in \mathcal{A}$ and m being arbitrary. These series can be added and multiplied together and multiplied by elements of \mathcal{A} . This turns the set of all such series into a module over \mathcal{A} , which is called the ring R of pseudodifferential operators (PDO). The *pseudo* comes from the

negative powers of ∂ , which act as an integration. We will use the notations R_+ and R_- to design the set of differential and integral operators, respectively:

$$R_+ = \left\{ \sum_{i \geq 0} X_i \partial^i \right\},$$

$$R_- = \left\{ \sum_{i < 0} X_i \partial^i \right\}.$$

Other notations such as

$$R_{(-\infty, -2)} = \left\{ \sum_{i=-\infty}^{-2} X_i \partial^i \right\}$$

will be also used. Projections of a PDO on its differential and integral parts will be also denoted by the subscripts $+$ and $-$.

We will frequently use the following formula, which is a generalization of Leibnitz's rule:

$$\partial^k f = \sum_{i=0}^{+\infty} \binom{k}{i} f^{(i)} \partial^{k-i}, \quad (6)$$

with $k \in \mathbb{Z}$ and $f \in \mathcal{A}$. Here

$$\binom{k}{i} = \frac{k(k-1) \cdots (k-i+1)}{i!}.$$

If $k \geq 0$, this series terminates because

$$\binom{k}{i} = 0 \quad \text{if } i > k.$$

For instance

$$\begin{aligned} \partial f &= f \partial + f', \\ \partial^2 f &= f \partial^2 + 2f' \partial + f''. \end{aligned}$$

For $k < 0$ the series is actually infinite and we have, for instance,

$$\partial^{-1} f = \sum_{i=0}^{+\infty} (-1)^i f^{(i)} \partial^{-1-i},$$

where we have used

$$\binom{-1}{i} = (-1)^i.$$

Using Leibnitz's rule we can rewrite any PDO in the *dual* left form

$$X = \sum_{i=-\infty}^m \partial^i X_i^*.$$

Definition 2.4 *The residue of a PDO is the coefficient of ∂^{-1} in its right form:*

$$\text{res} \left(\sum X_i \partial^i \right) = X_{-1}.$$

Lemma 2.5 *If a PDO X is rewritten in its left form one as $X_{-1}^* = X_{-1}$.*

PROOF: $\partial^{-1}X_{-1}^* = X_{-1}^*\partial^{-1} + o(\partial^{-2})$, and no other term can contribute to ∂^{-1} . \blacklozenge

Proposition 2.6 *For any PDOs X and Y there exists an $h \in \mathcal{A}$ such that*

$$\text{res}[X, Y] = \partial h.$$

PROOF: Due to the linearity of the above expression, it suffices to verify it for monoms $X = X_i\partial^i$ and $Y = Y_j\partial^j$ (no summation implied). Moreover, we may restrict ourselves to, say, $i \geq 0$, $j < 0$, $i + j \geq -1$, since otherwise we may take $h = 0$ (or interchange i and j). Then h is given by

$$h = \binom{i}{i+j+1} \sum_{\alpha=0}^{i+j} (-1)^\alpha X_i^{(\alpha)} Y_j^{(i+j-\alpha)}.$$

Indeed

$$\begin{aligned} [X, Y] &= X_i \sum_{k=0}^i \binom{i}{k} Y_j^{(k)} \partial^{j+i-k} - Y_j \sum_{k=0}^{+\infty} \binom{j}{k} X_i^{(k)} \partial^{i+j-k} \\ \text{res}[X, Y] &= X_i \binom{i}{j+i+1} Y_j^{(j+i+1)} - Y_j \binom{j}{j+i+1} X_i^{(j+i+1)}. \end{aligned}$$

But

$$\binom{j}{i+j+1} = (-1)^{i+j+1} \binom{i}{i+j+1}$$

so, finally,

$$\text{res}[X, Y] = \binom{i}{i+j+1} \left(X_i Y_j^{(j+i+1)} + (-1)^{i+j} Y_j X_i^{(j+i+1)} \right) = \partial h. \blacklozenge$$

Corollary 2.7

$$\int \text{res}(XY) = \int \text{res}(YX).$$

Proposition 2.8 *If $X = \sum_{i=-\infty}^m X_i \partial^i$ and $X_m = \text{const.} \neq 0$, then X^{-1} and $X^{1/m}$ exist, are unique, and commute with X .*

PROOF: We may put $X_m = 1$ without loss of generality. Let us write

$$X^{-1} = \partial^{-m} + Y_{-m-1} \partial^{-m-1} + Y_{-m-2} \partial^{-m-2} + \dots$$

Then $1 = XX^{-1}$ determines the unknown coefficients order by order. The first few orders are

$$\begin{aligned} 1 &= 1 \\ 0 &= Y_{-m-1} + X_{m-1} \\ 0 &= mY_{-m-1}' + X_{m-1}Y_{-m-1} + Y_{-m-2} + X_{m-2} \end{aligned}$$

We obtain a sequence of recurrence equations of the form $Y_{-m-k} = -X_{m-k} + Q_k$ where Q_k are differential polynomials in $\{X_i\}$ and $\{Y_j\}$ with $j > -m - k$. The right reciprocal can be computed and coincides with the left one. $X^{1/m}$ is constructed in the same way using $(X^{1/m})^m = 1$. Further, $[X, X^{-1}] = 1 - 1 = 0$. From $X = X^{1/m} \dots X^{1/m}$ we obtain, commuting both sides with $X^{1/m}$, $[X, X^{1/m}] = 0 + \dots + 0 = 0$. \blacklozenge

Corollary 2.9 *For arbitrary p , the PDO $X^{p/m}$ can be constructed. The highest term is ∂^p and $[X, X^{p/m}] = 0$.*

2.2 Lax pairs and KdV-type hierarchies

Let us introduce a coupling in the space of PDO's:

$$\langle X, Y \rangle = \int \text{res}(XY). \quad (7)$$

If $X = \sum_{i=0}^{n-2} X_i \partial^i \in R_{(0, n-2)}$, then ∂_X will denote the derivation of the type (3) corresponding to the set of coefficients $X = (X_0, \dots, X_{n-2})$ (beware of the double meaning of X).

Now we come back to the linear operator L defined in (2) and introduce, for any $f \in \mathcal{A}$,

$$\frac{\delta f}{\delta L} = \sum_{i=0}^{n-2} \partial^{-i-1} \frac{\delta f}{\delta u_i} \in R_- / R_{(-\infty, -n)}. \quad (8)$$

Let us explain the notation. $\delta f / \delta L$ is, in fact, an integral operator with unbounded negative indexes because the negative powers of ∂ can be indefinitely pushed to the right with the help of (6). Hence, it would be incorrect to write $\delta f / \delta L \in R_{(-n+1, -1)}$. However, the notation employed corresponds to the *dual* space of $R_{(0, n-2)}$ under the coupling (7). Indeed, it is easy to see that if $X \in R_{(0, n-2)}$ then no term of Y with a power of ∂^{-1} greater than $n-1$ will contribute to the coupling $\langle X, Y \rangle$. This is exploited in the following

Proposition 2.10

$$\partial_X \tilde{f} = \int \text{res} \left(X \frac{\delta f}{\delta L} \right) = \left\langle X, \frac{\partial f}{\partial L} \right\rangle.$$

PROOF:

$$\begin{aligned} \partial_X \tilde{f} &= \int \partial_X f = \int \sum_{i=0}^{n-2} \sum_{k=0}^{+\infty} X_i^{(k)} \frac{\partial f}{\partial u_i^{(k)}} = \int \sum_{i=0}^{n-2} X_i \frac{\delta f}{\delta u_i} \\ &= \int \sum_{i=0}^{n-2} \text{res} \left(X_i \partial^{-i-1} \frac{\delta f}{\delta u_i} \right) = \int \text{res} \left(X \frac{\delta f}{\delta L} \right). \quad \spadesuit \end{aligned}$$

Lemma 2.11

$$\partial_X L = X.$$

PROOF:

$$\partial_X L = \partial_X \left(\sum_{i=0}^{n-2} u_i \partial^i + \partial^n \right) = \sum_{i=0}^{n-2} (\partial_X u_i) \partial^i = \sum_{i,k=0}^{n-2} \sum_{l=0}^{+\infty} X_k^{(l)} \frac{\partial u_i}{\partial u_k^{(l)}} \partial^i = \sum_{i=0}^{n-2} X_i \partial^i. \quad \spadesuit$$

Let

$$P_m = \left(L^{m/n} \right)_+ \equiv L_+^{m/n} \in R_{(0, m)}. \quad (9)$$

Proposition 2.12 *The commutator $[P_m, L]$ belongs to $R_{(0, n-2)}$.*

PROOF: Obviously $[P_m, L] \in R_+$. Furthermore

$$[P_m, L] = [L^{m/n} - L_-^{m/n}, L] = -[L_-^{m/n}, L].$$

In the right-hand we have the commutator of two operators of orders -1 and n , so the leading term is, at most, of order $n + (-1) - 1 = n - 2$, which completes the proof. \spadesuit

Definition 2.13 A differential operator P with coefficients belonging to \mathcal{A} together with L make up a Lax pair if $[P, L] \in R_{(0, n-2)}$.

Thus, for any m we have constructed an operator P_m which, together with L , make up a Lax pair. Since $[P_m, L] \in R_{(0, n-2)}$, the derivation $\partial_{[P_m, L]}$ can be defined and, according to lemma 2.11, we get

$$\partial_{[P_m, L]}L = [P_m, L].$$

Now let $\{u_i\}$ depend on an additional parameter t . We write a differential equation

$$\partial_t L = \partial_{[P_m, L]}L,$$

or

$$\dot{L} = [P_m, L]. \quad (10)$$

This is equivalent to a system of partial differential equations for $\{u_i\}_{i=0, \dots, n-2}$ with independent variables t and x . The system is determined by two integers, m and n .

Definition 2.14 The system of equations (10) with fixed n and all m is called the n th KdV-type hierarchy of equations. The 2nd hierarchy is called the KdV hierarchy in the narrow sense. For $m = 3$ the KdV hierarchy in the narrow sense yields the KdV equation (1).

Let us consider some examples.

1. **The KdV equation.** For $n = 2$ we have $L = \partial^2 + u$. We get

$$\begin{aligned} L^{1/2} &= \partial + \frac{u}{2}\partial^{-1} - \frac{u'}{4}\partial^{-2} + o(\partial^{-3}) \\ P_3 &= L_+^{3/2} = \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u' \end{aligned}$$

and

$$[P_3, L] = \frac{1}{4}u''' + \frac{3}{2}uu',$$

so the equation obeyed by L is

$$\frac{d}{dt}(\partial^2 + u) = \frac{1}{4}u''' + \frac{3}{2}uu',$$

and we finally get the original KdV equation

$$4\dot{u} = u''' + 6uu'.$$

2. **The Boussinesq equation.** We consider now $n = 3$, with

$$L = \partial^3 + u\partial + v$$

and $m = 2$. We compute

$$\begin{aligned} L^{1/3} &= \partial + \frac{1}{3}u\partial^{-1} + o(\partial^{-2}) \\ P_2 &= L_+^{2/3} = \partial^2 + \frac{2}{3}u \end{aligned}$$

and

$$[P_2, L] = (2v' - u'')\partial + v'' - \frac{2}{3}u''' - \frac{2}{3}uu'.$$

One gets thus the system of equations

$$\begin{aligned}\dot{u} &= 2v' - u'', \\ \dot{v} &= v'' - \frac{2}{3}u''' - \frac{2}{3}uu'.\end{aligned}$$

One can eliminate v between the two equations and the result is the Boussinesq equation:

$$\ddot{u} = -\frac{1}{3}u''' - \frac{4}{3}(uu)'$$

2.3 First integrals

A first integral is a functional $\tilde{f} = \int f$, $f \in \mathcal{A}$, which is conserved by virtue of (10):

$$\begin{aligned}0 &= \partial_t \tilde{f} = \int \partial_t f = \int \sum_{i=0}^{n-2} \frac{\delta f}{\delta u_i} \dot{u}_i = \int \text{res} \left(\sum_{j=0}^{n-2} \dot{u}_j \partial^j \sum_{i=0}^{n-2} \partial^{-i-1} \frac{\delta f}{\delta u_i} \right) \\ &= \int \text{res} \left(\dot{L} \frac{\delta f}{\delta L} \right) = \int \text{res} \left([P_m, L] \frac{\delta f}{\delta L} \right).\end{aligned}$$

Lemma 2.15 For any $k \in$

$$\partial_t L^{k/n} = [P_m, L^{k/n}].$$

PROOF: Let $L^{1/n} = \partial + v_{-1}\partial^{-1} + \dots$, where all the v_i are differential polynomials in $\{u_j\}$ and viceversa. If we guess an equation of motion for the v_i , *i.e.* an expression for $\partial_t L^{1/n}$, which gives the correct evolution for L , this will prove the correctness of our assumption. Try $\partial_t L^{1/n} = [P_m, L^{1/n}]$. This works because both ∂_t and the commutator act as derivations on R :

$$\begin{aligned}\partial_t L &= \partial_t (L^{1/n})^n = \sum_{i=0}^{n-1} (L^{1/n})^i (\partial_t L^{1/n}) (L^{1/n})^{n-i-1} \\ &= \sum_{i=0}^{n-1} (L^{1/n})^i [P_m, L^{1/n}] (L^{1/n})^{n-i-1} = [P_m, (L^{1/n})^n] = [P_m, L]\end{aligned}$$

as desired. Now the rest is easy. For $k > 1$ the relation can be proved using the previous result and $L^{k/n} = (L^{1/n})^k$, while for $-k < 0$, using $\partial_t (L^{-k/n} L^{k/n}) = 0$, one gets

$$\begin{aligned}\partial_t L^{-k/n} &= -L^{-k/n} (\partial_t L^{k/n}) L^{-k/n} = -L^{-k/n} [P_m, L^{k/n}] L^{-k/n} \\ &= -[L^{-k/n}, P_m] = [P_m, L^{-k/n}]. \spadesuit\end{aligned}$$

Proposition 2.16 The functionals

$$J_k = \int \text{res} L^{k/n}, \quad k = 1, 2, 3, \dots$$

are first integrals of all the equations of the n th hierarchy.

PROOF:

$$\partial_t J_k = \int \text{res} \partial_t L^{k/n} = \int \text{res} [P_m, L^{k/n}] = 0$$

due to corollary 2.7. \spadesuit

Notice that if k is a multiple of n the first integral is trivial. As an example, we have, for the KdV equation,

$$\begin{aligned} J_1 &= \int u, \\ J_3 &= \int u^2, \\ J_5 &= \int (2u^3 - (u')^2). \end{aligned}$$

2.4 Soliton solutions

Besides having an infinite set of first integrals, equations (10) possess exact, analytic solutions the best known of which are the solitons. We will give a method to construct a differential operator L obeying (10) whose coefficients will be explicit functions of x and t . In this way, solutions $u_i(x, t)$ will be obtained.

Let N be an arbitrary natural number (it will turn out to represent the number of solitons of the solution). Let n and m be fixed and

$$y_k(x, t) = \sum_{\epsilon} a_{k,\epsilon} \exp(\epsilon \alpha_k x + \epsilon^m \alpha_k^m t) \quad k = 1, \dots, N. \quad (11)$$

Here the summation is over the n n th roots of unity, that is, $\epsilon^n = 1$. $\{\alpha_k\}$ and $\{a_{k,\epsilon}\}$ are complex numbers such that $\alpha_k \neq \alpha_l$ if $k \neq l$ and usually $a_{k,1} = 1$ (this is just an overall constant). We construct

$$\phi = \frac{1}{\Delta} \begin{vmatrix} y_1 & \cdots & y_N & 1 \\ y_1' & \cdots & y_N' & \partial \\ \vdots & & \vdots & \vdots \\ y_1^{(N-1)} & \cdots & y_N^{(N-1)} & \partial^{N-1} \\ y_1^{(N)} & \cdots & y_N^{(N)} & \partial^N \end{vmatrix} \quad (12)$$

where in the expansion of the determinant the differential operators of the last column do not act on the elements of the other columns. Δ is the Wronskian of $\{y_1, \dots, y_N\}$.

Lemma 2.17 *The functions $\{y_k\}$ have the following properties*

$$\partial_t y_k = \partial^m y_k, \quad \partial^n y_k = \alpha_k^n y_k, \quad \phi y_k = 0.$$

PROOF: The proof is obvious. \blacksquare

Now we construct L by dressing the operator ∂^n with the help of the operator ϕ and its inverse:

$$L = \phi \partial^n \phi^{-1}. \quad (13)$$

Proposition 2.18 *L defined by (13) is, in fact, a differential operator with highest term ∂^n and satisfies*

$$\dot{L} = [P_m, L]$$

with $P_m = L_+^{m/n}$.

PROOF: First let us show that the highest term is indeed ∂^n . We may rewrite (13) as $L = (\phi\partial^{-N})\partial^n(\phi\partial^{-N})^{-1}$. Since ϕ is a differential operator of order N we may write

$$\phi\partial^{-N} = 1 + a_{-1}\partial^{-1} + \dots + a_{-N}\partial^{-N}$$

and

$$(\phi\partial^{-N})^{-1} = \sum_{i=0}^{+\infty} b_{-i}\partial^{-i}, \quad b_0 = 1.$$

Then

$$L = (1 + a_{-1}\partial^{-1} + \dots + a_{-N}\partial^{-N})\partial^n \sum_{i=-\infty}^n u_i\partial^i$$

with $u_n = 1$. In order to show that L has no negative powers of ∂ , let us write $L = L_+ + L_-$. (13) becomes

$$L_+\phi - \phi\partial^n = -L_-\phi.$$

The right-hand side is an operator of order less than N , and thus so is the left-hand side, which, in addition, is obviously differential. It has the property

$$(L_+\phi - \phi\partial^n)y_k = 0, \quad k = 1, \dots, N.$$

But if a differential operator of order less than N sends to zero N linearly independent functions, it is identically zero. Thus $L_-\phi = 0$ and, since ϕ is invertible, $L_- = 0$. Hence $L = L_+$, a pure differential operator.

It remains to find the equation obeyed by L . Fiddling with (13) we get $L^{1/n} = \phi\partial\phi^{-1}$, $L^{m/n} = \phi\partial^m\phi^{-1}$ and

$$L_+^{m/n}\phi - \phi\partial^m = -L_-^{m/n}\phi.$$

Denote by Q the right-hand side of this equation. The order of Q is less than N and, looking at the left-hand side, it is a differential operator. Let $P_m = L_+^{m/n}$. We have

$$P_m\phi - \phi\partial^m = Q. \tag{14}$$

Now we derive the identity $\phi y_k = 0$ with respect to t and use lemma 2.17 and (14):

$$0 = \phi\dot{y}_k + \dot{\phi}y_k = \phi\partial^m y_k + \dot{\phi}y_k = \dot{\phi}y_k + P_m\phi y_k - Qy_k = (\dot{\phi} - Q)y_k.$$

The differential operator $\dot{\phi} - Q$ of order less than N sends to zero N independent functions y_k . Hence $\dot{\phi} = Q$ and (14) becomes

$$P_m\phi - \phi\partial^m = \dot{\phi}.$$

Finally

$$\begin{aligned} \dot{L} &= \dot{\phi}\partial^n\phi^{-1} - \phi\partial^n\phi^{-1}\dot{\phi}\phi^{-1} \\ &= (P_m\phi - \phi\partial^m)\partial^n\phi^{-1} - \phi\partial^n\phi^{-1}(P_m\phi - \phi\partial^m)\phi^{-1} \\ &= P_m\phi\partial^n\phi^{-1} - \phi\partial^n\phi^{-1}P_m = [P_m, L]. \blacksquare \end{aligned}$$

Let us give a couple of examples of this method of construction.

1. **A solitary soliton of the KdV equation.** We set $n = 2$, $m = 3$, $N = 1$ and

$$y(x, t) = e^{\alpha x + \alpha^3 t} + ae^{-\alpha x - \alpha^3 t}.$$

One gets

$$\phi = \partial - \frac{y'}{y}.$$

Setting $\varphi = y'/y$ we compute ϕ^{-1} to be

$$\phi^{-1} = \partial^{-1} + \varphi\partial^{-2} + (\varphi^2 - \varphi')\partial^{-3} + o(\partial^{-4}).$$

Then

$$\phi\partial^2\phi^{-1} = \partial^2 + 2\varphi'.$$

This is an exact calculation. The terms $o(\partial^{-4})$ in ϕ^{-1} do not contribute. Thus we have arrived at

$$u(x, t) = 2\varphi'.$$

An explicit computation gives

$$u(x, t) = \frac{2\alpha^2}{\cosh^2(\alpha x + \alpha^3 t - 1/2 \log a)}.$$

This is a typical *soliton* profile propagating with constant velocity, which depends on α . Notice that the amplitude also depends on α . This is quite meaningless in this situation, but not in the next example.

2. **A two-soliton solution of the KdV equation.** Again $n = 2$, $m = 3$ but now $N = 2$ and

$$\begin{aligned} y_1(x, t) &= e^A + ae^{-A} \\ y_2(x, t) &= e^B + be^{-B} \end{aligned}$$

where $A = \alpha x + \alpha^3 t$ and $B = \beta x + \beta^3 t$. Using the notations

$$\begin{aligned} A^\pm &= e^A \pm ae^{-A} \\ B^\pm &= e^B \pm be^{-B} \end{aligned}$$

the Wronskian can be expressed as $\Delta = \beta A^+ B^- - \alpha A^- B^+$ and

$$\phi = \frac{\alpha\beta}{\Delta}(\beta A^- B^+ - \alpha B^- A^+) - \frac{\beta^2 - \alpha^2}{\Delta} A^+ B^+ \partial + \partial^2 \equiv \partial^2 + C\partial + D.$$

Now

$$\phi^{-1} = \partial^{-2} - C\partial^{-3} + (C^2 + 2C' - D)\partial^{-4} + o(\partial^{-5})$$

and

$$\phi\partial^2\phi^{-1} = \partial^2 - 2C'.$$

Hence $u(x, t) = -2C'$ but we do not give the explicit form, which is quite unmanageable. Plotting this solution on a computer with convenient values of the constants (one has to avoid singularities in u for all values of A and B) one gets two solitary waves of the kind exhibited in the previous example, with velocities depending on the amplitude (now this has an absolute meaning) and with the taller wave overcoming the shorter, interacting and emerging again as two solitary waves with the same velocities as before the collision.

3 Elements of a hamiltonian formalism

In this section we present the algebraic framework underlying any hamiltonian formalism, without making any reference to manifolds. This is precisely what is needed for a hamiltonian treatment of infinite dimensional systems. In the next section we will implement these ideas to construct the bihamiltonian structure of the KdV hierarchy of equations. We omit most of the proofs, which are quite straightforward.

To define a hamiltonian structure we need the following elements:

1. \mathcal{G} , a Lie algebra.
2. Ω^0 a linear space in which the elements of \mathcal{G} act as left operators:

$$\begin{aligned} \xi : \Omega^0 &\longrightarrow \Omega^0 \\ f &\longmapsto \xi f \quad \forall \xi \in \mathcal{G}, \end{aligned}$$

such that

$$(\xi_1 \xi_2 - \xi_2 \xi_1) f = [\xi_1, \xi_2] f.$$

3. A set $\{\Omega^k\}_{k=1,2,\dots}$ of spaces of k -linear Ω^0 -valued functionals on \mathcal{G} :

$$w^k(\xi_1, \dots, \xi_k) \in \Omega^0$$

$$\forall \xi_1, \dots, \xi_k \in \mathcal{G}, \forall w^k \in \Omega^k.$$

4. An inner derivation $i(\xi) : \Omega^k \rightarrow \Omega^{k-1}$, $\forall \xi \in \mathcal{G}$, such that $i(\xi)\Omega^0 = \{0\}$ and

$$(i(\xi)\alpha)(\xi_1, \dots, \xi_{k-1}) = \alpha(\xi, \xi_1, \dots, \xi_{k-1})$$

$\forall \alpha \in \Omega^k$. In particular, if $\alpha \in \Omega^1$,

$$i(\xi)\alpha = \alpha(\xi).$$

We assume the following non-degeneracy conditions: $\forall \xi \in \mathcal{G}$, $\xi \neq 0$, there exists $\alpha \in \Omega^1$ such that $\alpha(\xi) \neq 0$, and $\forall \alpha \in \Omega^1$, $\alpha \neq 0$, there exists $\xi \in \mathcal{G}$ such that $\alpha(\xi) \neq 0$. We will use the notation $\langle \alpha, \xi \rangle$ for $\alpha(\xi)$, the coupling of elements of \mathcal{G} and Ω^1 .

5. An exterior derivation

$$d : \Omega^k \rightarrow \Omega^{k+1}$$

defined by the formula

$$\begin{aligned} (d\alpha)(\xi_1, \dots, \xi_{k+1}) &= \sum_{l=1}^{k+1} (-1)^{l-1} \xi_l \alpha(\xi_1, \dots, \hat{\xi}_l, \dots, \xi_{k+1}) \\ &+ \sum_{l,p=1}^{k+1} (-1)^{l+p} \alpha([\xi_l, \xi_p], \xi_1, \dots, \hat{\xi}_l, \dots, \hat{\xi}_p, \dots, \xi_{k+1}). \end{aligned} \quad (15)$$

In particular, if $f \in \Omega^0$,

$$df(\xi) = \xi f.$$

Now we can start to construct the hamiltonian structure.

Proposition 3.1 *The operator d has the property $d^2 = 0$.*

Definition 3.2 The operator

$$L_\xi = i(\xi)d + di(\xi)$$

is called the Lie derivative in the direction of $\xi \in \mathcal{G}$.

In particular, if $f \in \Omega^0$,

$$L_\xi f = i(\xi)df = df(\xi) = \xi f$$

would be, in a classical context, the derivative of the function f along ξ .

Proposition 3.3 $\forall \alpha \in \Omega^1$,

$$\begin{aligned} \xi(\alpha(\xi_1, \dots, \xi_k)) &= (L_\xi \alpha)(\xi_1, \dots, \xi_k) \\ &+ \sum_{l=1}^k \alpha(\xi_1, \dots, \xi_{l-1}, [\xi, \xi_l], \xi_{l+1}, \dots, \xi_k). \end{aligned}$$

Proposition 3.4 The following relations hold

1. $[L_\xi, L_\eta] = L_{[\xi, \eta]} \quad \forall \eta, \xi \in \mathcal{G}$,
2. $[L_\xi, i(\eta)] = i([\xi, \eta]) \quad \forall \eta, \xi \in \mathcal{G}$.

Consider now a linear skew-symmetric mapping $H : \Omega^1 \rightarrow \mathcal{G}$,

$$\langle H\alpha, \beta \rangle = -\langle \alpha, H\beta \rangle \quad \forall \alpha, \beta \in \Omega^1.$$

The Schouten bracket of two such mappings is the trilinear mapping

$$[H, K] : \Omega^1 \times \Omega^1 \times \Omega^1 \longrightarrow \Omega^0$$

defined by

$$[H, K](\alpha_1, \alpha_2, \alpha_3) = \langle KL_{H\alpha_1}\alpha_2, \alpha_3 \rangle + \langle HL_{K\alpha_1}\alpha_2, \alpha_3 \rangle + \text{cyclic permutations.} \quad (16)$$

Definition 3.5 A mapping H obeying $[H, H] = 0$ is called hamiltonian.

Proposition 3.6 The condition $[H, H] = 0$ is equivalent to

$$[H\alpha, H\beta] = H(i(H\alpha)d\beta - i(H\beta)d\alpha + di(H\alpha)\beta) \quad \forall \alpha, \beta \in \Omega^1.$$

PROOF: Let $\gamma \in \Omega^1$. Using 3.2 and the skewness of H we have

$$\begin{aligned} &\langle H(i(H\alpha)d\beta - i(H\beta)d\alpha + di(H\alpha)\beta) - [H\alpha, H\beta], \gamma \rangle \\ &= \langle HL_{H\alpha}\beta, \gamma \rangle + \langle i(H\beta)d\alpha, H\gamma \rangle - \langle [H\alpha, H\beta], \gamma \rangle \end{aligned} \quad (17)$$

But $\langle i(H\beta)d\alpha, H\gamma \rangle = (i(H\beta)d\alpha)(H\gamma) = (d\alpha)(H\beta, H\gamma)$ and, using (15), we get

$$\begin{aligned} (17) &= \langle HL_{H\alpha}\beta, \gamma \rangle + (H\beta)\alpha(H\gamma) - (H\gamma)\alpha(H\beta) - \alpha([H\beta, H\gamma]) \\ &\quad - \langle [H\alpha, H\beta], \gamma \rangle \\ &= \langle HL_{H\alpha}\beta, \gamma \rangle - (H\beta)\langle \gamma, H\alpha \rangle - (H\gamma)\langle \alpha, H\beta \rangle \\ &\quad - \langle [H\beta, H\gamma], \alpha \rangle - \langle [H\alpha, H\beta], \gamma \rangle \end{aligned} \quad (18)$$

Finally, using 3.3 to expand the second and third terms and using again the skewness of H , we obtain

$$\begin{aligned}
(18) &= \langle HL_{H\alpha}\beta, \gamma \rangle + \langle HL_{H\beta}\gamma, \alpha \rangle - \langle \gamma, [H\beta, H\alpha] \rangle \\
&\quad + \langle HL_{H\gamma}\alpha, \beta \rangle - \langle \alpha, [H\gamma, H\beta] \rangle - \langle \alpha, [H\beta, H\gamma] \rangle - \langle \gamma, [H\alpha, H\beta] \rangle \\
&= \langle HL_{H\alpha}\beta, \gamma \rangle + \text{cyclic permutations} = \frac{1}{2}[H, H](\alpha, \beta, \gamma). \spadesuit
\end{aligned}$$

Corollary 3.7 *If H is hamiltonian then $l = \text{Im}H$ is a Lie subalgebra.*

Definition 3.8 *In l we define a 2-form by the formula*

$$w(H\alpha, H\beta) = \langle H\alpha, \beta \rangle \quad \forall \alpha, \beta \in \Omega^1.$$

Proposition 3.9 *If H is hamiltonian then w is closed.*

PROOF: Let $\xi_i = H\alpha_i$, $i = 1, 2, 3$. Then, using (15) and the definition of ω ,

$$\begin{aligned}
d\omega(\xi_1, \xi_2, \xi_3) &= \xi_1\omega(\xi_2, \xi_3) - \omega([\xi_1, \xi_2], \xi_3) + \text{c.p.} \\
&= H\alpha_1\langle H\alpha_2, \alpha_3 \rangle - \langle [H\alpha_1, H\alpha_2], \alpha_3 \rangle + \text{c.p.}
\end{aligned}$$

Now using 3.3,

$$\begin{aligned}
d\omega(\xi_1, \xi_2, \xi_3) &= \langle H\alpha_2, L_{H\alpha_1}\alpha_3 \rangle + \text{c.p.} = -\langle HL_{H\alpha_1}\alpha_3, \alpha_2 \rangle + \text{c.p.} \\
&= \frac{1}{2}[H, H](\alpha_1, \alpha_2, \alpha_3). \spadesuit
\end{aligned} \tag{19}$$

Proposition 3.10 *If $l = \text{Im}H$ is a Lie subalgebra of \mathcal{G} and w is closed, then H is hamiltonian.*

PROOF: If l is a Lie subalgebra then (15) holds and the result follows from (19). \spadesuit

Notice that H and ω are inverses of each other in the following sense. We have, $\forall \alpha, \beta \in \Omega^1$,

$$\omega(H\alpha, H\beta) = i(H\beta)i(H\alpha)\omega = \langle H\beta, i(H\alpha)\omega \rangle,$$

but, on the other hand,

$$\omega(H\alpha, H\beta) = \langle H\alpha, \beta \rangle = -\langle \alpha, H\beta \rangle = \langle H\beta, -\alpha \rangle$$

which implies

$$\langle H\beta, \alpha + i(H\alpha)\omega \rangle = 0 \tag{20}$$

If $H\Omega^1 = \mathcal{G}$, the nondegeneracy would imply $\alpha = -i(H\alpha)\omega$, but this may not be the case. What can be shown with no restriction is the following

Lemma 3.11

$$H(i(H\alpha)\omega) = -H\alpha, \quad \forall \alpha \in \Omega^1.$$

PROOF: We have, $\forall \beta \in \Omega^1$,

$$\begin{aligned} \langle H(i(H\alpha)\omega), \beta \rangle &= -\langle i(H\alpha)\omega, H\beta \rangle \\ &= -i(H\beta)i(H\alpha)\omega = -\omega(H\alpha, H\beta) \\ &= \langle -H\alpha, \beta \rangle \end{aligned}$$

and the result follows from the nondegeneracy. \blacksquare

To each $f \in \Omega^0$ we may assign an element of \mathcal{G} according to

$$\xi_f = H(df) \equiv Hdf. \quad (21)$$

Definition 3.12 The Poisson bracket of two elements $f, g \in \Omega^0$ is defined by

$$\{f, g\} = \xi_f g = \langle dg, \xi_f \rangle = \langle Hdf, dg \rangle.$$

Proposition 3.13 The Poisson bracket has the following properties

1. $\{f, g\} = -\{g, f\}$,
2. $\xi_{\{f, g\}} = [\xi_f, \xi_g]$,
3. $\{f, \{g, h\}\} + \text{cyclic permutations} = 0$.

PROOF: The first item in the proposition is obvious. With respect to the second, the closeness of ω implies that $\forall \alpha \in \Omega^1$,

$$\begin{aligned} 0 &= (d\omega)(\xi_f, \xi_g, H\alpha) \\ &= \xi_f \omega(\xi_g, H\alpha) - \xi_g \omega(\xi_f, H\alpha) + H\alpha \omega(\xi_f, \xi_g) \\ &\quad - \omega([\xi_f, \xi_g], H\alpha) + \omega([\xi_f, H\alpha], \xi_g) - \omega([\xi_g, H\alpha], \xi_f) \\ &= -\xi_f H\alpha g + \xi_g H\alpha f + H\alpha \{f, g\} - \omega([\xi_f, \xi_g], H\alpha) + [\xi_f, H\alpha]g - [\xi_g, H\alpha]f \\ &= -H\alpha \xi_f g + H\alpha \xi_g f + H\alpha \{f, g\} - \omega([\xi_f, \xi_g], H\alpha) \\ &= -H\alpha \{f, g\} - (i([\xi_f, \xi_g])\omega)(H\alpha) \\ &= (-d\{f, g\} - i([\xi_f, \xi_g])\omega)(H\alpha) \\ &= \langle H(d\{f, g\} + i([\xi_f, \xi_g])\omega), \alpha \rangle \end{aligned}$$

and, due to the nondegeneracy of the coupling,

$$H(d\{f, g\} + i([\xi_f, \xi_g])\omega) = 0, \quad (22)$$

But $\xi_f, \xi_g \in l$ and, due to the fact that $l = \text{Im}H$ is a subalgebra, $[\xi_f, \xi_g] \in l$, so we can use lemma 3.11 to conclude that

$$H(d\{f, g\}) = [\xi_f, \xi_g]$$

and the result follows immediately.

With respect to the Jacoby identity, one has, using the previous result,

$$\begin{aligned} 0 &= (d\omega)(\xi_{h_1}, \xi_{h_2}, \xi_{h_3}) \\ &= \xi_{h_1} \omega(\xi_{h_2}, \xi_{h_3}) - \omega([\xi_{h_1}, \xi_{h_2}], \xi_{h_3}) + \text{c.p.} \\ &= \xi_{h_1} \{h_2, h_3\} - [\xi_{h_1}, \xi_{h_2}] h_3 + \text{c.p.} \\ &= \{h_1, \{h_2, h_3\}\} - \xi_{\{h_1, h_2\}} h_3 + \text{c.p.} \\ &= \{h_1, \{h_2, h_3\}\} - \{\{h_1, h_2\}, h_3\} + \text{c.p.} \\ &= 2\{h_1, \{h_2, h_3\}\} + \text{c.p.} \blacksquare \end{aligned}$$

4 Bihamiltonian structure of the KdV hierarchies

Here we want to present a concrete realization of the general scheme developed in the previous section. We will construct a couple of hamiltonian mappings such that the equations of the KdV hierarchies are hamiltonian with respect to the hamiltonian structures defined by both of them.

To start with, let us introduce an operator L more general than the one defined in (2):

$$L = \partial^n + u_{n-1}\partial^{n-1} + u_{n-2}\partial^{n-2} + \cdots + u_1\partial + u_0. \quad (23)$$

Later, we will discuss the reduction to $u_{n-1} = 0$ and the Virasoro algebra will appear as a by-product.

4.1 Hamiltonian elements for the KdV hierarchies

As a Lie algebra we consider the Lie algebra of “vector fields” $\xi = \partial_a$ defined in (3)

$$\partial_a \equiv \sum_{i=0}^{n-1} \sum_{k=0}^{+\infty} a_i^{(k)} \frac{\partial}{\partial u_i^{(k)}}, \quad (24)$$

with the Lie bracket

$$[\partial_a, \partial_b] = \partial_{\partial_a b - \partial_b a}. \quad (25)$$

In Appendix A we prove that this bracket satisfies the Jacobi identity and thus we are dealing indeed with a Lie algebra.

As the left module Ω^0 we take the space of functionals $\tilde{\mathcal{A}}$. The action of the elements of \mathcal{G} on $\tilde{\mathcal{A}}$ is defined by

$$\partial_a \tilde{f} = \int \partial_a f. \quad (26)$$

In the appendix it is also shown that this action provides a realization on $\tilde{\mathcal{A}}$ of the Lie bracket (25).

Now we have to construct the dual space Ω^1 . The elements of \mathcal{G} are determined by sets $a = (a_0, a_1, \dots, a_{n-2}, a_{n-1})$, $a_i \in \mathcal{A}$. Then the elements of Ω^1 are determined by sets $X = (X_0, \dots, X_{n-1})$, $X_i \in \mathcal{A}$. The coupling is

$$\langle X, \partial_a \rangle = X(\partial_a) = \int \sum_{i=0}^{n-1} a_i X_i \in \Omega^0. \quad (27)$$

This coupling can be expressed in a more convenient way. To each ∂_a we can attach a differential operator

$$\partial_a \iff a = a_{n-1}\partial^{n-1} + \cdots + a_0 \in R_{(0, n-1)}. \quad (28)$$

Similarly, to an element $X \in \Omega^1$ we can assign a PDO

$$X = \partial^{-1}X_0 + \partial^{-2}X_1 + \cdots + \partial^{-n}X_{n-1} \in R_-/R_{(-\infty, -n-1)}. \quad (29)$$

Then we can write

$$\langle X, \partial_a \rangle = \langle X, a \rangle = \int \text{res}(aX). \quad (30)$$

Let us stress that \mathcal{G} is not $R_{(0, n-1)}$. We resort to $R_{(0, n-1)}$ just to work with a sometimes more convenient expression of the coefficients a of ∂_a . Notice that $R_{(0, n-1)}$ with the natural bracket is not even a Lie algebra (except for the cases $n = 1, 2$). In an abuse of notation,

we may write something like $\mathcal{G} \sim R_{(0,n-1)}$, but the previous comment must be taken into account.

On account of lemma 2.11 we have

$$\partial_a L = a. \quad (31)$$

Now we have to construct the exterior derivative d . In fact, only

$$d : \Omega^0 \longrightarrow \Omega^1$$

is needed, and this is addressed by the following

Proposition 4.1 *If $\tilde{f} = \int f$ is an element of Ω^0 then $d\tilde{f} \in \Omega^1$ is given by*

$$d\tilde{f} = \frac{\delta f}{\delta L} = \sum_{i=0}^{n-1} \partial^{-i-1} \frac{\delta f}{\delta u_i}.$$

PROOF: If $\partial_a \in \mathcal{G}$ is arbitrary then

$$\langle \partial_a, d\tilde{f} \rangle = d\tilde{f}(\partial_a) = \partial_a \tilde{f} = \int \sum_{i=0}^{n-1} a_i \frac{\delta f}{\delta u_i}.$$

On the other hand,

$$\langle \partial_a, \frac{\delta f}{\delta L} \rangle = \int \text{res}(a \frac{\delta f}{\delta L}) = \int \sum_{i=0}^{n-1} a_i \frac{\delta f}{\delta u_i}. \quad \blacktriangleright$$

The crucial point of this scheme is the hamiltonian mapping $H : \Omega^1 \rightarrow \mathcal{G}$, which in this case is (remember the comment above)

$$H : R_- / R_{(-\infty, -n-1)} \longrightarrow R_{(0, n-1)}.$$

This is provided by the Adler mapping

$$\begin{aligned} X \longmapsto H(X) &= (LX)_+ L - L(XL)_+ \\ &= -(LX)_- L + L(XL)_- \end{aligned} \quad (32)$$

Let us show that this mapping is well defined. The first form shows that it does not matter which representative in the equivalence class we choose, because any term of order $n+1$ or higher in ∂^{-1} is killed by the $+$ projection. This also shows that the result is a differential operator. Finally, from the second form we deduce that $H(X)$ is at most of order $n-1$, because L , which is of order n , appears multiplied by the pieces of negative order provided by the $-$ projection.

A remark is in order. One may wonder why we have added to L the term in ∂^{n-1} . One could argue that it is possible to work with $\mathcal{G} \sim R_{(0, n-2)}$ and $\Omega^1 = R_- / R_{(-\infty, -n)}$ and the coupling (30) is still well defined. The answer is that in this case the Adler mapping would not work correctly because, in general, $H(X)$ would be in $R_{(0, n-1)}$ even though there is no term ∂^{n-1} in L .

Suppose now that we deform the operator L in the following way:

$$L \longrightarrow \hat{L} = L - \lambda, \quad (33)$$

where λ is a (complex) parameter. It is obvious that the modified Adler mapping still works (the deformation amounts to adding a constant to the zeroth order term of L) and so we may consider

$$\begin{aligned} X \longmapsto H(X) &= (\hat{L}X)_+\hat{L} - \hat{L}(X\hat{L})_+ \\ &= -(\hat{L}X)_-\hat{L} + \hat{L}(X\hat{L})_- \end{aligned} \quad (34)$$

Expanding in λ and using $X_+ = 0$ one gets

$$H(X) = H^{(0)}(X) + \lambda H^{(\infty)}(X), \quad (35)$$

where

$$H^{(0)}(X) = (LX)_+L - L(XL)_+ \quad (36)$$

$$H^{(\infty)}(X) = [X, L]_+ \quad (37)$$

All the properties of H will be valid for any value of λ and, in particular, for $\lambda = 0$ and $\lambda = \infty$, when H reduces to $H^{(0)}$ and $H^{(\infty)}$, respectively.

Now it rests to show that this mapping is indeed hamiltonian.

4.2 Hamiltonian property of the Adler mapping

In this subsection we will use Proposition 3.10 to show that the mapping defined in (35) is hamiltonian and a Poisson bracket can be constructed from it. We have to prove that H is skew symmetric, that $\text{Im}H$ is a subalgebra and that ω defined by 3.8 is closed.

Proposition 4.2 *The mapping H is skew symmetric.*

PROOF: For any $A, B \in R$ one has

$$\text{res}(A_+B) = \text{res}(A_+B_-) = \text{res}(AB_-).$$

Using this and corollary 2.7 one gets

$$\begin{aligned} \langle H(X), Y \rangle &= \int \text{res}(H(X)Y) = \int \text{res} \left((\hat{L}X)_+\hat{L} - \hat{L}(X\hat{L})_+ \right) Y \\ &= \int \text{res} \left((\hat{L}X)(\hat{L}Y)_- - (X\hat{L})_+Y\hat{L} \right) \\ &= \int \text{res} \left(X(\hat{L}Y)_-\hat{L} - (X\hat{L})(Y\hat{L})_- \right) \\ &= \int \text{res} \left(X \left((\hat{L}Y)_-\hat{L} - \hat{L}(Y\hat{L})_- \right) \right) \\ &= - \int \text{res}(XH(Y)) = -\langle X, H(Y) \rangle. \blacksquare \end{aligned}$$

Proposition 4.3 *$l = \text{Im}H$ is a Lie subalgebra:*

$$[\partial_{H(X)}, \partial_{H(Y)}] = \partial_{H([X, Y]_L + \partial_{H(X)}Y - \partial_{H(Y)}X)},$$

where

$$[X, Y]_L = \left(-X(\hat{L}Y)_+ + (X\hat{L})_-Y \right) - (X \leftrightarrow Y).$$

PROOF: See Appendix B. ¶

Proposition 4.4 *The 2-form defined on $\text{Im}H$ by*

$$\omega(\partial_{H(X)}, \partial_{H(Y)}) = \langle H(X), Y \rangle = \int \text{res}(H(X)Y)$$

is closed.

PROOF: See Appendix C. ¶

Propositions 4.3 and 4.4, together with proposition 3.10, imply that H is hamiltonian. In particular, the 2-forms

$$\omega^{(0)}(\partial_{H^{(0)}(X)}, \partial_{H^{(0)}(Y)}) = \int \text{res}(((LX)_+L - L(XL)_+)Y) \quad (38)$$

$$\omega^{(\infty)}(\partial_{H^{(\infty)}(X)}, \partial_{H^{(\infty)}(Y)}) = \int \text{res}([X, L]_+Y) \quad (39)$$

are closed. These forms are referred to as the second and first hamiltonian structures of the KdV hierarchy, respectively.

4.3 Poisson brackets for the KdV hierarchies

The elements of Ω^0 will be called hamiltonian functions or simply hamiltonians. According to (21), the vector field corresponding to a hamiltonian $\tilde{f} = \int f$ is

$$\xi_{\tilde{f}} = \partial_{H(\frac{\delta f}{\delta L})}.$$

Proposition 4.5 *The Poisson bracket of two hamiltonians is*

$$\{\tilde{f}, \tilde{g}\} = \xi_{\tilde{f}}\tilde{g} = \langle H(\frac{\delta f}{\delta L}), \frac{\delta g}{\delta L} \rangle = \int \text{res}\left(H(\frac{\delta f}{\delta L})\frac{\delta g}{\delta L}\right).$$

PROOF: Obvious from proposition 4.1 and the general theory. ¶

For $H^{(\infty)}$ it is not difficult to write down an explicit expression of the Poisson bracket in terms of the coefficients of the operators $X = \sum_{i=0}^{n-1} \partial^{-i-1} X_i$ and $L = \sum_{i=0}^n u_i \partial^i$, with $u_n = 1$.

Proposition 4.6

$$H^{(\infty)}(X) = \sum_{0 \leq \alpha + \beta \leq n-1} (l_{\beta\alpha} X_\alpha) \partial^\beta,$$

where $l_{\beta\alpha} = -\tilde{l}_{\beta\alpha} + \tilde{l}_{\alpha\beta}^*$ and

$$\begin{aligned} \tilde{l}_{\beta\alpha} &= \sum_{\gamma=0}^{n-1-\alpha-\beta} \binom{\gamma+\beta}{\beta} u_{\alpha+\beta+\gamma+1} \partial^\gamma \\ \tilde{l}_{\alpha\beta}^* &= \sum_{\gamma=0}^{n-1-\alpha-\beta} \binom{\gamma+\alpha}{\alpha} (-\partial)^\gamma u_{\alpha+\beta+\gamma+1}, \end{aligned}$$

and the operators l act only on the coefficients of X .

PROOF:

$$\begin{aligned}
H^{(\infty)}(X) &= [X, L]_+ = \sum_{i=0}^{n-1} \sum_{j=0}^n (\partial^{-i-1} X_i u_j \partial^j - u_j \partial^j \partial^{-i-1} X_i)_+ \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^n \left(\sum_{k=0}^{+\infty} \binom{-i-1}{k} (\partial^k (X_i u_j)) \partial^{j-i-1-k} - u_j \partial^{j-i-1} X_i \right)_+ \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^n \left(\sum_{k=0}^{j-i-1} \binom{-i-1}{k} (\partial^k (X_i u_j)) \partial^{j-i-1-k} \right. \\
&\quad \left. - u_j \sum_{k=0}^{+\infty} \binom{j-i-1}{k} (\partial^k X_i) \partial^{j-i-1-k} \right)_+ \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^n \sum_{k=0}^{j-i-1} \left(\binom{-i-1}{k} (\partial^k (X_i u_j)) \right. \\
&\quad \left. - \binom{j-i-1}{k} u_j (\partial^k X_i) \right) \partial^{j-i-1-k}_+ \\
&= \sum_{j=0}^n \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} \left(\binom{-i-1}{k} (\partial^k (X_i u_j)) - \binom{j-i-1}{k} u_j (\partial^k X_i) \right) \partial^{j-i-1-k}.
\end{aligned}$$

Writing $j - i - 1 - k = \beta$, $k = \gamma$, $i = \alpha$, which imply $\alpha = 0, \dots, n-1$, $\beta = 0, \dots, n-1$, with $0 \leq \alpha + \beta \leq n-1$, and $0 \leq \gamma \leq n-1 - \alpha - \beta$, and using

$$\begin{aligned}
\binom{\beta + \gamma}{\gamma} &= \binom{\beta + \gamma}{\beta} \\
\binom{-\alpha - 1}{\gamma} &= (-1)^\gamma \binom{\alpha + \gamma}{\alpha}
\end{aligned}$$

we get the desired result. \blacklozenge

A similar formula can be worked out for $H^{(0)}$, but it is quite cumbersome.

According to propositions 4.5 and 4.6, the Poisson bracket defined by $H^{(\infty)}$ is given by

$$\begin{aligned}
\{\tilde{f}, \tilde{g}\}^{(\infty)} &= \int \text{res} \left(\sum_{0 \leq \alpha + \beta \leq n-1} (l_{\beta\alpha} \frac{\delta f}{\delta u_\alpha}) \partial^\beta \sum_{k=0}^{n-1} \partial^{-k-1} \frac{\delta g}{\delta u_k} \right) \\
&= \int \sum_{0 \leq \alpha + \beta \leq n-1} \left(l_{\beta\alpha} \frac{\delta f}{\delta u_\alpha} \right) \frac{\delta g}{\delta u_\beta}.
\end{aligned} \tag{40}$$

For $n = 2$ we have $(\alpha, \beta) = (0, 0), (0, 1), (1, 0)$ and $\tilde{l}_{00} = u_1 + \partial$, $\tilde{l}_{01} = 1 = \tilde{l}_{10}$, $\tilde{l}_{00}^* = u_1 - \partial$, $\tilde{l}_{01}^* = 1 = \tilde{l}_{10}^*$, so $l_{00} = -2\partial$, $l_{01} = l_{10} = 0$ and the Poisson bracket is

$$\{\tilde{f}, \tilde{g}\} = -2 \int \left(\frac{\delta f}{\delta u_0} \right)' \frac{\delta g}{\delta u_0}. \tag{41}$$

4.4 Reduction to $u_{n-1} = 0$ and Virasoro algebra

We have extended the operator L by adding a term $u_{n-1} \partial^{n-1}$ and the hamiltonian mapping has been constructed under this assumption. What happens if we require $u_{n-1} = 0$? This condition must be compatible with the dynamical equations, *i.e.* $\partial_{H(X)} u_{n-1} = 0$. Things are easy for the first hamiltonian structure $H^{(\infty)}$:

Proposition 4.7 $H^{(\infty)}(X)$ is an operator of order $n - 2$ at most and so

$$\partial_{H(X)}u_{n-1} = 0$$

because $H^{(\infty)}(X)$ has no term in ∂^{n-1} . If $X = \sum_{i=0}^{n-1} \partial^{-i-1} X_i$, then $H^{(\infty)}(X)$ is independent of X_{n-1} . Thus the first hamiltonian structure reduces to $u_{n-1} = 0$ automatically.

PROOF: We have $H^{(\infty)}(X) = [L, X]_+$. The commutator of the operator L of order n with the operator X of order -1 has an order not greater than $n - 1 - 1 = n - 2$. The commutator of L with the term $\partial^{-n} X_{n-1}$ has an order not greater than $n - n - 1 = -1$, and hence it gives no contribution to $[L, X]_+$. \blacksquare

The situation with the second structure is more interesting.

Proposition 4.8 The vector field $\partial_{H^{(0)}(X)}$ is tangent to the “submanifold” $u_{n-1} = 0$ iff

$$\text{res}[L, X] = 0.$$

PROOF:

$$\begin{aligned} H^{(0)}(X) &= -(LX)_-L + L(XL)_- \\ &= -(\text{res}(LX)\partial^{-1} + o(\partial^{-2}))(\partial^n + o(\partial^{n-1})) \\ &\quad + (\partial^n + o(\partial^{n-1}))(\text{res}(XL)\partial^{-1} + o(\partial^{-2})) \\ &= -\text{res}[L, X]\partial^{n-1} + o(\partial^{n-2}). \end{aligned}$$

Therefore, $H^{(0)}(X)$ is of order not greater than $n - 2$ iff $\text{res}[L, X] = 0$. \blacksquare

Proposition 4.9 For $X = \sum_{i=0}^{n-1} \partial^{-i-1} X_i$ the condition $\text{res}[L, X] = 0$ is equivalent to

$$\sum_{k=1}^n \sum_{\alpha=1}^n \binom{k}{\alpha} (-1)^{\alpha-1} (u_k X_{k-\alpha})^{(\alpha-1)} = \text{constant},$$

from which X_{n-1} , which appears only once and without any derivative, can be expressed as a differential polynomial in $\{X_i\}_{i=0, \dots, n-2}$.

PROOF: Standard manipulations give

$$\text{res}[L, X] = - \sum_{i=1}^n \sum_{\alpha=1}^n (-1)^\alpha \binom{i}{\alpha} (X_{i-\alpha} u_i)^{(\alpha)}$$

and so, if $\text{res}[L, X] = 0$,

$$\left(\sum_{i=1}^n \sum_{\alpha=1}^n (-1)^{\alpha-1} \binom{i}{\alpha} (X_{i-\alpha} u_i)^{(\alpha-1)} \right)' = 0$$

from which the desired result follows immediately. \blacksquare

We have then the following procedure for computing the vector field associated to the second hamiltonian structure when $u_{n-1} = 0$. Given

$$\tilde{f} = \int f(u_0, \dots, u_{n-2}),$$

if we want to compute $H^{(0)}(d\tilde{f})$, then $X_i = \frac{\delta f}{\delta u_i}$, $i = 0, \dots, n-2$, as usual, while X_{n-1} is expressed in terms of the $\{X_i\}_{i=0, \dots, n-2}$ by means of proposition 4.9. Let us illustrate this in the case $n = 2$, $u_1 = 0$, $L = \partial^2 + u_0$. We have to compute

$$\{\tilde{f}, \tilde{g}\}^{(0)} = \int \text{res} \left(H^{(0)} \left(\frac{\delta f}{\delta L} \right) \frac{\delta g}{\delta L} \right),$$

with $f = f(u_0)$, $g = g(u_0)$ and

$$H^{(0)} \left(\frac{\delta f}{\delta L} \right) = (L \frac{\delta f}{\delta L})_+ L - L \left(\frac{\delta f}{\delta L} L \right)_+.$$

Let us denote $u = u_0$, $v = u_1 (= 0)$. Then

$$\frac{\delta g}{\delta L} = \sum_{i=0}^1 \partial^{-i-1} \frac{\delta g}{\delta u_i} = \partial^{-1} \frac{\delta g}{\delta u},$$

but

$$\frac{\delta f}{\delta L} = \sum_{i=0}^1 \partial^{-i-1} \frac{\delta f}{\delta u_i} = \partial^{-1} \frac{\delta f}{\delta u} + \partial^{-2} \frac{\delta f}{\delta v},$$

and to compute $H^{(0)}(\delta f / \delta L)$ we need to express $\delta f / \delta v$ using proposition 4.9. In the case at hand, that gives (we put the constant equal to zero)

$$\sum_{k=1}^2 \sum_{\alpha=1}^2 \binom{k}{\alpha} (-1)^{\alpha-1} \left(u_k \frac{\delta f}{\delta u_{k-\alpha}} \right)^{\alpha-1} = 0$$

and

$$v \frac{\delta f}{\delta u} + 2 \frac{\delta f}{\delta v} - \left(\frac{\delta f}{\delta u} \right)' = 0,$$

which, on $v = 0$, allows us to write

$$\frac{\delta f}{\delta v} = \frac{1}{2} \left(\frac{\delta f}{\delta u} \right)'.$$

Thus

$$\frac{\delta f}{\delta L} = \partial^{-1} \frac{\delta f}{\delta u} + \frac{1}{2} \partial^{-2} \left(\frac{\delta f}{\delta u} \right)'.$$

Using this expression one gets

$$\begin{aligned} \left(L \frac{\delta f}{\delta L} \right)_+ &= \frac{3}{2} \left(\frac{\delta f}{\delta u} \right)' + \frac{\delta f}{\delta u} \partial \\ \left(\frac{\delta f}{\delta L} L \right)_+ &= \frac{\delta f}{\delta u} \partial - \frac{1}{2} \left(\frac{\delta f}{\delta u} \right)' \end{aligned}$$

$$H^{(0)} \left(\frac{\delta f}{\delta L} \right) = 2u \left(\frac{\delta f}{\delta u} \right)' + \frac{\delta f}{\delta u} u' + \frac{1}{2} \left(\frac{\delta f}{\delta u} \right)'''$$

and

$$\text{res} \left(H^{(0)} \left(\frac{\delta f}{\delta L} \right) \frac{\delta g}{\delta L} \right) = \left(\frac{1}{2} \left(\frac{\delta f}{\delta u} \right)''' + 2u \left(\frac{\delta f}{\delta u} \right)' + \frac{\delta f}{\delta u} u' \right) \frac{\delta g}{\delta u}.$$

Finally

$$\{\tilde{f}, \tilde{g}\}^{(0)} = \int \left(\frac{1}{2} \left(\frac{\delta f}{\delta u} \right)''' + 2u \left(\frac{\delta f}{\delta u} \right)' + \frac{\delta f}{\delta u} u' \right) \frac{\delta g}{\delta u}. \quad (42)$$

Equation (42) is, in fact, a classical representation of the Virasoro algebra. In order to express it in a more familiar way, let's think now of u as an explicit function of a complex variable on the unit circle, $u = u(x)$, $|x| = 1$, and expand it in a Laurent series

$$u(x) = \sum_{k=-\infty}^{+\infty} T_k x^{-k-2} \quad (43)$$

$$T_k = \frac{1}{2\pi i} \oint dx u(x) x^{k+1} \quad (44)$$

Let's compute (42) for the local functionals $\tilde{f} = u(x)$, $\tilde{g} = u(y)$:

$$\begin{aligned} \{u(x), u(y)\} &= \oint dz \left(\frac{1}{2} \delta'''(z-x) + 2u(z) \delta'(z-x) + \delta(z-x) u'(z) \right) \delta(z-y) \\ &= -\frac{1}{2} \delta'''(x-y) - 2u(y) \delta'(x-y) + \delta(x-y) u'(y) \\ &= -\frac{1}{2} \delta'''(x-y) - 2u(x) \delta'(x-y) - u'(x) \delta(x-y) \end{aligned} \quad (45)$$

This implies, for the coefficients of the Laurent series,

$$\begin{aligned} \{T_r, T_s\} &= \left(\frac{1}{2\pi i} \right)^2 \oint dx dy x^{r+1} y^{s+1} \{u(x), u(y)\} \\ &= \frac{1}{2\pi i} \left((r-s) T_{r+s} + \frac{1}{2} (r^3 - r) \delta_{r+s} \right), \end{aligned} \quad (46)$$

that is, the well-known Virasoro algebra (with central charge $12 \times \frac{1}{2} = 6$). It must be noticed that the bracket corresponding to the first hamiltonian structure for this case is just a free oscillator. Indeed, one has, using (41),

$$\{u(x), u(y)\} = -2 \oint dz \delta'(z-x) \delta(z-y) = 2\delta'(x-y),$$

and expanding $u(x) = \sum_{k=-\infty}^{+\infty} a_k x^{-k-1}$, we arrive at

$$\{a_r, a_s\} = \frac{i}{\pi} r \delta_{r+s}.$$

4.5 Hamiltonians for the KdV hierarchies

Up to now we have constructed a bihamiltonian structure but the relation with the KdV equations has not been elucidated yet. In this subsection we will present the hamiltonians whose associated vector fields generate the equations of the KdV hierarchies.

Lemma 4.10

$$\frac{\delta}{\delta L} \int \text{res} L^{r/n} = \frac{r}{n} \left(L^{(r-n)/n} \right)_-.$$

PROOF: We have

$$\frac{\delta}{\delta L} \int \text{res} L^{r/n} = \sum_{i=0}^{n-1} \partial^{-i-1} \frac{\delta}{\delta u_i} \int \text{res} L^{r/n} = \sum_{i=0}^{n-1} \partial^{-i-1} \int \text{res} \left(\frac{\delta L^{r/n}}{\delta u_i} \right).$$

Now, using repeatedly corollary 2.7,

$$\begin{aligned}
\int \operatorname{res} \left(\frac{\delta L^{r/n}}{\delta u_i} \right) &= r \int \operatorname{res} \left(L^{(r-1)/n} \frac{\delta L^{1/n}}{\delta u_i} \right) \\
&= \frac{r}{n} \int \operatorname{res} \left(L^{(r-n)/n} \sum_{i=0}^{n-1} (L^{1/n})^i \frac{\delta L^{1/n}}{\delta u_i} (L^{1/n})^{n-i-1} \right) \\
&= \frac{r}{n} \int \operatorname{res} \left(L^{(r-n)/n} \frac{\delta L}{\delta u_i} \right) = \frac{r}{n} \operatorname{res} \left(L^{(r-n)/n} \partial^i \right) \\
&= \frac{r}{n} \left(L^{(r-n)/n} \right)_{-i-1}.
\end{aligned}$$

Hence

$$\frac{\delta}{\delta L} \int \operatorname{res} L^{r/n} = \frac{r}{n} \sum_{i=0}^{n-1} \partial^{-i-1} \left(L^{(r-n)/n} \right)_{-i-1} = \frac{r}{n} \left(L^{(r-n)/n} \right)_-. \blacktriangleright$$

Now we are in a position to show that the equations of the KdV hierarchies

$$\dot{L} = [P_m, L], \quad P_m = L_+^{m/n}, \tag{47}$$

can be given a hamiltonian form with respect both the first and second hamiltonian structures.

Proposition 4.11 *Equations (47) are hamiltonian with respect to the first hamiltonian structure. The hamiltonians are*

$$\tilde{h}_m = -\frac{n}{n+m} \int \operatorname{res} L^{(m+n)/n}.$$

PROOF: Using lemma 4.10 we have

$$\frac{\delta \tilde{h}_m}{\delta L} = -L_-^{m/n},$$

and then

$$\begin{aligned}
H^{(\infty)}(d\tilde{h}_m) &= H^{(\infty)}\left(\frac{\delta \tilde{h}_m}{\delta L}\right) = -[L_-^{m/n}, L]_+ \\
&= \left(L(L^{m/n} - L_+^{m/n}) \right)_+ - \left((L^{m/n} - L_+^{m/n})L \right)_+ \\
&= [L, L^{m/n}]_+ + [L_+^{m/n}, L]_+ = [L_+^{m/n}, L]_+ = [L_+^{m/n}, L] = [P_m, L].
\end{aligned}$$

Hence the equations of motion are

$$\dot{L} = H^{(\infty)}\left(\frac{\delta \tilde{h}_m}{\delta L}\right) = [P_m, L]. \blacktriangleright$$

Proposition 4.12 *Equations (47) are hamiltonian with respect to the second hamiltonian structure, with hamiltonians*

$$\tilde{g}_m = \frac{n}{m} \int \operatorname{res} L^{m/n}.$$

PROOF: Now we have $\delta\tilde{g}_m/\delta L = L_-^{(m-n)/n}$. Then

$$\begin{aligned}
H^{(0)}(L_-^{(m-n)/n}) &= \left(L(L^{(m-n)/n} - L_+^{(m-n)/n}) \right)_+ L - L \left((L^{(m-n)/n} - L_+^{(m-n)/n}) \right)_+ \\
&= (LL^{(m-n)/n})_+ L - L(L^{(m-n)/n}L)_+ \\
&\quad - (LL_+^{(m-n)/n})_+ L + L(L_+^{(m-n)/n}L)_+ \\
&= L_+^{m/n}L - LL_+^{m/n} - LL_+^{(m-n)/n}L + LL_+^{(m-n)/n}L \\
&= [P_m, L]. \blacksquare
\end{aligned}$$

Proposition 4.13 *The preceding hamiltonians are in involution, that is, for fixed n ,*

$$\{\tilde{h}_p, \tilde{h}_q\}^{(\infty)} = \{\tilde{g}_p, \tilde{g}_q\}^{(0)} = 0.$$

PROOF:

$$\begin{aligned}
\{\tilde{h}_p, \tilde{h}_q\}^{(\infty)} &= \langle H^{(\infty)}\left(\frac{\delta\tilde{h}_p}{\delta L}, \frac{\delta\tilde{h}_q}{\delta L}\right) \rangle \\
&= - \int \text{res} \left([L_+^{p/n}, L] L_-^{q/n} \right) \\
&= - \int \text{res} \left(L_+^{p/n} L L^{q/n} - L L_+^{p/n} L^{q/n} \right) \\
&= - \int \text{res} \left(L_+^{p/n} L^{q/n} L - L_+^{p/n} L^{q/n} L \right) = 0,
\end{aligned}$$

and similarly for $\{\tilde{g}_p, \tilde{g}_q\}^{(0)}$. \blacksquare

For fixed n , $\tilde{g}_{m+n} = -\tilde{h}_m$ and so, in fact,

$$\{\tilde{h}_p, \tilde{g}_q\}^{(\infty)} = \{\tilde{h}_p, \tilde{g}_q\}^{(0)} = \{\tilde{g}_p, \tilde{g}_q\}^{(\infty)} = \{\tilde{h}_p, \tilde{h}_q\}^{(0)} = 0. \quad (48)$$

Notice that the hamiltonians with fixed n are nothing but the first integrals of a given (n, m) system of equations. For instance

$$\tilde{g}_m = \frac{n}{m} J_m. \quad (49)$$

Thus, every first integral of a given equation in the n th hierarchy generates the hamiltonian evolution of an equation of the hierarchy.

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A Lie algebra defined by $[\partial_a, \partial_b] = \partial_{\partial_a b - \partial_b a}$

First, we want to show that

$$[\partial_a, [\partial_b, \partial_c]] + \text{c.p.} = 0.$$

Consider for instance the piece

$$[\partial_a, [\partial_b, \partial_c]] = [\partial_a, \partial_{\partial_b c - \partial_c b}] = \partial_{\partial_a(\partial_b c - \partial_c b) - \partial_{\partial_b c - \partial_c b} a}.$$

We have to prove that

$$\partial_a(\partial_b c - \partial_c b) - \partial_{\partial_b c - \partial_c b} a + \text{c.p.} = 0.$$

Taking into account the cyclic permutations, the pieces proportional to, let's say, c_i , are (summation over repeated indices is implicit)

$$\begin{aligned} (\partial_a(\partial_b c))_i &= a_j^{(k)} \frac{\partial b_m^{(l)}}{\partial u_j^{(k)}} \frac{\partial c_i}{\partial u_m^{(l)}} + a_j^{(k)} b_m^{(l)} \frac{\partial^2 c_i}{\partial u_j^{(k)} \partial u_m^{(l)}} \\ -(\partial_b(\partial_a c))_i &= -b_j^{(k)} \frac{\partial a_m^{(l)}}{\partial u_j^{(k)}} \frac{\partial c_i}{\partial u_m^{(l)}} - b_j^{(k)} a_m^{(l)} \frac{\partial^2 c_i}{\partial u_j^{(k)} \partial u_m^{(l)}} \\ -(\partial_{\partial_a b} c)_i &= -\left(a_m^{(l)} \frac{\partial b_j}{\partial u_m^{(l)}} \right)^{(k)} \frac{\partial c_i}{\partial u_j^{(k)}} \\ (\partial_{\partial_b a} c)_i &= \left(b_m^{(l)} \frac{\partial a_j}{\partial u_m^{(l)}} \right)^{(k)} \frac{\partial c_i}{\partial u_j^{(k)}} \end{aligned}$$

The second order derivatives of the first and second pieces cancel each other while the surviving terms are canceled by the third and fourth pieces due to the fact that, using $[\partial_a, \partial] = 0$, we have, for instance,

$$\left(b_m^{(l)} \frac{\partial a_j}{\partial u_m^{(l)}} \right)^{(k)} = (\partial_b a)^{(k)} = \partial_b(a^{(k)}) = b_m^{(l)} \frac{\partial a_j^{(k)}}{\partial u_m^{(l)}}.$$

Next we want to prove that the action of ∂_a on $\tilde{\mathcal{A}}$ is a representation of the algebra. Explicitly

$$[\partial_a, \partial_b] \tilde{f} = (\partial_a \partial_b - \partial_b \partial_a) \tilde{f}.$$

It suffices to show that this holds for the generators of the differential algebra \mathcal{A} . We have

$$\begin{aligned} [\partial_a, \partial_b] u_i^{(j)} &= \partial_{\partial_a b - \partial_b a} u_i^{(j)} = (\partial_a b - \partial_b a)_i^{(j)} \\ &= \left(a_l^{(k)} \frac{\partial b_i}{\partial u_l^{(k)}} - b_l^{(k)} \frac{\partial a_i}{\partial u_l^{(k)}} \right)^{(j)}, \end{aligned}$$

and

$$(\partial_a \partial_b - \partial_b \partial_a) u_i^{(j)} = \partial_a(b_i^{(j)}) - \partial_b(a_i^{(j)}) = a_l^{(k)} \frac{\partial b_i^{(j)}}{\partial u_l^{(k)}} - b_l^{(k)} \frac{\partial a_i^{(j)}}{\partial u_l^{(k)}},$$

and the equality follows again from $[\partial_a, \partial] = 0$.

B Proof of Proposition 4.3

We have

$$[\partial_{H(X)}, \partial_{H(Y)}] = \partial_{\partial_{H(X)}H(Y) - \partial_{H(Y)}H(X)}.$$

Then, using $\partial_{H(X)}\hat{L} = H(X)$,

$$\begin{aligned} a &\equiv \partial_{H(X)}H(Y) - \partial_{H(Y)}H(X) \\ &= \partial_{H(X)}(-(\hat{L}Y)_-\hat{L} + \hat{L}(Y\hat{L})_-) - (X \leftrightarrow Y) \\ &= -(H(X)Y)_-\hat{L} - (\hat{L}Y)_-H(X) + H(X)(Y\hat{L})_- + \hat{L}(YH(X))_- \\ &\quad + H(\partial_{H(X)}Y) - (X \leftrightarrow Y) \\ &= \underbrace{((\hat{L}X)_-\hat{L}Y - \hat{L}(X\hat{L})_-Y)}_1 - \hat{L} + (\hat{L}Y)_- \underbrace{((\hat{L}X)_-\hat{L} - \hat{L}(X\hat{L})_-)}_1 - \underbrace{\hat{L}(X\hat{L})_-}_3 \\ &\quad - \underbrace{((\hat{L}X)_-\hat{L} - \hat{L}(X\hat{L})_-)}_3 - \underbrace{\hat{L}(X\hat{L})_-}_2 (Y\hat{L})_- - \hat{L}(Y \underbrace{((\hat{L}X)_-\hat{L} - \hat{L}(X\hat{L})_-)}_2) - \\ &\quad + H(\partial_{H(X)}Y) - (X \leftrightarrow Y). \end{aligned}$$

Pieces marked by 3 cancel out when terms coming from $(X \leftrightarrow Y)$ are taken into account. Moreover, terms marked by 1 and 2 give rise to the following pieces:

$$\begin{aligned} 1 &\rightsquigarrow (((\hat{L}X)_-\hat{L}Y)_- + (\hat{L}Y)_-(\hat{L}X)_-)\hat{L} - (X \leftrightarrow Y) \\ &= ((\hat{L}X)_-\hat{L}Y - (\hat{L}X)_-(\hat{L}Y)_-)_-\hat{L} - (X \leftrightarrow Y) \\ &= ((\hat{L}X)_-(\hat{L}Y)_+)_-\hat{L} - (X \leftrightarrow Y) = (\hat{L}X(\hat{L}Y)_+)_-\hat{L} - (X \leftrightarrow Y), \end{aligned}$$

and

$$\begin{aligned} 2 &\rightsquigarrow \hat{L}(X\hat{L})_-(Y\hat{L})_- + \hat{L}(Y\hat{L}(X\hat{L})_-)_- - (X \leftrightarrow Y) \\ &= \hat{L}((X\hat{L})_-(Y\hat{L})_- + Y\hat{L}(X\hat{L})_-)_- - (X \leftrightarrow Y) \\ &= \hat{L}(-(Y\hat{L})_-(X\hat{L})_- + Y\hat{L}(X\hat{L})_-)_- - (X \leftrightarrow Y) \\ &= \hat{L}((Y\hat{L})_+(X\hat{L})_-)_- - (X \leftrightarrow Y) = -\hat{L}((X\hat{L})_+(Y\hat{L})_-)_- - (X \leftrightarrow Y) \\ &= -\hat{L}((X\hat{L})_+Y\hat{L})_- - (X \leftrightarrow Y). \end{aligned}$$

Then

$$\begin{aligned} a &= (\hat{L}X(\hat{L}Y)_+)_-\hat{L} - \hat{L}((X\hat{L})_+Y\hat{L})_- - (\hat{L}(X\hat{L})_-Y)_-\hat{L} \\ &\quad + \hat{L}(X(\hat{L}Y)_-\hat{L})_- + H(\partial_{H(X)}Y) - (X \leftrightarrow Y) \\ &= (\hat{L}(X(\hat{L}Y)_+ - (X\hat{L})_-Y))_-\hat{L} - \hat{L}(((X\hat{L})_+Y - X(\hat{L}Y)_-)\hat{L})_- \\ &\quad + H(\partial_{H(X)}Y) - (X \leftrightarrow Y). \end{aligned}$$

Now consider the term

$$\begin{aligned} (X\hat{L})_+Y - X(\hat{L}Y)_- &= (X\hat{L} - (X\hat{L})_-)Y - X(\hat{L}Y - (\hat{L}Y)_+) \\ &= X(\hat{L}Y)_+ - (X\hat{L})_-Y. \end{aligned}$$

Putting this into the above expression one gets finally

$$\begin{aligned} a &= (\hat{L}(X(\hat{L}Y)_+ - (X\hat{L})_-Y))_-\hat{L} - \hat{L}((X(\hat{L}Y)_+ - (X\hat{L})_-Y)\hat{L})_- \\ &\quad + H(\partial_{H(X)}Y) - (X \leftrightarrow Y) \\ &= -(\hat{L}[X, Y]_L)_-\hat{L} + \hat{L}([X, Y]_L\hat{L})_- + H(\partial_{H(X)}Y - \partial_{H(Y)}X) \\ &= H([X, Y]_L + \partial_{H(X)}Y - \partial_{H(Y)}X). \blacksquare \end{aligned}$$

C Proof of Proposition 4.4

Let $X, Y, Z \in \Omega^1$. Then

$$d\omega(\partial_{H(X)}, \partial_{H(Y)}, \partial_{H(Z)}) = \partial_{H(X)}\omega(\partial_{H(Y)}, \partial_{H(Z)}) - \omega([\partial_{H(X)}, \partial_{H(Y)}], \partial_{H(Z)}) + \text{c.p.}$$

Let's compute

$$\begin{aligned} (1) &\equiv \partial_{H(X)}\omega(\partial_{H(Y)}, \partial_{H(Z)}) = \partial_{H(X)} \int \text{res}(H(Y)Z) \\ &= \partial_{H(X)} \int \text{res}(((\hat{L}Y)_+ \hat{L} - \hat{L}(Y\hat{L})_+)Z) \\ &= \int \text{res}(((H(X)Y)_+ \hat{L} - \hat{L}(YH(X))_+)Z) \\ &\quad + \int \text{res}(((\hat{L}Y)_+ H(X) - H(X)(Y\hat{L})_+)Z) \\ &\quad + \int \text{res}(H(\partial_{H(X)}Y)Z) + \int \text{res}(H(Y)\partial_{H(X)}Z) \\ &= \int \text{res}(H(X)(Y(\hat{L}Z)_- - (Z\hat{L})_- Y + Z(\hat{L}Y)_+ - (Y\hat{L})_+ Z)) \\ &\quad + \int \text{res}(-H(Z)\partial_{H(X)}Y + H(Y)\partial_{H(X)}Z) \\ &= \int \text{res}(H(X)(-Y(\hat{L}Z)_+ + (Y\hat{L})_- Z - (Z\hat{L})_- Y + Z(\hat{L}Y)_+)) \\ &\quad + \int \text{res}(-H(Z)\partial_{H(X)}Y + H(Y)\partial_{H(X)}Z) \\ &= \int \text{res}(H(X)[Y, Z]_L) + \int \text{res}(-H(Z)\partial_{H(X)}Y + H(Y)\partial_{H(X)}Z), \end{aligned}$$

and

$$\begin{aligned} (2) &\equiv -\omega([\partial_{H(X)}, \partial_{H(Y)}], \partial_{H(Z)}) \\ &= \omega(\partial_{H(Z)}, \partial_{H([X, Y]_L + \partial_{H(X)}Y - \partial_{H(Y)}X)}) \\ &= \int \text{res}(H(Z)(H([X, Y]_L + \partial_{H(X)}Y - \partial_{H(Y)}X))). \end{aligned}$$

Hence

$$d\omega(\partial_{H(X)}, \partial_{H(Y)}, \partial_{H(Z)}) = (1) + (2) + \text{c.p.} = 2 \int \text{res}(H(X)[Y, Z]_L) + \text{c.p.}$$

Furthermore,

$$(3) \equiv \int \text{res}(H(X)[Y, Z]_L) + \text{c.p.} = \int \text{res}(H(X)(Y(\hat{L}Z)_+ - (Y\hat{L})_- Z)) + \text{p.}$$

where we have rearranged the terms and now p. means all the (signed) permutations. Expanding $H(X)$ in the above expression one gets

$$\begin{aligned} (3) &= \int \text{res}((-\hat{L}X)_- \hat{L} + \underbrace{\hat{L}(X\hat{L})_-}_{\text{p.}})Y(\hat{L}Z)_+ \\ &\quad + \int \text{res}((-\hat{L}X)_+ \hat{L} + \underbrace{\hat{L}(X\hat{L})_+}_{\text{p.}})(Y\hat{L})_- Z) + \text{p.} \end{aligned}$$

The underbraced terms are

$$\begin{aligned}
(4) &\equiv \int \text{res}(\hat{L}(X\hat{L})_+ Y(\hat{L}Z)_+ - (\hat{L}X)_+ \hat{L}(Y\hat{L})_- Z) + \text{p.} \\
&= \int \text{res}((\hat{L}Z)_+ \hat{L}(X\hat{L})_- Y - (\hat{L}X)_+ \hat{L}(Y\hat{L})_- Z) + \text{p.} \\
&= 0,
\end{aligned}$$

while the rest of the terms are

$$(5) \equiv \int \text{res}(-(\hat{L}X)_- \hat{L}Y(\hat{L}Z)_+) + \int \text{res}(Z\hat{L}(X\hat{L})_+(Y\hat{L})_-) + \text{p.}$$

This expression vanishes if the following lemma is used:

Lemma C.1 For any $A, B, C \in R$,

$$\int \text{res}(AB_+C_-) + \text{c.p.} = \int \text{res}(ABC).$$

PROOF:

$$\begin{aligned}
&\int \text{res}(AB_+C_-) + \text{c.p.} = \int \text{res}(A_+B_+C_- + A_-B_+C_-) + \text{c.p.} \\
&= \int \text{res}(A_+B_+C + A_-BC_- - A_-B_-C_- - A_+B_+C_+) + \text{c.p.} \\
&= \int \text{res}(A_+B_+C + A_-BC_-) + \text{c.p.} \\
&= \frac{1}{3} \int \text{res}(2AB_+C_- + A_+B_+C + A_-BC_-) + \text{c.p.} \\
&= \frac{1}{3} \int \text{res}(2AB_+C_- + CA_+B_+ + BC_-A_-) + \text{c.p.} \\
&= \frac{1}{3} \int \text{res}(2AB_+C_- + AB_+C_+ + AB_-C_-) + \text{c.p.} \\
&= \frac{1}{3} \int \text{res}(AB_+C + ABC_-) + \text{c.p.} = \frac{1}{3} \int \text{res}(AB_+C + BC_-A) + \text{c.p.} \\
&= \frac{1}{3} \int \text{res}(AB_+C + AB_-C) + \text{c.p.} = \frac{1}{3} \int \text{res}(ABC) + \text{c.p.} \\
&= \int \text{res}(ABC). \blacksquare
\end{aligned}$$

Using this one gets indeed

$$\begin{aligned}
(5) &= - \int \text{res}(\hat{L}Y(\hat{L}Z)_+(\hat{L}X)_-) + \int \text{res}(Z\hat{L}(X\hat{L})_+(Y\hat{L})_-) + \text{p.} \\
&= - \int \text{res}(\hat{L}Y\hat{L}Z\hat{L}X) + \int \text{res}(Z\hat{L}X\hat{L}Y\hat{L}) + \text{u.p.} \\
&= - \int \text{res}(Z\hat{L}X\hat{L}Y\hat{L}) + \int \text{res}(Z\hat{L}X\hat{L}Y\hat{L}) + \text{u.p.} \\
&= 0 + \text{u.p.} \\
&= 0,
\end{aligned}$$

where here u.p. denotes the (signed) uncyclic permutations.