



## Brief Paper

On the approximation of delay elements by feedback<sup>☆</sup>Carles Batlle<sup>1a,\*</sup>, Alicia Miralles<sup>b</sup><sup>a</sup>Dept. de Matemàtica Aplicada i Telemàtica, EUPVG, UPC Av. V. Balaguer s/n, Vilanova i la Geltrú, 08800 Spain<sup>b</sup>Dept. de Matemàtica Aplicada i Telemàtica, ETSETB, UPC Mòdul C3, Campus Nord, Barcelona, 08034 Spain

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**Abstract**

In a recent paper (Beghi, Lepschy & Viaro (1997) *IEEE Transactions on Circuits and System-I*, 44, 824–828) a procedure for obtaining proper rational approximants of the transfer function of a delay was suggested. We study a generalization of the feedback problem posed in the cited paper and obtain its general solution. Explicit computations of the generalized approximants are obtained in terms of Bernoulli numbers and it is found that they correspond to iterate resummations of the approximants in (Beghi et al., 1997) before truncation. The properties and frequency performance of the new rational approximants are studied and compared to those of the original ones. © 2000 Elsevier Science Ltd. All rights reserved.

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**1. Introduction**

The study of the rational approximations of the transfer function of a delay,  $e^{-Ts}$ , is still a matter of interest (Glader, Högnäs, Mäkilä, Jr. & Toivonen, 1991; Lam, 1991; Leonard, 1998). The most popular method is based on Padé approximants (Baker, Jr. & Graves-Morris, 1996), widely used in simulation software. In Beghi et al. (1997) it was argued that consideration of the closed-loop characteristics where the delay is generally embedded can lead to approximations better suited for the closed-loop system, and, based on this idea, a new method to obtain rational approximations to the delay was proposed, yielding two families of proper rational functions with a better frequency response than that of the corresponding Padé approximants.

The central idea of Beghi et al. (1997) is to use a constant test function and to feed it to a closed loop having the delay in its direct path. If  $W(s)$  represents the closed-loop transfer function and  $G(s) = e^{-sT}$ , then

$W(s) = G(s)/(1 \pm G(s))$ , where the  $\pm$  sign corresponds to  $\mp$  gain in the feedback loop. The authors of Beghi et al. (1997) observed that, both with the positive and negative feedback, the output to a constant test function was a polynomial plus a periodic function (with period  $T$  for positive feedback and period  $2T$  for negative feedback). Expanding the periodic part in Fourier series, truncating it and computing the Laplace transform one gets a rational function which, upon division by the transfer function of the input, yields a rational approximation to  $W(s)$  and hence to  $G(s)$ . The approximations can be labeled by the place where the Fourier series is truncated. One may ask whether the approximants so obtained depend on the fact that a constant test function is used, and, if the answer is affirmative, what happens if one uses arbitrary powers of  $t$  as test functions, and this is the question that we address in this paper.

The paper is organized as follows. In Section 2 we generalize the procedure in Beghi et al. (1997) and we obtain the exact analytical solution to the feedback problem for arbitrary power test functions. In Section 3 we obtain, for negative and positive feedback, the Fourier series of the periodic part of the solution, and show how its truncations are related to resummations of the series in Beghi et al. (1997). In Section 4 we compare the frequency response of our approximants to that of those in Beghi et al. (1997). Finally, we state our conclusions in Section 5.

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## 2. The feedback problem and its solution

The functional equation for the output  $y(t)$  of a closed-loop system with a time-delay  $T$  in the direct path, a gain  $a$  in the feedback loop, and input  $u(t)$  is

$$y(t) = u(t - T) + ay(t - T). \quad (1)$$

We pose the following problem: *Assuming that the input is zero for  $t < 0$  and a power of  $t$  for  $t \geq 0$ , find a solution of (1) of the form*

$$y(t) = q(t) + p(t) \quad (2)$$

for  $t \geq 0$ , such that  $q(t)$  is a polynomial and  $p(t)$  is periodic.

The idea behind the problem is to obtain an output which can be meaningfully truncated and such that its Laplace transform is a rational function. This is the case if the output is a polynomial plus a periodic function, which can be expanded in Fourier series.

Let  $p(t)$  have period  $\tau$ . Substitution of (2) into (1) yields

$$q(t) - aq(t - T) - u(t - T) = ap(t - T) - p(t). \quad (3)$$

The left-hand side is a polynomial, while the right-hand side is periodic. Hence the later must be a constant

$$ap(t - T) - p(t) = \text{constant}. \quad (4)$$

If we expand  $p(t)$  in Fourier series with period  $\tau$

$$p(t) = \frac{a_0}{2} + \sum_{k \geq 1} \left( a_k \cos \frac{2\pi kt}{\tau} + b_k \sin \frac{2\pi kt}{\tau} \right) \quad (5)$$

and substitute in (4) one gets that, for a given  $k$ , either  $a_k = b_k = 0$  or

$$a = \cos \frac{2\pi kT}{\tau} \pm \sqrt{-\sin^2 \frac{2\pi kT}{\tau}}. \quad (6)$$

This shows that  $a = \pm 1$  and, furthermore,  $\tau$  can only take the values

$$\tau = \frac{2k}{n}T, \quad k, n \in \mathbb{N}. \quad (7)$$

However, one can show that, on any interval  $[kT, (k+1)T]$ ,  $k = 0, 1, 2, \dots$ ,  $p(t)$  is a polynomial. This forbids all the periods which are not a multiple of  $T$ . Furthermore, using the same techniques as below, one can see that the solutions corresponding to  $\tau = 3T, 4T, 5T, \dots$  are exactly the same as in the cases  $\tau = T$  and  $\tau = 2T$ .

Summing up, the only nontrivial solutions to our feedback problem are (1)  $\tau = T$  and  $a = 1$ , or (2)  $\tau = 2T$ ,  $a = -1$  and  $p(t)$  has odd terms only, which coincide with the kind of solutions found in Beghi et al. (1997) for the special case  $u(t) = 1 \cdot \theta(t - 0)$ , where  $\theta$  denotes the step function.

One can eliminate  $p(t)$  from (3) using its known periodicity. For  $\tau = T$  and  $a = 1$  one immediately gets, with the shift  $t \rightarrow t + T$ ,

$$q(t + T) - q(t) = u(t). \quad (8)$$

For  $\tau = 2T$  and  $a = -1$  one must iterate (1) in order to relate outputs  $2T$  apart:  $y(t) = u(t - T) - u(t - 2T) + y(t - 2T)$ . Use of (2) then gives, after shifting  $t$  by  $2T$ ,

$$q(t + 2T) - q(t) = u(t + T) - u(t). \quad (9)$$

If we set  $u(t) = t^m$  in (9) and look for polynomial solutions of the form

$$q(t) = \sum_{k=1}^m a_k t^k, \quad (10)$$

we get, after a reordering of the sums and comparison of powers of  $t$ , the  $m$  equations

$$\sum_{k=l+1}^m a_k \binom{k}{l} (2T)^{k-l} = \binom{m}{l} T^{m-l}, \quad l = 0, 1, \dots, m-1. \quad (11)$$

Note that in the expansion for  $q(t)$  we have explicitly set  $a_0 = 0$ , since it cancels in (9) anyway. The above system of equations is upper-triangular with nonzero elements  $2T(l+1)$  in the diagonal. Hence it has a unique solution and, using several properties of the Bernoulli numbers  $B_k$  (Gradshteyn & Ryzhik, 1980), it is a simple computation to check that

$$a_k = -\frac{m!}{k!} T^{m-k} \frac{2^{m-k+1} - 1}{(m-k+1)!} B_{m-k+1}, \quad k = 1, \dots, m, \quad (12)$$

is a solution to (11). Notice that  $B_{2k+1} = 0$  except for  $B_1 = -\frac{1}{2}$ . Associated to the Bernoulli numbers one may introduce the Bernoulli polynomials

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}, \quad (13)$$

whose properties will be useful later.

For positive feedback, one must note, see (8), that a polynomial of degree  $m+1$  is needed in order to compensate for  $u(t) = t^m$ :

$$q(t) = \sum_{k=1}^{m+1} a_k t^k. \quad (14)$$

Substitution into (8) and rearranging of sums yields the  $m$  equations

$$\sum_{k=l+1}^{m+1} a_k \binom{k}{l} T^{k-l} = 0, \quad l = 0, 1, \dots, m-1. \quad (15)$$

Again, this is a compatible and determined system, and its solution is

$$a_k = \frac{m!}{k!(m-k+1)!} T^{m-k} B_{m-k+1}, \quad k = 1, \dots, m+1. \tag{16}$$

### 3. Computation of the two families of approximants

Our aim in this section is to compute the Fourier series of  $p(t)$ , truncate it, add  $q(t)$ , Laplace transform the sum and finally divide by the Laplace transform of  $u(t) = t^m$ . The end result will be a rational approximation to  $W(s)$  and, hence, a rational approximant to  $G(s)$  will be obtained.

In order to compute the Fourier series of  $p(t)$ , it is useful first to show that, for negative feedback, the  $2T$ -periodic  $p(t)$  is even (odd) for  $m$  odd (even) with respect to  $t = T$ .

The functional equation for  $q(t)$  for negative feedback is

$$q(t + 2T) - q(t) = (t + T)^m - t^m \tag{17}$$

and the  $2T$ -periodic function  $p(t)$  is given by

$$p(t) = \begin{cases} -q(t) & \text{if } t \in [0, T], \\ (t - T)^m - q(t) & \text{if } t \in [T, 2T]. \end{cases} \tag{18}$$

For  $m$  odd,  $m = 2l + 1$ , we have

$$q(t) = \frac{1}{2}t^{2l+1} + q_e(t), \tag{19}$$

where  $q_e$  is an even polynomial. Using this in combination with (17) and (18) it is easy to prove that  $p(T - \varepsilon) = p(T + \varepsilon)$ ,  $\forall \varepsilon$ .

For  $m$  even,  $m = 2l$ , the relation  $p(T - \varepsilon) = -p(T + \varepsilon)$  follows easily from

$$q(t) = \frac{1}{2}t^{2l} + q_o(t), \tag{20}$$

where  $q_o(t)$  is an odd polynomial in  $t$ .

Using the symmetry and (18) we have the Fourier series

$$p_{2n+1}(t) = \frac{\hat{A}_0}{2} + \sum_{k \geq 1} \hat{A}_k \cos \frac{\pi kt}{T}, \quad p_{2n}(t) = \sum_{k \geq 1} \hat{B}_k \sin \frac{\pi kt}{T} \tag{21}$$

for  $m = 2n + 1$  odd and  $m = 2n$  even, respectively, and where, due to the symmetry, the coefficients can be computed integrating over  $[0, T]$  and using  $p(t) = -q(t)$ . A straightforward integration yields

$$\hat{A}_0 = -\frac{2}{m+1} T^m (2^{m+1} - 1) B_{m+1}, \quad m \text{ odd}, \tag{22}$$

while  $\hat{A}_0 = 0$  for  $m$  even.

In order to compute the rest of coefficients, one may take advantage of the known Fourier series of the Bernoulli polynomials

$$B_{2n}(x) = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \sum_{k \geq 1} \frac{\cos 2k\pi x}{k^{2n}}, \tag{23}$$

$$n = 1, 2, \dots, \quad x \in [0, 1]$$

and

$$B_{2n+1}(x) = \frac{(-1)^{n+1} 2(2n+1)!}{(2\pi)^{2n+1}} \sum_{k \geq 1} \frac{\sin 2k\pi x}{k^{2n+1}}, \tag{24}$$

$$n \geq 0, \quad x \in [0, 1].$$

In fact, our  $q(t)$  is not a Bernoulli polynomial, but

$$q'_m(t) = -2(2T)^{m-1} B_m\left(\frac{t}{2T}\right) + T^{m-1} B_m\left(\frac{t}{T}\right) \tag{25}$$

and we may use (23) or (24) because  $t \leq T$  in our integrals.

For  $m = 2n + 1$ , integrating by parts yields

$$\hat{A}_k = \frac{2}{\pi k} \int_0^T \sin \frac{\pi kt}{T} q'_{2n+1}(t) dt. \tag{26}$$

Using (25) and (24) in this integral shows that, for  $k$  even, the contributions coming from the two Bernoulli polynomials cancel each other, while for  $k$  odd only the first polynomial does contribute, in accordance with the general result established in Section 2. Plugging in all the factors one gets

$$\hat{A}_{2k+1} = 2(-1)^n \frac{T^{2n+1} (2n+1)!}{\pi^{2n+2} (2k+1)^{2n+2}}, \quad m = 2n+1, \tag{27}$$

$$n = 0, 1, 2, \dots$$

and  $\hat{A}_{2k} = 0$ . For  $m = 2n$ , integration by parts now gives

$$\hat{B}_k = \frac{2}{\pi k} ((-1)^k q_{2n}(T) - q_{2n}(0)) - \frac{2}{\pi k} \int_0^T \cos \frac{\pi kt}{T} q'_{2n}(t) dt. \tag{28}$$

One has  $q_{2n}(0) = 0$  and  $q_{2n}(T) = -(T^{2n}/2n + 1)(2^{2n+1} B_{2n+1}(1/2) - B_{2n+1}(1)) = 0$  due to the vanishing of the odd-order Bernoulli polynomials at  $\frac{1}{2}$  and 1. Hence

$$\hat{B}_k = -\frac{2}{\pi k} \int_0^T \cos \frac{\pi kt}{T} q'_{2n}(t) dt, \tag{29}$$

and, again, use of (25) and (23) shows that the contributions from the two Bernoulli polynomials cancel each other if  $k$  is even while only the first one is nonzero if  $k$  is odd. Thus  $\hat{B}_{2k} = 0$  and

$$\hat{B}_{2k+1} = 2(-1)^{n-1} \frac{T^{2n} (2n)!}{\pi^{2n+1} (2k+1)^{2n+1}}, \quad n = 1, 2, \dots \tag{30}$$

Using the expression for  $q(t)$  in terms of the Bernoulli numbers and the above obtained Fourier series of  $p(t)$ , we can write down the expansions for the output  $y(t)$  associated to our negative feedback loop and the input  $u(t) = t^m$ . Then we can compute the Laplace transform of  $y(t)$  and divide by  $\mathcal{L}(t^m) = m!/s^{m+1}$ , to obtain the corresponding expansions for the transfer function  $W(s)$ . For  $m = 2n$  this process yields

$$W(s) = - \sum_{l=1}^{2n} T^{2n-l} \frac{2^{2n-l+1} - 1}{(2n-l+1)!} B_{2n-l+1} s^{2n-l} - 2(-1)^n s^{2n+1} \frac{T^{2n+1}}{\pi^{2n}} \times \sum_{k \geq 0} \frac{1}{(2k+1)^{2n} s^2 T^2 + \pi^2 (2k+1)^2}, \tag{31}$$

while for  $m = 2n + 1$  one gets

$$W(s) = - \sum_{l=1}^{2n+1} T^{2n+1-l} \frac{2^{2n+2-l} - 1}{(2n+2-l)!} B_{2n+2-l} s^{2n+1-l} - \frac{T^{2n+2}}{(2n+2)!} (2^{2n+2} - 1) B_{2n+2} s^{2n+1} + 2(-1)^n s^{2n+3} \frac{T^{2n+3}}{\pi^{2n+2}} \times \sum_{k \geq 0} \frac{1}{(2k+1)^{2n+2} s^2 T^2 + \pi^2 (2k+1)^2}. \tag{32}$$

Any truncation of the series in either (31) or (32) will yield a rational approximant for  $W(s)$  and hence for  $G(s) = \exp(-Ts)$ . Comparing (31) and (32) it is easy to see that the expansions corresponding to  $2n - 1$  and  $2n$  are the same. The explicit expression for  $m = 1, 2$  is

$$W(s) = \frac{1}{2} - \frac{1}{4} Ts + \frac{2}{\pi^2} T^3 s^3 \times \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2 s^2 T^2 + (2n+1)^2 \pi^2}. \tag{33}$$

This is to be compared with the expression for  $m = 0$  (Beghi et al., 1997):

$$W(s) = \frac{1}{2} - 2sT \sum_{n=0}^{+\infty} \frac{1}{s^2 T^2 + (2n+1)^2 \pi^2}. \tag{34}$$

We can define a two-parameter family of approximations, corresponding to the various values of  $m$  and the number of terms of the series that we keep. Taking as reference the even values of  $m$ , we set

$$W_{p,q}^o(s) = - \sum_{l=1}^{2p} T^{2p-l} \frac{2^{2p-l+1} - 1}{(2p-l+1)!} B_{2p-l+1} s^{2p-l} - 2(-1)^p s^{2p+1} \frac{T^{2p+1}}{\pi^{2p}} \times \sum_{k=0}^q \frac{1}{(2k+1)^{2p} s^2 T^2 + \pi^2 (2k+1)^2} \tag{35}$$

with  $p = 1, 2, \dots, q = 0, 1, 2, \dots$ . It is easy to see that, as a rational function, the highest degree of  $s$  in  $W_{p,q}^o(s)$  is  $2(q+1) + (2p-1) = 2(p+q) + 1$ , which is odd. Hence  $G_{p,q}^o(s) = W_{p,q}^o(s)/(1 - W_{p,q}^o(s))$  will also be a rational function of odd degree. In comparison, the approximants of Beghi et al. (1997) for negative feedback are all even rational functions.

The exact expression for  $W(s)$  corresponding to  $m = 0$  can be written as  $W(s) = \frac{1}{2} - 2sTS_0(s, T)$ , where

$$S_{2p}(s, T) = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^{2p} s^2 T^2 + (2n+1)^2 \pi^2}. \tag{36}$$

It is immediate to see that

$$\pi^2 S_{2p}(s, T) + T^2 s^2 S_{2p+2}(s, T) = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^{2p+2}} = (-1)^p \frac{\pi^{2(p+1)} (2^{2(p+1)} - 1)}{2(2(p+1))!} B_{2p+2}. \tag{37}$$

Using (37) one can relate  $S_0(s, T)$  to  $S_{2p}(s, T)$  for any  $p$ . The expressions so obtained are resummations of  $S_0(s, T)$  in terms of series with higher rates of convergence. When this is translated to  $W(s)$ , it is found that the successive resummations from  $m = 0$  yield precisely the expressions corresponding to higher values of  $m$ . Hence, using higher powers of  $t$  as test functions in the feedback problem corresponds to resummations of the original series in Beghi et al. (1997).

Finally, we will present the end results corresponding to positive feedback without the proofs. In fact the computations are less involved than for negative feedback. Notice that now  $p(t)$  is given by a uniform expression, namely  $p(t) = -q(t)$  all over the period  $[0, T]$ . It is possible to show that  $p(t)$  has odd (even) symmetry around  $t = T/2$  for  $m$  even (odd). The key element to compute the Fourier coefficients is to notice that now  $q'(t) = T^{m-1} B_m(t/T)$ .

Following the same steps as in the negative feedback case, it is found out that the closed-loop transfer functions  $W(s)$  corresponding to  $m = 2n - 1$  and  $m = 2n, n = 1, 2, \dots$ , are the same, and a new two-parameter family of approximants can be constructed.

$$W_{p,q}^e(s) = \frac{1}{sT} - \frac{1}{2} + \sum_{l=1}^{2p-1} \frac{T^{2p-l}}{(2p+1-l)!} B_{2p+1-l} s^{2p-l} + 2(-1)^p s^{2p+1} \frac{T^{2p+1}}{(2\pi)^{2p}} \sum_{k=1}^q \frac{1}{k^{2p}} \frac{1}{T^2 s^2 + 4\pi^2 k^2} \tag{38}$$

with  $p = 1, 2, \dots$  and  $q = 1, 2, \dots$ . As a rational function, the highest power of  $s$  in  $W_{p,q}^e(s)$  is  $1 + (2p-1) + 2q = 2(p+q)$  which is always even. Thus, the approximants to the exponential,  $G_{p,q}^e(s) = W_{p,q}^e(s)/(1 + W_{p,q}^e(s))$ , will also be rational functions with even highest degree. For  $m = 0$ , the highest power is  $1 + 2q$ , which is always odd.

As in the negative feedback case, it can be shown that using higher powers of  $t$  as test functions corresponds to resummations of the expressions in Beghi et al. (1997).

#### 4. Numerical study of the new approximants

One may expect that the families corresponding to  $m \geq 1$  would yield better approximations, for a given order of the rational function, than those corresponding to  $m = 0$ , due to the higher rate of convergence of the series that are eventually truncated. The problem is that, for  $m \geq 1$ , the alternating sign of the Bernoulli numbers makes the approximations in general nonstable, i.e.  $G_{p,q}(s)$  develops poles with positive real parts. In fact, we have numerically tested that this is the case for  $p > 1$ , while for  $p = 1$ , which corresponds to  $m = 1, 2$ , it is shown in the appendix that it yields stable approximations. Hence, from a practical point of view, any improvement with respect to the approximants obtained in Beghi et al. (1997) is to be found with  $p = 1$ . We will see presently that this is the case, and that in fact  $p = 1$  also improves the Padé approximants of the same order.

A measure of the goodness of an approximation can be obtained by comparing the frequency response, i.e.  $G_{p,q}(i\omega)$ , with that of the exact delay,  $\exp(-i\omega T)$ . In fact, it is easy to see that  $|G_{p,q}(i\omega)| = 1$ , so the error only affects in fact the phase. To be precise, we will compare our even approximant of order 6  $G_{1,2}^e(s)$ , with the corresponding approximant for  $m = 0$ , which we will call  $G_3^e(s)$ , obtained taking three terms in the series, and with the standard all-pass Padé approximant  $[6,6](s)$ . Fig. 1 shows  $|G_{1,2}^e(i\omega) - \exp(-i\omega)|$ ,  $|G_3^e(i\omega) - \exp(-i\omega)|$  and

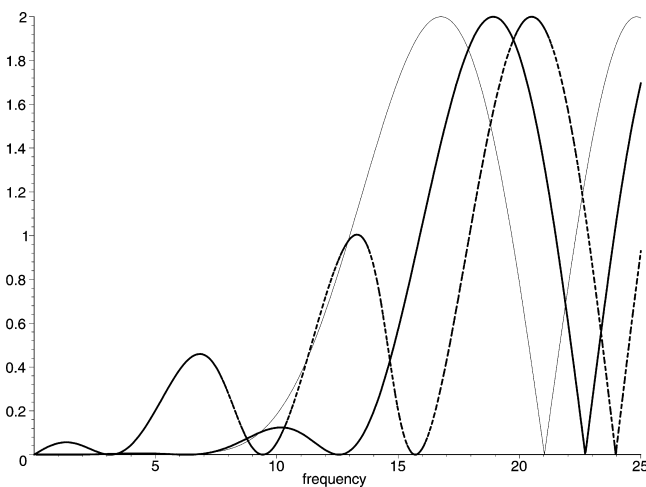


Fig. 1. Absolute value of the error with respect to the exact delay for our order 6 approximation (continuous thick line), the order 6 approximant for  $m = 0$  (dotted thick line), and the standard Padé approximant  $[6,6]$  (continuous thin line). In the horizontal axis  $\omega T$  runs from 0 to 25.

$|[6,6](i\omega) - \exp(-i\omega)|$ . Note that the maximum error is 2, since all the approximants are on the unit circle when  $s = i\omega$ . It is observed that the Padé approximation gives the best results for low frequencies, but it suddenly explodes at  $\omega T \sim 10$ , while the error of the approximant of Beghi et al. (1997) increases smoothly. Our approximant retains the best characteristics of both approximants, namely, nearly zero error at low frequencies and a slow increase of the error from  $\omega T \sim 10$  onwards. This is true for all the other approximants with  $p = 1$  when compared with the Padé ones and the approximants corresponding to  $m = 0$ .

#### 5. Conclusions

In this paper we have generalized the method proposed in Beghi et al. (1997) to obtain a series of all-pass rational approximants of a delay element. Our generalization consists in considering an arbitrary input  $u(t) = t^m$ ,  $m \in \mathbb{N}$ , as auxiliary function instead of just a constant. We have shown why the periodic component of the output can have only period  $T$  or  $2T$  and why the feedback gain can only be  $+1$  or  $-1$ . We have been able to solve the equation satisfied by the polynomial, nonperiodic part of the output for arbitrary  $m$  in terms of the Bernoulli numbers. Also, the Fourier series of the periodic part has been computed using the known expansions of the Bernoulli polynomials.

We have seen that, both for negative and positive feedback, the expressions obtained with  $m = 2p$  and  $m = 2p - 1$  are the same, a fact that is related to the vanishing of the odd index Bernoulli numbers  $B_p$  for  $p > 1$ . Truncation of the series obtained for a given  $m$  yields a rational approximation to the closed-loop transfer function and hence an approximation to the delay. Since we can vary both  $m$  and the number of terms taken in the series, we get two biparametric families of approximants, one for each feedback sign. However, it turns out that only for  $m = 1, 2$ , that is  $p = 1$ , are the rational approximations thus obtained stable.

The new stable families keep the good properties of the families corresponding to  $m = 0$ , namely an error smoothly increasing with frequency, but at the same time have the nearly zero error at low frequency characteristic of the all-pass Padé approximants. Thus our new approximants are a real improvement and can be used over a larger rank of frequencies with smaller distortion.

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## Appendix A. Proof of the stability of $G_{1,q}^o$ and $G_{1,q}^c$

For  $p = 1$  the family of odd order approximants  $G_{1,q}^o$  can be written in terms of  $z = sT$  as  $G_{1,q}^o(z) = A_q(z)/(B_q(z) - A_q(z))$ , where

$$A_q(z) = (2 - z)\pi^2 \prod_{k=0}^q ((2k + 1)^2(z^2 + \pi^2(2k + 1)^2)) \\ + 8z^3 \sum_{k=0}^q \prod_{l=0, l \neq k}^q ((2l + 1)^2(z^2 + \pi^2(2l + 1)^2))$$

and

$$B_q(z) = 4\pi^2 \prod_{k=0}^q ((2k + 1)^2(z^2 + \pi^2(2k + 1)^2)).$$

We want to show that all the zeros of  $P_q(z) = B_q(z) - A_q(z)$  have strictly negative real part. First of all, it is easy to see that none of the zeros of  $\prod_k (2k + 1)^2(z^2 + \pi^2(2k + 1)^2)$  is a zero of  $P_q(z)$ . Hence, we may factor out this term and  $P_q(z) = 0$  is equivalent to

$$\frac{\pi^2}{8}(z + 2) = z \sum_{k=0}^q \frac{z^2}{(2k + 1)^2(z^2 + \pi^2(2k + 1)^2)},$$

which after some simple algebra becomes

$$\frac{\pi^2}{4} + z \left( \frac{\pi^2}{8} - \sum_{k=0}^q \frac{1}{(2k + 1)^2} \right) \\ = -\pi^2 \sum_{k=0}^q \frac{z}{z^2 + \pi^2(2k + 1)^2}. \quad (\text{A.1})$$

The key point is to notice that  $\sum_{k=0}^{\infty} 1/(2k + 1)^2 = \pi^2/8$ . Hence the left-hand side of (A.1) has strictly positive real part for  $\Re(z) \geq 0$ , while the right-hand side, due to the global minus sign, has nonpositive real part for  $\Re(z) \geq 0$  (the fractions in the sum are positive-real functions). This shows that no zero of  $P_q(z)$  can have  $\Re(z) \geq 0$  and concludes the proof.

The same result can be proved for  $G_{1,q}^c(s)$ . This time, however, the essential point is to compare the imaginary parts and to use  $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$  in order to get the contradiction.

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