Balanced model order reduction method for systems depending on a parameter

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Introduction

- Order reduced models are useful to simulate very large models using less computational resources, allowing, for instance, the exploration of parameter regions.
- They should be easily computable, preserving some of the structural properties of the full model and yielding an error with respect to the original model that can be bounded in terms of the complexity of the approximating model.
- For linear time-invariant MIMO systems, model order reduction (MOR) based on the truncation of balanced realizations preserves the stability, controllability and observability of the full model, and provides bounds for the norm of the error system.

- Computation of a balanced realization for a linear system relies on numerical linear algebra algorithms, and does not allow for the presence of symbolic parameters in the model.
- Our goal is to present a balanced realization MOR for systems depending on a parameter, with a final reduced order model which depends also on the parameter, up to a certain power.
- We deduce expressions to arbitrary order in the parameter, except for the last step of the procedure, where we only have explicit results up to second order.
- As a by-product, we obtain results for the second order correction to the singular subspaces of the Singular Value Decomposition (SVD).

The rest of the presentation is organized as follows

- Review of the balanced realization procedure.
- Power series expansion for the balanced realization.
- Application: a system of masses and springs.
- Conclusions.

- Detailed computations can be found in the proceedings paper and in Batlle, C. and Roqueiro, N., Balanced model order reduction for systems depending on a parameter, `http://arxiv.org/abs/1604.08086`.
- Extensive presentations of model order reduction techniques and its applications can be found in
    - Antoulas, A.C. (2005), *Approximation of large-scale dynamical systems*, Advances in Design and Control. SIAM,
    - Schilders, W.H.A., van der Vorst, H.A., and Rommes, J. (eds.) (2008), *Model order reduction: theory, research aspects and applications*, volume 13 of *Mathematics in industry; The European Consortium for Mathematics in Industry*, Elsevier,
    - Benner, P., Gugercin, S., and Willcox, K. (2015), A Survey of Projection-Based Model Reduction Methods for Parametric Dynamical Systems, *SIAM Review*, 57(4), 483–531,

  which contain references to the original literature.

Balanced realization procedure

- Consider the nonlinear control system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned}$$

with $x \in \mathbb{R}^N$, $u \in \mathbb{R}^M$, $y \in \mathbb{R}^P$ and $f(0) = 0$.

- The controllability function $L_c(x)$ is the solution of the optimal control problem

$$L_c(x) = \inf_{u \in L^2((-\infty,0),\mathbb{R}^M)} \frac{1}{2} \int_{-\infty}^{0} ||u(t)||^2 \mathrm{d}t$$

subject to $x(-\infty) = 0$, $x(0) = x$ and $\dot{x} = f(x) + g(x)u$.

- Roughly speaking, $L_c(x)$ measures the minimum 2-norm of the input signal necessary to bring the system to the state x from the origin.

- The observability function $L_o(x)$ is proportional to the 2-norm of the output signal obtained when the system is relaxed from the state x

$$L_o(x) = \frac{1}{2}\int_0^\infty ||y(t)||^2 \mathrm{d}t = \frac{1}{2}\int_0^\infty ||h(x(t))||^2 \mathrm{d}t,$$

with $x(0) = x$ and subjected to $\dot{x} = f(x)$, that is, with $u = 0$.
- For linear control systems,

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

assumed to be observable, controllable and Hurwitz, both $L_c(x)$ and $L_o(x)$ are quadratic functions

$$\begin{aligned} L_c(x) &= \frac{1}{2}x^T W_c^{-1} x, \\ L_0(x) &= \frac{1}{2}x^T W_o x. \end{aligned}$$

- The matrices $W_c > 0$ and $W_o > 0$, the controllability and observability Gramians, are the solutions to the matrix Lyapunov equations

$$AW_c + W_cA^T + BB^T = 0, \quad A^TW_o + W_oA + C^TC = 0.$$

- They are explicitly given by

$$W_c = \int_0^\infty e^{At}BB^Te^{A^Tt}\,\mathrm{d}t, \quad W_o = \int_0^\infty e^{A^Tt}C^TCe^{At}\,\mathrm{d}t,$$

but the numerical solution of the Lyapunov equations if computationally preferred.

- Notice that $W_c > 0$ and $W_o > 0$ means that the uncontrollable or unobservable subspaces have been already dropped from the system (Kalman decomposition).

- The matrix W_c provides information about the states that are easy to control, in the sense that signals u of small norm can be used to reach them.
- W_o allows to find the states that are easily observable, *i.e.* they produce outputs of large norm.
- From the point of view of the input-output map associated to the linear control system, one would like to select the states that score well on both counts, and this leads to the concept of balanced realization, for which $W_c = W_o$.
- The balanced realization is obtained by means of a linear transformation $x = Tz$, with T computed in a series of steps.
- This is a well known algorithm, implemented for instance with the MATLAB command `balred` or, with greater flexibility and with some state-of-the-art improvements, in the MORLAB toolbox.

1. Solve the Lyapunov equations for $W_c > 0$, $W_o > 0$.
2. Perform Cholesky factorizations of the Gramians:

$$W_c = XX^T, \quad W_o = YY^T.$$

   Notice that $X > 0$ and $Y > 0$.
3. Compute the SVD of $Y^T X$:

$$Y^T X = U \Sigma V^T,$$

   with U and V orthogonal and

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_N), \quad \sigma_1 \geq \cdots \geq \sigma_N > 0.$$

4. The balancing transformation is given then by

$$T = XV\Sigma^{-1/2}, \quad \text{with } T^{-1} = \Sigma^{-1/2} U^T Y^T.$$

5. The balanced realization is given by the linear system
$$\tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B, \quad \tilde{C} = CT.$$
6. In the new coordinates,
$$\tilde{W}_c = T^{-1}W_cT^{-T} = \Sigma, \quad \tilde{W}_o = T^TW_oT = \Sigma,$$
and the controllability and observability functions are
$$\begin{aligned} \tilde{L}_c(z) &= \frac{1}{2}\sum_{i=1}^{N}\frac{z_i^2}{\sigma_i} = \frac{1}{2}z^T\Sigma^{-1}z, \\ \tilde{L}_o(z) &= \frac{1}{2}\sum_{i=1}^{N}\sigma_i z_i^2 = \frac{1}{2}z^T\Sigma z, \end{aligned}$$
so that the state with only nonzero coordinate z_i is both easier to control and easier to observe than the state corresponding to z_{i+1}, for $i = 1, 2, \ldots, N-1$.

- If, for a given r, $1 \leq r < N$, one has $\sigma_r \gg \sigma_{r+1}$, it may be sensible to keep just the states corresponding to the coordinates $z_1, z_2, \ldots, z_r$.
- The σ_i are the Hankel singular values of the control system.
- $\mathcal{H}_\infty$-norm lower and upper error bounds of the balanced truncation method are given by

$$\sigma_{r+1} \leq \|G(s) - G_r(s)\|_{\mathcal{H}_\infty} \leq 2 \sum_{i=r+1}^{N} \sigma_i.$$

- From these inequalities it follows that, in order to get the smallest error for the truncated system, one should disregard the states associated with the smallest Hankel singular values.

- If we denote by $\tilde{A}_r$ the upper-left square block of $\tilde{A}$ formed by the first r rows and columns, and by $\tilde{B}_r$ and $\tilde{C}_r$ the matrices obtained from the first r rows or columns of $\tilde{B}$ or $\tilde{C}$, respectively, the reduced system of order r obtained by balanced truncation is given by

$$\dot{Z}_r = \tilde{A}_r Z_r + \tilde{B}_r u, \quad y = \tilde{C}_r Z_r,$$

with $Z_r = (z_1, \ldots, z_r)$.

Power series expansion for the balanced realization

- The balanced realization algorithm relies on numerical algebra and hence cannot be carried out as described if the system contains a symbolic parameter.
- In this paper we address this issue, assuming that the linear system is given by matrices $A(m)$, $B(m)$ and $C(m)$ which depend analytically on the parameter m.
- The parameter m may represent an uncertain physical coefficient, or it may appear by considering an unspecified working point in the linearisation of a non-linear system.
- Goal: to develop a power series expansion in m of the balanced model order reduction algorithm for the linear input/output system given by $A(m)$, $B(m)$, $C(m)$.

- Assume that $A(m)$, $B(m)$ and $C(m)$ are analytic in m,

$$
\begin{aligned}
A(m) &= \sum_{k=0}^{\infty} A_k m^k, \\
B(m) &= \sum_{k=0}^{\infty} B_k m^k, \\
C(m) &= \sum_{k=0}^{\infty} C_k m^k,
\end{aligned}
$$

- Our goal is to find power expansions for the matrices $W_c(m)$, $W_o(m)$, $X(m)$, $Y(m)$, $U(m)$, $V(m)$, $\Sigma(m)$ and $T(m)$ which appear in the balanced realization MOR procedure.

- For instance, for $W_c(m)$ one has $W_c(m) = \sum_{k=0}^{\infty} W_k^c m^k$, and substituting into the Lyapunov equation for the controllability Gramian one gets the set of Lyapunov equations

$$\begin{aligned} A_0 W_0^c + W_0^c A_0^T + B_0 B_0^T &= 0, \\ A_0 W_r^c + W_r^c A_0^T + P_r &= 0, \; r = 1, 2, \ldots, \end{aligned}$$

with $P_r = B_0 B_r^T + \sum_{s=0}^{r-1} \left(A_{r-s} W_s^c + W_s^c A_{r-s}^T + B_{r-s} B_s^T \right)$.

- These equations can be solved recursively to the desired order, starting with the zeroth order Lyapunov equation $A_0 W_0^c + W_0^c A_0^T + B_0 B_0^T = 0$.
- Notice that the internal dynamics is always given by A_0, and that it is only the effective control term P_r the one that changes with the order.

- Similarly, one can obtain recursive relations for $W_o(m)$, $X(m)$, $Y(m)$ and $R(m) = Y^T(m)X(m)$.
- The final step in the algorithm is the power expansion of the matrices of the SVD of $R(m)$.
- This is a more involved step, and in the paper we only provide the results up to second order in m.
- Let

$$\begin{aligned} R(m) &= R_0 + mR_1 + m^2 R_2, \\ U(m) &= U_0 + mU_1 + m^2 U_2, \\ V(m) &= V_0 + mV_1 + m^2 V_2, \\ \Sigma(m) &= \Sigma_0 + m\Sigma_1 + m^2 \Sigma_2, \end{aligned}$$

and let $u_j^{(k)}$ denote the jth column vector of U_k, and $v_j^{(k)}$ the one of V_k, and $\sigma_j^{(k)}$ the jth element of the diagonal matrix Σ_k.

- The first order corrections to the singular values are given by

$$\sigma_i^{(1)} = \left\langle u_i^{(0)}, R_1 v_i^{(0)} \right\rangle = \left\langle v_i^{(0)}, R_1^T u_i^{(0)} \right\rangle, \quad i = 1, \ldots, N,$$

  and they can be computed from the zeroth order SVD and the first order data R_1. Here $\langle \ , \ \rangle$ denotes the Euclidean inner product.

- The first order correction to the singular vectors u_j and v_j are the solutions $u_i^{(1)}$ and $v_i^{(1)}$ to the systems

$$\begin{pmatrix} R_0 R_0^T - (\sigma_i^{(0)})^2 \mathbb{I} \\ (u_i^{(0)})^T \end{pmatrix} u_i^{(1)} = \begin{pmatrix} Q_i^{(1)} \\ 0 \end{pmatrix}, \quad i = 1, \ldots, N,$$

$$\begin{pmatrix} R_0^T R_0 - (\sigma_i^{(0)})^2 \mathbb{I} \\ (v_i^{(0)})^T \end{pmatrix} v_i^{(1)} = \begin{pmatrix} P_i^{(1)} \\ 0 \end{pmatrix}, \quad i = 1, \ldots, N,$$

  with $Q_i^{(1)}$ and $P_i^{(1)}$ column vectors that can again be obtained from the zeroth order solution and the first order data R_1.

- The second order correction to the singular value σ_i, $i = 1, \ldots, N$, is given by

$$\sigma_i^{(2)} = \frac{1}{2}\sigma_i^{(0)}\left(||u_i^{(1)}||^2 - ||v_i^{(1)}||^2\right) + \left\langle u_i^{(0)}, R_1 v_i^{(1)} + R_2 v_i^{(0)}\right\rangle,$$

with the right-hand side depending only on data from the zeroth and first order approximations, plus the second order perturbation R_2.

- Finally, the second order correction to the singular vectors are the solutions of the systems

$$\begin{pmatrix} R_0 R_0^T - (\sigma_i^{(0)})^2 \mathbb{I} \\ (u_i^{(0)})^T \end{pmatrix} u_i^{(2)} = \begin{pmatrix} Q_i^{(2)} \\ -\frac{1}{2}||u_i^{(1)}||^2 \end{pmatrix},$$

$$\begin{pmatrix} R_0^T R_0 - (\sigma_i^{(0)})^2 \mathbb{I} \\ (v_i^{(0)})^T \end{pmatrix} v_i^{(2)} = \begin{pmatrix} P_i^{(2)} \\ -\frac{1}{2}||v_i^{(1)}||^2 \end{pmatrix}.$$

- The matrices appearing on the left hand-sides are the same as those of the first order correction, and the data on the right-hand side can be computed from the first order solution and the second order perturbation data.
- To our knowledge, this result about the second order perturbation of the singular vectors has not been reported in the litrature.
- Using the above results,one can compute

$$T(m) = X(m)V(m)\Sigma^{-1/2}(m),$$
$$T^{-1}(m) = \Sigma^{-1/2}(m)U(m)^T Y^T(m)$$

  up to second order in m, and then the matrices $A_2(m)$, $B_2(m)$, $C_2(m)$ which yield the balanced realization of the system again up to powers m^2.
- From the balanced realization, a suitable order reduced system can then be obtained.

Application: a system of masses and springs

- We consider a system of N masses m_i and (linear)springs with constants k_i and natural lengths d_i, so that the ith spring lies between masses m_i and m_{i+1}, $i = 1, \ldots, N-1$, and the last spring connects mass m_N to a fixed wall. We also add a linear damper to each mass, with coefficients γ_i and, furthermore, act with an external force F on the first mass.
- After redefining the coordinates to absorb the lengths d_i and introducing the canonical momenta $p_i = \dot{x}_i/m_i$, the system can be put in the first order form

$$\dot{X} = \left(\begin{array}{c|c} 0_{N\times N} & \text{diag}(1/m_1, \ldots, 1/m_N) \\ & \\ K_{N\times N} & -\text{diag}(\gamma_1/m_1, \ldots, \gamma_N/m_N) \end{array} \right) X + BF,$$

- The state is $X = (x_1, \dots, x_N, p_1, \dots, p_N)^T$. and the input vector is

$$B = (\underbrace{0, \dots, 0}_{N}, 1, 0, \dots, 0)^T.$$

- K is the stiffness matrix

$$K = \begin{pmatrix} -k_1 & k_1 & \cdots & 0 & 0 \\ k_1 & -(k_1+k_2) & \cdots & 0 & 0 \\ 0 & k_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -(k_{N-2}+k_{N-1}) & k_{N-1} \\ 0 & 0 & \cdots & k_{N-1} & -(k_{N-1}+k_N) \end{pmatrix}$$

- If we measure the velocity of the first mass, we have the output $y = CZ$ with C

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- In order to obtain a test of our whole algorithm, we consider the set of physical constants given by

$$\begin{aligned} k_i &= 100(i+1),\ i = 1, \ldots, N, \\ m_i &= i(1+m),\ i = 1, \ldots, N, \\ \gamma_i &= 1,\ i = 1, \ldots, N, \end{aligned}$$

with m the parameter of the Taylor expansion. We set $N = 10$, which yields a system with 20 states, and consider reduced systems with four states.

Our procedure, which we have implemented entirely in `Matlab`, yields the reduced system, parametrized by m, given by

$$A_4 = \begin{pmatrix} -0.28\,m^2 + 0.255\,m - 0.218 & 0.504\,m^2 - 0.84\,m + 2.06 \\ -0.504\,m^2 + 0.84\,m - 2.06 & -0.0393\,m^2 + 0.0548\,m - 0.0799 \\ 0.198\,m^2 - 0.193\,m + 0.181 & 1.01\,m^2 - 1.05\,m + 1.07 \\ 0.648\,m^2 - 0.745\,m + 0.862 & 0.0653\,m^2 - 0.0808\,m + 0.103 \end{pmatrix.$$

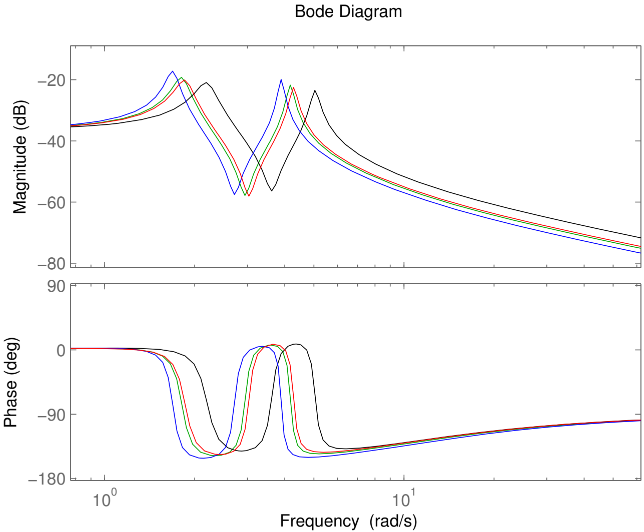
$$\begin{matrix} 0.198\,m^2 - 0.193\,m + 0.181 & -0.648\,m^2 + 0.745\,m - 0.862 \\ -1.01\,m^2 + 1.05\,m - 1.07 & 0.0653\,m^2 - 0.0808\,m + 0.103 \\ -0.143\,m^2 + 0.149\,m - 0.155 & 1.39\,m^2 - 2.14\,m + 4.91 \\ -1.39\,m^2 + 2.14\,m - 4.91 & -0.106\,m^2 + 0.119\,m - 0.134 \end{matrix}\right),$$

$$B_4 = \begin{pmatrix} -0.0362\,m^2 + 0.0505\,m - 0.143 \\ 3.95 \cdot 10^{-4}\,m^2 + 0.00639\,m - 0.0813 \\ 0.0135\,m^2 - 0.0239\,m + 0.102 \\ 0.00731\,m^2 - 0.0167\,m + 0.0922 \end{pmatrix},$$

and

$$C_4 = \begin{pmatrix} -0.0362\,m^2 + 0.0505m - 0.143 \\ -3.95 \cdot 10^{-4}\,m^2 - 0.00639\,m + 0.0813 \\ 0.0135\,m^2 - 0.0239\,m + 0.102 \\ -0.00731\,m^2 + 0.0167\,m - 0.0922 \end{pmatrix}^T .$$

- The next figure shows a detail of the Bode diagrams for $m = 0.5$ computed using the polynomial approximations of degree zero (black), one (blue) and two (red), together with the exact reduced system (green).
- It is clearly seen that the results improve as the order of the polynomial approximation is increased.
- Notice that the zeroth order polynomial approximation is equivalent to considering $m = 0$.


Bode Diagram
Magnitude (dB)
−20
−40
−60
−80
Phase (deg)
90
0
−90
−180
10^0
10^1
Frequency (rad/s)

Conclusions

- We have developed a parameter dependent model order reduction algorithm based on the balanced realization approximation. The algorithm yields a reduced order model which can be used to design a controller valid for a range of values of the parameter. As a by-product, we have obtained an expression for the second order perturbation of the singular subspaces.
- Some extensions of our work include considering several parameters instead of just one, or computing some further higher order corrections of the parametrized SVD.
- We have not addressed the issue of the estimation of the error of the reduced model. This error involves both the truncation errors of the different steps of the algorithm and the error which comes from the truncation of the balanced realization.
- In its present form, our algorithm can only be applied to stable systems.