Applications of passive control to electromechanical systems

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Modelling and Control of Complex Dynamical Systems: from Ports to Robotics

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Contents

1 Preamble

This lecture is devoted to some applications of passive control for electromechanical systems in the framework of port controlled hamiltonian systems, which in the simplest, explicit version are of the form

$$\dot{x} = (\mathcal{J}(x) - \mathcal{R}(x))(\nabla H(x))^T + g(x)u,$$

where $x \in \mathbb{R}^n$, $\mathcal{J}$ is antisymmetric, $\mathcal{R}$ is symmetric and positive semi-definite and $u \in \mathbb{R}^m$ is the control. The function $H(x)$ is the hamiltonian, or energy, of the system. The natural outputs in this formulation are

$$y = g^T(x)(\nabla H(x))^T.$$

This kind of systems enjoy nice properties when interconnected, and yield themselves to passive control strategies quite naturally. It is assumed that the reader is already acquainted with the essential underpinnings of port controlled hamiltonian systems and passivity (see, for instance, [21] or elsewhere in this Summer School).

We start with an elementary account of how to compute the generalized force in an electromechanical system. Next we model several systems in the port hamiltonian framework, and apply the IDA-PBC methodology to them. Finally, we include an explicit example of interconnection of port hamiltonian systems using some of the models presented. An appendix about how to solve the kind of PDE that appear in the examples is also included for completeness. Some proposed exercises are scattered around the text. Most of them are straightforward but some may require consulting the cited literature for hints. The reader is encouraged to model the systems and controls proposed with 20-sim, preferably using the bond graph formalism.

2 Electromechanical energy conversion

From the most general point of view, electromechanical systems are electromagnetic systems, *i.e.* systems with electric and/or magnetic fields, with moving parts. The archetypical electromechanical system is the electric machine, which can be used either as a generator, converting the mechanical energy of a primary mover into electric energy, or as an electric motor, taking the electric power to yield mechanical work. In this Section we will learn how to compute the energy contained in an electromechanical system and how to compute the (generalized) force it can yield, and we will illustrate the results with several examples.

Our presentation follows [11, 12], which is standard electrical engineering material. Notice however that there is some discussion in the literature [4] about the way that Maxwell equations are used when dealing with the concept of magnetic flux in systems with moving parts, and in particular in the case of rotating machines [2, 17, 10], where noninertial reference frames are used. We will not enter into this in these lectures and will assume the standard electrical engineering model.

When we change something in a system we generally change the energy it contains. If we apply an external force and change the geometry of the system, this change must overcome the internal forces holding the system, and this changes its internal energy. If we increase the charges in an electric system, say a capacitor, this increases the mechanical forces needed to keep the system together and, if we let the distance between plates vary, we can use the displacement to produce mechanical work.

Mathematically, this change in the energy of the system can be expressed as

$$W_f(q,\lambda,x;q_0,\lambda_0,x_0) = \int_{(q_0,\lambda_0,x_0)}^{(q,\lambda,x)} \left\{ \sum_{j=1}^{E} v_j \mathrm{d}\tilde{q}_j + \sum_{j=1}^{M} i_j \mathrm{d}\tilde{\lambda}_j - \sum_{j=1}^{G} f_j \mathrm{d}\tilde{x}_j \right\} \tag{1}$$

where we have G generalized coordinates x_j describing the geometry of the system, E electric elements (capacitors) containing charges q_j at voltage v_j, and M magnetic elements (inductors) through which currents i_j flow and containing magnetic fluxes λ_j. We denote by f_j the (generalized) force **done by the system** along the coordinate x_j. In general, the dependent variables v_j, i_j and f_j depend on all of the independent variables q_j, λ_j and x_j. For **linear electromagnetic systems**, v is a linear function of q and i is a linear function of λ, while the dependence on x is generally nonlinear.

Now, (1) is a line integral in $\mathbb{R}^{G+E+M}$ (at least locally; the x can live in other manifolds if rotating parts are present). In general its value will depend not only on the initial and final values, but also on the trajectory between them. However, we will assume that **the system is conservative** or, for the more mathematically inclined, that the $1-$form inside the integral in (1) is exact. Physically, this means that phenomena as eddy current losses, magnetic hysteresis cycles or mechanical friction are neglected. To include these into the formalism, the thermal domain should be added.

Under this assumption, W_f is a state function and v_j, i_j or f_j can be obtained by taking the derivatives with respect to q_j, λ_j or (minus) x_j, respectively. In particular, the generalized force, which is our main objective, is given by

$$f_i = -\frac{\partial W_f(q,\lambda,x)}{\partial x_i}. \tag{2}$$

However, it seems that we cannot get any further since W_f, as defined by (1), depends on f_i, which is what we are trying to compute. At this point the fact that W_f is a state function comes to our help. It allows us to reach the final point (q,λ,x) using a particular path and in such a way that f_i does not appear explicitly. First of all, we can assume that the initial state has $q_0 = 0$, $\lambda_0 = 0$, since the change in energy when going from $(0,0,x_0)$ to (q_0,λ_0,x_0) does not contribute to (2). Then, in the first leg of our path, we go from x_0 to x, without changing the zero initial values of q or λ; since the system has no electromagnetic field, no work is done and $W_{f1} = 0$. In the second leg we keep the geometry fixed, so no mechanical work is done, and go to the final values of charge and magnetic flux:

$$W_{f2} = \int_{(q_0,\lambda_0,x)}^{(q,\lambda,x)} \left\{ \sum_{j=1}^{E} v_j(\tilde{q},\tilde{\lambda},x) \mathrm{d}\tilde{q}_j + \sum_{j=1}^{M} i_j(\tilde{q},\tilde{\lambda},x) \mathrm{d}\tilde{\lambda}_j \right\},$$

and since $W_f = W_{f2}$ we only need to know v and i in terms of (q,λ,x).

2.1 Electric capacitor

Figure 1 shows a general plane capacitor, where the dielectric has a nonlinear constitutive relation. As usual, we assume that the transversal dimensions of the plates are much

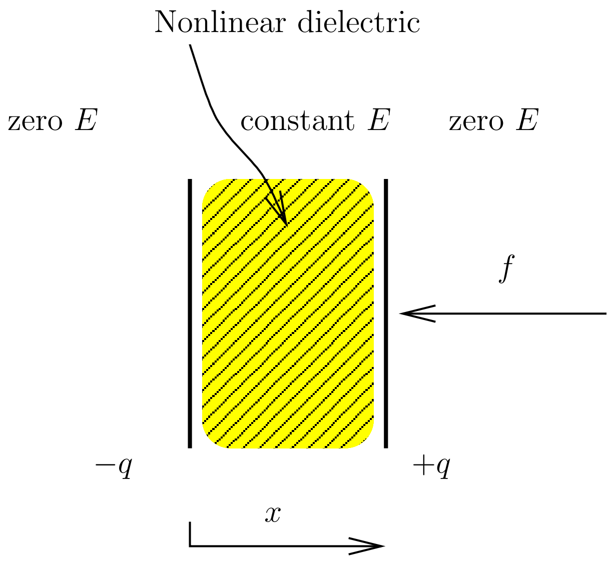

Figure 1: A nonlinear capacitor.

bigger than the plate's separation x, so that the electric field is zero outside the capacitor and constant inside of it. In the vacuum or in linear isotropic dielectrics, it can be shown that the voltages of a set of conductors depend linearly on the charges. Here we assume a more general situation where the voltage difference depends nonlinearly on q. To make contact with the linear case, we introduce a capacitance-like function $C(x,q)$ so that

$$v(x,q) = \frac{q}{C(x,q)}.$$

According to our general discussion, we have to compute

$$W_f(q,x) = \int_{(0,x_0)}^{(q,x)} v \; \mathrm{d}q - f\mathrm{d}x$$

Going first with $q=0$ from x_0 to x and then from $q=0$ to q with x fixed, one gets

$$W_f(x,q) = \int_0^q \frac{\xi}{C(x,\xi)}\mathrm{d}\xi.$$

Finally, we derive with respect to x to obtain the force on the plates. An integration by parts helps to make contact with the linear case:

$$\begin{aligned} f(x,q) &= -\frac{\partial W_f}{\partial x}(x,q) = -\int_0^q \xi \frac{\partial}{\partial x}\left(\frac{1}{C(x,\xi)}\right)\mathrm{d}\xi \\ &= -\frac{q^2}{2}\frac{\partial}{\partial x}\left(\frac{1}{C(x,q)}\right) + \int_0^q \frac{\xi^2}{2}\frac{\partial^2}{\partial\xi\partial x}\left(\frac{1}{C(x,\xi)}\right)\mathrm{d}\xi \\ &= q\frac{E(x,q)}{2} + \frac{1}{2}\int_0^q \xi^2 \frac{\partial^2}{\partial\xi\partial x}\left(\frac{1}{C(x,\xi)}\right)\mathrm{d}\xi. \end{aligned}$$

For a linear system, $C(x,q) = C(x)$,

$$f(x,q) = q\frac{E(x,q)}{2}$$

which can be easily obtained by simple arguments from the electric field seen by the $+q$ plate, and

$$W_f(x,q) = \int_0^q \frac{\xi}{C(x)}\mathrm{d}\xi = \frac{1}{2}\frac{q^2}{C(x)},$$

which is the well-known elementary result.

2.2 Magnetic stationary system

Figure 2 shows two coils coupled by the magnetic field they generate. Although this is not an electromechanical system (there are no moving parts) it allows us to introduce several concepts which will be used in the next example.

The current of any of the coils produces a magnetic field which goes through the other coil. Variations in any of the currents change the flux of the magnetic induction field and this, according to Lenz's law, originates an induced voltage in both coils. The iron core is introduced to "bend" the magnetic field lines so that the coupling is tighter. The coils have N_1 and N_2 turns, respectively. Any turn of any of the two coils has three contributions to the magnetic flux that goes through it:

- the flux due to the lines of the induction magnetic field generated by the current of the same coil and which close through the iron core,
- the flux due to the lines of the induction magnetic field generated by the current of the same coil and which do not close through the iron core, and
- the flux due to the lines of the induction magnetic field generated by the current of the other coil and which close through the iron core.

This can be written as

$$\begin{aligned}\Phi_1 &= \Phi_{l1} + \Phi_{m1} + \Phi_{m2} \\ \Phi_2 &= \Phi_{l2} + \Phi_{m2} + \Phi_{m1}\end{aligned}$$

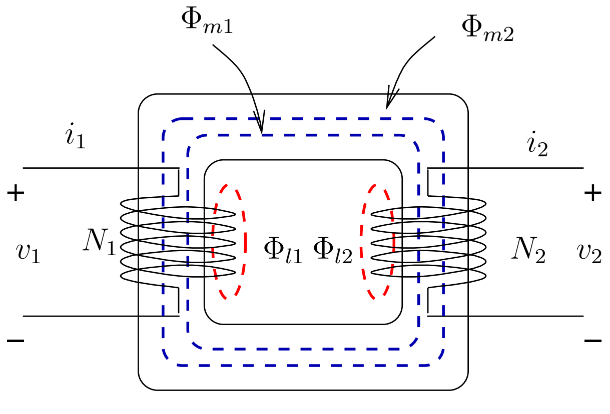

Figure 2: Magnetically coupled coils.

Φ_{l1} and Φ_{l2} are known as the *leakage* fluxes, while Φ_{m1} and Φ_{m2} are the *magnetizing* fluxes. The total flux through the coils, which is the quantity which enters the electrical equations, is then

$$\begin{aligned}\lambda_1 &= N_1\Phi_1,\\ \lambda_2 &= N_2\Phi_2.\end{aligned}$$

For a linear magnetic system, fluxes can be expressed in terms of path reluctances, number of turns and currents as

$$\Phi_{l1} = \frac{N_1 i_1}{\mathcal{R}_{l1}},\ \Phi_{m1} = \frac{N_1 i_1}{\mathcal{R}_m},\ \Phi_{l2} = \frac{N_2 i_2}{\mathcal{R}_{l2}},\ \Phi_{m2} = \frac{N_2 i_2}{\mathcal{R}_m},$$

and one finally gets the well-known linear relation in terms of inductances:

$$\begin{aligned}\lambda_1 &= L_{11} i_1 + L_{12} i_2,\\ \lambda_2 &= L_{21} i_1 + L_{22} i_2.\end{aligned}$$

For a system with a variable geometry, the mutual inductances L_{12} and L_{21} depend on the geometrical parameters. If the system is nonlinear, the inductances also depend on the various currents. Usually, only the core experiences nonlinear effects (*i.e.* saturation), and hence only the mutual inductances are current-dependent.

The dynamical equations of the system are now given by

$$v_1 = r_1 i_1 + \frac{\mathrm{d}\lambda_1}{\mathrm{d}t}, \quad v_2 = r_2 i_2 + \frac{\mathrm{d}\lambda_2}{\mathrm{d}t}$$

where r_1, r_2 are the electric resistances of the respective windings.

An equivalent circuit interpretation, which is often used in the electrical engineering literature, can be given if we introduce

$$i_2^{'} = \frac{N_2}{N_1} i_2,\ v_2^{'} = \frac{N_1}{N_2} v_2,\ \lambda_2^{'} = \frac{N_1}{N_2}\lambda_2,\ r_2^{'} = \left(\frac{N_1}{N_2}\right)^2 r_2,$$

$$L_{l2}^{'} = \left(\frac{N_1}{N_2}\right)^2 L_{l2}, L_{m1} = \frac{N_1}{N_2} L_{12}.$$

This is called *referring the winding 2 to winding 1.* The equations keep the same form as before the change

$$v_1 = r_1 i_1 + \frac{\mathrm{d}\lambda_1}{\mathrm{d}t}, \quad v_2^{'} = r_2^{'} i_2^{'} + \frac{\mathrm{d}\lambda_2^{'}}{\mathrm{d}t},$$

but now

$$\begin{aligned}\lambda_1 &= L_{l1} i_1 + L_{m1}(i_1 + i_2^{'})\\ \lambda_2^{'} &= L_{l2}^{'} i_2^{'} + L_{m1}(i_1 + i_2^{'})\end{aligned}$$

which can be seen as a T-circuit (see Figure 3).

As an special case we consider the **ideal transformer**. Set $r_1 = r_2 = L_{l1} = L_{l2} = 0$. Then $v_1 = v_2^{'}$, *i.e.*

$$v_2 = \frac{N_2}{N_1} v_1.$$

If, additionally, $L_{m1} = N_1^2/\mathcal{R}_m$ is large enough so that $i_1 + i_2^{'}$ is negligible, we get

$$i_2 = -\frac{N_1}{N_2} i_1.$$

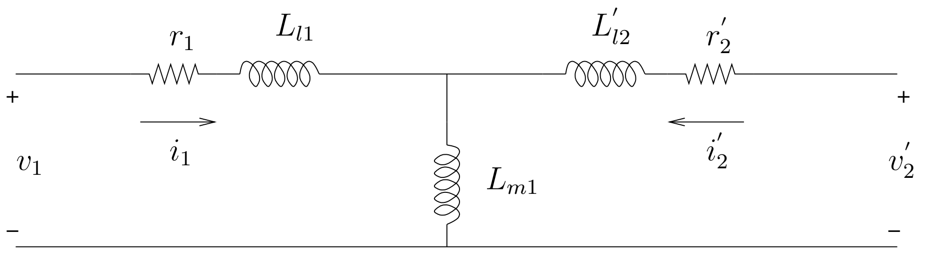

Figure 3: An equivalent circuit for the magnetic system.

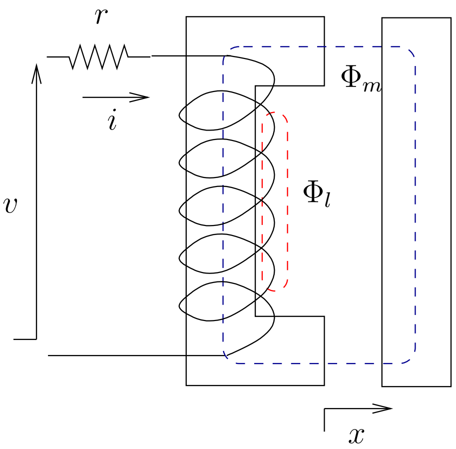

Figure 4: An elementary electromagnet: a magnetic system with a moving part.

2.3 Elementary electromagnet

Figure 4 shows an elementary electromagnet, a magnetic system with a moving part the flux linkage λ through the coil depends on a geometry variable, the "air gap" x.

The electrical equation of motion is

$$v = ri + \frac{\mathrm{d}\lambda}{\mathrm{d}t},$$

where the flux linkage can be computed from the number of turns, N, and the magnetic induction flux, Φ, as

$$\lambda = N\Phi.$$

In turn, Φ has a leakage, Φ_l, and a magnetizing, Φ_m, parts, $\Phi = \Phi_l + \Phi_m$, which can be computed in terms of the reluctances of the respective paths:

$$\Phi_l = \frac{Ni}{\mathcal{R}_l}, \qquad \Phi_m = \frac{Ni}{\mathcal{R}_m}.$$

The reluctance of the magnetizing path has a fixed contribution, the part of the iron path, and a variable one, the part of the air gap:

$$\mathcal{R}_m = \frac{l_i}{\mu_{ri}\mu_0 A_i} + \frac{2x}{\mu_0 A_g},$$

where μ_{ri}, the relative magnetic permeability of the iron core, is of the order of 10^3. Assuming that the sections of the iron and air gap paths are the same, $A_i = A_g = A$, one gets

$$\mathcal{R}_m = \frac{1}{\mu_0 A}\left(\frac{l_i}{\mu_{ri}} + 2x\right).$$

The relation between the current and the flux linkage can finally be written as

$$\lambda = \left(\frac{N^2}{\mathcal{R}_l} + \frac{N^2}{\mathcal{R}_m}\right) i = (L_l + L_m)i$$

with

$$L_m = \frac{N^2}{\mathcal{R}_m} = \frac{N^2\mu_0 A}{\frac{l_i}{\mu_{ri}} + 2x} \equiv \frac{b}{c+x}.$$

EXERCISE Compute the electromechanical energy W_f and the force necessary to move x.

2.4 Coenergy

Assume that, from $v = \partial_q W_f$ and $i = \partial_\lambda W_f$, we can express q and λ as functions of v and i (and x). Then the **coenergy**, W_c, is the Legendre transform of the energy W_f:

$$W_c(v,i,x) = \left(\sum_{j=1}^{E} q_j v_j + \sum_{j=1}^{M} \lambda_j i_j - W_f(q,\lambda,x)\right)_{q=q(v,x),\lambda=\lambda(i,x)}.$$

For linear electromagnetic systems, *i.e.* $q = C(x)v$, $\lambda = L(x)i$, energy equals coenergy:

$$W_c(v,i,x) = W_f(q,\lambda,x)\big|_{q=q(v,x),\lambda=\lambda(i,x)}. \tag{3}$$

In particular, for linear magnetic systems,

$$W_f(\lambda,x) = \frac{1}{2}\lambda^T L^{-1}(x)\lambda, \quad W_c(i,x) = \frac{1}{2} i^T L(x) i.$$

Notice that the coupling force can be obtained from the coenergy as

$$\partial_x W_c = \partial_x \lambda^T i - \left(\partial_x \lambda^T \partial_\lambda W_f + \partial_x W_f\right) = -\partial_x W_f = f.$$

EXERCISE Prove (3).

EXERCISE Consider an electromechanical system with a nonlinear magnetic material such that

$$\lambda = (a + bx^2)i^2,$$

where a and b are constants and x is a variable geometric parameter. Compute W_f, W_c and f, and check all the relevant relations.

3 Port hamiltonian system modelling

In this Section we will introduce the port hamiltonian model of a wide class of electromechanical systems, for which a passive controller will be discussed in the next Section. We also consider power converters which, although not by themselves electromechanical systems, are an integral part of the control of electromechanical devices [13].

3.1 General electromechanical systems

We consider the class of electromechanical systems [18] described by

$$\begin{aligned} \dot{\lambda} + Ri &= Bv \\ J\ddot{\theta} &= -r_m\dot{\theta} + T_e(i,\theta) + T_m \end{aligned} \tag{4}$$

where, assuming a linear magnetic system (with no permanent magnet), $\lambda = L(\theta)i$. The generalized electrical force is given by

$$T_e(i,\theta) = \frac{1}{2} i^T \partial_\theta L \, i.$$

Both i and θ can contain an arbitrary number of degrees of freedom and, although we are adopting a "rotational notation", translational motion is also included. T_m is the (external) mechanical force, J is the moment of inertia (or the mass) of the moving parts and r_m is a friction coefficient. The matrix B describes which parts of the electrical system are actuated by the control voltages v. This general model includes most of the classical electrical machines, as well as linear motors and levitating systems.

Introducing $p = J\dot{\theta}$ and $x = (\lambda \ \theta \ p)^T$, the port hamiltonian description is given by

$$\dot{x} = \begin{pmatrix} -R & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -r_m \end{pmatrix} \partial_x H + \begin{pmatrix} B & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ T_m \end{pmatrix} \tag{5}$$

with the Hamiltonian

$$H(x) = \frac{1}{2}\lambda^T L^{-1}(\theta)\lambda + \frac{1}{2J}p^2.$$

EXERCISE Show that indeed (5) reduces to (4).

EXERCISE Write the port hamiltonian model (5) for the electromagnet.

3.2 Power converters

Power converters are of great value in many growing application areas such as the control of electromechanical devices, portable battery-operated equipment or uninterruptible power sources. In particular, dc-to-dc power converters are used to reduce or elevate a given dc voltage, storing the energy in intermediate ac elements and delivering it as needed. The essential trick is the use of switches, which are operated in a periodic manner and which make the system to alternate between several dynamics. Generally, the individual dynamics are linear and can be solved analytically; the action of the switches yields a nonlinear dynamics which can display a rich behavior (see [8, 1] and references therein).

Figure 5 shows a functional model of the boost (or elevator) converter (the detailed electronics of how the switches are implemented is not shown). The switches s_1 and s_2 are complementary: when s_1 is closed ($s_1 = 1$), s_2 is open ($s_2 = 0$), and viceversa. Thus, the different circuit topologies can be described with a single boolean variable $S = s_2$.

The port hamiltonian modelling of electric circuits can be done in a systematic way using tools from graph theory [3], but since we are dealing here with a circuit of very small size we will adopt a more pedestrian approach and concentrate on the problems

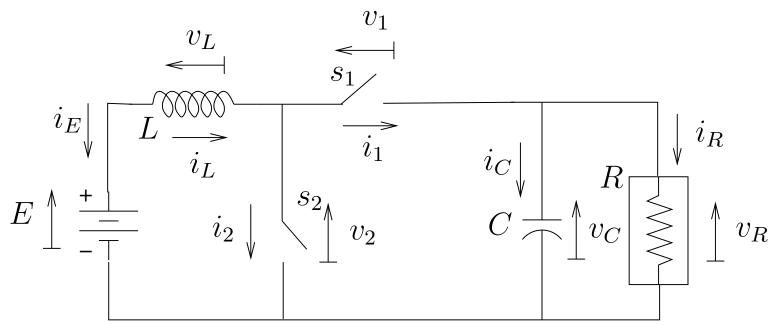

Figure 5: A functional description of the boost converter.

presented by the switches, using the ideas of [7]. A more in-deep conceptual analysis of the switches can be found in [9].

The hamiltonian dynamical variables of the boost converter are the magnetic flux at the coil, ϕ_L, and the charge of the capacitor, q_C. Hence we have two one-dimensional hamiltonian subsystems, with a global hamiltonian $H = H_C + H_L$,

$$\frac{\mathrm{d}q_C}{\mathrm{d}t} = i_C, \quad v_C = \frac{\partial H}{\partial q_C}, \tag{6}$$

and

$$\frac{\mathrm{d}\phi_L}{\mathrm{d}t} = v_L, \quad i_L = \frac{\partial H}{\partial \phi_L}, \tag{7}$$

connected by Kirchoff's laws

$$\begin{aligned} i_L &= i_1 + i_2 \\ i_1 &= i_C + i_R \\ v_2 + v_L &= E \\ v_C + v_1 &= v_2 \\ v_C &= v_R \\ i_E + i_L &= 0 \end{aligned} \tag{8}$$

Here we treat the switches as ports, with their correspondent effort and flow variables. For the time being we do not terminate the resistive port, *i.e.* we do not use $v_R = Ri_R$.

EXERCISE Write the EF-representation [5] of the Dirac structure associated to (8).

Using (6) and (7), the first four equations of (8) can be written as

$$\begin{aligned} \frac{\partial H}{\partial \phi_L} &= i_1 + i_2 \\ i_1 &= \frac{\mathrm{d}q_C}{\mathrm{d}t} + i_R \\ v_2 + \frac{\mathrm{d}\phi_L}{\mathrm{d}t} &= E \\ \frac{\partial H}{\partial q_C} + v_1 &= v_2 \end{aligned} \tag{9}$$

The second and third equations in (9) yield a hamiltonian system with four inputs and $\mathcal{J} = \mathcal{R} = 0_{2\times 2}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q_C \\ \phi_L \end{pmatrix} = 0(\nabla H)^T + \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} i_1 \\ v_2 \\ i_R \\ E \end{pmatrix}. \tag{10}$$

Next we will use the constraints imposed by the switches to absorb the ports s_1 and s_2 into the hamiltonian structure:

- $S = 0 \Rightarrow s_1 = 1, s_2 = 0 \Rightarrow v_1 = 0, i_2 = 0,$
- $S = 1 \Rightarrow s_1 = 0, s_2 = 1 \Rightarrow i_1 = 0, v_2 = 0.$

Hence, when $S = 1$ we already have the values of the port variables i_1, v_2 in (10), while if $S = 0$, using the first and fourth equations in (9),

$$i_1 = \frac{\partial H}{\partial \phi_L}, \quad v_2 = \frac{\partial H}{\partial q_C}.$$

We can put together both results as

$$\begin{aligned} i_1 &= (1-S)\frac{\partial H}{\partial \phi_L}, \\ v_2 &= (1-S)\frac{\partial H}{\partial q_C}. \end{aligned} \tag{11}$$

Now

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q_C \\ \phi_L \end{pmatrix} &= \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} (1-S)\ \partial H/\partial \phi_L \\ (1-S)\ \partial H/\partial q_C \\ i_R \\ E \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1-S \\ -(1-S) & 0 \end{pmatrix}\begin{pmatrix} \partial H/\partial q_C \\ \partial H/\partial \phi_L \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} i_R \\ E \end{pmatrix}, \end{aligned} \tag{12}$$

which is a port hamiltonian system with outputs

$$y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} \partial H/\partial q_C \\ \partial H/\partial \phi_L \end{pmatrix} = \begin{pmatrix} -v_C \\ i_L \end{pmatrix} = \begin{pmatrix} -v_R \\ -i_E \end{pmatrix}.$$

Finally, we terminate the resistive port using

$$i_R = \frac{v_R}{R} = \frac{v_C}{R} = \frac{1}{R}\frac{\partial H}{\partial q_C}$$

and get our final port hamiltonian representation of the boost converter

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q_C \\ \phi_L \end{pmatrix} = \left[\begin{pmatrix} 0 & 1-S \\ -(1-S) & 0 \end{pmatrix} - \begin{pmatrix} 1/R & 0 \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} \partial H/\partial q_C \\ \partial H/\partial \phi_L \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} E, \tag{13}$$

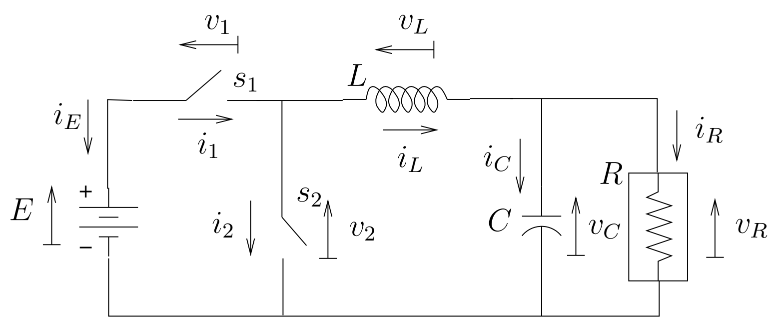

Figure 6: The buck converter.

with natural output

$$y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} \partial H/\partial q_C \\ \partial H/\partial \phi_L \end{pmatrix} = i_L = -i_E.$$

Notice that the interconnection structure $\mathcal{J}$ is modulated by the boolean variable S. Designing a control for this system means choosing S as a function of the state variables.

EXERCISE Figure 6 shows an scheme of the buck (or step-down) power converter. Show that the final port hamiltonian structure is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q_C \\ \phi_L \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 1/R & 0 \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} \partial H/\partial q_C \\ \partial H/\partial \phi_L \end{pmatrix} + \begin{pmatrix} 0 \\ 1-S \end{pmatrix} E.$$

4 Passive control examples

Remember ([16] and elsewhere in this Summer School) that the central objective of the IDA-PBC methodology for a given port hamiltonian system

$$\dot{x} = (\mathcal{J}(x,u) - \mathcal{R}(x))(\nabla H)^T(x) + g(x,u),$$

is to find additional structure $\mathcal{J}_a$ and resistive $\mathcal{R}_a$ matrices, a control function $u = \beta(x)$ and a vector $K(x)$, gradient of a function $H_a(x)$, such that

$$(\mathcal{J}(x,\beta(x))+\mathcal{J}_a(x)-\mathcal{R}(x)-\mathcal{R}_a(x))K(x) = -(\mathcal{J}_a(x)-\mathcal{R}_a(x))(\nabla H)^T(x)+g(x,\beta(x)) \quad (14)$$

is satisfied, and such that $H_d(x) = H(x)+H_a(x)$ has a minimum at the desired equilibrium point. Equation (14) is a complicated PDE for H_a and several techniques have been proposed in the literature [16, 18, 15] to facilitate its solution. Here we are going to illustrate some of the basic techniques with concrete examples. Appendix A contains a very simple account of the method of characteristics, which is the basic tool for solving the kind of PDE that arises from (14).

4.1 Magnetic levitation system

Figure 7 shows a very simplified model of a magnetic levitation system.

The flux lines generated by the current at the coil close through the air gap and the iron ball. Since the air gap has a variable reluctance, the system tries to close it, and this counteracts the gravity.

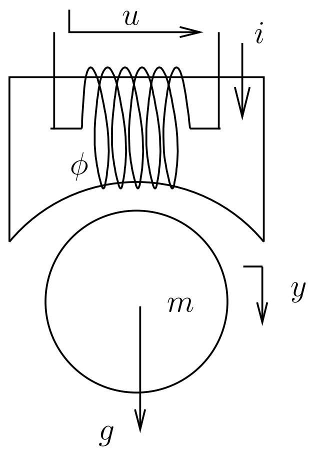

Figure 7: A magnetic levitation system.

The equations of motion are

$$\begin{aligned}\dot{\phi} &= -Ri + u \\ \dot{y} &= v \\ m\dot{v} &= F_m + mg\end{aligned}$$

with $\phi = L(y)i$ the linkage flux, R the resistance of the coil, and F_m the magnetic force, given by

$$F_m = \frac{\partial W_c}{\partial y},$$

where the magnetic coenergy is (we assume a linear magnetic system)

$$W_c = \frac{1}{2}\frac{\partial L}{\partial y}\, i^2.$$

In general, L is a complicated function of the air gap, y. A classical approximation for L for this kind of systems for small y is

$$L(y) = \frac{k}{a+y}$$

with k, a constants, which is essentially the relation used for our model of the electromagnet, without the leakage term.

Taking $x_1 = \phi$, $x_2 = y$, $x_3 = mv$, this can be written as a PCH

$$\dot{x} = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]\frac{\partial H}{\partial x} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u$$

with Hamiltonian

$$H(x) = \frac{1}{2k}(a + x_2)x_1^2 + \frac{1}{2m}x_3^2 - mgx_2.$$

Note that the gravity term could also have been included as an external mechanical force.

The gradient of the hamiltonian is

$$(\nabla H)^T = \begin{pmatrix} x_1 \frac{a+x_2}{k} \\ \frac{1}{2k}x_1^2 - mg \\ \frac{x_3}{m} \end{pmatrix}.$$

Given a desired y^*, the equilibrium point is

$$x^* = \begin{pmatrix} \sqrt{2kmg} \\ y^* \\ 0 \end{pmatrix},$$

with an equilibrium control

$$u^* = \frac{R}{k} x_1^*(a + x_2^*).$$

We will first try to compute the IDA-PBC control without using the specific theory presented for general electromechanical systems.

Taking first $J_a = R_a = 0$, the IDA-PBC equation $(J - R)K(x) = g\beta(x)$ yields in this case

$$\begin{aligned} -RK_1(x) &= \beta(x) \\ K_3(x) &= 0 \\ -K_2(x) &= 0, \end{aligned}$$

and we see that $H_a(x) = H_a(x_1)$, which can be chosen so that

$$H_d(x) = H(x) + H_a(x_1)$$

has a critical point at $x = x^*$.

Unfortunately

$$\frac{\partial^2 H_d}{\partial x^2}(x) = \begin{pmatrix} \frac{1}{k}(a + x_2) + H_a''(x_1) & \frac{x_1}{k} & 0 \\ \frac{x_1}{k} & 0 & 0 \\ 0 & 0 & \frac{1}{m} \end{pmatrix}$$

has at least one negative eigenvalue no matter what H_a we choose, so x^* will not be asymptotically stable. The source of the problem is the lack of coupling between the mechanical and magnetic part in the interconnection matrix J. To solve this, we aim at

$$J_d = \begin{pmatrix} 0 & 0 & -\alpha \\ 0 & 0 & 1 \\ \alpha & -1 & 0 \end{pmatrix}, \qquad i.e. \quad J_a = \begin{pmatrix} 0 & 0 & -\alpha \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}.$$

Taking $R_a = 0$, the IDA-PBC equation now becomes

$$\begin{aligned} -\alpha K_3 - RK_1(x) &= \frac{\alpha}{m} x_3 + \beta(x) \\ K_3(x) &= 0 \\ \alpha K_1(x) - K_2(x) &= -\frac{\alpha}{k}(a + x_2)x_1, \end{aligned}$$

Now $H_a = H_a(x_1, x_2)$. Using the second equation, the first equation yields the control

$$u = \beta(x) = RK_1 - \alpha\frac{x_3}{m},$$

while the third equation is a PDE for $H_a(x_1, x_2)$:

$$\alpha\frac{\partial H_a}{\partial x_1} - \frac{\partial H_a}{\partial x_2} = -\alpha\frac{x_1(a+x_2)}{k}. \tag{15}$$

EXERCISE Solve (15) by the method of characteristics. It is better to give the initial condition as $(0, s, \Phi(s))$ instead of $(s, 0, \Phi(s))$.

The way in which (14) has been solved in this case seems to be quite model-dependent. In [18] a more general method for systems of the form (5) with $T_m = \partial_\theta V(\theta)$ is proposed. The central idea is to aim for a H_d given by

$$H_d(x) = \frac{1}{2}(\lambda - \mu_d(\theta,p))^T L^{-1}(\theta)(\lambda - \mu_d(\theta,p)) + V_d(\theta) + \frac{1}{2J}p^2,$$

where $\mu_d(\theta, p)$ is a kind of desired permanent magnet, and consider a hamiltonian structure of the form

$$\mathcal{J}_d - \mathcal{R}_a = \begin{pmatrix} -R & \alpha(x) & \beta(x) \\ -\alpha^T(x) & 0 & 1 \\ -\beta^T(x) & -1 & -r_a(p) \end{pmatrix}.$$

It can be seen then that the method boils down to an algebraic equation for

$$i_d = L^{-1}(\theta)\mu_d(\theta,p),$$

namely

$$\frac{1}{2}i_d^T\partial_\theta L(\theta)i_d + \partial_\theta V_d(\theta) - \partial_\theta V(\theta) - (r_m - r_a(p))\frac{p}{J} = 0, \tag{16}$$

and that α, β and u can be computed from it.

EXERCISE Try to derive (16). See [18] for details.

EXERCISE Solve (16) for the levitating ball system. Use $V_d(y) = K_p\frac{(y-y^*)^2}{\sqrt{1+(y-y^*)^2}}$, $r_a(p) = r_{a1} + r_{a2}\frac{p^2}{1+p^2}$, with $K_p > 0$, $r_{a1} > 0$, $r_{a2} > 0$.

4.2 Power converters

We retake the boost converter, or rather, its averaged model, where $u = 1 - S$, instead of taking values in the set $\{0, 1\}$, varies over $[0, 1]$ (see [6] for a discussion of this averaging process; essentially, S is changed periodically with a frequency much higher than the highest natural frequency of the system).

We have

$$\mathcal{J}(u) = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} 1/R & 0 \\ 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and, assuming a linear electromagnetic system,

$$H(x_1, x_2) = \frac{1}{2C}x_1^2 + \frac{1}{2L}x_2^2.$$

We set as control objective the regulation of the load voltage at V_d, so that the equilibrium point is $x^* = (CV_d, \frac{LV_d^2}{RE})$ and the equilibrium value of the control is

$$u^* = \frac{E}{V_d}.$$

Notice that this makes sense (*i.e.* $u^* \in [0,1]$) since this is a boost converter and $V_d \geq E$.

To solve (14) we take $\mathcal{J}_a = \mathcal{R}_a = 0$ and get

$$\begin{pmatrix} -1/R & u \\ -u & 0 \end{pmatrix} \frac{\partial H_a}{\partial x} = gE,$$

i.e.

$$\begin{aligned} -u\frac{\partial H_a}{\partial x_1} &= E, \\ -\frac{1}{R}\frac{\partial H_a}{\partial x_1} + u\frac{\partial H_a}{\partial x_2} &= 0. \end{aligned}$$

The standard way to solve this system [16] is to solve for the derivatives of H_a and then impose the identity of the second order cross derivatives. One gets

$$\begin{aligned} \frac{\partial H_a}{\partial x_1} &= -\frac{E}{u}, \\ \frac{\partial H_a}{\partial x_2} &= -\frac{E}{R}\frac{1}{u^2}. \end{aligned}$$

Imposing the identity of the cross derivatives we arrive at the PDE

$$\frac{2}{R}\frac{\partial u}{\partial x_1} - u\frac{\partial u}{\partial x_2} = 0. \tag{17}$$

EXERCISE Solve (17). As initial condition for the characteristic's method, take $(0, k_1 s, as + b)$, with k_1, a and b constants. In [19, 20] it is discussed how to choose the values of k_1, a and b to impose the stability of the equilibrium point, as well as how to improve the resulting controller so that it is robust with respect to E and R.

EXERCISE Repeat the above analysis for the buck converter. See the beginning of Section 3.2 of [16] for some hints.

EXERCISE Implement the buck and boost systems and controllers in 20-sim. To test them, use $L = 20$ mH, $C = 20$ μF, $E = 15$ V and $R = 30$ Ω.

5 Connecting systems

In this Section we will present an example of a hamiltonian system appearing as the result of interconnecting two of them by means of a Dirac structure (in its Kirchoff's law form).

The systems we are going to consider are the magnetic levitation system and the boost converter without the resistive port. The hamiltonian structures are given by

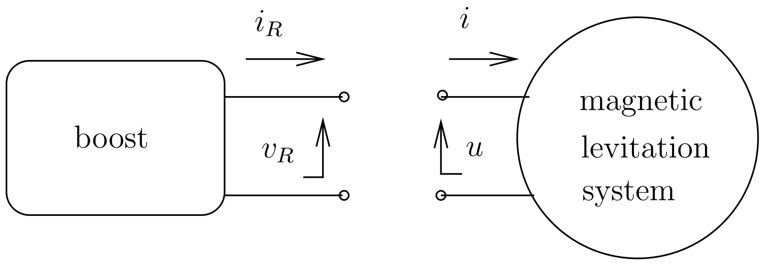

Figure 8: Interconnection of the boost converter with the magnetic levitation system.

- magnetic levitation system:

$$\begin{aligned}
\dot{w} &= \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] (\nabla H)^T + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u, && (18) \\
H(w_1, w_2, w_3) &= \frac{1}{2k}(a + w_2) w_1^2 + \frac{1}{2m} w_3^2 - mgw_2, && (19) \\
y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} (\nabla H)^T = i. && (20)
\end{aligned}$$

- open boost converter (we set $\eta = 1 - S$):

$$\begin{aligned}
\dot{z} &= \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} (\nabla \tilde{H})^T + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i_R \\ E \end{pmatrix}, && (21) \\
\tilde{H}(z_1, z_2) &= \frac{1}{2C} z_1^2 + \frac{1}{2L} z_2^2, && (22) \\
\tilde{y} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} (\nabla \tilde{H})^T = \begin{pmatrix} -v_R \\ -i_E \end{pmatrix}. && (23)
\end{aligned}$$

Now we interconnect both systems as shown in Figure 8. Kirchoff's laws are

$$i_R = i, \quad v_R = u,$$

or, using (20) and (23),

$$i_R = \frac{\partial H}{\partial w_1}, \quad u = \frac{\partial \tilde{H}}{\partial z_1}. \tag{24}$$

These two relations can be substituted into (18) and (21), so that the input terms get some hamiltonian gradients. The general theory of interconnection of port hamiltonian systems tells us that these terms can be absorbed into the structure part if both systems are taken together, and this is indeed what happens:

$$\dot{x} = \left[\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & -\eta & 0 \end{pmatrix} - \begin{pmatrix} R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right] (\nabla H_T)^T + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} E$$

where $x = (w_1\ w_2\ w_3\ z_1\ z_2)^T$ and $H_T = H + \tilde{H}$, and where the natural output of the interconnected system is

$$y_T = (0\ 0\ 0\ 0\ 1)(\nabla H_T)^T = i_L = -i_E.$$

EXERCISE Write the port hamiltonian model of the interconnection of the open (non resistance-terminated) buck converter and the electromagnet.

EXERCISE - OPEN QUESTION Can you write the magnetic levitating ball model as the interconnection of the mechanical part (the ball) and the electrical one (the coil)?

A Solving quasilinear PDEs

Equations of the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u), \tag{25}$$

where $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, appear frequently in modern control theory, and in particular in the IDA-PBC scheme. Equation (25) is called a quasilinear PDE because the derivatives in u appear linearly, although in general the dependence on u is nonlinear.

Geometrically, the solution to (25) is a surface $u = u(x, y)$ whose normal $(u_x, u_y, -1)$ is constrained by (25). This simple fact allows the explicit construction of a solution for this low dimension case, although the resulting method, called characteristics's method in the literature, can be generalized to higher order cases as well as to fully nonlinear PDEs.

Let $(x_0, y_0.u_0)$ be a point on a solution surface and let $(x(\tau), y(\tau), u(\tau))$ be a curve on the surface through it at $\tau = 0$. This means that the tangent vector $(x'(0), y'(0), u'(0))$ must be tangent to the surface at the point. Let us see how can we impose this condition.

Let $(p = u_x(x_0, y_0), q = u_y(x_0, y_0), -1)$ be a normal at the point. According to the preceding discussion, it must satisfy

$$c(x_0, y_0, u_0) = a(x_0, y_0, u_0)p + b(x_0, y_0, u_0)q. \tag{26}$$

The set of all planes through the point (x_0, y_0, u_0) is given by

$$u - u_0 = p(x - x_0) + q(y - y_0), \tag{27}$$

and p and q must satisfy (26) for this plane to be tangent to the surface. We can be sure that the tangent vector to the curve is indeed tangent to the surface if we impose that it belongs to the whole family of planes. Now it can be shown that the equation of the line common to all the planes in the set is

$$\frac{x - x_0}{a(x_0, y_0, u_0)} = \frac{y - y_0}{b(x_0, y_0, u_0)} = \frac{u - u_0}{c(x_0, y_0, u_0)}. \tag{28}$$

EXERCISE Show that (28) yields the equation of the line common to the family of planes (27).

Using the vector along the line of (28), we impose

$$\begin{aligned} x'(0) &= a(x_0, y_0, u_0), \\ y'(0) &= b(x_0, y_0, u_0), \\ u'(0) &= c(x_0, y_0, u_0), \end{aligned}$$

or, taking into account that this must be valid for any point on the curve,

$$\begin{aligned} x'(\tau) &= a(x(\tau), y(\tau), u(\tau)), \\ y'(\tau) &= b(x(\tau), y(\tau), u(\tau)), \\ u'(\tau) &= c(x(\tau), y(\tau), u(\tau)). \end{aligned} \tag{29}$$

The solutions to this system of ODE are called *characteristic curves* of the PDE, while their projections on the plane $u = 0$ are simply called *characteristics*. To generate a solution surface, one must start with a curve of initial conditions $(x(0, s), y(0, s), u(0, s))$ and solve (29) for each point on the curve. This way one gets $(x(\tau, s), y(\tau, s), u(\tau, s))$. If the curve of initial conditions does not lie on a characteristic curve, it is possible to solve for τ and s in terms of x and y, and finally get $u(x, y)$.

As a (manifestly trivial) example, let us consider

$$3u_x + 5u_y = u,$$

with an initial curve $(s, 0, f(s))$ where f is arbitrary. We have to solve

$$\begin{aligned} x' &= 3, \\ y' &= 5, \\ u' &= u. \end{aligned}$$

The solution satisfying the initial conditions is

$$\begin{aligned} x &= 3\tau + s, \\ y &= 5\tau, \\ u &= f(s)e^{\tau}. \end{aligned}$$

From the first two equations we get $\tau = y/5$ and $s = x - 3y/5$, and the corresponding solution surface is

$$u(x, y) = f(x - \frac{3y}{5})e^{\frac{y}{5}}.$$

EXERCISE For the above example, show that we cannot get the solution surface if the initial condition is given on $(3s, 5s, f(s))$.

Finally, let us remark that when several PDE for the same function are involved, some compatibility conditions must be met for the system to be solvable. See Chapter 2 of [14] for an account in terms of Fröbenius theorem and the integrability of a distribution of vector fields.

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