Partial differential equations

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Summary

Most equations in physics deal with quantities which depend on several space, or space-time, coordinates, and hence contain partial derivatives, and they satisfy partial derivatives equations (PDE).

In spite of being more complex than ODE, many of the ideas of the previous lecture, like those related to self-adjoint operators and their spectrum, completeness of eigenfunctions, or Green functions, can still be discussed for PDE.

In this lecture we will study the three basic kinds of second order lineal PDE, mainly in two variables : the wave equation in $1+1$, the heat equation in $1+1$ and the potential (or Laplace-Poisson) equation in 2 dimensions, extending the presentation to a higher number of independent variables when appropriate. We will also present the method of characteristics for linear first order PDE.

References

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Definitions. Characteristics of a linear PDE

- Let $\phi(x)$ be a function of d variables, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. One of the variables can be the time t, but not necessarily.
- We will denote the partial derivatives of ϕ as

$$\frac{\partial \phi}{\partial x_\mu} = \partial_\mu \phi, \quad \frac{\partial^2 \phi}{\partial x_\mu \, \partial x_\nu} = \partial^2_{\mu\nu} \phi$$

  and so on. Sometimes the notations ϕ_μ, $\phi_{\mu\nu}$ are also used.
- A PDE of order n for ϕ is an equation involving partial derivatives of ϕ, with the highest one being of order n.
- For instance, for $d = 3$,

$$\frac{\partial^3 \phi}{\partial x_1 \partial x_2 \partial x_3} - \phi \frac{\partial^2 \phi}{\partial x_1^2} - \frac{\partial \phi}{\partial x_2} - \phi = x_1 \cos x_2$$

  is a PDE of third order. It is nonlinear due to the $\phi \partial^2_{x_1^2} \phi$ term.

- We will only consider PDE of order one and two, and mostly for $d = 2$, and we will deal only with linear equations.
- The most general first order linear PDE for $d = 2$ is

$$\sum_{\mu=1}^{2} a_\mu(x)\partial_\mu\phi + b(x)\phi = c(x).$$

- Likewise, for second order,

$$\sum_{\mu=1}^{2}\sum_{\nu=1}^{2} a_{\mu\nu}(x)\partial_\mu\partial_\nu\phi + \sum_{\mu=1}^{2} a_\mu(x)\partial_\mu\phi + b(x)\phi = c(x).$$

- The **principal part** of a PDE is the part containing the higher order derivatives. For the above PDE they are

$$\sum_{\mu=1}^{2} a_\mu(x)\partial_\mu\phi, \quad \sum_{\mu=1}^{2}\sum_{\nu=1}^{2} a_{\mu\nu}(x)\partial_\mu\partial_\nu\phi,$$

respectively.

- The general solution of an ordinary differential equation of order n contains n arbitrary constants. For PDE, one has instead arbitrary functions.
- To see this, consider the second order PDE for $\phi(x, y)$

$$\frac{\partial^2 \phi}{\partial x \, \partial y} = 0.$$

- It is obvious that

$$\phi(x, y) = \varphi_1(x) + \varphi_2(y),$$

for any differentiable but otherwise arbitrary functions φ_1, φ_2, is a solution.

- As a less trivial case, consider

$$\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} = 1.$$

- If we perform the change of variables $z = x + y$, $w = x$ we obtain, using the chain rule, $\partial_x = \partial_z + \partial_w$, $\partial_y = \partial_z$.
- Let $\tilde{\phi}$ be the function of z, w such that $\tilde{\phi}(z, w) = \phi(x, y)$. Noticing that $\partial_x - \partial_y = \partial_w$, the PDE becomes

$$\frac{\partial \tilde{\phi}}{\partial w} = 1,$$

with trivial solution $\tilde{\phi}(z, w) = w + \varphi(z)$, φ arbitrary.
- Finally, in terms of the original variables,

$$\phi(x, y) = x + \varphi(x + y).$$

One can check that this is indeed a solution of the PDE (do it!).

- How can these arbitrary functions be determined? For ODE we know that one needs a number of initial conditions at a given point (or boundary conditions at several points). However, it may happen that not all the points are suitable to provide these conditions.
- As an example, consider the first order ODE for $y(x)$

$$a(x)y'(x) + b(x)y = c(x),$$

  with initial condition $y(x_0) = y_0$.
- One knows the value of y at $x = x_0$ and using the ODE one tries to extend the knowledge of y to an open set around x_0.
- Suppose that we want to compute $y(x_0 + \epsilon)$ for small ϵ. Using a first order Taylor expansion one has

$$y(x_0 + \epsilon) = y(x_0) + \epsilon y'(x_0) + O(\epsilon^2).$$

- We know that $y(x_0) = y_0$ and, since $y(x)$ must be a solution of the ODE,

$$y'(x_0) = \frac{c(x_0) - b(x_0)y(x_0)}{a(x_0)} = \frac{c(x_0) - b(x_0)y_0}{a(x_0)}.$$

- Finally, to first order in ϵ,

$$y_0(x + \epsilon) \approx y_0 + \frac{c(x_0) - b(x_0)y_0}{a(x_0)},$$

and the approximation can be made as good as desired for ϵ sufficiently small.

- Once we have $y_0(x + \epsilon)$ we can compute $y(x_0 + 2\epsilon) = y(x_0 + \epsilon + \epsilon)$ using

$$y(x_0 + \epsilon + \epsilon) = y(x_0 + \epsilon) + \epsilon y'(x_0 + \epsilon)$$

and so on (do it!).

- But what if $a(x_0) = 0$? Obviously our procedure is dead from the start, but any other way to solve the ODE will also run into problems and, in fact, one cannot ensure the existence or uniqueness of solutions in this case.
- As a more complex example, consider the second order ODE

$$p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x),$$

with initial conditions at $x = a$ given by $y(a)$, $y'(a)$.

- We can compute an approximate value of the solution at $x = a + \epsilon$ using a Taylor expansion at first order,

$$y(a + \epsilon) = y(a) + \epsilon y'(a).$$

- This is actually independent of the second order ODE, but if want to repeat the procedure to get a value of y at $x = a + 2\epsilon$,

$$y(a + 2\epsilon) = y(a + \epsilon) + \epsilon y'(a + \epsilon)$$

we will need $y'(a + \epsilon)$, and here is where the specific ODE must be used.

- We can apply the same strategy to compute $y'(a+\epsilon)$,

$$y'(a+\epsilon) = y'(a) + \epsilon y''(a).$$

- The initial conditions do not include $y''(a)$, but we can use the ODE to compute its value for a point on the solution curve $y(x)$,

$$y''(a) = \frac{1}{p_2(a)}\left(f(a) - p_1(a)y'(a) - p_0(a)y(a)\right),$$

  where we assumed that $p_2(a) \neq 0$.

- By repeating this scheme we can obtain the solution $y(x)$, $y'(x)$, $x > a$, in an approximate way, but with the desired precision, provided that we do not encounter a point, and in particular the initial point, where $p_2(x)$ becomes zero.
- For a PDE one must provide information about $\phi(x)$ on a manifold of co-dimension 1 of $\mathbb{R}^d$ (a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$), instead of a point.

- For PDE one must take into account that
  - not all the curves or surfaces can be used to provide this information.
  - the kind of information depends on the type of PDE.
- To illustrate the first point, let us return to the PDE

$$\frac{\partial\phi}{\partial x}-\frac{\partial\phi}{\partial y}=1, \quad \text{with general solution } \phi(x,y)=x+\varphi(x+y).$$

- Let us take the line $x+y=1$ to give the information about ϕ

$$\phi(x,y)|_{x+y=1}=\phi(x,1-x)=f(x),$$

  for a given $f(x)$.
- One has

$$f(x)=\phi(x,1-x)=x+\varphi(x+1-x)=x+\varphi(1),$$

  from which one cannot obtain the function $\varphi(x)$ (besides having a contradiction unless f is of the form $f(x)=x+k$).

- Instead, if one considers the line $x - y = 2$, that is, $y = x - 2$, and specifies on it $\phi(x, x-2) = f(x)$, one gets

$$f(x) = \phi(x, x-2) = x + \varphi(x + x - 2) = x + \varphi(2x - 2)$$

and $\varphi(2x-2) = f(x) - x$.

- It follows from this that, at an arbitrary point $t = 2x - 2$,

$$\varphi(t) = f\left(1 + \frac{t}{2}\right) - 1 - \frac{t}{2}, \quad \varphi(x+y) = f\left(1 + \frac{x+y}{2}\right) - 1 - \frac{x+y}{2}.$$

- Finally, the solution to the PDE that satisfies the given condition is (check both the PDE and the condition!)

$$\phi(x, y) = f\left(1 + \frac{x+y}{2}\right) - 1 - \frac{x-y}{2}.$$

- How does one know whether the given curve on which the condition is given is a good one? It turns out that this depends on the principal part of the PDE.
- The curves on which one can give the condition on ϕ for a linear first order PDE with principal part

$$a(x,y)\frac{\partial \phi}{\partial x} + b(x,y)\frac{\partial \phi}{\partial y}$$

are those whose normal $\vec{n} = (n_x, n_y)$ at no point satisfies

$$a(x,y)n_x + b(x,y)n_y = 0.$$

- Curves such that their normal obeys the above equation everywhere are called **characteristics**. Hence, an admissible curve must be nowhere tangent to a characteristic.

- In our example, the equation of the caracteristics is $n_x - n_y = 0$, from which $n_x = n_y$ and the normal at an arbitrary point is proportional to $\vec{n} = (1, 1)$.
- This means that the tangent is proportional to $\vec{t} = (1, -1)$ and the characteristics are the lines $x + y = k$, with slope -1.
- Our curve $x + y = 1$ was itself a characteristic, and hence the problems. Instead, the second curve $x - y = 2$, having slope equal to 1 everywhere, was not tangent to a characteristic at any point.
- In two dimensions, $d = 2$, the principal part of the PDE is

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy}.$$

- The characteristics will be curves in $\mathbb{R}^2$ such that, if the normal is $\vec{n} = (n_x, n_y)$,

$$a(x, y)n_x^2 + 2b(x, y)n_x n_y + c(x, y)n_y^2 = 0.$$

- Let us consider, for instance,

$$-3u_{xx} + 2u_{xy} + 3u_x = 0.$$

- The equation for the characteristics is

$$-3n_x^2 + 2n_x n_y = 0.$$

- This has solutions $n_x = 0$ and $-3n_x + 2n_y = 0$.
- The case $n_x = 0$ means that the normal is pointing along the Y direction everywhere, and hence that the curves are constant lines $y = C$, of which we have one for each point in the plane.

- The solutions to $-3n_x + 2n_y = 0$ will be also lines, for which $n_x = 2/3n_y$ or, in terms of the tangent $\vec{t} = (t_x, t_y)$,

$$-t_y = \frac{2}{3}t_x,$$

with solution $y = -2/3x + K$, $K \in \mathbb{R}$, which again provides a line for each point in the plane.
- We have found that there are two different characteristics through each point for the PDE $-3u_{xx} + 2u_{xy} + 3u_x = 0$.

Classification of second order linear PDE

- The three most important PDE of classical physics are the wave equation, the heat equation and the Poisson-Laplace equation. For two independent variables they are
- one-dimensional **wave equation**. The function $\phi(x,t)$ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = f(x,t), \quad c > 0,$$

  which describes the propagation of the field ϕ in one space dimension with constant velocity c, and where f is an external source (of force or charge, for instance).
- one dimensional **heat equation**. The function $\phi(x,t)$ obeys

$$\frac{\partial^2 \phi}{\partial x^2} - k\frac{\partial \phi}{\partial t} = f(x,t), \quad k > 0,$$

  where ϕ is proportional to the temperature and f describes a heat source.

- two-dimensional **potential** or **Poisson equation**. The function $\phi(x, y)$ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y),$$

  where $\phi(x, y)$ is typically a time-independent electrical or gravitational potential, and $f(x, y)$ is its source. For $f = 0$ one has the Laplace equation.
- These three equations are the canonical representatives of the hyperbolic, parabolic and elliptic kinds.
- These equations can be extended to a higher number of independent variables ($\phi(\vec{r}, t)$, $\phi(\vec{r})$, $\vec{r} \in \mathbb{R}^n$).

- A quadratic equation in the plane

$$ax^2 + 2bxy + cy^2 + fx + gy + h = 0$$

describes an hyperbole, a parabola or an ellipsis if the discriminant $ac - b^2$ is negative, zero or positive, respectively.

- By analogy, the PDE with principal part

$$a(x,y)\frac{\partial^2\phi}{\partial x^2} + 2b(x,y)\frac{\partial^2\phi}{\partial x\,\partial y} + c(x,y)\frac{\partial^2\phi}{\partial y^2}$$

is, at a given point (x,y),

$$\begin{aligned} \text{hyperbolic} \quad & \text{if} \quad a(x,y)c(x,y) - b^2(x,y) < 0, \\ \text{parabolic} \quad & \text{if} \quad a(x,y)c(x,y) - b^2(x,y) = 0, \\ \text{elliptic} \quad & \text{if} \quad a(x,y)c(x,y) - b^2(x,y) > 0. \end{aligned}$$

- For the wave equation one has $a(x,t) = 1$, $b(x,t) = 0$, $c(x,t) = -1/c^2$, and hence $ac - b^2 = -1/c^2 < 0$, so it is hyperbolic everywhere.
- For the heat equation, $a(x,t) = 1$, $b(x,t) = 0$, $c(x,t) = 0$, $ac - b^2 = 0$ and it is parabolic at all points.
- For the potential equation $a(x,t) = 1$, $b(x,t) = 0$, $c(x,t) = 1$ and $ac - b^2 = 1 > 0$, and the equation is elliptic at all points.
- A given PDE can have different characters at different regions of the plane.
- Notice that, for the heat equation, the term with the time derivative does not belong to the principal part.
- It can be shown that an PDE is hyperbolic, parabolic or elliptic at a point if 2, 1 or no characteristics pass through that point, respectively.

- We are interested in solving our PDE in on a given region $\Omega \subset \mathbb{R}^2$, with boundary $\partial\Omega$, part or all of which can be at infinity.
- In order to obtain a solution, one must impose boundary conditions for ϕ on $\partial\Omega$.
- There are three main classes of boundary conditions, which can be combined and imposed on parts or the whole of $\partial\Omega$.
- Dirichlet conditions. The value of ϕ on $\partial\Omega$ is determined by a given function g,

$$\phi|_{\partial\Omega} = g(x, y), \quad (x, y) \in \partial\Omega.$$

- Neumann conditions. Now what is given by g is the normal derivative $\partial_n \phi$ on $\partial\Omega$,

$$\partial_n \phi|_{\partial\Omega} = g(x,y), \quad (x,y) \in \partial\Omega, \quad \partial_n \phi = \vec{n} \cdot \vec{\nabla} \phi.$$

- Cauchy conditions. Both ϕ and $\partial_n \phi$ are fixed, but only on a part of $\partial\Omega$,

$$\begin{aligned} \phi|_{\partial\Omega} &= g_1(x,y), \ (x,y) \in \partial_1\Omega \subset \partial\Omega, \\ \partial_n \phi|_{\partial\Omega} &= g_2(x,y), \ (x,y) \in \partial_1\Omega \subset \partial\Omega. \end{aligned}$$

These are usually associated to second order PDE in time, and the Cauchy conditions, or Cauchy data, correspond to initial conditions.

- One can also consider Robin boundary conditions, for which a linear combination of ϕ and $\partial_n\phi$ is given at the boundary,

$$\alpha(x,y)\phi(x,y) + \beta(x,y)\partial_n\phi(x,y) = \gamma(x,y), \quad (x,y) \in \partial\Omega.$$

- Each kind of second order PDE requires certain classes of boundary conditions in order to guarantee both existence and uniqueness of solutions and smooth variation of the solution with the boundary data.
    - Hyperbolic PDE require Cauchy conditions on part of the boundary, and Dirichlet or Neumann on the rest.
    - Parabolic PDE require Dirichlet conditions on part of the boundary, and Dirichlet or Neumann on the rest.
    - Elliptic PDE need Dirichlet or Neumann conditions (or appropriate asymptotic behavior if the region is not bounded).

The method of characteristics

- The most general linear first order equation for $u(x, y)$ is

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = f(x,y).$$

- Consider the vector field in $\mathbb{R}^2$ $\vec{v}$ given by

$$\vec{v}(x,y) = (a(x,y), b(x,y)),$$

and let $\vec{\gamma}(\hat{t}) = (x(\hat{t}), y(\hat{t}))$ be one of its integral curves, parameterized by $\hat{t}$.

- At each point the components of the integral curve satisfy

$$\begin{aligned} \dot{x} &= a(x(\hat{t}), y(\hat{t})), \\ \dot{y} &= b(x(\hat{t}), y(\hat{t})), \end{aligned}$$

where the dot indicates the derivative with respect to $\hat{t}$.

- We impose now that this curve is on the surface $u = u(x, y)$ in $\mathbb{R}^3$ that describes the solution to the PDE. This means that the points $u(x(\hat{t}), y(\hat{t}))$ should be on the surface for all $\hat{t}$.
- Going to the PDE and using $a = \dot{x}$, $b = \dot{y}$ we obtain

$$\begin{aligned}\dot{x}(\hat{t})\frac{\partial u}{\partial x}(x(\hat{t}), y(\hat{t})) \quad &+ \quad \dot{y}(\hat{t})\frac{\partial u}{\partial y}(x(\hat{t}), y(\hat{t})) \\ &+ \quad c(x(\hat{t}), y(\hat{t}))u(x(\hat{t}), y(\hat{t})) = f(x(\hat{t}), y(\hat{t})).\end{aligned}$$

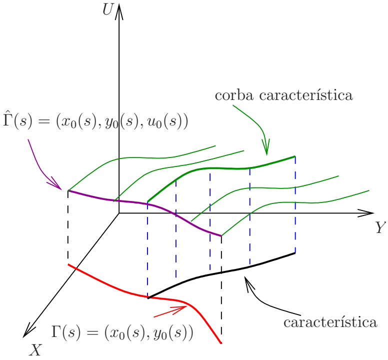
- Understanding that everything is now a function of $\hat{t}$ and using the chain rule, this can be rewritten as

$$\dot{u} + c(\hat{t})u = f(\hat{t}).$$

- Putting together the two equations that we have introduced for $x(\hat{t})$ and $y(\hat{t})$ and the one that we have obtained for $u(\hat{t})$ we get the system of ordinary differential equations

$$\begin{aligned} \dot{x} &= a, \\ \dot{y} &= b, \\ \dot{u} + cu &= f, \end{aligned}$$

which is called the characteristic system associated to the PDE. Its solutions are called the characteristic curves, and its projections on the XY plane are, precisely, the characteristics that we introduced earlier. The next figure illustrates this.



The method of characteristics. The solution surface is constructed by knitting together characteristic curves (green). Different curves correspond to different values of s, which parameterizes the initial condition (purple). The characteristics are the black curves, and in order to obtain a surface the curve on which the initial condition is given (red) should not be tangent to the characteristics.

- If we parameterize the initial condition for a characteristic curve with s, each characteristic curve will be written as

$$(x(s_0,\hat{t}), y(s_0,\hat{t}), u(s_0,\hat{t})),$$

for a given s_0, with

$$(x(s_0,0), y(s_0,0), u(s_0,0)) = (x_0(s), y_0(s), u_0(s))$$

on the curve $\hat{\Gamma}(s)$.

- If we now consider all the curves obtained by varying s in an open set, we will get the equations of a surface parameterized by $(s,\hat{t})$,

$$\begin{aligned} x &= x(s,\hat{t}), \\ y &= y(s,\hat{t}), \\ u &= u(s,\hat{t}). \end{aligned}$$

- From the first two equations one can try to obtain $(s, \hat{t})$ in terms of (x, y). Using this information in the third equation, one gets then the solution $u = u(x, y)$ to the PDE.
- It may happen, though, that those first two equations do not define, not even locally, s and $\hat{t}$ in terms of x and y, due to a failure of the conditions for the result of the inverse function theorem to hold. It can be shown, in fact, that this happens precisely at points where the curve Γ on which the initial conditions are defined is tangent to a characteristic, and this is a situation that should be avoided.
- Since the characteristics are integral equations of the vector field (a, b), which is determined solely by the PDE, it might be convenient to find those before giving the initial conditions.

- Consider, as a first example

$$u_x + u_y = 2,$$

with initial conditions given by $u(x,0) = x^2$.

- This means that Γ is the X axis, and choosing $s = x$ as parameter we have $\hat{\Gamma}(s) = (s, 0, s^2)$, which is a parabola in XU.
- The characteristic system is $(a = 1,\ b = 1,\ c = 0,\ f = 2)$,

$$\begin{aligned} \dot{x} &= 1, \\ \dot{y} &= 1, \\ \dot{u} &= 2, \end{aligned}$$

which we should solve with initial conditions $x(s,0) = s$, $y(s,0) = 0$, $u(s,0) = s^2$.

- The general solution is

$$\begin{aligned} x(s,\hat{t}) &= \hat{t} + f_1(s), \\ y(s,\hat{t}) &= \hat{t} + f_2(s), \\ u(s,\hat{t}) &= 2\hat{t} + f_3(s). \end{aligned}$$

- Imposing the initial conditions one gets

$$\begin{aligned} x(s,\hat{t}) &= \hat{t} + s, \\ y(s,\hat{t}) &= \hat{t}, \\ u(s,\hat{t}) &= 2\hat{t} + s^2. \end{aligned}$$

- From the first two equations one has $\hat{t} = y$ and $s = x - y$, and going with this to the third equation one obtains the desired solution,

$$u(x,y) = 2y + (x-y)^2.$$

- Notice that, if we do not fully give the initial conditions, from the first two equations one has

$$x - y = f_1(s) - f_2(s).$$

- From here, assuming $f_1'(s) \neq f_2'(s)$, it is possible, at least locally, to obtain s as a function of $x - y$, $s = g(x - y)$. Substituting into the third equation one gets then

$$u(x, y) = 2y + \Psi(x - y).$$

- This is the general solution of $u_x + u_y = 2$, which contains an arbitrary function Ψ.

- The condition $f_1'(s) \neq f_2'(s)$ is, in terms of the curve of initial conditions,

$$x'(s,0) \neq y'(s,0),$$

where the prime indicates derivation with respect to s. This means that Γ has nowhere slope equal to 1 which is precisely the condition of not being tangent to the characteristics, which have tangent vector $(\dot{x},\dot{y}) = (1,1)$.

- As a second example, consider the PDE $u_x = 1$, with IC $u(x, 0) = h(x)$. The characteristic system is $\dot{x} = 1$, $\dot{y} = 0$, $\dot{u} = 1$,and one immediately obtains

$$
\begin{aligned}
x(s, \hat{t}) &= \hat{t} + s, \\
y(s, \hat{t}) &= 0, \\
u(s, \hat{t}) &= \hat{t} + h(s).
\end{aligned}
$$

- Now it is not possible to express $(s, \hat{t})$ in terms of (x, y): the characteristics have tangent vector $(1, 0)$, and hence the curve Γ of initial conditions $\Gamma(s) = (s, 0)$ is a characteristic.

D'Alembert's solution

- Let $\phi(x,t)$, with $t \geq 0$ and $-\infty < x < +\infty$, obey the unforced wave equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0.$$

- At each point there are two characteristics $x = \pm ct + k_{\pm}$, and we can assume Cauchy data on $\Gamma = \{x \in \mathbb{R},\ t = 0\}$ given by

$$\phi(x,0) = \phi_0(x), \quad \dot{\phi}(x,0) = v_0(x),$$

and we want to propagate these for $t > 0$.

- This is an hyperbolic equation, and it can be solved by a two-fold application of the method of characteristics for first order PDE.

- If we factorize the wave equation as

$$0 = \frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial \phi}{\partial x} - \frac{1}{c}\frac{\partial \phi}{\partial t}\right),$$

we get

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right) U_1 = 0, \quad U_1 = \left(\frac{\partial \phi}{\partial x} - \frac{1}{c}\frac{\partial \phi}{\partial t}\right).$$

- Factorizing in reverse order

$$0 = \frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial \phi}{\partial x} + \frac{1}{c}\frac{\partial \phi}{\partial t}\right),$$

one gets instead

$$\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right) U_2 = 0, \quad U_2 = \left(\frac{\partial \phi}{\partial x} + \frac{1}{c}\frac{\partial \phi}{\partial t}\right).$$

- Solving $\partial_x U_1 + 1/c\partial_t U_1 = 0$ using the method of characteristics leads to $\dot{x} = 1$, $\dot{t} = 1/c$ and $\dot{U}_1 = 0$, from where

$$
\begin{aligned}
x(s,\hat{t}) &= \hat{t} + f_1(s), \\
t(s,\hat{t}) &= \frac{1}{c}\hat{t} + f_2(s), \\
U_1(s,\hat{t}) &= f_3(s).
\end{aligned}
$$

- From the two first equations one has $x - ct = f_1(s) - f_2(s) \equiv \hat{f}(s)$, so that $s = \hat{f}^{-1}(x - ct)$ and

$$U_1(x,t) = (f_3 \circ \hat{f}^{-1})(x - ct) \equiv f(x - ct).$$

- In the same way, one gets

$$U_2(x,t) = g(x + ct).$$

- We should now relate f and g to the Cauchy data.
- From the relation between ϕ and U_1 and U_2 it follows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} - \frac{1}{c}\frac{\partial \phi}{\partial t} &= f(x-ct), \\ \frac{\partial \phi}{\partial x} + \frac{1}{c}\frac{\partial \phi}{\partial t} &= g(x+ct). \end{aligned}$$

- Adding and subtracting these equations one gets then

$$\begin{aligned} \frac{\partial \phi}{\partial x}(x,t) &= \frac{1}{2}\left(f(x-ct)+g(x-ct)\right), \quad \text{(Eq1)} \\ \frac{\partial \phi}{\partial t}(x,t) &= \frac{c}{2}\left(g(x+ct)-f(x-ct)\right). \quad \text{(Eq2)} \end{aligned}$$

- Setting $t = 0$ in these equations one can get the initial waveform of the gradient

$$\frac{\partial \phi}{\partial x}(x,0) = \frac{1}{2}(f(x) + g(x)), \quad \text{(Eq3)}$$

as well as that of the velocity

$$v_0(x) = \frac{\partial \phi}{\partial t}(x,0) = \frac{c}{2}(g(x) - f(x)). \quad \text{(Eq4)}$$

- Integrating (Eq2) we obtain

$$\begin{aligned} \phi(x,t) &= \phi(x,0) + \int_0^t \frac{\partial \phi}{\partial \tau}(x,\tau)\ \mathrm{d}\tau \\ &= \phi_0(x) + \frac{c}{2}\int_0^t (g(x+c\tau) - f(x-c\tau))\ \mathrm{d}\tau \\ &= \phi_0(x) + \frac{1}{2}\int_x^{x+ct} g(\xi)\ \mathrm{d}\xi - \frac{1}{2}\int_{x-ct}^{x} f(\xi)\ \mathrm{d}\xi. \ \text{(Eq5)} \end{aligned}$$

- From (Eq3) and (Eq4) one can obtain the desired relation to the Cauchy data,

$$\begin{aligned} g(x) &= \frac{\partial \phi}{\partial x}(x,0) + \frac{1}{c} v_0(x), \\ f(x) &= \frac{\partial \phi}{\partial x}(x,0) - \frac{1}{c} v_0(x). \end{aligned}$$

- Putting now these in (Eq5) we obtain the solution to the wave equation

$$\begin{aligned} \phi(x,t) &= \phi_0(x) + \frac{1}{2}\int_x^{x+ct} \left(\frac{\partial \phi}{\partial x}(\xi,0) + \frac{1}{c} v_0(\xi) \right) \, \mathrm{d}\xi \\ &- \frac{1}{2}\int_{x-ct}^{x} \left(\frac{\partial \phi}{\partial x}(\xi,0) - \frac{1}{c} v_0(\xi) \right) \, \mathrm{d}\xi \end{aligned}$$

- This can be further rewritten as

$$\begin{aligned}\phi(x,t) &= \phi_0(x) + \frac{1}{2}\phi(x+ct,0) - \frac{1}{2}\phi(x,0) + \frac{1}{2c}\int_x^{x+ct} v_0(\xi)\ \mathrm{d}\xi \\ &- \frac{1}{2}\phi(x,0) + \frac{1}{2}\phi(x-ct,0) + \frac{1}{2c}\int_{x-ct}^{x} v_0(\xi)\ \mathrm{d}\xi \\ &= \frac{1}{2}\phi_0(x+ct) + \frac{1}{2}\phi_0(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\ \mathrm{d}\xi.\end{aligned}$$

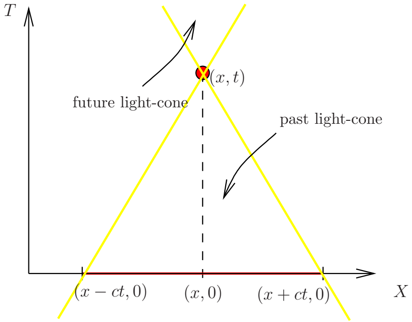
d'Alembert's (1747) solution of the wave equation

$$\phi(x,t) = \frac{1}{2}\phi_0(x+ct) + \frac{1}{2}\phi_0(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\ \mathrm{d}\xi$$

- If the initial velocity is zero v_0 is zero between $x - ct$ and $x + ct$ d'Alembert's solution boils down to

$$\phi(x,t) = \frac{1}{2}\phi_0(x+ct) + \frac{1}{2}\phi_0(x-ct)$$

and the solution is just a superposition of the initial waveforms moving to the right and to the left with velocity c.
- D'Alembert's expression shows that only those points that at $t = 0$ are at a distance less than or equal to ct of the point x have an influence on the solution at x at time t
- This situation is reflected in the next figure. The closed interval $[x - ct, x + ct]$, the intersection of the past light-cone of the point (x,t) with $t = 0$, contains the initial data that has an influence on the value of ϕ at (x,t).



Past light-cone of a point in space-time. Only the data inside the past light-cone of (x, t) can influence the value of $\phi(x, t)$. Yellow lines have slope $\pm 1/c$, and are characteristics through (x, t) . The name light-cone comes from electromagnetism in the vacuum, where c is the speed of light (or from special relativity, if one wants to think in terms of time-like space-time intervals). Notice however that this description applies to any wave equation, with c the appropriate velocity.

Causal Green function

- Now we want to solve

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = q(x,t),$$

for $-\infty < x < +\infty$, $-\infty < t < +\infty$.

- We assume that $\phi(x,t)$ and $\partial_t\phi(x,t)$ are zero at $t = -\infty$, and we will use Fourier transform techniques.
- The Green function associated to this problem, $G(x,t;\xi,\tau)$, will be the solution to

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)G(x,t;\xi,\tau) = \delta(x-\xi)\delta(t-\tau).$$

- The doubly Fourier transformed Green function is defined by

$$\hat{G}(k,\omega;\xi,\tau) = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}t \; e^{-ikx-i\omega t} \; G(x,t;\xi,\tau).$$

- Transforming the equation for $G(x,t;\xi,\tau)$ yields then

$$\left(\frac{1}{c^2}(i\omega)^2 - (ik)^2\right)\hat{G}(k,\omega;\xi,\tau) = e^{-ik\xi}e^{-i\omega\tau}.$$

- From this one has that the Fourier transform of the Green function is

$$\hat{G}(k,\omega;\xi,\tau) = -c^2\frac{e^{-ik\xi}e^{-i\omega\tau}}{\omega^2 - c^2k^2}.$$

- The Green function is given by the inverse Fourier transform

$$\begin{aligned} G(x,t;\xi,\tau) &= \frac{1}{(2\pi)^2}(-c^2)\int_{-\infty}^{+\infty}\mathrm{d}k\int_{-\infty}^{+\infty}\mathrm{d}\omega\; e^{ikx}e^{i\omega t}\frac{e^{-ik\xi}e^{-i\omega\tau}}{\omega^2-c^2k^2} \\ &= -\frac{c^2}{(2\pi)^2}\int_{-\infty}^{+\infty}\mathrm{d}k\; e^{ik(x-\xi)}\int_{-\infty}^{+\infty}\mathrm{d}\omega\;\frac{e^{i\omega(t-\tau)}}{\omega^2-c^2k^2}. \end{aligned}$$

- Let us compute the ω integral, for a given k .
- The integrand has poles on the real line at $\omega = \pm c|k|$. If we want to obtain a causal Green function we will have to move them to the upper half-plane, to

$$\omega = \pm c|k| + i\epsilon, \quad \epsilon > 0.$$

- Since

$$\frac{1}{\omega^2 - c^2k^2} = \frac{1}{2c|k|}\left(\frac{1}{\omega - c|k|} - \frac{1}{\omega + c|k|}\right)$$

one has

$$\int_{-\infty}^{+\infty} \mathrm{d}\omega \ \frac{e^{i\omega(t-\tau)}}{\omega^2 - c^2k^2} =$$

$$\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty} \mathrm{d}\omega \ \frac{1}{2c|k|}e^{i\omega(t-\tau)}\left(\frac{1}{\omega - c|k| - i\epsilon} - \frac{1}{\omega + c|k| - i\epsilon}\right).$$

- Due to the analyticity in the lower half-plane, the integral is zero if $t - \tau < 0$, while for $t - \tau > 0$ we obtain the result

$$\frac{1}{2c|k|}2\pi i\left(e^{i(c|k|+i\epsilon)(t-\tau)} - e^{i(-c|k|+i\epsilon)(t-\tau)}\right).$$

- In the $\epsilon \to 0^+$ this is

$$-\frac{2\pi}{c|k|}\sin c|k|(t-\tau) = -\frac{2\pi}{ck}\sin ck(t-\tau).$$

- Putting together the results for $t-\tau<0$ and $t-\tau>0$ we get

$$\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \frac{e^{i\omega(t-\tau)}}{\omega^2 - c^2k^2} = -\theta(t-\tau)\frac{2\pi}{ck}\sin ck(t-\tau).$$

- Substituting this into the expression for the Green function we obtain the causal Green function

$$G_+(x,t;\xi,\tau) = \frac{c}{2\pi}\theta(t-\tau)\int_{-\infty}^{+\infty} \mathrm{d}k\, \frac{\sin ck(t-\tau)}{k} e^{ik(x-\xi)}.$$

- The function of k which appears in the last integral is actually analytic everywhere. To compute the integral, one has to use the fact that the Fourier transform of the square pulse

$$p_a(x) = \theta(x+a) - \theta(x-a)$$

is

$$\hat{p}_a(k) = \int_{-a}^{a} e^{-ikx} \, \mathrm{d}x = \frac{2}{k} \sin ka.$$

- This implies that

$$\theta(x+a) - \theta(x-a) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2}{k} \sin ka \; e^{ikx} \, \mathrm{d}k.$$

- Using this result with $a = c(t-\tau)$, one gets

$$G_+(x,t;\xi,\tau) = \frac{c}{2}\theta(t-\tau)\left(\theta(x+c(t-\tau)) - \theta(x-c(t-\tau))\right)\big|_{x\to x-\xi}$$
$$= \frac{c}{2}\theta(t-\tau)\left(\theta(x-\xi+c(t-\tau)) - \theta(x-\xi-c(t-\tau))\right)$$

- This is different from zero when $t > \tau$, and when, furthermore,

$$\xi - c(t-\tau) < x < \xi + c(t-\tau),$$

that is, $|x-\xi| < c(t-\tau)$.

- The second condition is again a consequence of the fact that only space-time points inside the past light-cone can affect the solution at (x,t), and hence x can, at most, be at a distance $c(t-\tau)$ of ξ.

- Causality for the wave equation has thus two components: one purely temporal and a second one related to the distance in space-time.
- Using this Green function, the causal solution to

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = q(x,t),$$

is

$$\begin{aligned}\phi(x,t) &= \int_{-\infty}^{+\infty} \mathrm{d}\tau \int_{-\infty}^{+\infty} \mathrm{d}\xi\ G_+(x,t;\xi,\tau)q(\xi,\tau) \\ &= \frac{c}{2}\int_{-\infty}^{t} \mathrm{d}\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} \mathrm{d}\xi\ q(\xi,\tau).\end{aligned}$$

Separation of variables

- Consider now the homogeneous wave equation for $x \in [0, L]$.

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0,\ x \in [0, L],\ t \geq 0.$$

- We assume Cauchy data at $t = 0$ and homogeneous Dirichlet boundary conditions at $x = 0, L$,

$$\begin{aligned} \phi(x,0) &= \phi_0(x),\ x \in [0,L], & \phi(0,t) &= 0, & t \geq 0, \\ \partial_t(x,0) &= v_0(x),\ x \in [0,L]. & \phi(L,t) &= 0, & t \geq 0. \end{aligned}$$

- Cauchy and boundary conditions should be compatible

$$\phi_0(0) = \phi_0(L) = 0,\ v_0(0) = v_0(L) = 0,$$

This means that, seen in the (x, t) plane, the conditions are continuous at the points $(0, 0)$ and $(L, 0)$ where the space and time boundaries meet.

- The separation of variables method tries to find a solution of the form

$$\phi(x,t) = X(x)T(t).$$

- Substituting this into the wave equation one gets

$$\frac{1}{c^2}\ddot{T}(t)X(x) - X''(x)T(t) = 0,$$

or

$$\frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

- The left hand-side of this equation is a function of t only, but it is equal to a function of x. The only possibility that this is true is if both members of the equation are actually equal to a constant.
- Due to the boundary conditions on x, this constant must be negative, and we choose it as $-k^2$, $k \in \mathbb{R}$.
- We have thus

$$\begin{aligned} \frac{X''(x)}{X(x)} &= -k^2, \\ \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} &= -k^2. \end{aligned}$$

.

- The general solution of the equation for $X(x)$ is

$$X(x) = Ae^{-ikx} + Be^{ikx}.$$

- Imposing the boundary conditions in $x = 0$ and $x = L$ leads to the system $A + B = 0$, $Ae^{-ikL} + Be^{ikL} = 0$, which has a nontrivial solution only if $k = \frac{n\pi}{L}$, $n \in \mathbb{Z}$, for which $B = -A$.
- The solution to the spatial part is then

$$\begin{aligned} X_n(x) &= A_n \left(e^{-i\frac{n\pi}{L}x} - e^{i\frac{n\pi}{L}x} \right) = -2iA_n \sin \frac{n\pi x}{L} \\ &\equiv C_n \sin \frac{n\pi x}{L}, \quad n = 1, 2, \ldots \end{aligned}$$

  where we have retained only the positive values of n, because the negative ones yield solutions proportional to the corresponding positive values and for $n = 0$ we have a solution identically zero.

- The equation for $T(t)$ is, for the allowed values of k,

$$\frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} = -\frac{n^2\pi^2}{L^2},\ n = 1, 2, \ldots$$

- This has general solution

$$T_n(t) = D_n e^{-i\frac{n\pi c}{L}t} + E_n e^{i\frac{n\pi c}{L}t}, \quad n = 1, 2, \ldots$$

- The total solution $\phi_n(x, T) = X_n(x)T_n(t)$ is, finally,

$$\phi_n(x, t) = \sin\frac{n\pi x}{L}\left(\alpha_n e^{-i\frac{n\pi c}{L}t} + \beta_n e^{i\frac{n\pi c}{L}t}\right), \quad n = 1, 2, \ldots$$

- These are solutions that obey the boundary conditions, but we have to impose now the initial conditions. In general, a single or a finite combination of the X_n will not satisfy the Cauchy data, but since we have an infinite number of solutions, we can form a series and use the infinite number of coefficients to try to fit the data:

$$\phi(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(\alpha_n e^{-i\frac{n\pi c}{L}t} + \beta_n e^{i\frac{n\pi c}{L}t} \right) .$$

- Leaving aside mathematical considerations about the convergence of the series, the homogeneity of the boundary conditions is essential for this to work, as well as the linearity of the wave equation.

- Evaluating the series, as well as its time derivative, at $t = 0$ we have

$$\begin{aligned} \phi_0(x) &= \phi(x,0) = \sum_{n=1}^{\infty}(\alpha_n + \beta_n)\sin\frac{n\pi x}{L}, \\ v_0(x) &= \partial_t\phi(x,0) = \sum_{n=1}^{\infty}(-i\frac{n\pi c}{L}\alpha_n + i\frac{n\pi c}{L}\beta_n)\sin\frac{n\pi x}{L} \\ &= -i\frac{\pi c}{L}\sum_{n=1}^{\infty}n(\alpha_n - \beta_n)\sin\frac{n\pi x}{L}. \end{aligned}$$

- These sine functions have period $2L$, and therefore $\alpha_n + \beta_n$ are the Fourier coefficients of the skew-symmetric extension $\hat{\phi}_0(x)$ of $\phi_0(x)$ to $[-L, L]$; likewise, the $-i\frac{n\pi c}{L}n(\alpha_n - \beta_n)$ are those of the extension $\hat{v}_0(x)$ of $v_0(x)$.

One has then

$$\begin{aligned}
\alpha_n + \beta_n &= \frac{2}{2L}\int_{-L}^{L} \hat{\phi}_0(x)\sin\frac{2\pi n x}{2L}\,\mathrm{d}x \\
&= \frac{2}{L}\int_{0}^{L} \phi_0(x)\sin\frac{\pi n x}{L}\,\mathrm{d}x = I_n, \\
I_n &= \frac{2}{L}\int_{0}^{L} \phi_0(x)\sin\frac{\pi n x}{L}\,\mathrm{d}x, \\
-i\frac{n\pi c}{L}n(\alpha_n - \beta_n) &= \frac{2}{2L}\int_{-L}^{L} \hat{v}_0(x)\sin\frac{2\pi n x}{2L}\,\mathrm{d}x \\
&= \frac{2}{L}\int_{0}^{L} v_0(x)\sin\frac{\pi n x}{L}\,\mathrm{d}x = J_n, \\
J_n &= \frac{2}{L}\int_{0}^{L} v_0(x)\sin\frac{\pi n x}{L}\,\mathrm{d}x.
\end{aligned}$$

- The system for α_n, β_n is

$$\alpha_n + \beta_n = I_n, \quad \alpha_n - \beta_n = i\frac{L}{c\pi n}J_n,$$

with solution

$$\alpha_n = \frac{1}{2}I_n + \frac{1}{2}i\frac{L}{c\pi n}J_n, \quad \beta_n = \frac{1}{2}I_n - \frac{1}{2}i\frac{L}{c\pi n}J_n.$$

- The solution to the Dirichlet problem for the wave equation is thus

$$\begin{aligned}\phi(x,t) &= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[\left(\frac{1}{2}I_n + i\frac{1}{2}\frac{L}{c\pi n}J_n\right)e^{-i\frac{n\pi c}{L}t}\right. \\ &\quad \left. + \left(\frac{1}{2}I_n - i\frac{1}{2}\frac{L}{c\pi n}J_n\right)e^{i\frac{n\pi c}{L}t}\right]\end{aligned}$$

- Combining the complex exponentials one finally gets

$$\phi(t,x)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(I_n\cos\frac{n\pi ct}{L}+\frac{L}{c\pi n}J_n\sin\frac{n\pi ct}{L}\right).$$

with

$$I_n=\frac{2}{L}\int_0^L\phi_0(x)\sin\frac{\pi nx}{L}\,\mathrm{d}x,\quad J_n=\frac{2}{L}\int_0^L v_0(x)\sin\frac{\pi nx}{L}\,\mathrm{d}x.$$

- A similar solution can be worked out for Neumann conditions; the final result has cosine functions of x instead of sine, and the series starts at $n=0$.

Review of solutions of the wave equation

- We have obtained solutions to the wave equation for
  - Homogeneous equation, $x \in \mathbb{R}$, initial conditions at $t = 0$ (d'Alembert's solution).
  - Non-homogeneous equation, $x \in \mathbb{R}$, $t \in \mathbb{R}$ (Green function solution).
  - Homogeneous equation, $x \in [0, L]$ with boundary conditions, initial conditions at $t = 0$ (separation of variables + Fourier series solution).
- The first and third solutions can be written in the form

$$\phi(x,t) = F(x - ct) + G(x + ct)$$

  for appropriate functions F and G, depending on the 2 characteristics $x \pm ct$ of the wave equation at each point.
- The Green function case has an extra $\theta(t - \tau)$ that does not allow to write the solution in this form.

The heat equation

- The heat equation
$$\frac{\partial \phi}{\partial t}(x,t) = \kappa \frac{\partial^2 \phi}{\partial x^2}(x,t), \quad \kappa > 0,$$
is the parabolic equation *par excellence*.
- At each point (x,t) there is a single characteristic, given by $n_x = 0$, that is, $x = \text{constant}$. Hence initial conditions can be given at $t = 0$,
$$\phi(x,0) = \phi_0(x).$$
- Furthermore, if the domain of x is bounded, Dirichlet or Neumann conditions are also provided at the spatial boundary. The data at $t = 0$ are also Dirichlet. They are not of Cauchy type, since the value of $\partial_t \phi(x, t = 0)$ is not given: the equation is of first order in time, and $\partial_t \phi(x, t = 0)$ is computed from the PDE.

- Since $\kappa > 0$, if in a given region one has a spatial profile with negative curvature
$$\frac{\partial^2 \phi}{\partial x^2} < 0,$$
then
$$\frac{\partial \phi}{\partial t} < 0$$
and the other way around.
- This means that the evolution in time tends to iron out the spatial profile, erasing the spatial gradients of the solution. Non-constant solutions in space can only be maintained by externally imposing a gradient using the boundary conditions or, in the non-homogeneous case, by extracting or injecting heat at each point.

- If we run the heat equation in reverse time or, equivalently, set $\kappa < 0$, the opposite behavior happens. Any initial kink in the spatial profile is amplified and the solution diverges exponentially in time. Physically, this would correspond to heat flowing from lower to higher temperature regions, hence violating the second principle of thermodynamics.
- The heat equation is also known as the *diffusion equation* when what one is considering is the flow of some chemical species from regions of higher to lower concentration. In this case it is written as

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2}, \quad D > 0.$$

The heat kernel

- Let us consider an unbounded spatial domain of the form $-\infty < x < +\infty$, hence without boundary conditions, and an initial condition of the form $\phi(x,0) = \phi_0(x)$.
- The Fourier transform with respect to x is

$$\hat{\phi}(k,t) = \int_{-\infty}^{+\infty} \phi(x,t) e^{-ikx} \, \mathrm{d}k,$$

- Transforming the heat equation with respect to this, considering t as a parameter, yields

$$\frac{\partial \hat{\phi}}{\partial t}(k,t) = -\kappa k^2 \hat{\phi}(k,t).$$

- For fixed k this is an ODE of first order in t for $\hat{\phi}(k,t)$, with general solution

$$\hat{\phi}(k,t) = \hat{\phi}(k,0)e^{-\kappa k^2 t},$$

where $\hat{\phi}(k,0)$ is the Fourier transform of the initial condition $\phi(x,0) = \phi_0(x)$.

- We can now anti-transform to obtain

$$\begin{aligned} \phi(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}k \; e^{ikx}\hat{\phi}(k,0)e^{-\kappa k^2 t} \\ &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi}\hat{\phi}(k,0)e^{ikx-\kappa k^2 t}. \end{aligned}$$

- Expressing $\hat{\phi}(k,0)$ in terms of its anti-transform

$$\hat{\phi}(k,0) = \int_{-\infty}^{+\infty} \phi_0(\xi) e^{-ik\xi} \mathrm{d}\xi,$$

we obtain, changing the order of integration

$$\begin{aligned}\phi(x,t) &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \left(\int_{-\infty}^{+\infty} \phi_0(\xi) e^{-ik\xi} \mathrm{d}\xi \right) e^{ikx-\kappa k^2 t} \\ &= \int_{-\infty}^{+\infty} \mathrm{d}\xi\ \phi_0(\xi) \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} e^{ik(x-\xi)-\kappa k^2 t}.\end{aligned}$$

- This can be written in the form

$$\phi(x,t) = \int_{-\infty}^{+\infty} G(x,\xi,t)\phi_0(\xi)\ \mathrm{d}\xi,$$

where we have defined the heat kernel as

$$G(x,\xi,t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} e^{ik(x-\xi)-\kappa k^2 t}.$$

- Notice that

$$G(x,\xi,0) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} e^{ik(x-\xi)} = \delta(x-\xi)$$

and the initial condition is satisfied,

$$\phi(x,0) = \int_{-\infty}^{+\infty} G(x,\xi,0)\phi_0(\xi)\ \mathrm{d}\xi = \int_{-\infty}^{+\infty} \delta(x-\xi)\phi_0(\xi)\ \mathrm{d}\xi = \phi_0(x).$$

- Completing a square in k in the exponential which appears in the heat kernel and using

$$\int_{-\infty}^{+\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}$$

one can show that

$$G(x, \xi, t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{1}{4\kappa t}(x-\xi)^2}.$$

- Since $t > 0$ (and $\kappa > 0$ too) the root in the denominator is real and different from zero. Using this form of the heat kernel the solution of the heat equation becomes

$$\phi(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{+\infty} e^{-\frac{1}{4\kappa t}(x-\xi)^2} \phi_0(\xi) \, \mathrm{d}\xi.$$

- In this last expression, the heat kernel seems to play the role of a Green function, but with respect to the initial condition instead of an external forcing.
- This can be actually understood if one considers that an initial condition can be replaced with a delta function at $t = 0$.
- Using the same double Fourier transform as in the wave equation computation, one can see that the causal Green function of the heat equation is

$$G_+(x,t;\xi,\tau) = \theta(t-\tau)\frac{1}{\sqrt{4\pi\kappa(t-\tau)}}e^{-\frac{1}{4\kappa(t-\tau)}(x-\xi)^2}.$$

- Notice that $G_+(x,t;\xi,0) = \theta(t)G(x,\xi,t) = G(x,\xi,t)$, since we are only considering $t > 0$.

- Consider now, for a given initial profile $\phi_0(x)$, the non-homogeneous heat equation without initial conditions.

$$\partial_t \phi - \kappa \partial_x^2 \phi = \phi_0(x)\delta(t).$$

- Using the above Green function, the solution to this will be given by

$$\begin{aligned}\phi(x,t) &= \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\infty} \mathrm{d}\tau \; G_+(x,\xi;t,\tau)\phi_0(\xi)\delta(\tau) \\ &= \int_{-\infty}^{\infty} \mathrm{d}\xi \; G_+(x,\xi;t,0)\phi_0(\xi) = \int_{-\infty}^{\infty} G(x,\xi,t)\phi_0(\xi) \; \mathrm{d}\xi.\end{aligned}$$

- The heat kernel is therefore the causal Green function of the problem with the time delta located at $\tau = 0$.

- To complete the argument, if one integrates

$$\partial_t \phi - \kappa \partial_x^2 \phi = \phi_0(x)\delta(t)$$

with respect to time between $t = -\epsilon$ and $t = \epsilon$ one gets

$$\phi(x, +\epsilon) = \phi(x, -\epsilon) + \phi_0(x) = \phi_0(x)$$

since $\phi(x, t) = 0$ for $t < 0$. This shows that, indeed, the delta function introduces the initial condition at $t = 0$ if one considers the problem for $t \in \mathbb{R}$.

- Techniques related to a generalization of the heat kernel play a fundamental role in many areas of physics and differential geometry.

Separation of variables

- Separation of variables for the homogeneous heat equation can be discussed in the same way that we did for the wave equation, considering the PDE for $(x,t) \in (0,L) \times (0,+\infty)$, with initial condition $\phi(x,0) = \phi_0(x)$ and either Dirichlet or Neumann homogeneous conditions at $x = 0$ and $x = L$.
- The Dirichlet homogeneous conditions $\phi(0,t) = \phi(L,t) = 0$ correspond to fixed (zero in the considered temperature scale) constant temperature at the boundary and the Neumann ones $-\partial_x \phi(0,t) = \partial_x \phi(L,t) = 0$ to zero heat flow at the boundary.
- After separation of variables the solutions to the time equation are decreasing exponentials for $t > 0$.
- A single Fourier series for the initial data has to be computed in this case, since the equation is of first order in time.

Laplace and Poisson equations

- The Poisson equation in $\mathbb{R}^n$

$$-\Delta\phi(\vec{r}) = f(\vec{r}),\ \vec{r} = (x_1, x_2, \ldots, x_n),\quad \Delta = \vec{\nabla}^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

  or the Laplace equation

$$\Delta\phi(\vec{r}) = 0,$$

  are the object of study of potential theory.

- The name comes from the fact that these equations are those of the electrostatic potential or of the gravitational field,

$$-\Delta\phi = \frac{\rho}{\epsilon_0},\quad -\Delta\phi = -4\pi G\rho,$$

  where the $\rho(\vec{r})$ are the charge and the mass densities, respectively

- For a point charge q_0 at $\vec{r}_0$, the Poisson equation becomes

$$-\Delta\phi = \frac{q_0}{\epsilon_0}\delta(\vec{r}-\vec{r}_0),$$

and hence, except for the constant factor q_0/ϵ_0, the potential of a point charge is the Green function of the operator $-\Delta$.

- The vector delta which appears here is, in Cartesian coordinates, the product of unidimensional deltas

$$\delta(\vec{r}-\vec{r}_0) = \prod_{i=1}^{n}\delta(x_i - x_{0i}).$$

- Since the integral of a delta is equal to the dimensionless constant 1, the physical dimension of a delta must be the inverse of its argument. In our case

$$[\delta(\vec{r}-\vec{r}_0)] = L^{-n}.$$

- It is sometimes also useful to be able to express the delta of a function in terms of elementary deltas,

$$\delta(f(x)) = \sum_{\alpha} \frac{1}{|f'(\alpha)|}\delta(x-\alpha),$$

where the sum is over the zeros α of f, assuming that all of them are simple, so that $f'(\alpha) \neq 0$.

- If the PDE is defined on a domain (open, bounded, connected set) Ω with boundary $\partial\Omega$, one can impose on $\partial\Omega$ Dirichlet

$$\phi|_{\partial\Omega} = g,$$

or Neumann

$$\left(\vec{n}\cdot\vec{\nabla}\right)\Big|_{\partial\Omega} = g,$$

conditions.

- A solution to the Laplace equation

$$\Delta\phi = 0$$

in a domain Ω is called an harmonic function on Ω.

- In two dimensions, an harmonic function $\phi(x, y)$ is the solution to

$$\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

- The Laplacian can be expressed in terms of $z = x + iy$, $\bar{z} = x - iy$ as

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial\bar{z}},$$

and the zero Laplacian condition becomes

$$\frac{\partial^2\phi}{\partial z\,\partial\bar{z}}(z, \bar{z}) = 0.$$

- This means that an harmonic function, when expressed in terms of complex variables, is only a function of z or $\bar{z}$.
- With the additional requirement of the existence of derivatives this means that harmonic functions are analytic or anti-analytic. In fact, since the Laplacian is real, both the real and imaginary parts of these functions are harmonic.
- Consider, for instance, $\phi_1 = z^2$ and $\phi_2 = \bar{z}^3$. One has

$$\begin{aligned} z^2 &= (x+iy)^2 = x^2 - y^2 - 2ixy, \\ \bar{z}^3 &= (x-iy)^3 = x^3 - 3xy^2 - i(3x^2y - y^3), \end{aligned}$$

  and $x^2 - y^2$, xy, $x^3 - 3xy^2$ and $3x^2y - y^3$ are all harmonic functions.
- In contrast, $\phi_3 = z\bar{z} = x^2 + y^2$ is not an harmonic function.

Cauchy data for elliptic PDE: an ill-posed problem

- One could be tempted to think of the Poisson equation as just a kind of wave equation with a sign change. That this is not possible is already pointed out by the non-existence of characteristics for elliptic PDE, but if one tries to impose Cauchy data on an elliptic PDE, one runs into an ill-posed problem, in the sense that the solutions are not continuous functions of the initial data.
- To see this, consider the Laplace equation in two dimensions s with the following Cauchy data at $y = 0$,

$$\partial_{xx}\phi + \partial_{yy}\phi = 0, \ -\infty < x < +\infty, \ y > 0,$$
$$\phi(x,0) = 0, \quad \frac{\partial\phi}{\partial y}(x,0) = \frac{1}{n}\sin nx,$$

for given $n = 1, 2, \ldots$

- For n large enough, the stated problem can be made as close as desired to the limit case

$$\partial_{xx}\phi + \partial_{yy}\phi = 0, \ -\infty < x < +\infty, \ y > 0,$$
$$\phi(x,0) = 0, \quad \frac{\partial \phi}{\partial y}(x,0) = 0,$$

which has solution $\phi(x,y) = 0$.

- It is immediate to verify that

$$u_n(x,y) = \frac{1}{n^2}\sin nx \sinh ny$$

is, for any n, a solution to the original problem with finite n.

- One has, for any $y > 0$,

$$\lim_{n\to\infty} u_n(x, y) = \infty.$$

- We have that an arbitrarily small modification of the Cauchy data brings in a solution that differs as much as we want from the solution corresponding to unmodified Cauchy data.
- This is a general feature of elliptic equations, and shows that Cauchy data cannot be assigned to them.

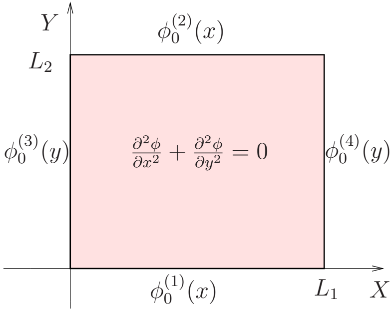
Separation of variables

- Separation of variables for elliptic problems requires sometimes some ingenuity to construct the Fourier series that match the boundary conditions. We illustrate this for the case of a rectangle.
- Let $\Omega = (0, L_1) \times (0, L_2)$, and consider the Dirichlet problem

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

$$\phi(x,0) = \phi_0^{(1)}(x), \quad \phi(x,L_2) = \phi_0^{(2)}(x), \quad x \in [0,L_1],$$
$$\phi(0,y) = \phi_0^{(3)}(y), \quad \phi(L_1,y) = \phi_0^{(4)}(y), \quad y \in [0,L_2],$$

that is described in the next figure.



DIRICHLET PROBLEM FOR THE LAPLACE EQUATION ON A RECTANGLE. We will construct four series which are solutions of the Laplace equation and such each of them is zero on three of the boundaries. Boundary conditions are assumed to be continuous on the corners, *i.e.* $\phi_0^{(1)}(L_1)=\phi_0^{(4)}(0)$ and so on.

- Looking for solutions of the fom $\phi(x,y) = X(x)Y(y)$ leads to the equation

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{constant}.$$

- The idea is to construct functions that are zero on 3 of the 4 sides of the rectangle. Depending on the sides, the constant that appears in the separation of variables equation should be positive or negative, because with a sinus we can get a zero on opposing sides, but with an hyperbolic sinus only on one.
- If the constant is positive we can write

$$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2.$$

- The solutions for X will be combinations of sine and cosine functions, of which the ones that are zero at $x = 0$ and $x = L_1$ are

$$X(x) = \sin \frac{n\pi x}{L_1}, \;\; k = k_n = \frac{n\pi}{L_1}, \quad n = 1, 2, \ldots$$

- For these values of k the equation for Y is

$$\frac{Y''}{Y} = \frac{n^2\pi^2}{L_1^2},$$

with solution

$$Y(y) = A \cosh \frac{n\pi y}{L_1} + B \sinh \frac{n\pi y}{L_1}.$$

- We cannot make this to be zero at both $y = 0$ and $y = L_2$, but we can impose $Y(0) = 0$ or $Y(L_2) = 0$.
- In the first case $A = 0$ and the solution is, with $B = 1$,

$$Y(y) = \sinh \frac{n\pi y}{L_1}.$$

- In the second case

$$B = -A \frac{\cosh \frac{n\pi L_2}{L_1}}{\sinh \frac{n\pi L_2}{L_1}}$$

and, choosing $A = \sinh \frac{n\pi L_2}{L_1}$,

$$Y(y) = \sinh \frac{n\pi L_2}{L_1} \cosh \frac{n\pi y}{L_1} - \cosh \frac{n\pi L_2}{L_1} \sinh \frac{n\pi y}{L_1}$$
$$= \sinh \frac{n\pi}{L_1}(L_2 - y).$$

- The product of $X(x)$ and the two $Y(y)$ yields the solutions

$$\phi_{2,n}(x,y) = \sin\frac{n\pi x}{L_1}\sinh\frac{n\pi y}{L_1},$$
$$\phi_{1,n}(x,y) = \sin\frac{n\pi x}{L_1}\sinh\frac{n\pi(L_2-y)}{L_1},$$

  where the index indicates the boundary where the function does not vanish.
- Similarly, taking the separation constant positive, one can construct the other two sets of solutions

$$\phi_{4,n}(x,y) = \sinh\frac{n\pi x}{L_2}\sin\frac{n\pi y}{L_2},$$
$$\phi_{3,n}(x,y) = \sinh\frac{n\pi(L_1-x)}{L_2}\sin\frac{n\pi y}{L_2}.$$

- Out of the four sets of solutions one can construct the total solution

$$\phi(x, y) = \sum_{m=1}^{4} \sum_{n=1}^{\infty} a_{m,n} \phi_{m,n}(x, y),$$

on which the boundary conditions can be imposed.

- For instance, imposing the condition at $y = 0$, one has that all the functions with $m = 2, 3, 4$ vanish at $y = 0$ and

$$\phi(x, 0) = \phi_0^{(1)}(x) = \sum_{n=1}^{\infty} a_{1,n} \phi_{1,n}(x, 0) = \sum_{n=1}^{\infty} a_{1,n} \sinh \frac{n\pi L_2}{L_1} \sin \frac{n\pi x}{L_1}.$$

The $a_{1,n} \sinh \frac{n\pi L_2}{L_1}$ are the sine coefficients of the skew-symmetric extension of $\phi_0^{(1)}$ and therefore

$$a_{1,n} \sinh \frac{n\pi L_2}{L_1} = \frac{2}{L_1} \int_0^{L_1} \phi_0^{(1)}(x) \sin \frac{n\pi x}{L_1} \, \mathrm{d}x.$$

- The other coefficients are determined in a similar way and the desired solution can be computed.
- Solving the Laplace equation by separation of variables in other geometries requires expressing first the Laplacian in the appropriate coordinates.
- For problems in the plane with rotational symmetry (for instance, Dirichlet conditions on two concentric circles, with the Laplace equation being satisfied in the space between them) it is convenient to use polar coordinates, for which

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Self-adjointness

- The operator $-\Delta$, defined on a bounded domain $\Omega \subset \mathbb{R}^n$, is Hermitic with respect to the scalar product

$$\langle \phi, \varphi \rangle = \int_\Omega \phi(\vec{r})\varphi(\vec{r})\ \mathrm{d}^n r.$$

- Indeed

$$\begin{aligned}
\langle \phi, -\Delta\varphi \rangle &= \int_\Omega \phi(-\vec{\nabla}^2\varphi)\ \mathrm{d}^n r = -\int_\Omega \phi\vec{\nabla}\cdot(\vec{\nabla}\varphi)\ \mathrm{d}^n r \\
&= -\int_\Omega \vec{\nabla}\cdot(\phi\vec{\nabla}\varphi)\ \mathrm{d}^n r + \int_\Omega \vec{\nabla}\phi\cdot\vec{\nabla}\varphi\ \mathrm{d}^n r \\
&= -\int_{\partial\Omega} \phi\vec{\nabla}\varphi\cdot\vec{n}\ \mathrm{d}S + \int_\Omega \vec{\nabla}\cdot(\vec{\nabla}\phi\ \varphi)\mathrm{d}^n r - \int_\Omega (\vec{\nabla}^2\phi)\varphi \mathrm{d}^n r \\
&= -\int_{\partial\Omega} \left(\phi\vec{\nabla}\varphi - \vec{\nabla}\phi\ \varphi\right)\cdot\vec{n}\ \mathrm{d}S + \int_\Omega (-\vec{\nabla}^2\phi)\varphi \mathrm{d}^n r \\
&= -\int_{\partial\Omega} \left(\phi\vec{\nabla}\varphi - \vec{\nabla}\phi\ \varphi\right)\cdot\vec{n}\ \mathrm{d}S + \langle -\Delta\phi, \varphi \rangle.
\end{aligned}$$

- The term
$$\int_{\partial\Omega}\left(\phi\vec{\nabla}\varphi-\vec{\nabla}\phi\;\varphi\right)\cdot\vec{n}\;\mathrm{d}S$$
can be set to zero by imposing Dirichlet homogeneous conditions on φ and ϕ ($\varphi=0$, $\phi=0$ in $\partial\Omega$), or Neumann homogeneous conditions ($\partial_n\varphi=0$, $\partial_n\phi=0$ a $\partial\Omega$).
- Since the conditions are the same for φ and ϕ, the domains of $-\Delta$ and $(-\Delta)^\dagger$ are also the same, and $-\Delta$ is not only Hermitic, but self-adjoint.
- The hyperbolic and parabolic operators appearing in the respective PDEs are not self-adjoint, because they are given initial data (say at $t=0$) and there are no conditions on the other time boundary (which is, in fact, not defined). Furthermore, parabolic operators, such as the one of the heat equation, have a real first order derivative, and they are not even Hermitic.

- the above means that all the known results for self-adjoint operators can be applied to $-\Delta$ with Dirichlet or Neumann conditions on a bounded domain.
- Consider for instance $\Omega = (0, L_1) \times (0, L_2)$, with Dirichlet homogeneous conditions $\phi(x, 0) = \phi(x, L_2) = 0$, $x \in [0, L_1]$, $\phi(0, y) = \phi(L_1, y) = 0$, $y \in [0, L_2]$.
- We look for eigenfunctions $-\vec{\nabla}^2\phi = \lambda\phi$ by means of separation of variables, $\phi(x, y) = X(x)Y(y)$.
- One immediately gets

$$-\frac{X''}{X} - \frac{Y''}{Y} = \lambda$$

and thus

$$-\frac{X''}{X} = \frac{Y''}{Y} + \lambda = \text{constant}.$$

- In order to satisfy the boundary conditions for X, we choose the constant as k^2. Then $X(x) = A\cos kx + B\sin kx$ and $X(0) = 0$, $X(L_1) = 0$ leads to $A = 0$ and $k = \frac{n\pi}{L_1}$, $n = 1, 2, \ldots$
- The solution is

$$X_n(x) = \sin\frac{n\pi x}{L_1}, \quad k_n = \frac{n\pi}{L_1}, \quad n = 1, 2, \ldots$$

- Using these values of k in the equation for Y one gets

$$\frac{Y''}{Y} = \frac{n^2\pi^2}{L_1^2} - \lambda.$$

- The constant in the right hand-side of this equation must be negative if we want $Y(0) = Y(L_2) = 0$. Hence

$$\frac{n^2\pi^2}{L_1^2} - \lambda = -p^2.$$

- One gets then $Y(y) = A\cos py + B\sin py$ and the boundary conditions lead to

$$Y_m(y) = \sin\frac{m\pi y}{L_2}, \quad p_m = \frac{m\pi}{L_2}, \quad m = 1, 2, \ldots$$

- Putting everything together, the normalized eigenfunctions are

$$\phi_{n,m}(x, y) = \sqrt{\frac{4}{L_1 L_2}}\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}, \quad n, m = 1, 2, \ldots$$

with real eigenvalues

$$\lambda_{n,m} = \frac{n^2\pi^2}{L_1^2} + \frac{m^2\pi^2}{L_2^2}, \quad n, m = 1, 2, \ldots$$

- These eigenfunctions are orthogonal

$$\int_0^{L_1} \mathrm{d}x \int_0^{L_2} \mathrm{d}y \; \phi_{n,m}(x,y)\phi_{m',n'}(x,y) = \delta_{n,n'}\delta_{m,m'}.$$

- They also form a complete set

$$f(x,y) = \sum_{n,m=1}^{\infty} A_{n,m}\phi_{n,m}(x,y),$$

with

$$A_{n,m} = \int_0^{L_1} \mathrm{d}x \int_0^{L_2} \mathrm{d}y \; \phi_{n,m}(x,y) f(x,y),$$

for any function f that vanishes at the boundary of $[0,L_1] \times [0,L_2]$.

- This means that they obey the completeness condition, which in this case reads

$$\sum_{n,m=1}^{\infty} \phi_{n,m}(x,y)\phi_{n,m}(\tilde{x},\tilde{y}) = \delta(x-\tilde{x})\delta(y-\tilde{y}).$$

- In a system with arbitrary coordinates χ_i it is possible to find the complete set of eigenfunctions as a product of functions of each coordinate provided that the homogeneous boundary conditions are on sets of the form $\chi_i = \text{constant}$. In general, though, only in the case of Cartesian coordinates can one obtain these in terms of elementary functions.

Green functions

- Once one has the eigenfunctions, one can write down the Green function, the solution to

$$-\Delta G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0)$$

corresponding to the same boundary conditions, by means of

$$G(\vec{r},\vec{r}_0) = \sum_{n\in I} \frac{1}{\lambda_n}\phi_n(\vec{r})\phi_n^*(\vec{r}_0),$$

where I is the (multi)index set of the eigenvalues and where zero modes are assumed to not exist.

- For bounded domains this is the preferred method to obtain the Green function, rather than solving its differential equation from scratch.

- In our example this will be

$$G(x,y;x_0,y_0)=\sum_{n,m=1}^{\infty}\frac{\frac{4}{L_1L_2}}{\frac{n^2\pi^2}{L_1^2}+\frac{m^2\pi^2}{L_2^2}}\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\sin\frac{n\pi x_0}{L_1}\sin\frac{m\pi y_0}{L_2}$$

$$=\sum_{n,m=1}^{\infty}\frac{4}{\pi^2}\frac{L_1L_2}{n^2L_2^2+m^2L_1^2}\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\sin\frac{n\pi x_0}{L_1}\sin\frac{m\pi y_0}{L_2}.$$

- Using the one-dimensional result

$$\sum_{n=1}^{\infty}\frac{2}{a}\sin\frac{n\pi x}{a}\sin\frac{n\pi\tilde{x}}{a}=\delta(x-\tilde{x}),\quad x,\tilde{x}\in(0,a),$$

it is immediate to see that this is indeed the desired Green function.

- Using the Green function one can obtain a solution to the Poisson equation with the same homogeneous boundary conditions,

$$-\Delta u = f(x,y), \quad u(x,y) = 0 \text{ for } x = 0, L_1 \text{ or } y = 0, L_2,$$

as

$$u(x,y) = \int_0^{L_1} \mathrm{d}\tilde{x} \int_0^{L_2} \mathrm{d}\tilde{y}\; G(x,y;\tilde{x},\tilde{y}) f(\tilde{x},\tilde{y}).$$

- As was the case for operators of a single variable, one can, playing with the Lagrange identity, obtain also the solution for non-homogeneous boundary conditions using the same Green function (see Section 6.5.5 of [1]).

Coulomb potential

- We want to compute the electrostatic potential of a point charge q_0 in the vacuum, placed at $\vec{r}_0 \in \mathbb{R}^3$,

$$-\vec{\nabla}^2 V(\vec{r}, \vec{r}_0) = \frac{q_0}{\epsilon_0}\delta(\vec{r} - \vec{r}_0), \quad \vec{r}, \vec{r}_0 \in \mathbb{R}^3.$$

  Except for q_0/ϵ_0, this is the Green function of $-\Delta$ in $\mathbb{R}^3$.
- We will do the computation using a Fourier transform in $\mathbb{R}^3$,

$$V(\vec{r}, \vec{r}_0) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \hat{V}(\vec{k}, \vec{r}_0) e^{i\vec{k}\cdot\vec{r}}\, \mathrm{d}^3 k.$$

- The transform of our Poisson equation is

$$-(i\vec{k})\cdot(i\vec{k})\hat{V}(\vec{k}, \vec{r}_0) = \frac{q_0}{\epsilon_0} e^{-i\vec{k}\cdot\vec{r}_0},$$

  from which

$$\hat{V}(\vec{k}, \vec{r}_0) = \frac{q_0}{\epsilon_0}\frac{1}{k^2} e^{-i\vec{k}\cdot\vec{r}_0}, \quad k^2 = \vec{k}\cdot\vec{k} = |\vec{k}|^2.$$

- Using this $\hat{V}$ in the anti-transform we get

$$V(\vec{r},\vec{r}_0) = \frac{1}{(2\pi)^3}\frac{q_0}{\epsilon_0}\int_{\mathbb{R}^3}\frac{1}{k^2}e^{-i\vec{k}\cdot(\vec{r}-\vec{r}_0)}\,\mathrm{d}^3k.$$

- The easiest way to compute this integral is using spherical coordinates for $\vec{k}$, $k_1 = k\sin\theta\cos\phi$, $k_2 = k\sin\theta\sin\phi$, $k_3 = k\cos\theta$, with the Z direction in $\vec{k}$ space pointing in the direction of $\vec{r}-\vec{r}_0$, so that

$$\vec{k}\cdot(\vec{r}-\vec{r}_0) = k|\vec{r}-\vec{r}_0|\cos\theta.$$

Then

$$
\begin{aligned}
V(\vec{r},\vec{r}_0) &= \frac{1}{(2\pi)^3}\frac{q_0}{\epsilon_0}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \int_0^{\infty} k^2 \mathrm{d}k \, \frac{1}{k^2} e^{-ik|\vec{r}-\vec{r}_0|\cos\theta} \\
&= \frac{1}{(2\pi)^2}\frac{q_0}{\epsilon_0}\int_0^{\infty} \mathrm{d}k \, \frac{-1}{-ik|\vec{r}-\vec{r}_0|} e^{-ik|\vec{r}-\vec{r}_0|\cos\theta}\Big|_{\theta=0}^{\theta=\pi} \\
&= \frac{1}{2\pi^2}\frac{q_0}{\epsilon_0}\frac{1}{|\vec{r}-\vec{r}_0|}\int_0^{\infty} \frac{\sin k|\vec{r}-\vec{r}_0|}{k}\mathrm{d}k \\
&= \frac{1}{2\pi^2}\frac{q_0}{\epsilon_0}\frac{1}{|\vec{r}-\vec{r}_0|}\int_0^{\infty} \frac{\sin k}{k}\mathrm{d}k \qquad \int_0^{\infty} \frac{\sin k}{k}\mathrm{d}k = \frac{\pi}{2} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_0}{|\vec{r}-\vec{r}_0|},
\end{aligned}
$$

which is Coulomb's law for the electrical potential of a point charge in three dimensions of space.

- In Section 6.5.4 of [1] it is shown that the electrostatic potential created by a point charge q_0 placed at $\vec{r}_0 \in \mathbb{R}^n$ is

$$V(\vec{r}, \vec{r}_0) = \frac{q_0}{\epsilon_0} \frac{1}{(n-2)S_{n-1}} \frac{1}{|\vec{r} - \vec{r}_0|^{n-2}},$$

  where S_{n-1} is the surface of a unit sphere in $\mathbb{R}^n$,

$$S_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

  with Γ the Gamma function, which has the properties $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(1/2) = \sqrt{\pi}$.

- For $n = 3$ this reproduces our Coulomb potential. As n grows, the potential goes to zero with a higher power, $n-2$, of the distance to the charge. This is due to the field lines having more spatial dimensions to spread out.

- For $n = 2$ the general formula diverges. It can be seen that it can be given a finite value by extracting an infinite quantity which does not depend on $\vec{r}$ and hence does not contribute to the electric field. The result is the electrostatic potential of a point charge in the plane,

$$V(\vec{r}, \vec{r}_0) = -\frac{1}{2\pi\epsilon_0} q_0 \log|\vec{r} - \vec{r}_0|.$$

- Finally, for $n = 1$ one gets

$$V(r, r_0) = -\frac{q_0}{2\epsilon_0}|r - r_0|,$$

which corresponds to a piecewise constant electrical field, except at $r = r_0$, where it flips sign,

$$E(r) = -\frac{\mathrm{d}}{\mathrm{d}r} V(r, r_0) = \mathsf{sign}(r - r_0)\frac{q_0}{2\epsilon_0}.$$

- The results for the electrical field for a given n can also be obtained using Gauss theorem and appropriate (hyper)surfaces.