Modeling of electromechanical systems

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Contents of this lecture

Basic ideas of electromechanical energy conversion
from a network modeling point of view

Port Hamiltonian formulation of electromechanical systems

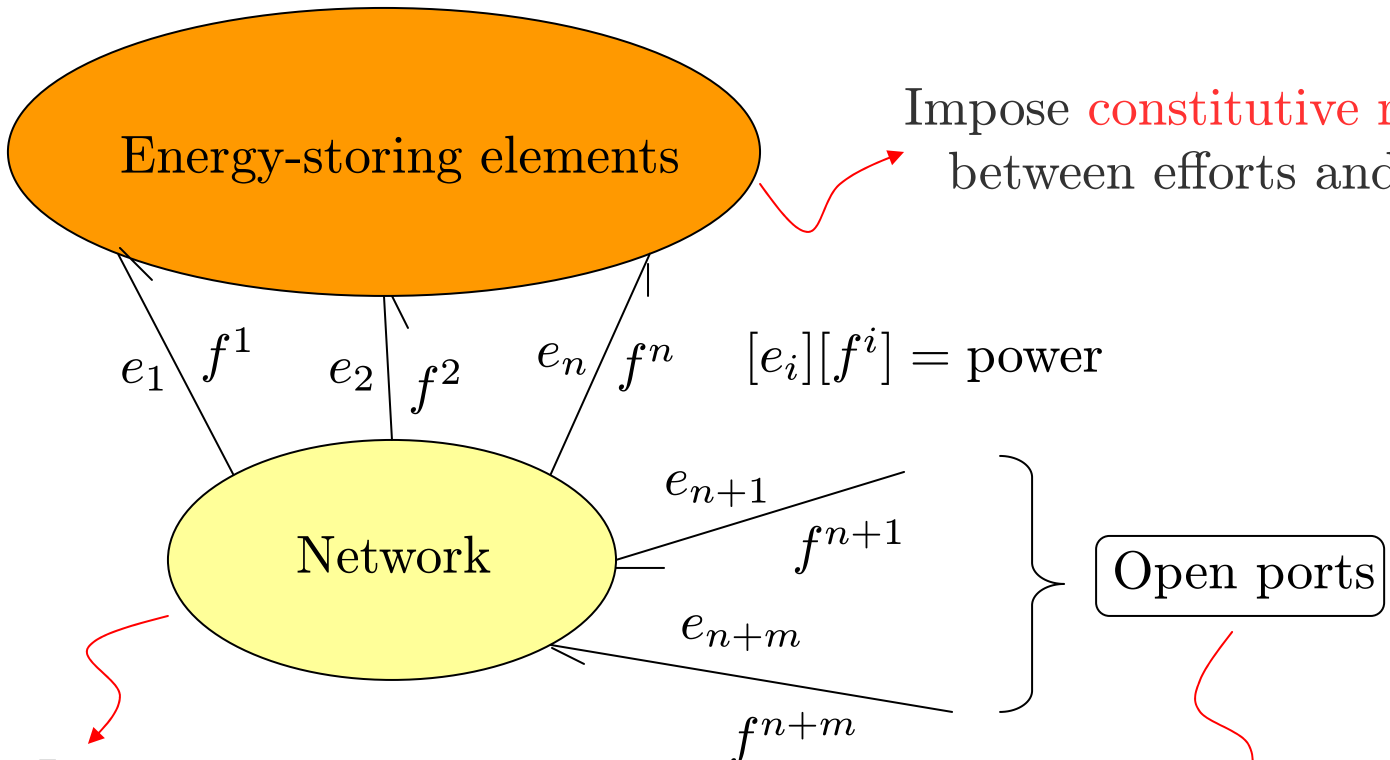
Very simple examples: lumped parameter only, no complex
induction machines, no piezoelectric devices, ...

... but it can be done

Variable structure systems: power converters

A more complex example

Network description of systems



Impose constitutive relations
between efforts and flows

$[e_i][f^i] = \text{power}$

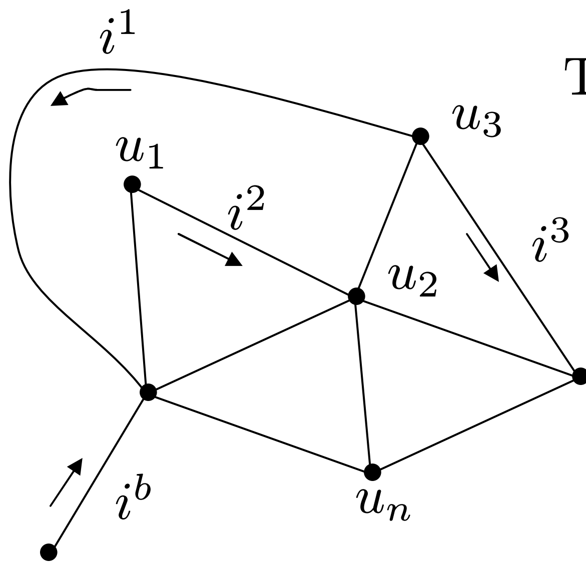
Is power continuous

$$\sum_{i=1}^{n} e_i \cdot f^i = \sum_{i=n+1}^{n+m} e_i \cdot f^i$$

Can be connected
to other systems
or to dissipative elements

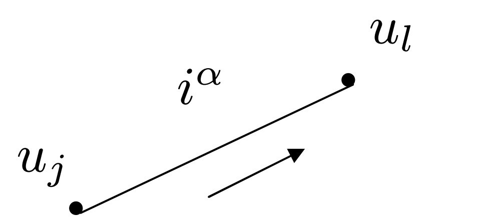
Example: Tellegen's theorem

Circuit with b branches and n nodes



To each node we assign a voltage u_j, $j = 1, \ldots, n$

To each branch we assign
a current i^α, $\alpha = 1, \ldots, b$,
and this gives an orientation to the branch



For each branch we define
the voltage drop v_α, $\alpha = 1, \ldots, b$:

$$v_\alpha = u_j - u_l$$

This is KVL!

Mathematically, the circuit, with the orientation induced by the currents, is a digraph (directed graph)

We can define its $n \times b$ adjacency matrix A by

$$A^i_\alpha = \begin{cases} -1 & \text{if branch } \alpha \text{ is incident on node } i \\ +1 & \text{if branch } \alpha \text{ is anti-incident on node } i \\ 0 & \text{otherwise} \end{cases}$$

Then, KCL states that

$$\sum_{\alpha=1}^{b} A^i_\alpha i^\alpha = 0, \qquad \forall\ i = 1, \ldots, n$$

In fact, KVL can also be stated in terms of A:

$$v_\alpha = \sum_{i=1}^{n} A^i_\alpha u_i$$

The sum contains only two terms, because each branch connects only two nodes

Tellegen's theorem. Let $\{v_{(1)\alpha}(t_1)\}_{\alpha=1,\ldots,b}$ be a set of branch voltages satisfying KVL at time t_1, and let $\{i^{\alpha}_{(2)}(t_2)\}_{\alpha=1,\ldots,b}$ be a set of currents satisfying KCL at time t_2. Then

$$\sum_{\alpha=1}^{b} v_{(1)\alpha}(t_1) i^{\alpha}_{(2)}(t_2) \equiv \langle v_{(1)}(t_1), i_{(2)}(t_2)\rangle = 0$$

Proof:

$$\sum_{\alpha=1}^{b} v_{(1)\alpha}(t_1) i^{\alpha}_{(2)}(t_2) \overset{\text{KVL}}{=} \sum_{\alpha=1}^{b}\left(\sum_{i=1}^{n} A^{i}_{\alpha} u_{(1)i}(t_1)\right) i^{\alpha}_{(2)}(t_2)$$

$$= \sum_{i=1}^{n}\left(\sum_{\alpha=1}^{b} A^{i}_{\alpha} i^{\alpha}_{(2)}(t_2)\right) u_{(1)i}(t_1) \underset{\text{KCL}}{=} \sum_{i=1}^{n} 0 \cdot u_{(1)i}(t_1) = 0$$

Notice that $\{v_{(1)\alpha}(t_1)\}$ and $\{i^{\alpha}_{(2)}(t_2)\}$ may correspond to different times and they may even correspond to different elements for the branches of the circuit.

The only invariant element is the topology of the circuit *i.e.* the adjacency matrix.

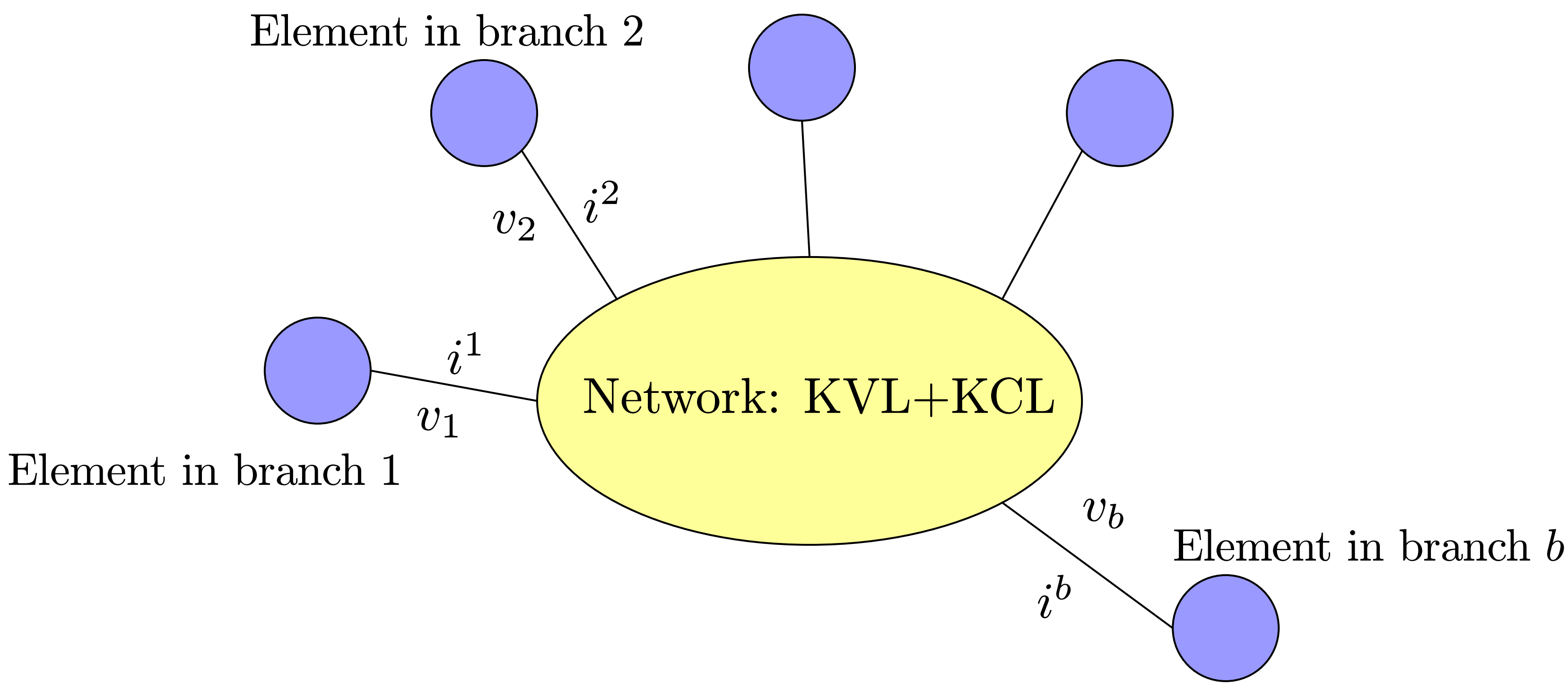
Corollary. Under the same conditions as for Tellegen‘s theorem,

$$\left\langle \frac{\mathrm{d}^r}{\mathrm{d}t_1^r} v_{(1)}(t_1), \frac{\mathrm{d}^s}{\mathrm{d}t_2^s} i_{(2)}(t_2) \right\rangle = 0$$

for any $r, s \in \mathbb{N}$.

In fact, even duality products between voltages and currents in different domains (time or frequency) can be taken and the result is still zero.

In terms of abstract network theory, a circuit can be represented as follows

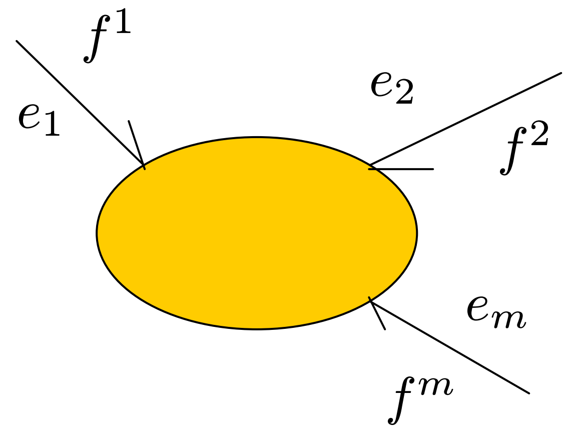


The kth branch element imposes a constitutive relation between v_k and i^k.

May be linear or nonlinear, algebraic or differential, ...

Energy storing elements

Consider a system with m power ports



$$W_f(t) - W_f(t_0) = \int_{t_0}^{t} \sum_{i=1}^{m} e_i(\tau) f^i(\tau) \mathrm{d}\tau$$

Assume the state of the system is given by $x \in \mathbb{R}^n$, and that

$$\dot{x} = G(x) + g(x)e$$

Internal dynamics

Dynamics due to the inputs

Assume also that outputs can be computed from the state

$$\boxed{f = g^T(x)\phi(x)}$$

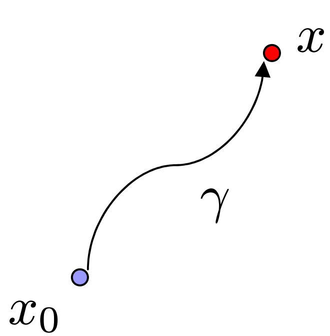
$$W_f(t) - W_f(t_0) = \int_{t_0}^{t} \langle e(\tau), f(\tau) \rangle \mathrm{d}\tau = \int_{t_0}^{t} \langle e(\tau), g^T(x(\tau))\phi(x(\tau)) \rangle \mathrm{d}\tau$$

$$= \int_{t_0}^{t} \langle g(x(\tau))e(\tau), \phi(x(\tau)) \rangle \mathrm{d}\tau \quad = \int_{t_0}^{t} \langle \dot{x}(\tau) - G(x(\tau)), \phi(x(\tau)) \rangle \mathrm{d}\tau$$

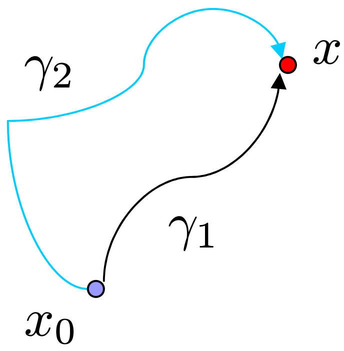
Assume the internal dynamics is

$$G(x) = J(x)\phi(x) \quad \textbf{with} \quad J^T = -J$$

$$W_f(t) - W_f(t_0) = \int_{t_0}^{t} \langle \dot{x}_{(}\tau), \phi(x(\tau)) \rangle \ \mathrm{d}\tau \quad = \int_{\gamma(x_0, x)} \phi(z) \ \mathrm{d}z$$



To avoid violating energy conservation
(First Principle of Thermodynamics)
we cannot let this depend on
the particular γ connecting x_0 and x



Suppose $\Delta W(\gamma_2) > \Delta W(\gamma_1)$

$$x_0 \xrightarrow[\Delta W(\gamma_1)]{\gamma_1} x \xrightarrow[-\Delta W(\gamma_2)]{-\gamma_2} x_0$$

We input $\Delta W(\gamma_1)$ and then obtain $\Delta W(\gamma_2)$
while the system returns to the same state

free generator of energy!

To avoid this $\phi(x)\mathrm{d}x$ must be an exact form

$$\phi_i(x) = \frac{\partial \Phi(x)}{\partial x_i}, \quad i = 1, \ldots, n$$

where $\Phi(x)$ is an state space function

$$W_f(x) - W_f(x_0) = \Phi(x)$$

or, taking
appropriate references,

$$W_f(x) = \int_{\gamma(x)} \phi(z)\mathrm{d}z$$

$$\phi_i(x) = \frac{\partial \Phi(x)}{\partial x_i}, \quad i = 1, \ldots, n$$

$$\boxed{\frac{\partial \phi_i}{\partial x_j}(x) = \frac{\partial \phi_j}{\partial x_i}(x), \quad i, j = 1, \ldots, n}$$

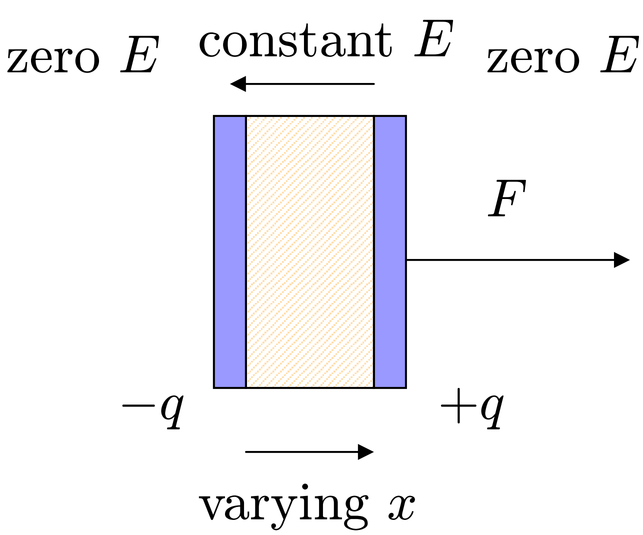
MAXWELL´S RECIPROCITY RELATIONS

$$\phi_i(x) = \frac{\partial W_f(x)}{\partial x_i}$$

we are using an all-input power convention

a $-$ sign is used in EE literature to compute
the mech force done by the system

Electric capacitor



$$n = m = 2,\ g = \mathbb{I}_2,\ G = 0$$

$$e_1 = i \text{ (current)},\ e_2 = V \text{ (velocity)}$$

$$(x, q),\ \phi_1 = F(x, q),\ \phi_2 = v(x, q)$$

$$v(x, q) = \frac{q}{C(x, q)}$$

nonlinear dielectric

$$\frac{\partial F}{\partial q} = \frac{\partial v}{\partial x} = q \frac{\partial}{\partial x}\left(\frac{1}{C(x, q)}\right) = -\frac{q}{C^2(x, q)} \partial_x C(x, q)$$

$$F(x,q) = -\int_0^q \xi \frac{\partial}{\partial x}\left(\frac{1}{C(x,\xi)}\right) \mathrm{d}\xi$$

$$= \frac{q^2}{2}\frac{\partial}{\partial x}\left(\frac{1}{C(x,q)}\right) - \int_0^q \frac{\xi^2}{2}\frac{\partial^2}{\partial \xi \partial x}\left(\frac{1}{C(x,\xi)}\right) \mathrm{d}\xi$$

in the linear case, $C(x,q) = C(x)$

$$F(x,q) = \frac{q^2}{2}\frac{\partial}{\partial x}\left(\frac{1}{C(x)}\right) = \frac{q}{2}\frac{\partial}{\partial x}\left(\frac{q}{C(x)}\right) = \frac{q}{2}\frac{\partial}{\partial x} v(x,q)$$

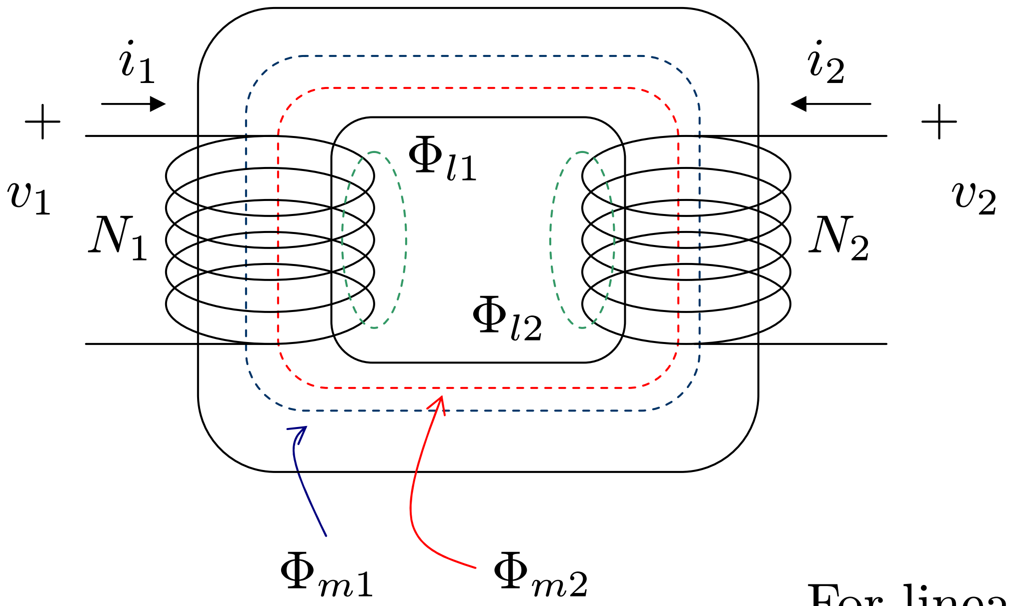
$$= -q \boxed{\frac{E(x,q)}{2}}$$

electric field seen by the + plate

Furthermore, going from $(0,0)$ to (x,q)
first with $q=0$ and then to q
with the final x, one gets the energy

$$W_f(x,q) = \int_0^q \frac{\xi}{C(x)} \mathrm{d}\xi = \frac{1}{2}\frac{q^2}{C(x)}$$

Magnetic stationary system



$$\begin{aligned}\Phi_1 &= \Phi_{l1} + \Phi_{m1} + \Phi_{m2} \\ \Phi_2 &= \Phi_{l2} + \Phi_{m2} + \Phi_{m1}\end{aligned}$$

$$\begin{aligned}\lambda_1 &= N_1\Phi_1, \\ \lambda_2 &= N_2\Phi_2.\end{aligned}$$

For linear magnetic systems

$$\Phi_{l1} = \frac{N_1 i_1}{\mathcal{R}_{l1}} \equiv L_{l1} i_1, \;\; \Phi_{m1} = \frac{N_1 i_1}{\mathcal{R}_m} \equiv L_{m1} i_1,$$

$$\Phi_{l2} = \frac{N_2 i_2}{\mathcal{R}_{l2}} \equiv L_{l2} i_2, \;\; \Phi_{m2} = \frac{N_2 i_2}{\mathcal{R}_m} \equiv L_{m2} i_2,$$

$$\begin{aligned} \lambda_1 &= L_{11} i_1 + L_{12} i_2 \\ \lambda_2 &= L_{21} i_1 + L_{22} i_2 \end{aligned}$$

$$L_{11} = N_1(L_{l1} + L_{m1}), \quad L_{22} = N_2(L_{l2} + L_{m2})$$

$$L_{12} = N_1 L_{m2} = \frac{N_1 N_2}{\mathcal{R}_m} = N_2 L_{m1} = L_{21}$$

$$n = m = 2,\ g = \mathbb{I}_2,\ G = 0,\ x = (\lambda_1, \lambda_2),\ e = (v_1, v_2),$$
$$i_1 = \phi_1(\lambda_1, \lambda_2),\ i_2 = \phi_2(\lambda_1, \lambda_2)$$

$L_{12} = L_{21}$ implies that $i = L^{-1}\lambda$ with L^{-1} $\boxed{\text{symmetric}}$

Maxwell´s reciprocity relations

adding parasitic resistances
means that G is no longer zero

$$v_1 = r_1 i_1 + \frac{\mathrm{d}\lambda_1}{\mathrm{d}t}, \quad v_2 = r_2 i_2 + \frac{\mathrm{d}\lambda_2}{\mathrm{d}t}$$

Assume now that $L_{l1} = L_{l2} = 0$

$$\begin{aligned} \lambda_1 &= \boxed{\frac{N_1 N_2}{\mathcal{R}_m}} \left(\frac{N_1}{N_2} i_1 + i_2 \right) \\ \lambda_2 &= \boxed{\frac{N_1 N_2}{\mathcal{R}_m}} \left(i_1 + \frac{N_2}{N_1} i_2 \right) \end{aligned}$$

$$\lambda_1 \frac{N_2}{N_1} = \lambda_2$$

Furthermore, if $r_1 = r_2 = 0$, it follows from the dynamics, that

$$\boxed{v_1 \frac{N_2}{N_1} = v_2}$$

This is half the ideal transformer relationships!

The final assumption is this

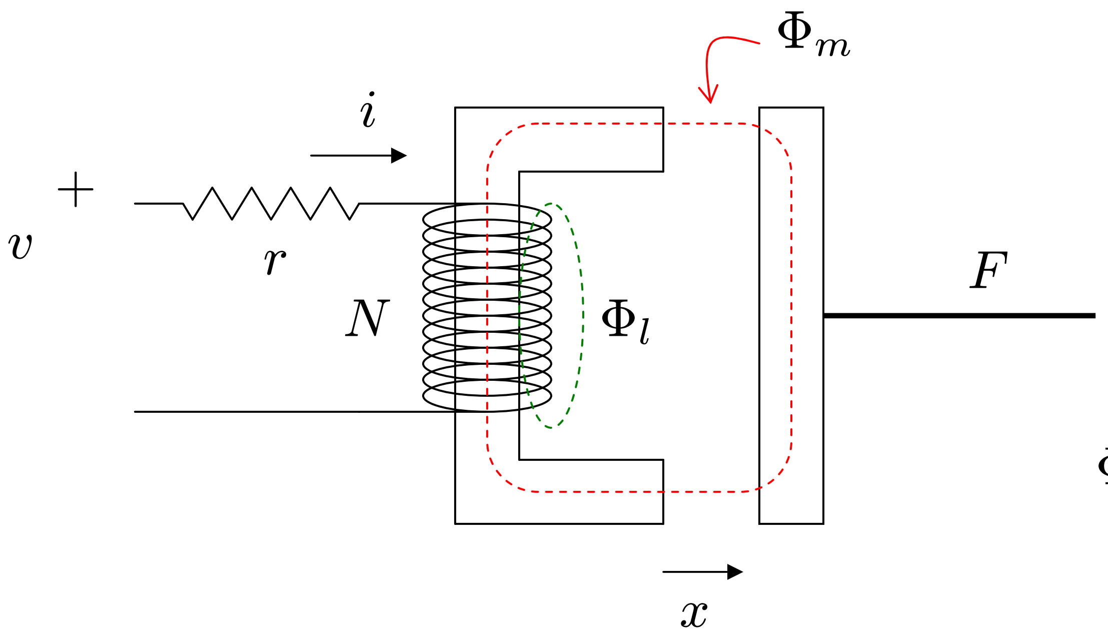
Let $N_1 N_2 / \mathcal{R}_m$ become very large while N_1/N_2 remains finite

The only way for λ to remain finite is

$$\boxed{i_2 = - \frac{N_1}{N_2} i_1}$$

The minus sign is due to the all-input power convention

Elementary electromagnet



$$v = ri + \frac{\mathrm{d}\lambda}{\mathrm{d}t}$$

$$\lambda = N\Phi$$

$$\Phi = \Phi_l + \Phi_m$$

$$\Phi_l = \frac{Ni}{\mathcal{R}_l}, \qquad \Phi_m = \frac{Ni}{\mathcal{R}_m}$$

but now $\mathcal{R}_m$ depends on the air gap x:

$$\mathcal{R}_m(x) = \frac{1}{\mu_0 A}\left(\frac{l_i}{\mu_{ri}} + 2x\right)$$

This again a 2-port system with state variables λ, x, inputs v (voltage) and V (velocity), and outputs $i = \phi_1(\lambda, x)$, $F = \phi_2(\lambda, x)$.

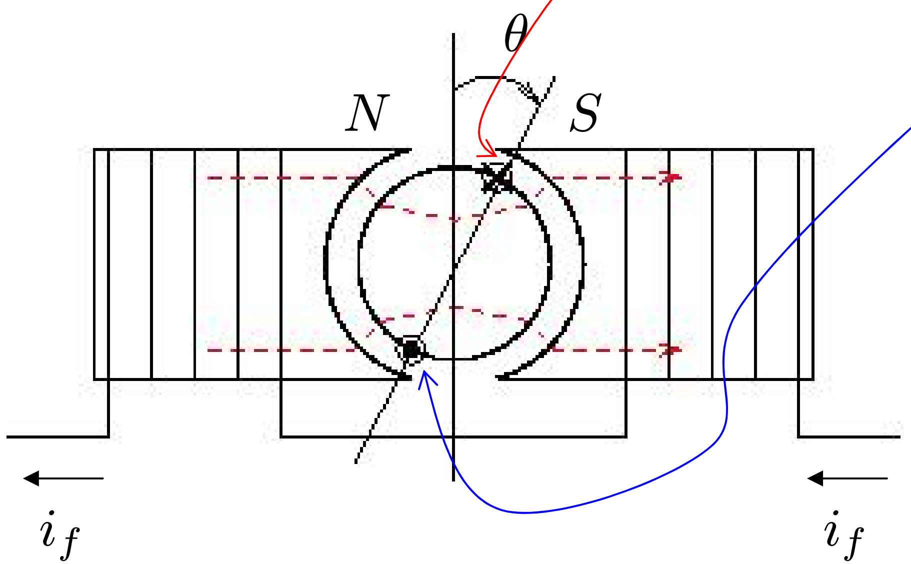
$$\lambda = \left(\frac{N^2}{\mathcal{R}_l} + \frac{N^2}{\mathcal{R}_m(x)} \right) i = (L_l + L_m(x)) i$$

$$L_m(x) = \frac{N^2}{\mathcal{R}_m} = \frac{N^2 \mu_0 A}{\frac{l_i}{\mu_{ri}} + 2x} \equiv \frac{b}{c + x}$$

From this $i = \phi_1(\lambda, x)$ can be computed.

Exercise: Compute the force $F(\lambda, x)$ and the electromechanical energy $W_f(\lambda, x)$.

DC motor



i_a current into the page

i_a current out of the page

a special mechanism
with slip rings and brushes
is necessary

The main effect is direct conversion of energy without storing it in *coordinate dependent* elements

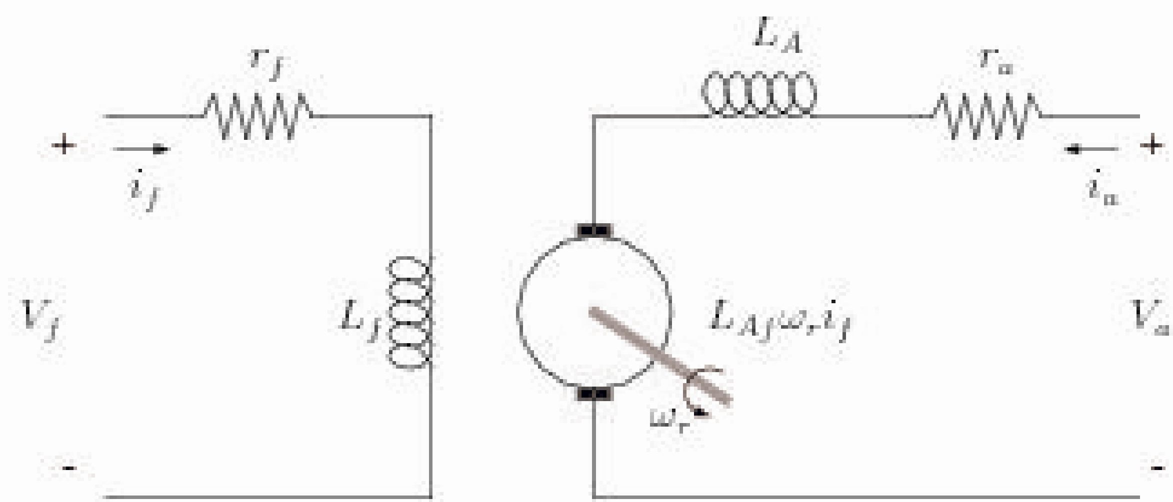
$\tau = l_1 l_2 B i_a$

$v_{emf} = -l_1 l_2 B \dot{\theta} = -l_1 l_2 B \omega_r$

Assuming linear magnetic materials

$$B = \frac{L_{Af}}{l_1 l_2} i_f$$

$$\begin{aligned} \tau &= L_{Af} i_f i_a \\ v_{emf} &= -L_{Af} i_f \omega_r \end{aligned}$$



$$\begin{aligned} \dot{\lambda}_f &= -r_f i_f + V_f \\ \dot{\lambda}_a &= -L_{Af} i_f \omega_r - r_a i_a + V_a \\ \dot{p}_m &= L_{Af} i_f i_a - B_r \omega_r - T_L \end{aligned}$$

$\lambda_f = L_f i_f$ (the field flux)

$\lambda_a = L_A i_a$ (the armature flux)

$p_m = J_m \omega_r$ (the mech angular momentum)

$-T_L$ is the external mech torque

Exercise: Write the bond graph corresponding to the dc motor presented

Exercise: Show that, neglecting the field port, in the limit when L_A and J_m go to zero (define this formally!), and dissipations disapear, the DC motor becomes a pure gyrator.

Co-energy for electromech systems

Suppose an electromechanical system with electrical state variables $x_e \in \mathbb{R}^n$ and geometrical state variables $\theta \in \mathbb{R}^m$

We have $W_f = W_f(x_e, \theta)$

associated flows $$f_i(x_e, \theta) = \frac{\partial W_f}{\partial x_{ei}}(x_e, \theta), \quad i = 1, \ldots, n$$

Co-energy is defined as $$\boxed{W_c(f, x_e, \theta) = \sum_{i=1}^{n} f_i x_{ei} - W_f(x_e, \theta)}$$

... but $$\frac{\partial W_c}{\partial x_{ei}} = f_i - \frac{\partial W_f}{\partial x_{ei}}(x_e, \theta) = f_i - f_i = 0, \quad i = 1, \ldots, n$$

Hence, W_c can be written as a function of only f (and θ)

Legendre transformation

Widely used in analytical mechanics and thermodynamics

While variables x are known as *energy* or *Hamiltonian* variables, transformed variables f are the *co-energy* or *Lagrangian* variables.

Mechanical forces can be computed also from the co-energy

$$W_c(f,\theta) = f \cdot x_e(f,\theta) - W_f(x_e(f,\theta),\theta)$$

$$\partial_\theta W_c(f,\theta) \;\; = f\partial_\theta x_e - \underbrace{\boxed{\partial_{x_e} W_f}}_{f} \partial_\theta x_e - \partial_\theta W_f \; = -\partial_\theta W_f = -F$$

So mechanical forces can be computed as

$$\boxed{F = -\partial_\theta W_c}$$

notice again a reversed sign with respect to standard EE literature

For linear electromagnetic systems, $q = C(x)v$, $\lambda = L(x)i$,

$$\mathrm{W}_c(v, i, x) = W_f(q, \lambda, x)\big|_{q=q(v,x),\lambda=\lambda(i,x)}$$

In particular, for linear magnetic systems,

$$W_f(\lambda, x) = \frac{1}{2}\lambda^T L^{-1}(x)\lambda, \quad \boxed{W_c(i, x) = \frac{1}{2} i^T L(x) i}$$

Exercise: Prove the above equation.

Exercise: Consider an electromechanical system with a nonlinear magnetic material such that

$$\lambda = (a + bx^2)i^2,$$

where a and b are constants and x is a variable geometric parameter. Compute W_f, W_c and f, and check all the relevant relations.

Port Hamiltonian modeling of electromechanical systems

Cast the previous systems as (explicit) port Hamiltonian models

$$\dot{x} = (\mathcal{J}(x) - \mathcal{R}(x))(\nabla H(x))^T + g(x)u$$

$x \in \mathbb{R}^n$, $\mathcal{J}$ is antisymmetric, $\mathcal{R}$ is symmetric and positive semi-definite and $u \in \mathbb{R}^m$ is the control

$H(x)$ is the energy $(= W_f(x))$

$$y = g^T(x)(\nabla H(x))^T$$

General electromechanical model

$$H(\lambda, p, \theta) = \frac{1}{2}\lambda^T L^{-1}(\theta)\lambda + \frac{1}{2}p^T J_m^{-1} p$$

λ are the generalized electrical energy variables
(they may be charges or magnetic fluxes)
p are the generalized mechanical momenta
(linear or angular, or associated to any other generalized coordinate)
θ are the generalized geometric coordinates

$$\begin{aligned} \dot{\lambda} + R_e i &= Bv \\ \dot{p} &= -R_m\omega - T_e(\lambda, \theta) + T_m \\ \dot{\theta} &= J_m^{-1} p \end{aligned}$$

B indicates how the input voltages are connected to the electrical devices

T_e is the electrical torque
T_m is the external applied mechanical torque

$$i = L^{-1}(\theta)\lambda = \partial_\lambda H, \quad \omega = J_m^{-1} p = \partial_p H$$

Notice that the port yielding T_e is internal
(connected to the mechanical inertia)

$$T_e = \frac{\partial H}{\partial \theta} = \frac{1}{2}\lambda^T \partial_\theta L^{-1}(\theta)\lambda \quad = -\frac{1}{2}\lambda^T L^{-1}\partial_\theta L L^{-1}\lambda$$

$$\boxed{\partial_\theta L^{-1} = -L^{-1}\partial_\theta L L^{-1}}$$

$$\begin{aligned} \dot{\lambda} + R_e i &= Bv \\ \dot{p} &= -R_m\omega - T_e(\lambda, \theta) + T_m \\ \dot{\theta} &= J_m^{-1} p \end{aligned}$$

$$\dot{x} = \begin{pmatrix} -R_e & 0 & 0 \\ 0 & -R_m & -1 \\ 0 & 1 & 0 \end{pmatrix} \partial_x H + \begin{pmatrix} B & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ T_m \end{pmatrix}$$

$$x = (\lambda \; p \; \theta)^T$$

This general model includes many of the classical electrical machines, as well as linear motors and levitating systems.

Exercise: Write the general port Hamiltonian model for the electromagnet and for the variable geometry capacitor

The dc motor needs a special formulation

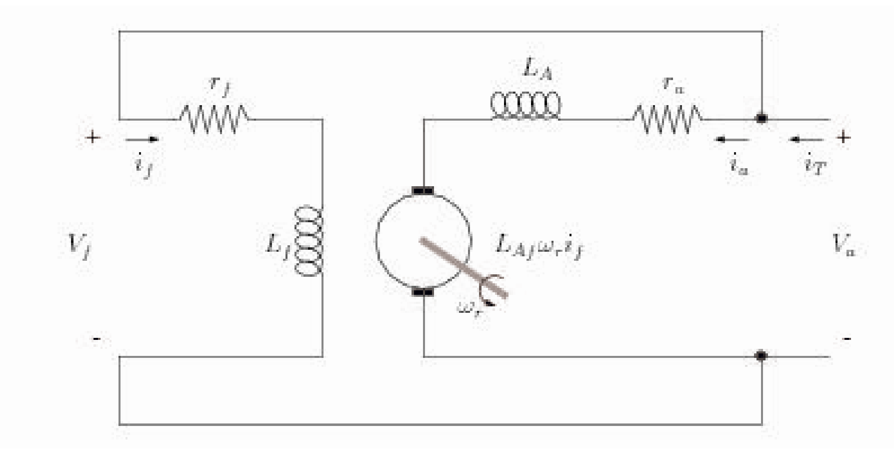
$$x^T = \begin{pmatrix} \lambda_f & \lambda_a & p = J_m \omega_r \end{pmatrix}$$

$$H(x) = \frac{1}{2}\lambda L^{-1}\lambda^T + \frac{1}{2J_m}p^2, \quad L = \begin{pmatrix} L_f & 0 \\ 0 & L_A \end{pmatrix}$$

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -L_{Af} i_f \\ 0 & L_{Af} i_f & 0 \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} r_f & 0 & 0 \\ 0 & r_a & 0 \\ 0 & 0 & B_r \end{pmatrix}$$

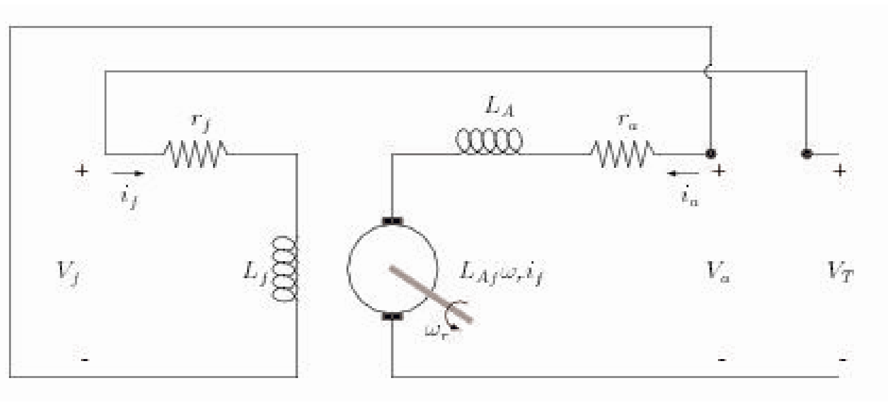
$$g = \mathbb{I}_3 \qquad u^T = \begin{pmatrix} V_f & V_a & -T_L \end{pmatrix}$$

The input voltages V_f and V_a can be obtained from the same source in several ways



Shunt connection

$$V_a = V_f \text{ and } i_T = i_a + i_f$$



Series connection

$$i_a = i_f \text{ and } V_T = V_a + V_f$$

Exercise: Obtain the port Hamiltonian models of the shunt-connected and of the series-connected dc machines.
The later is in fact an implicit port Hamiltonian model.
Check this by writing the corresponding bond graph and identifying the differential causality assignment.

Power converters

Control of electromechanical systems

Portable battery-powered equipment

UPS

Power conversion systems

(*e.g.* conditioning from fuel-cell, wind-based,...)

$\Bigg\}$ Power converters

dc-to-dc: to elevate or reduce dc voltages
in a load-independent way

ac-to-dc: rectifiers

dc-to-ac: inverters

dc-to-dc store the energy
in intermediate ac elements
an deliver it as needed

trick: use of switches,
operated in a periodic manner

system alternates
between several dynamics

Variable Structure Systems (VSS)

VSS are non-smooth dynamical systems

difficulties in simulation and control design

averaging

Assume the VSS cycles over several dynamics in a periodic way

Compute the average of relevant variables over a cycle

State Space Averaging (SSA)

$$\langle x \rangle(t) = \frac{1}{T} \int_{t-T}^{t} x(\tau)\ \mathrm{d}\tau$$

VSS in PHDS form

$$\dot{x} = \left[\mathcal{J}(S, x) - \mathcal{R}(S, x)\right]\left(\nabla H(x)\right)^T + g(S, x)u$$

S is a (multi)-index, with values on a finite, discrete set, enumerating the different structure topologies

For simplicity, $S \in \{0, 1\}$

$$S = 0 \quad \Rightarrow \quad \dot{x} = (\mathcal{J}_0(x) - \mathcal{R}_0(x))(\nabla H(x))^T + g_0(x)u$$
$$S = 1 \quad \Rightarrow \quad \dot{x} = (\mathcal{J}_1(x) - \mathcal{R}_1(x))(\nabla H(x))^T + g_1(x)u$$

if x does not vary too much during a given topology

$$S = 0 \quad \Rightarrow \quad \dot{x} \approx (\mathcal{J}_0(\langle x\rangle) - \mathcal{R}_0(\langle x\rangle))(\nabla H(\langle x\rangle))^T + g_0(\langle x\rangle)u$$
$$S = 1 \quad \Rightarrow \quad \dot{x} \approx (\mathcal{J}_1(\langle x\rangle) - \mathcal{R}_1(\langle x\rangle))(\nabla H(\langle x\rangle))^T + g_1(\langle x\rangle)u$$

the length of time in a given cycle with the system in a given topology is determined by a function of the state variables

in our approximation, a function of the averages, $t_0(\langle x\rangle)$, $t_1(\langle x\rangle)$, with $t_0 + t_1 = T$

$$\begin{aligned} x(t) = x(t-T) \quad &+ \quad t_0(\langle x\rangle)\left[(\mathcal{J}_0(\langle x\rangle) - \mathcal{R}_0(\langle x\rangle))(\nabla H(\langle x\rangle))^T + g_0(\langle x\rangle)u\right] \\ &+ \quad t_1(\langle x\rangle)\left[(\mathcal{J}_1(\langle x\rangle) - \mathcal{R}_1(\langle x\rangle))(\nabla H(\langle x\rangle))^T + g_1(\langle x\rangle)u\right] \end{aligned}$$

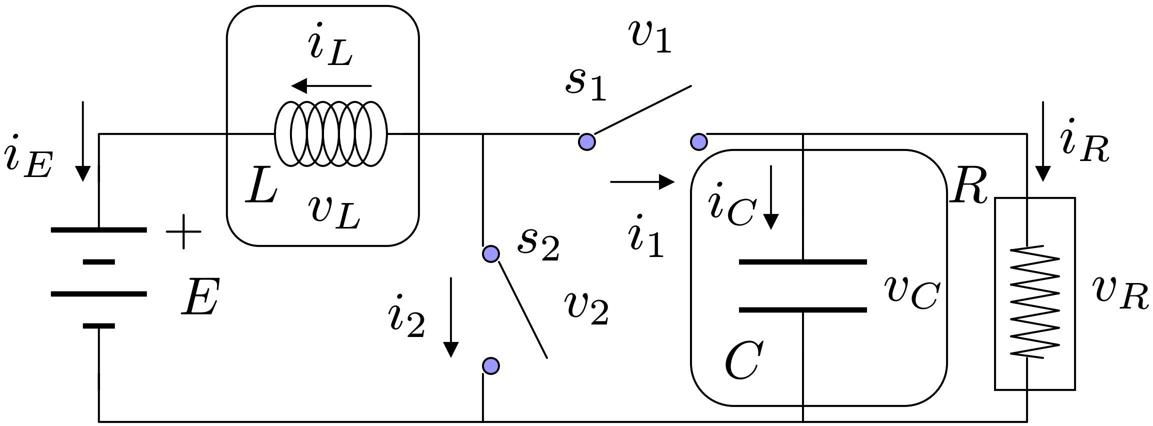
$$\boxed{\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle x\rangle \quad &= \quad d_0(\langle x\rangle)\left[(\mathcal{J}_0(\langle x\rangle) - \mathcal{R}_0(\langle x\rangle))(\nabla H(\langle x\rangle))^T + g_0(\langle x\rangle)u\right] \\ &+ \quad d_1(\langle x\rangle)\left[(\mathcal{J}_1(\langle x\rangle) - \mathcal{R}_1(\langle x\rangle))(\nabla H(\langle x\rangle))^T + g_1(\langle x\rangle)u\right] \end{aligned}}$$

$$d_0 + d_1 = 1$$

$$d_{0,1}(\langle x\rangle) = \frac{t_{0,1}(\langle x\rangle)}{T}$$

duty cycle

Boost (step-up) dc-to-dc converter



s_1 closed $(s_1 = 1)$
s_2 open $(s_2 = 0)$
and viceversa

only a single
boolean variable $S = s_2$

Two energy-storing elements: $H = H_C + H_L$

Two 1-d PHDS

$$\boxed{\frac{\mathrm{d}q_C}{\mathrm{d}t} = i_C, \quad v_C = \frac{\partial H}{\partial q_C}}$$

$$\boxed{\frac{\mathrm{d}\phi_L}{\mathrm{d}t} = v_L, \quad i_L = \frac{\partial H}{\partial \phi_L}}$$

Connected by Kirchoffs laws

$$\begin{aligned} i_L &= i_1 + i_2 \\ i_1 &= i_C + i_R \\ v_2 + v_L &= E \\ v_C + v_1 &= v_2 \\ v_C &= v_R \\ i_E + i_L &= 0 \end{aligned}$$

Using the PHD equations, the first four KL are written as

$$
\begin{aligned}
\frac{\partial H}{\partial \phi_L} &= i_1 + i_2 \\
i_1 &= \frac{\mathrm{d}q_C}{\mathrm{d}t} + i_R \\
v_2 + \frac{\mathrm{d}\phi_L}{\mathrm{d}t} &= E \\
\frac{\partial H}{\partial q_C} + v_1 &= v_2
\end{aligned}
$$

Internal ports

The second and third yield a 2-d PCH

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q_C \\ \phi_L \end{pmatrix} = 0(\nabla H)^T + \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} i_1 \\ v_2 \end{pmatrix} \\ \begin{pmatrix} i_R \\ E \end{pmatrix} \end{pmatrix}
$$

True ports

The switches impose the following conditions

$$S = 0 \Rightarrow s_1 = 1, s_2 = 0 \Rightarrow \boxed{v_1 = 0}, \boxed{i_2 = 0}$$

$$S = 1 \Rightarrow s_1 = 0, s_2 = 1 \Rightarrow i_1 = 0, v_2 = 0$$

For $S = 1$ we have the values of v_2 and i_1

For $S = 0$

$$\begin{aligned} \boxed{\frac{\partial H}{\partial \phi_L} = i_1 + i_2} \\ i_1 &= \frac{\mathrm{d}q_C}{\mathrm{d}t} + i_R \\ v_2 + \frac{\mathrm{d}\phi_L}{\mathrm{d}t} &= E \\ \boxed{\frac{\partial H}{\partial q_C} + v_1 = v_2} \end{aligned}$$

$$i_1 = \frac{\partial H}{\partial \phi_L}, \quad v_2 = \frac{\partial H}{\partial q_C}$$

In compact form

$$\begin{aligned} i_1 &= (1 - S)\frac{\partial H}{\partial \phi_L} \\ v_2 &= (1 - S)\frac{\partial H}{\partial q_C} \end{aligned}$$

Substituting i_1 and v_2 back in the 2-d PHS

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q_C \\ \phi_L \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} (1-S)\,\partial H/\partial \phi_L \\ (1-S)\,\partial H/\partial q_C \\ i_R \\ E \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \boxed{\begin{matrix} 0 & 1-S \\ -(1-S) & 0 \end{matrix}} \end{pmatrix}}_{\mathcal{J}} \begin{pmatrix} \partial H/\partial q_C \\ \partial H/\partial \phi_L \end{pmatrix} + \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{g} \begin{pmatrix} i_R \\ E \end{pmatrix}$$

Skew symmetric

Not a coincidence!

Is a consequence of Kirchoff´s laws being an instantiation of Dirac structures and $FE^T + EF^T = 0$

$$y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} \partial H/\partial q_C \\ \partial H/\partial \phi_L \end{pmatrix} = \begin{pmatrix} -v_C \\ i_L \end{pmatrix} = \begin{pmatrix} -v_R \\ -i_E \end{pmatrix}$$

last two KL

Now we can terminate the resistive port and introduce some dissipation

$$i_R = \frac{v_R}{R} = \frac{v_C}{R} = \frac{1}{R}\frac{\partial H}{\partial q_C}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q_C \\ \phi_L \end{pmatrix} = \left[\begin{pmatrix} 0 & 1-S \\ -(1-S) & 0 \end{pmatrix} - \boxed{\begin{pmatrix} 1/R & 0 \\ 0 & 0 \end{pmatrix}}\right]\begin{pmatrix} \partial H/\partial q_C \\ \partial H/\partial \phi_L \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} E$$

symmetric,
positive semidefinite matrix

Only a single natural output

$$y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} \partial H/\partial q_C \\ \partial H/\partial \phi_L \end{pmatrix} = i_L = -i_E$$

So the boost converter, a VSS, has a PHDS representation.

The three basic second order dc-to-dc converters have this representation

$$\dot{x} = \begin{pmatrix} -1/R & \alpha - \beta S \\ -(\alpha - \beta S) & 0 \end{pmatrix} (\nabla H(x))^T + \begin{pmatrix} 0 \\ 1 - \gamma S \end{pmatrix} E$$

Converter	α	β	γ
buck	1	0	1
boost	1	1	0
buck-boost	0	1	1

$$x = (q_C \ \ \phi_L)^T$$

Assuming linear elements, $H(q_c, \phi_L) = \frac{1}{2C} q_C^2 + \frac{1}{2L} \phi_L^2$

$$\dot{x} = \begin{pmatrix} -1/R & \alpha - \beta S \\ -(\alpha - \beta S) & 0 \end{pmatrix} (\nabla H(x))^T + \begin{pmatrix} 0 \\ 1 - \gamma S \end{pmatrix} E$$

Let us compute the averaged equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle x \rangle &= d_0(\langle x \rangle) \left[\begin{pmatrix} -1/R & \alpha \\ -\alpha & 0 \end{pmatrix} (\nabla H(\langle x \rangle))^T + \begin{pmatrix} 0 \\ 1 \end{pmatrix} E \right] \\ &+ d_1(\langle x \rangle) \left[\begin{pmatrix} -1/R & \alpha - \beta \\ -(\alpha - \beta) & 0 \end{pmatrix} (\nabla H(\langle x \rangle))^T + \begin{pmatrix} 0 \\ 1 - \gamma \end{pmatrix} E \right] \end{aligned}$$

$$\boxed{d_0 + d_1 = 1}$$

$$= \begin{pmatrix} -1/R & \alpha - \beta d_1(\langle x \rangle) \\ -(\alpha - \beta d_1(\langle x \rangle)) & 0 \end{pmatrix} (\nabla H(\langle x \rangle))^T + \begin{pmatrix} 0 \\ 1 - \gamma d_1(\langle x \rangle) \end{pmatrix} E$$

$$\boxed{\langle S \rangle = 0 \cdot d_0 + 1 \cdot d_1}$$

VSS averaged PHDS model

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x \rangle = \begin{pmatrix} -1/R & \alpha - \beta \langle S \rangle \\ -(\alpha - \beta \langle S \rangle) & 0 \end{pmatrix} (\nabla H(\langle x \rangle))^T + \begin{pmatrix} 0 \\ 1 - \gamma \langle S \rangle \end{pmatrix} E$$

Better averaged descriptions of VSS can be obtained by considering higher order Fourier-like coefficients, instead of just $\langle x \rangle$.

k-phasors

$$\langle x \rangle_k(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)e^{-jk\omega\tau}\,\mathrm{d}\tau$$

The phasor averages of VSS PHDS are also PHDS

See printed notes for details

A storing and conditioning energy system

Many metropolitan electrical-based vehicles use dc-motors which draw their power from segmented dc power lines

Vehicles are able to brake in a regenerative way, returning power to the line

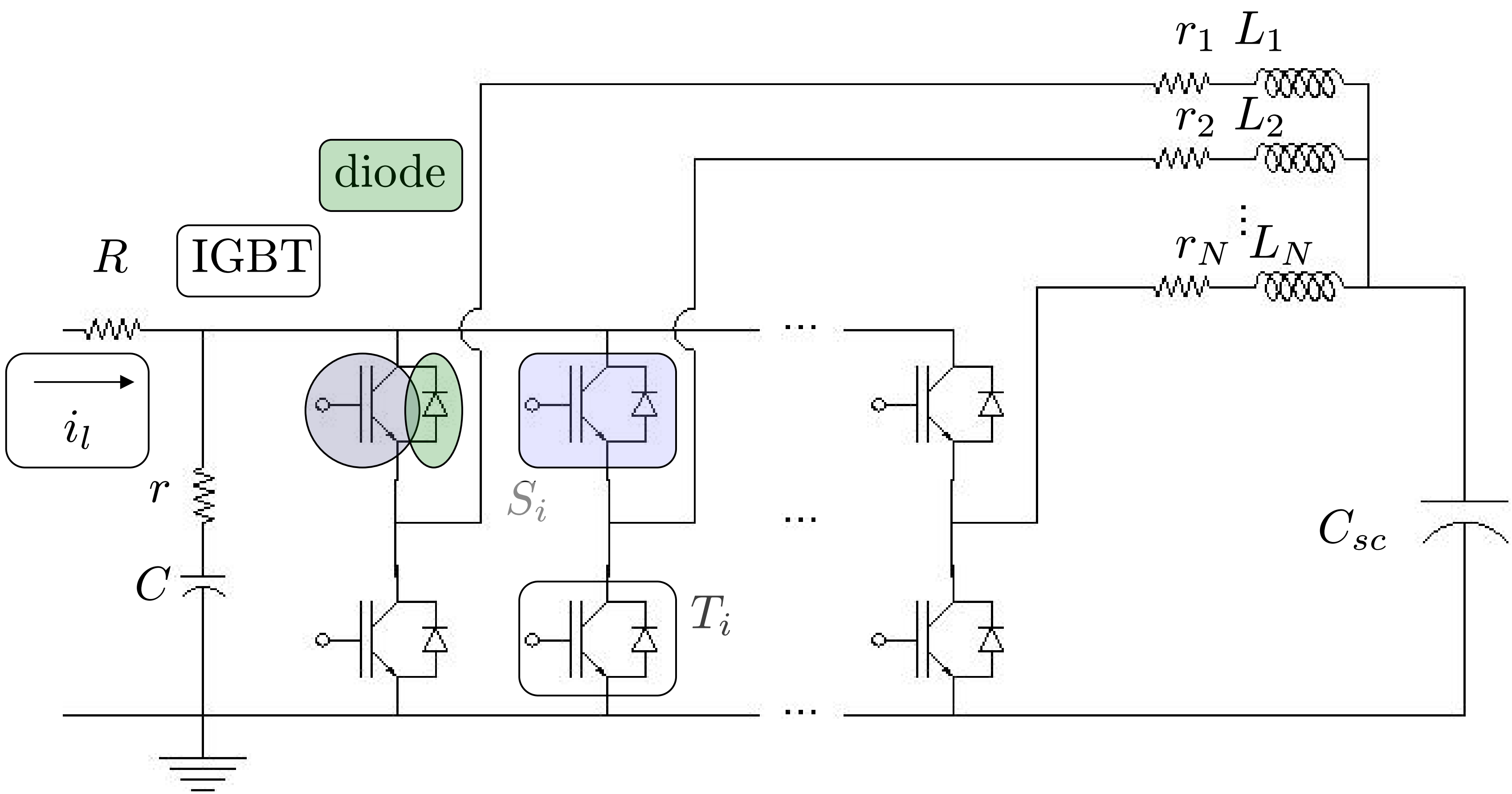
Power can be reused only if another vehicle is accelerating in the same segment of the power line

if not, this power is dissipated in special resistors and lost

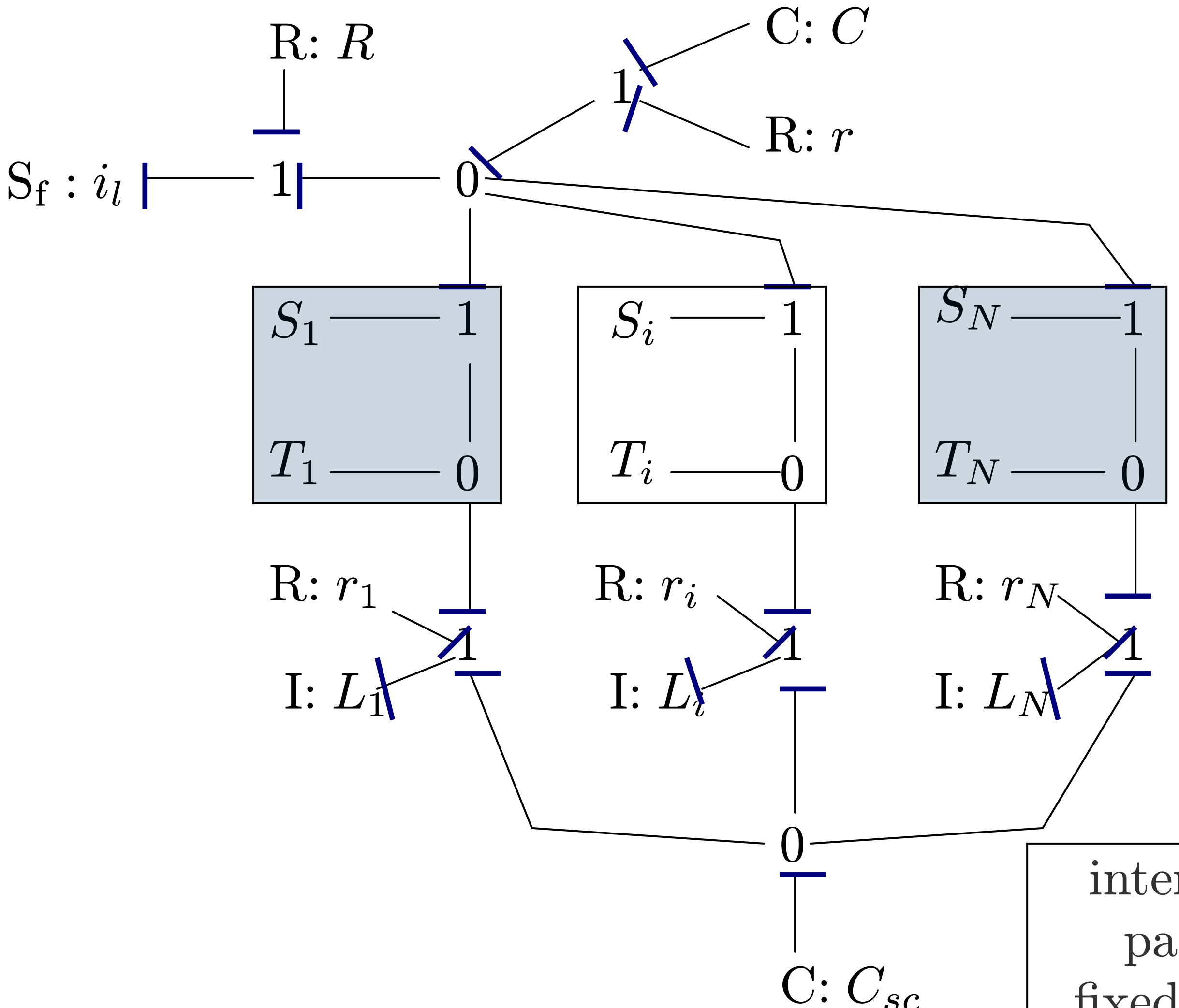
Problem: install in each power segment a device to store the excess energy and return it when needed at the required voltage

Due to several reasons (technical and economical) the storage elements must be either supercapacitors or mechanical flywheels

A power conversion system must be put in the middle to deliver the energy in the required form

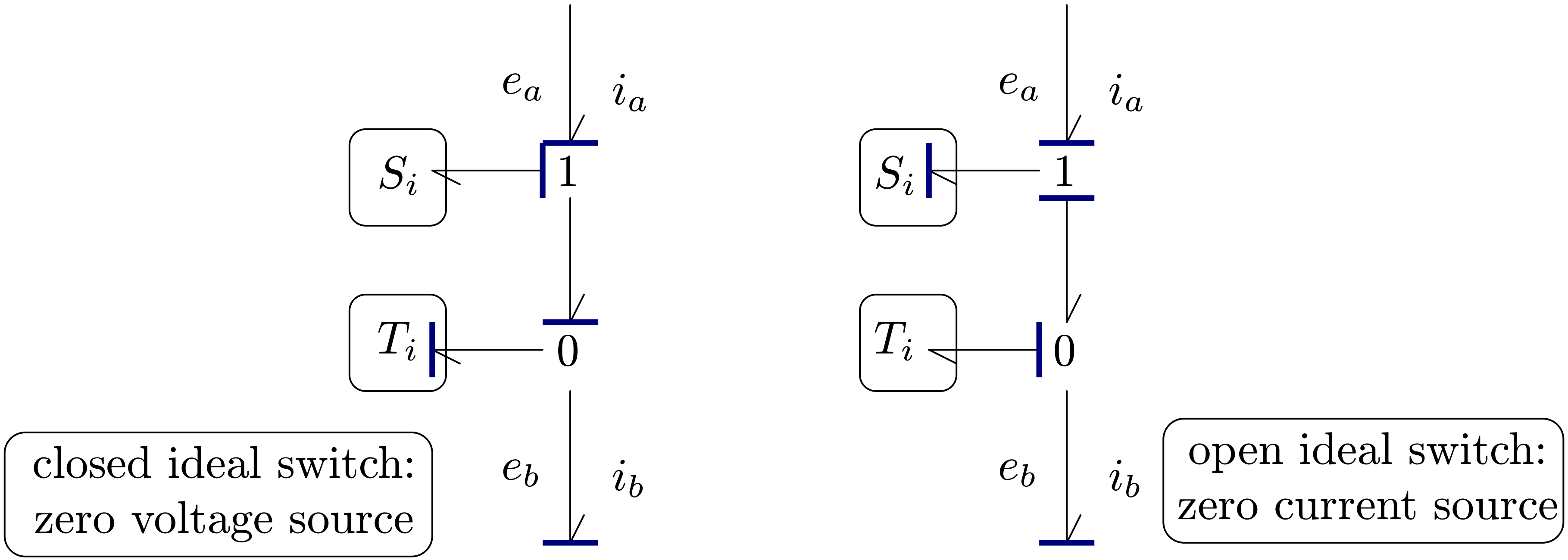


The causal bond graph of the system is (power conventions are not displayed)



internal causality of each pair of switches is not fixed by external causality

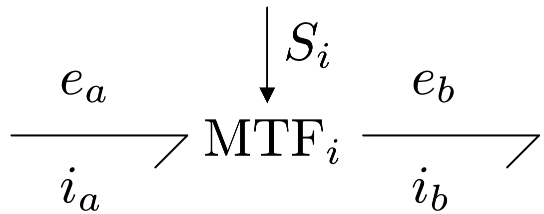
Two combinations of internal causality are possible



If $S_i = 1$ when S_i is closed and $S_i = 0$ when S_i is open, then

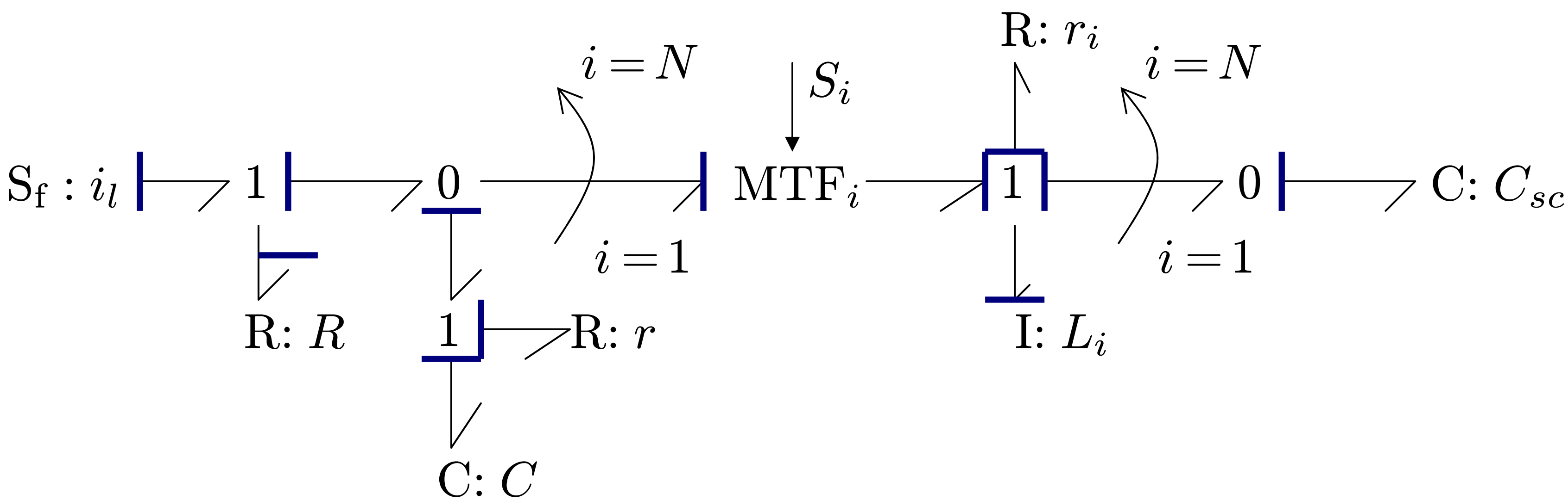
$$
\begin{aligned}
e_b &= S_i e_a \\
i_a &= S_i i_b
\end{aligned}
$$

Each pair of switches can be modeled as a modulated transformer



$$
\begin{aligned}
e_b &= S_i e_a \\
i_a &= S_i i_b
\end{aligned}
$$

and then



Exercise: Write the state equations for the above bond graph.

If you solve the above exercise, you will see that the resulting equations can be written in PHDS form with state variables q (charge of C), Q (charge of C_{sc}) and λ_i (flux of L_i, $i = 1, \ldots, N$)

$$\dot{x} = (\mathcal{J} - \mathcal{R})\partial_x H + g i_l \qquad H(Q, q, \lambda_i) = \frac{1}{2C_{sc}}Q^2 + \frac{1}{2C}q^2 + \sum_{i=1}^{N} \frac{1}{2L_i}\lambda_i^2$$

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & -S_1 & \cdots & -S_N \\ -1 & S_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & S_N & 0 & \cdots & 0 \end{pmatrix} \qquad \mathcal{R} = \begin{pmatrix} 0_{2\times 2} & 0_{2\times N} \\ 0_{N\times 2} & \delta_{ij} r_j + r S_i S_j \end{pmatrix}$$

$$g = \begin{pmatrix} 0 \\ 1 \\ rS_1 \\ rS_2 \\ \vdots \\ rS_N \end{pmatrix}$$

This can be obtained also from the interconnection of the individual PHD subsystems, as we did for the boost.

The model can be expanded to include the power line + the electrical vehicles, or the supercapacitor can be substituted by a normal one attached to a dc-motor and a flywheel

Important modeling remarks

The transformer model obtained is dependent on the external causality imposed upon each pair of switches.
It has to be recomputed if something different is connected that changes that.

The model has produced four-quadrant switches:
there is no constraint on the signs of currents and voltages through them.
In practice, the type of switch displayed is two-quadrant:
it can sustain any current but only non-zero voltages in one way.