Dirac structures and port-Hamiltonian systems. Interconnection and model order reduction

Seminar 1 - Dirac structures and port-Hamiltonian systems

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Key references

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Definition of a Dirac structure (I)

- Spaces:
  - $\mathcal{F}$, a finite-dimensional vector space, called the space of *flows*.
  - $\mathcal{F}^*$, its dual space, called the space of *efforts*.
  - $\mathcal{F} \times \mathcal{F}^*$, the space of *power variables*.
- The *power* associated to a pair $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ is

$$P(f, e) = \langle e | f \rangle,$$

  with $\langle e | f \rangle$ the action of the form e on the vector f.
- (Indefinite) symmetric *bilinear form* on $\mathcal{F} \times \mathcal{F}^*$:

$$\lessdot (f^a, e^a) | (f^b, e^b) \gtrdot = \langle e^a | f^b \rangle + \langle e^b | f^a \rangle$$

Definition of a Dirac structure (II)

- This bilinear form is indefinite, since

$$\lessdot(f,e)|(f,e)\gtrdot = 2\langle e|f\rangle$$

can have any sign.

- However, it is non-degenerate. Suppose that

$$\lessdot(f,e)|(\tilde{f},\tilde{e})\gtrdot = \langle e|\tilde{f}\rangle + \langle\tilde{e}|f\rangle = 0$$

for all $(\tilde{f},\tilde{e})$.
Setting $\tilde{f}=0$ one gets first $f=0$ (by definition, the only vector whose image is zero for all forms is the zero vector), and then this implies $e=0$ (a form that sends all the vectors to zero must be the zero form).

Definition of a Dirac structure (III)

- A constant *Dirac structure* on $\mathcal{F} \times \mathcal{F}^*$ is a subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ such that

$$\textbf{(D1)} \quad \mathcal{D} = \mathcal{D}^\perp,$$

with $^\perp$ the orthogonal complement with respect to $\lessdot \mid \gtrdot$.

- If the components of vectors and forms are given with respect to dual basis, then

$$\langle e|f\rangle = e^T f$$

and we can write

$$\mathcal{D}^\perp = \left\{(\tilde{f}, \tilde{e}) \in \mathcal{F} \times \mathcal{F}^* \text{ s.t. } \tilde{e}^T f + e^T \tilde{f} = 0 \;\forall\; (f, e) \in \mathcal{D}\right\}.$$

Definition of a Dirac structure (IV)

- If $(f, e) \in \mathcal{D}$ then

$$\langle e|\tilde{f}\rangle + \langle \tilde{e}|f\rangle = 0 \ \forall(\tilde{f}, \tilde{e}) \in \mathcal{D}$$

so that $\mathcal{D} \subset \mathcal{D}^{\perp}$, and one only has to prove that $\mathcal{D}^{\perp} \subset \mathcal{D}$.

- If $\mathcal{D}$ is a Dirac structure in $\mathcal{F} \times \mathcal{F}^*$, one has, using the non-degeneracy of the bilinear form $\lessdot|\gtrdot$, that

$$\dim \mathcal{D} = \dim \mathcal{D}^{\perp} = \dim \mathcal{F} \times \mathcal{F}^* - \dim \mathcal{D} = 2 \dim \mathcal{F} - \dim \mathcal{D},$$

from which

$$\dim \mathcal{D} = \dim \mathcal{F}.$$

Definition of a Dirac structure (V)

- *Power conservation*. If $(f,e) \in \mathcal{D}$ then $(f,e) \in \mathcal{D}^\perp$ and hence

$$0 = \langle\!\langle (f,e) | (f,e) \rangle\!\rangle = \langle e|f \rangle + \langle e|f \rangle = 2\langle e|f \rangle$$

so that $\mathcal{D}$ is power-preserving or *power continuous*:

$$\langle e|f \rangle = 0 \quad \text{for any } (f,e) \in \mathcal{D}.$$

- When describing the dynamics on a manifold $\mathcal{M}$, $\mathcal{D}$ is actually modulated by the *state variables* $x \in \mathcal{M}$. Then

$$\mathcal{F}(x) = T_x\mathcal{M}$$

and $\mathcal{D}(x) \subset T_x\mathcal{M} \times T_x^*\mathcal{M}$, which is assumed to depend smoothly on x. $\mathcal{D}(x)$ might be constant for certain coordinates in $\mathcal{M}$.

Example

$\mathcal{F} = \mathbb{R}^2$, $\mathcal{F}^* = \mathbb{R}^2$, $\mathcal{F} \times \mathcal{F}^* = \mathbb{R}^4$. Let

$$\mathcal{D} = \text{span}\{(1,0,0,1)^T, (0,1,-1,0)^T\}.$$

The $(f,e) \in \mathcal{D}$ are of the form $\alpha(1,0,0,1)^T + \beta(0,1,-1,0)^T$ *i.e.*

$$f = (\alpha, \beta)^T, \; e = (-\beta, \alpha)^T.$$

The $(\tilde{f}, \tilde{e})$ in $\mathcal{D}^\perp$ are such that, for all α, β,

$$\tilde{e}^T f + e^T \tilde{f} = (\alpha \tilde{e}_1 + \beta \tilde{e}_2) + (-\beta \tilde{f}_1 + \alpha \tilde{f}_2) = 0.$$

With $\alpha = 0$, $\beta = 1$ one gets $\tilde{e}_2 = \tilde{f}_1$, and $\alpha = 1$, $\beta = 0$ yields $\tilde{e}_1 = -\tilde{f}_2$, so that $\mathcal{D}^\perp \subset \mathcal{D}$ and $\mathcal{D}$ is indeed a Dirac structure. It is also obvious that $e^T f = 0$ for $(f,e) \in \mathcal{D}$.

Power-preserving definition of a Dirac structure (I)

- Dirac structures have built-in power-continuity. It turns out that, conversely, this is nearly enough to have a Dirac structure.
- A Dirac structure on $\mathcal{F} \times \mathcal{F}^*$ is a subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ satisfying the following two conditions:

  (D2) $\langle e|f \rangle = 0$ for all $(f, e) \in \mathcal{D}$.
  (D3) $\dim \mathcal{D} = \dim \mathcal{F}$.
- This alternative definition is somehow more intuitive, but cannot be generalized to the infinite dimensional case.
- **(D1)** $\Rightarrow$ **(D2), (D3)** has already been proved.

Power-preserving definition of a Dirac structure (II)

- **(D2), (D3) $\Rightarrow$ (D1).** Let (f,e) and $(\tilde{f},\tilde{e})$ belong to $\mathcal{D}$. Since $\mathcal{D}$ is a subspace, then $(f+\tilde{f}, e+\tilde{e}) \in \mathcal{D}$ and

$$
\begin{aligned}
0 &\overset{(D2)}{=} \langle e+\tilde{e} | f+\tilde{f}\rangle = \langle e|f\rangle + \langle e|\tilde{f}\rangle + \langle \tilde{e}|f\rangle + \langle \tilde{e}|\tilde{f}\rangle \\
&\overset{(D2)}{=} \langle e|\tilde{f}\rangle + \langle \tilde{e}|f\rangle = \lessdot(f,e)|(\tilde{f},\tilde{e})\gtrdot .
\end{aligned}
$$

Given $(f,e)\in\mathcal{D}$ we get thus that $\lessdot(f,e)|(\tilde{f},\tilde{e})\gtrdot = 0$ for all $(\tilde{f},\tilde{e})\in\mathcal{D}$, and hence $\mathcal{D}\subset\mathcal{D}^\perp$. Furthermore, using (D3) and the non-degeneracy of $\lessdot\ |\ \gtrdot$,

$$
\dim \mathcal{D}^\perp = 2\dim\mathcal{F} - \dim\mathcal{D} \overset{(D3)}{=} \dim\mathcal{D}
$$

and thus $\mathcal{D}^\perp = \mathcal{D}$, which ends the proof.

Power-preserving definition of a Dirac structure (III)

From the preceding proof it follows that (D2) by itself implies that $\mathcal{D} \subset \mathcal{D}^\perp$. From the non-degeneracy of the bilinear form we also have $\dim \mathcal{D}^\perp = 2 \dim \mathcal{F} - \dim \mathcal{D}$. Therefore, any subspace $\mathcal{D}$ satisfying (D2), be it a Dirac structure or not, has the property that

$$\dim \mathcal{D} \leq \dim \mathcal{D}^\perp = 2 \dim \mathcal{F} - \dim \mathcal{D},$$

from which

$$\dim \mathcal{D} \leq \dim \mathcal{F}.$$

Hence, we can also say that

a Dirac structure is any subspace of maximal dimension satisfying power-continuity.

Kernel and image representations (I)

- Every Dirac structure $\mathcal{D}$ admits a *kernel representation*

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid Ff + Ee = 0\},$$

  for linear maps $F : \mathcal{F} \to \mathcal{V}$ and $E : \mathcal{F}^* \to \mathcal{V}$ satisfying
  1. $EF^* + FE^* = 0$,
  2. rank $(F + E) = \dim \mathcal{F}$,

  where $\mathcal{V}$ is a vector space with the same dimension as $\mathcal{F}$.
- Adjoint linear maps $F^* : \mathcal{V}^* \to \mathcal{F}^*$ and $E^* : \mathcal{V}^* \to \mathcal{F}$ are defined by

$$\langle f | F^* u \rangle = \langle Ff | u \rangle, \; \langle e | E^* u \rangle = \langle Ee | u \rangle,$$

  for all $f \in \mathcal{F}, e \in \mathcal{F}^*, u \in \mathcal{V}^*$.

Kernel and image representations (II)

- Given a basis $\{f_1, \dots, f_n\}$ in $\mathcal{F}$, the corresponding dual basis $\{e_1, \dots, e_n\}$ in $\mathcal{F}^*$, and any basis in $\mathcal{V}$, with $\dim \mathcal{V} = m \geq n$, the linear maps F and E are represented by $m \times n$ matrices (which we denote by the same symbol as the maps) satisfying
  1. $EF^T + FE^T = 0$,
  2. rank $[F \mid E] = n$.
- Dirac structures can also be given by means of an *image representation*

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid \exists u \in \mathcal{V}^* \text{ s.t. } f = E^* u, \; e = F^* u\}$$

  with the same maps and spaces of the kernel representation.
- If $\dim \mathcal{V} > \dim \mathcal{F}$ one has a *relaxed kernel representation* or a *relaxed image representation*.

Kernel and image representations (III)

To proof the existence of the kernel and image representations we will work in some given basis. Let $n = \dim \mathcal{F}$, $m = \dim \mathcal{V}$, $m \geq n$. Since $\mathcal{D}$ is a subspace of dimension n of $\mathcal{F} \times \mathcal{F}^*$, there exist $m \times n$ matrices E and F such that

$$\mathcal{D} = \mathrm{Im} \begin{bmatrix} E^T \\ F^T \end{bmatrix}$$

with rank $[F|E] = n$, which means that any element $(f, e) \in \mathcal{D}$ can be written as $f = E^T u$, $e = F^T u$ for a certain $u \in \mathcal{V}^*$. From $\langle e|f \rangle = 0$, one gets $u^T F E^T u = 0$ for any $u \in \mathcal{V}^*$ or

$$EF^T + FE^T = 0.$$

Conversely, any subset defined by the kernel representation is a Dirac structure, since it satisfies $\langle e|f \rangle = 0$ and has the appropriate dimension.

Kernel and image representations (IV)

Consider again the Dirac structure of the exemple. It can be written as

$$\begin{aligned} f &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E^T u, \\ e &= \begin{pmatrix} -\beta \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = F^T u. \end{aligned}$$

from which

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and all the properties *i.e.* rank$[F|E] = 2$, $EF^T + FE^T = 0$ and $Ff + Ee = 0$, can be checked out.

Kirchhoff laws (I)

Let $\mathcal{C}$ be a circuit with N nodes and B branches, and let us consider its associated digraph. The incidence matrix D is an $N \times B$ matrix whose elements are given by

$$d_\alpha^j = \begin{cases} +1 & \text{if branch } j \text{ is anti-incident on node } \alpha, \\ -1 & \text{if branch } j \text{ is incident on node } \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

with $\alpha = 1, \ldots, N$ and $j = 1, \ldots, B$.
In terms of D, Kirchhoff laws for the admissible currents $i \in \mathbb{R}^b$ and the admissible voltage drops $v \in (\mathbb{R}^B)^* = \mathbb{R}^B$ are formulated as

$$Di = 0, \quad v \in \mathrm{Im}\ D^T.$$

Kirchhoff laws (II)

Let us define $\mathcal{F} = \mathbb{R}^B$, $\mathcal{V} = \mathbb{R}^N$ and

$$\mathcal{K} = \{(i, v) \in \mathbb{R}^b \times \mathbb{R}^B \mid Di = 0,\ v \in \text{Im } D^T\}.$$

This defines a Dirac structure. Indeed, writing $v = D^T u$ for some $u \in \mathbb{R}^N$ one gets (D2), which is known as Tellegen theorem in circuit theory:

$$\langle v|i\rangle = v^T i = u^T Di = u^T \cdot 0 = 0.$$

Furthermore, since D is a matrix representing a linear map from $\mathbb{R}^B$ to $\mathbb{R}^N$,

$$\dim \text{Ker} D + \dim \text{Im} D = B$$

which, upon using $\dim \text{Im} D^T = \dim \text{Im} D$, yields (D3).

Port-Hamiltonian systems (I)

- Consider a lumped-parameter physical system defined on a manifold $\mathcal{M}$, with local coordinates $x \in \mathbb{R}^n$.
- The total energy of the system is given by the Hamiltonian $H(x)$, and the system is assumed to have m open ports.
- For each x we consider $\mathcal{F}_x = T_x\mathcal{M} \times \mathbb{R}^m$, $\mathcal{F}_x^* = T_x^*\mathcal{M} \times \mathbb{R}^m$, and define a smooth Dirac structure $\mathcal{D}(x) \subset \mathcal{F}_x \times \mathcal{F}_x^*$.
- It is assumed that the Dirac structure varies smoothly over $\mathcal{M}$, which means, essentially, that the matrices of any given representation are smooth functions of x.
- We will write the elements of $\mathcal{F}_x \times \mathcal{F}_x^*$ as (f_x, f_b, e_x, e_b), where b stands for "boundary", and the f_x and e_x are power variables associated to state ports, *i.e.* ports connected to energy storing elements.

Port-Hamiltonian systems (II)

- A port-Hamiltonian system (PHS) on $\mathcal{M}$ is defined by

$$(-\dot{x}, f_b, \partial_x H, e_b) \in \mathcal{D}(x) \; \forall \; x \in \mathcal{M}.$$

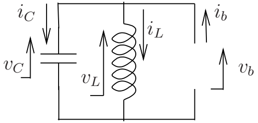
- This is, in general, a set of differential and algebraic equations (DAE), and may include the definition of some boundary variables as inputs or outputs.
- The minus sign in $f_x = -\dot{x}$ is consistent with the common convention that power flows from the boundary ports into the system and from the internal network into the energy storing elements:

$$0 = \langle e|f \rangle = \langle e_x|f_x \rangle + \langle e_b|f_b \rangle = -\partial_x H \dot{x} + e_b^T f_b,$$

from which $\dot{H} = e_b^T f_b$.
- Source or dissipative terms can be added to the system through the boundary ports.

Port-Hamiltonian systems (III)

- The following system has $\mathcal{M} = \mathbb{R}^2$ and a single port:



- The vector space $\mathcal{F}$ is in this case $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ and the Dirac structure in $\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$ is provided by Kirchhoff laws

$$\begin{aligned} v_b &= v_C, \\ v_b &= v_L, \\ i_b &= i_C + i_L. \end{aligned}$$

Port-Hamiltonian systems (IV)

- The resulting subspace has dimension $6-3=3$ and a kernel representation is given by

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix}}_{F_K} \begin{pmatrix} -i_C \\ i_L \\ i_b \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}}_{E_K} \begin{pmatrix} v_C \\ -v_L \\ v_b \end{pmatrix} = 0.$$

- One has rank $[F_K|E_K] = 3$ and $E_K F_K^T + F_K E_K^T = 0$.
- The state variables of the system are the charge q and the linked magnetic flux ϕ, with flows $i_C = \dot{q}$, $v_L = \dot{\phi}$.

Port-Hamiltonian systems (V)

- Since we have to put all the flows $\dot{x}$ into the f vector, the above result does not conform to the PHS formalism that we have presented.
- We should exchange v_l and i_L, which corresponds to swapping the second columns of E_K and F_K. To this end, the following trivial result is useful:
- Consider a kernel representation given by E, F, and a given index l, and define

$$\hat{E}_{ij} = \begin{cases} E_{ij} & j \neq l, \\ F_{ij} & j = l \end{cases}, \quad \hat{F}_{ij} = \begin{cases} F_{ij} & j \neq l, \\ E_{ij} & j = l \end{cases}, \quad \text{for all } i.$$

Then $\hat{E}$, $\hat{F}$ is also a kernel representation of the Dirac structure.

Port-Hamiltonian systems (VI)

- Using this result we obtain the representation

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}}_{F} \underbrace{\begin{pmatrix} -i_C \\ -v_L \\ i_b \end{pmatrix}}_{f} + \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} v_C \\ i_L \\ v_b \end{pmatrix}}_{e} = 0.$$

- The replacement

$$i_C = \dot{q},\ v_L = \dot{\phi},\ v_C = \partial_q H(q,\phi),\ i_L = \partial_\phi H(q,\phi)$$

results in the expected equations $\dot{q} = i_b - \partial_\phi H$, $\dot{\phi} = \partial_q H$, $v_b = \partial_q H$, with i_b and v_b the natural input and output, respectively.

Port-Hamiltonian systems (VII)

- Consider a kernel representation of a PHS

$$F(x)\begin{pmatrix}-\dot{x}\\ f_b\end{pmatrix}+E(x)\begin{pmatrix}\partial_x H\\ e_b\end{pmatrix}=0.$$

- As said before, this is generally a DAE, that is, not all the x are true state variables and the dynamics lives in a submanifold of $\mathcal{M}$. It will not be a DAE if the $\dot{x}$ can be explicitly solved for.
- Assume that F and E can be split (non necessarily in a unique way) as

$$F=(F_x\ F_{b1}\ F_{b2}),\quad E=(E_x\ E_{b1}\ E_{b2})$$

with rank $F_x=n$ and rank $(F_x\ F_{b1}\ E_{b2})=n+m$.

Port-Hamiltonian systems (VIII)

- Separating correspondingly the components of f_b and e_b

$$(F_x \;\; F_{b1} \;\; E_{b2}) \begin{pmatrix} -\dot{x} \\ f_{b1} \\ e_{b2} \end{pmatrix} + (E_x \;\; E_{b1} \;\; F_{b2}) \begin{pmatrix} \partial_x H \\ e_{b1} \\ F_{b2} \end{pmatrix} = 0.$$

- Setting $\tilde{F} = (F_x \;\; F_{b1} \;\; E_{b2})$, $\tilde{E} = (E_x \;\; E_{b1} \;\; F_{b2})$, $y = (f_{b1}^T e_{b2}^T)^T$, $u = (e_{b1}^T f_{b2}^T)^T$ this can be written as

$$\tilde{F} \begin{pmatrix} -\dot{x} \\ y \end{pmatrix} + \tilde{E} \begin{pmatrix} \partial_x H \\ u \end{pmatrix} = 0,$$

with $\tilde{F}\tilde{E}^T + \tilde{E}\tilde{F}^T = 0$ but now rank $\tilde{F} = n + m$, so that it is invertible.

Port-Hamiltonian systems (IX)

- Operating with $\tilde{F}^{-1}$

$$\begin{pmatrix} -\dot{x} \\ y \end{pmatrix} = -\tilde{F}^{-1}\tilde{E}\begin{pmatrix} \partial_x H \\ u \end{pmatrix}.$$

- Because of $\tilde{F}\tilde{E}^T + \tilde{E}\tilde{F}^T = 0$, $\tilde{F}^{-1}\tilde{E}$ is skew-symmetric:

$$\tilde{E}^T(\tilde{F}^{-1})^T = -\tilde{F}^{-1}\tilde{E}.$$

- Writing, with g arbitrary and J and B both skew-symmetric,

$$\tilde{F}^{-1}(x)\tilde{E}(x) = \begin{pmatrix} J(x) & g(x) \\ -g^T(x) & -B(x) \end{pmatrix}$$

we obtain an explicit input/output PHS model

$$\boxed{\begin{aligned} \dot{x} &= J(x)\partial_x H(x) + g(x)u, \\ y &= g^T(x)\partial_x H(x) + B(x)u. \end{aligned}}$$

Port-Hamiltonian systems (X)

- We split the ports into open ports (u, y) and resistive-terminated ports (u_r, y_r), and write accordingly

$$
\begin{aligned}
\dot{x} &= J\partial_x H + gu + g_r u_r, \\
y &= g^T \partial_x H + Bu, \\
y_r &= g_r^T \partial_x H,
\end{aligned}
$$

  where some possible extra terms in u, u_r for y and y_r have been omitted in order to simplify the discussion.

- Ports (u_r, y_r) are now terminated by means of linear resistors

$$
u_r = -R_r y_r, \quad R_r = R_r^T, \; R_r \geq 0.
$$

  The minus sign is due to the fact that $u_r^T y_r$ is the power flowing from the port, and this cannot be positive for a true dissipation.

Port-Hamiltonian systems (XI)

- One has

$$u_r = -R_r y_r = -R_r g_r^T \partial_x H$$

and hence $\dot{x} = J\partial_x H + gu - g_r R_r g_r^T \partial_x H$.

- Collecting the terms one gets the expression for an explicit input/output PHS with dissipation

$$\begin{aligned} \dot{x} &= (J(x) - R(x))\partial_x H(x) + g(x)u, \\ y &= g^T(x)\partial_x H(x) + B(x)u, \end{aligned}$$

where

$$R(x) = g_r(x) R_r g_r^T(x)$$

satisfies $R^T = R$ and $R^T \geq 0$.

Port-Hamiltonian systems (XII)

- The addition of dissipation can be done even if the PHS is not in explicit form. If one considers open ports (e_b, f_b) and resistive ports (e_r, f_r), the dissipative PHS is defined as

$$(-\dot{x}, f_b, f_r, \partial_x H, e_b, e_r) \in \mathcal{D}(x) \; \forall \; x \in \mathcal{M}.$$

- One can terminate the dissipative ports with a general nonlinear relationship $e_r = -F_r(f_r)$, with F_r a function such that $-f_r^T e_r = f_r^T F_r(f_r) \geq 0$.
- Then, using $e^T f = 0$,

$$\dot{H} = \partial_x H \dot{x} = e_b^T f_b + e_r^T f_r = e_b^T f_b - f_r^T F_r(f_r) \leq e_b^T f_b.$$

- If $H(x)$ is bounded from below, it follows from this that the energy that can be extracted from the system, $-\int_0^t e_b^T f_b$, is bounded, *i.e.* the system is passive.

Exercises

1. Let $\mathcal{F} = \mathbb{R}^2$, $\mathcal{F}^* = \mathbb{R}^2$, $\mathcal{F} \times \mathcal{F}^* = \mathbb{R}^4$. Find out which of the following subspaces define Dirac structures:
   1. $\mathcal{D}_1 = \text{span}\{(0,1,1,0)^T, (0,-1,1,0)^T\}$.
   2. $\mathcal{D}_2 = \text{span}\{(0,0,1,0)^T, (0,0,0,1)^T\}$.
   3. $\mathcal{D}_3 = \text{span}\{(1,0,1,0)^T, (0,-1,0,0)^T\}$.
2. For arbitrary finite-dimensional $\mathcal{F}$, show that the following are Dirac structures, and find corresponding kernel and image representations:
   1. $\mathcal{D}_e = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* \mid e = 0\}$.
   2. $\mathcal{D}_f = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* \mid f = 0\}$.
   3. $\mathcal{D}_J = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* \mid f = Je\}$, $J = -J^*$.

Exercises (cont'd)

3. Prove the equivalence of the kernel and image representations, that is, if E and F satisfy $EF^T + FE^T = 0$, rank $(F \;\; E) = n$ then

$$Ee + Ff = 0 \iff \exists \, \mu \in \mathbb{R}^n \text{ s.t. } e = F^T\mu, \;\; f = E^T\mu$$

for $f \in \mathbb{R}^n$, $e \in \mathbb{R}^n$.

4. Prove the following extremely useful linear algebra result

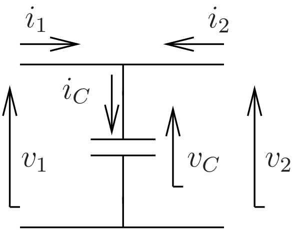
$$Ax = b \iff A^T\alpha = 0 \Rightarrow \alpha^T b = 0.$$

Hint: use one of the orthogonal decompositions associated to any matrix $A \in \mathbb{R}^{m\times n}$ ($\mathcal{R}$ denotes the image or column space and $\mathcal{N}$ the kernel)

$$\mathcal{R}(A) \oplus \mathcal{N}(A^T) = \mathbb{R}^m, \;\; \mathcal{R}(A^T) \oplus \mathcal{N}(A) = \mathbb{R}^n.$$

Exercises (cont'd)

5. Find the kernel and image representations of the Dirac structure associated to Kirchhoff laws.
   Hint: Consider $\bar{D}$, a matrix with any $N-1$ rows of D (which are then independent), and the matrix N made of the $B-(N-1)$ independent vectors of $\mathcal{N}(D)$.
6. Obtain the Dirac structure and the PHS model of the following circuit (assume an arbitrary $H(q)$ for the capacitor).



Exercises (cont'd)

7. Prove that if F is invertible and satisfies $FE^T + EF^T = 0$, then $G = F^{-1}E$ is skew-symmetric, *i.e.* $G^T = -G$.
8. Obtain the Dirac structure and the PHS model of the system formed by a point mass m under the action of a linear spring k and an external force F.
9. Write down the explicit PHS model with dissipation of the system obtained when the second port of the circuit of Exercise 6 is terminated with a linear resistor r.
10. Obtain the Dirac structure and the PHS model of the system formed by two masses m_1 and m_2 attached rigidly together by a massless rod and under the action of external forces F_1 and F_2, respectively.

Exercises (cont'd)

11. Let $R \geq 0$ and let A be any matrix of appropriate dimensions. Prove that $ARA^T \geq 0$. Remember that a matrix B is $B \geq 0$ if $x^T B x \geq 0$ for all x.
12. Consider a PHS system (with or without dissipation) with Hamiltonian $H(x)$ and open ports (e_b, f_b). Prove that, if $H(x) \geq a$, then the energy that can be extracted from the system in $[t_0, t]$

$$-\int_{t_0}^{t} e_b^T(\tau) f_b(\tau) \, \mathrm{d}\tau$$

is bounded. Construct a physical example where the bounded from below condition is not fulfilled.