Generalized discontinuous conduction modes in the complementarity formalism

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Introduction

Modeling of variable structure systems using complementary variables.

Linear complementarity systems with sources:

$$\begin{aligned} \dot{x} &= Ax + Bu + E, \quad x \in \mathbb{R}^n \\ y &= Cx + Du + F, \quad y, u \in \mathbb{R}^p \\ & 0 \leq y \perp u \geq 0 \end{aligned}$$

Since y is linked to x, the evolution of x may force y to violate $y \geq 0$ after reaching $y = 0$.

To avoid this, u might be forced to *jump* from 0 to > 0 to change the dynamics of x.

In turn, this can keep $y = 0$ for a while until u returns to 0.

$$\begin{aligned} \dot{x} &= Ax + Bu + E, \quad x \in \mathbb{R}^n \\ y &= Cx + Du + F, \quad y, u \in \mathbb{R}^p \\ & 0 \leq y \perp u \geq 0 \end{aligned}$$

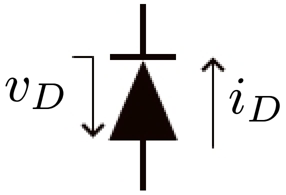
If $D = 0$, and under suitable rank conditions on C, this will force x (or a combination of components) to a constant value, effectively reducing the dynamics.

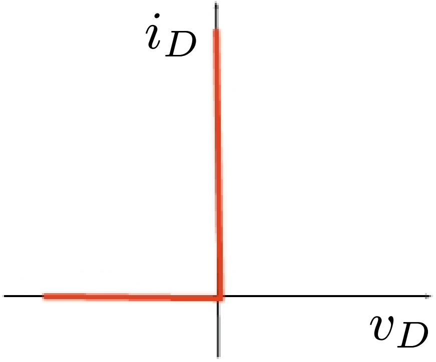
This is called discontinuous conduction mode in the power converter literature, due to the fact that a current remains fixed to zero for a while.

Since this may involve other kind of variables, we call this Generalized Discontinuous Conduction Mode (GDCM).

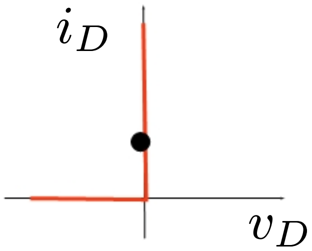
Characteristic curve of an ideal diode

$$0 \leq -v_D \perp i_D \geq 0$$



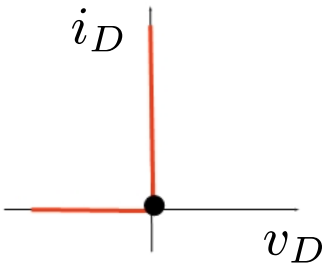


Assume the diode is conducting,
i.e. $i_D > 0$ and hence $v_D = 0$:

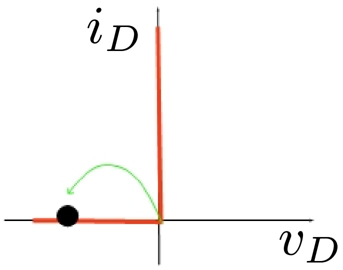


This means that the diode is acting as a (zero)voltage source for the rest of the system.

Suppose that i_D evolves to zero,
and would go below zero if $v_D = 0$:



This may force a jump in $-v_D$,
and i_D will remain at zero until $-v_D$ returns to zero:



Now the diode is acting as a (zero)current source.

However, whether the jump happens to occur or not depends on the state of the system to which the diode is connected.

Although special reasonings can be done for diode circuits, more general results can be obtained applying the theory in A. van der Schaft & J. Schumacher, *Complementarity modeling of hybrid systems*, IEEE Trans. Automat. Control **43**, pp. 483-490, 1998, generalizing previous work of the same authors together with M. Çamlibel and W. Heemels.

A series of dynamical complementarity problems (DCP) must be solved.

Each DCP may involve the solution of an LCP. For systems with a single diode (or with a single pair of complementary variables, *i.e.* $p = 1$), this can be done analytically.

The DCP algorithm assumes right-analyticity of solutions, *i.e.* a quantity can be computed in $(t_0, t_0 + \epsilon)$ for some $\epsilon > 0$ if the quantity and all its time derivatives are known at t_0.

The DCP algorithm allows the computation of right-smooth continuations from an initial condition.

The DCP algorithm

Consider a couple of complementary variables, u and y, and let

$$\begin{array}{rcl}\mathcal{U} & \equiv & (u^{(0)}, u^{(1)}, u^{(2)}, \ldots), \\ \mathcal{Y} & = & (y^{(0)}, y^{(1)}, y^{(2)}, \ldots)\end{array}$$

denote the values of u and y and their successive right-time derivatives at $t = t_0$.

To ensure $0 \leq u \perp y \geq 0$ on $[t_0, t_0 + \epsilon)$, one of the following must be true

$$\mathcal{U} \succeq 0 \text{ and } \mathcal{Y} = 0, \qquad (I)$$

or

$$\mathcal{Y} \succeq 0 \text{ and } \mathcal{U} = 0, \qquad (J)$$

where $\succeq$ means lexicographic nonnegativity, *i.e.* all the terms are zero or the first nonzero term is positive.

If the terms of the sequence are the Taylor coefficients of an analytic function, lexicographic nonnegativity ensures nonnegativity of the function in an open interval.

We will also consider finite sequences

$$\begin{aligned}\mathcal{U}_k &\equiv (u^{(0)}, u^{(1)}, \ldots, u^{(k)}),\\ \mathcal{Y}_k &\equiv (y^{(0)}, y^{(1)}, \ldots, y^{(k)}).\end{aligned}$$

A pair of sequences, finite or not, satisfying either (I) or (J), will be called *valid*.

Consider a dynamical system of the form

$$\dot{x} = f(x, z) + au, \qquad (X)$$

$$\dot{z} = g(x, z, u), \qquad (Z)$$

$$y = \beta x + \alpha, \qquad (Y)$$

with $x \in \mathbb{R}$, $z \in \mathbb{R}^{n-1}$, $a \neq 0$, $\beta \neq 0$, and where $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are complementary variables, $0 \leq u \perp y \geq 0$.

This is a relative degree $\rho = 1$ system, and from (X), (Z) and (Y) one can compute the equations linking the values of u, y and their successive time derivatives at $t = t_0$.

$$\dot{x} = f(x,z) + au, \dot{z} = g(x,z,u),$$
$$y = \beta x + \alpha,$$

$$y^{(0)} = \beta x_0 + \alpha \equiv \beta\gamma_0, \tag{$Y0$}$$
$$y^{(1)} = \beta f(x_0, z_0) + a\beta u^{(0)} \equiv \beta\gamma_1 + a\beta u^{(0)}, \tag{$Y1$}$$
$$\begin{aligned} y^{(2)} &= \beta\partial_x f(x_0,z_0)(f(x_0,z_0) + au^{(0)}) \\ &+\beta\partial_z f(x_0,z_0)g(x,z,u^{(0)}) + a\beta u^{(1)} \\ &\equiv \beta\gamma_2 + a\beta u^{(1)}, \end{aligned} \tag{$Y2$}$$

and, in general,

$$y^{(k)} = \beta\gamma_k + a\beta u^{(k-1)}, \tag{Yk}$$

where γ_k depends on x_0, z_0 and the time derivatives of u at $t = t_0$ up to order $k-2$.

DCP(k) consists in finding valid sequences $\mathcal{U}_k$ and $\mathcal{Y}_k$ satisfying relations $y^{(0)} = \beta\gamma_0$, $y^{(1)} = \beta\gamma_1 + a\beta u^{(0)}$, …, up to $y^{(k)} = \beta\gamma_k + a\beta u^{(k-1)}$.

DCP(k) may have many solutions; for instance, $u^{(0)}$ does not appear in $y^{(0)} = \beta\gamma_0$, and hence it is free, apart from being non-negative, for DCP(0).

Since the conditions of DCP(k) are a subset of those of DCP($k+1$), the solutions of DCP($k+1$) must be chosen among those of DCP(k). This is called the *nesting property* of the DCP.

We say that the system is in mode J_k if *all* the solutions of DCP(k) satisfy $\mathcal{Y}_k \succ 0$, and that the system is in mode I_k if *all* the solutions of DCP(k) verify $\mathcal{U}_k \succ 0$; otherwise, the system is said to be in mode K_k.

Due to the nesting property, if the system is in mode J_k (I_k) it will be in mode J_l (resp. I_l) for any $l > k$.

We will assume that

1. $y^{(0)} \geq 0$,
2. $a\beta > 0$.

These conditions ensure in our case existence and uniqueness of smooth solutions starting from x_0, z_0 (theorems 3.1 and 3.2 of van der Schaft & Schumacher paper).

Then

DCP(0): Find a valid pair $(u^{(0)})$, $(y^{(0)})$ such that $y^{(0)} = \beta\gamma_0$ holds.

Two situations are possible:

Case 1: $y^{(0)} > 0$. This forces $u^{(0)} = 0$. The system is in mode J_0.
Case 2: $y^{(0)} = 0$. In this case $u^{(0)} \geq 0$ is still free. The system is in mode K_0.

DCP(1): Find a valid pair $(u^{(0)}, u^{(1)})$, $(y^{(0)}, y^{(1)})$ such that $y^{(0)} = \beta\gamma_0$ and $y^{(1)} = \beta\gamma_1 + a\beta u^{(0)}$ hold.

Since the conditions of DCP(0) are a subset of these, we start with the solutions obtained there.

Case 1: Since $y^{(0)} > 0$ we must have $u^{(1)} = 0$ for the pair to be valid. One also gets $y^{(1)} = \beta\gamma_1$. Notice that the sign of $y^{(1)}$ does not matter since already $y^{(0)} > 0$. The system is in mode J_1.

Case 2: In this case $y^{(1)} = \beta\gamma_1 + a\beta u^{(0)}$ becomes a 1-dimensional LCP for $y^{(1)}$, $u^{(0)}$, which always has solution due to $a\beta > 0$.

Three subcases are possible.

Case 2.1: $\beta\gamma_1 > 0$. The only solution to the LCP is $u^{(0)} = 0$, $y^{(1)} = \beta\gamma_1 > 0$; one must choose $u^{(1)} = 0$ and the system is in mode J_1.

Case 2.2: $\gamma_1 = 0$. Now we have $u^{(0)} = 0$, $y^{(1)} = 0$; any $u^{(1)} \geq 0$ is valid and the system is in mode K_1.

Case 2.3: $\beta\gamma_1 < 0$. The solution is $u^{(0)} = -\frac{\gamma_1}{a} = -\frac{\beta\gamma_1}{a\beta} > 0$, $y^{(1)} = 0$; any $u^{(1)} \in \mathbb{R}$ is valid and the system is in mode I_1.

DCP(2): Find a valid pair $(u^{(0)}, u^{(1)}, u^{(2)})$, $(y^{(0)}, y^{(1)}, y^{(2)})$ such that $y^{(0)} = \beta\gamma_0$, $y^{(1)} = \beta\gamma_1 + a\beta u^{(0)}$ and $y^{(2)} = \beta\gamma_2 + a\beta u^{(1)}$ hold.

As is the transition to DCP(0) to DCP(1), solutions coming from modes I_1 or J_1 will yield solutions in I_2 and J_2, respectively, so the only case worth studying is 2.2, for which now $y^{(2)} = \beta\gamma_2 + a\beta u^{(1)}$ is an LCP for $y^{(2)}$, $u^{(1)}$.

Again this always has solution due to $a\beta > 0$, and three situations can be encountered.

Case 2.2.1: $\beta\gamma_2 > 0$. The solution is $u^{(1)} = 0$, $y^{(2)} = \beta\gamma_2 > 0$; one must choose $u^{(2)} = 0$ and the system is in mode J_2.

Case 2.2.2: $\gamma_2 = 0$. Now $u^{(1)} = 0$, $y^{(2)} = 0$; any $u^{(2)} \geq 0$ is valid and the system is in mode K_2.

Case 2.2.3: $\beta\gamma_2 < 0$. The solution is $u^{(1)} = -\frac{\gamma_2}{a} = -\frac{\beta\gamma_2}{a\beta} > 0$, $y^{(2)} = 0$; any $u^{(2)} \in \mathbb{R}$ is valid and the system is in mode I_2.

Successive DCPs can be solved, and we assume that after a finite number of steps the system ends up in a J or I mode.

The general case can be summarized as follows:

Proposition 1. Under the conditions of the preceding discussion,

- ★ if the above procedure enters a J mode for the first time when solving DCP(k), then $u^{(l)} = 0$ for all $l \in \mathbb{N}$, $y^{(l)} = 0$ for $l = 0, \ldots, k-1$, and $y^{(k)} = \beta\gamma_k > 0$.
- ★ if the above procedure enters a I mode for the first time when solving DCP(k), then $y^{(l)} = 0$ for all $l \in \mathbb{N}$, $u^{(l)} = 0$ for $l = 0, \ldots, k-2$, and $u^{(k-1)} = -\frac{\gamma_k}{a} > 0$.

Now assume that $y > 0$ before t_0. This means that all the derivatives of u are zero on an open interval to the left of t_0. Then

Proposition 2. Assume that $y(t) > 0$, and hence $u(t) = 0$, for $t \in (t_0 - \tilde{\epsilon}, t_0)$, for some $\tilde{\epsilon} > 0$. Then, if for some $k \geq 1$,

$$\gamma_0 = 0, \text{ and } \gamma_l = 0, \forall l = 1, \ldots, k-1, \text{ but } \beta\gamma_k < 0 \tag{GDCMC}$$

one has that

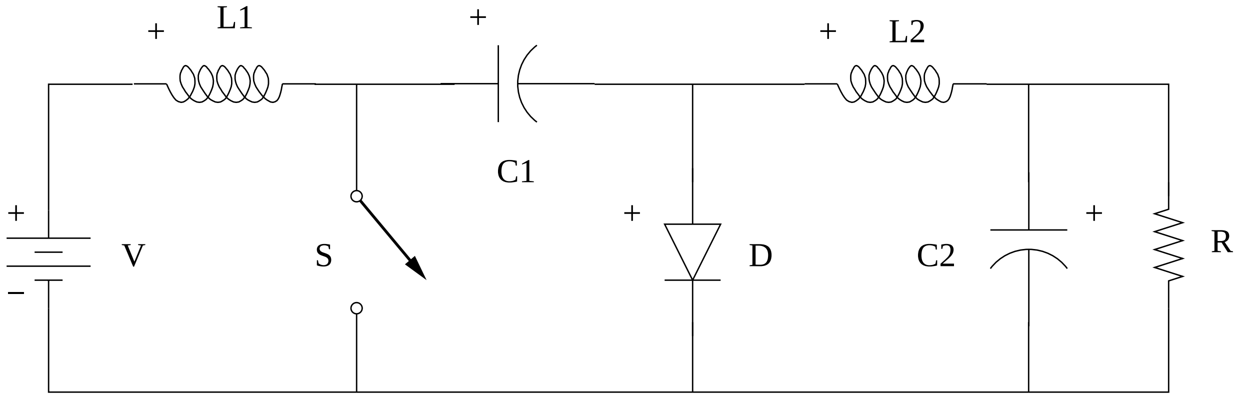
- ★ the $(k-1)$th time derivative of u has a jump at t_0, going from 0 to $-\gamma_k/a > 0$.
- ★ there exists $\epsilon > 0$ such that $y(t) = 0$, $u(t) > 0$ for $t \in (t_0, t_0 + \epsilon)$, and hence we have a GDCM.

We call (GDCMC) the kth order Generalized Discontinuous Conduction Mode Condition.

Proposition 3. Assume $y(t) > 0$ for $t \in (t_0 - \tilde{\epsilon}, t_0)$, for some $\tilde{\epsilon} > 0$, and that $y(t_0) = 0$ and $\beta\gamma_1 < 0$. Then u has a discontinuity at $t = t_0$, from 0 to $-\gamma_1/a > 0$, and there exists $\epsilon > 0$ such that $y(t) = 0$ and $u(t) > 0$ for $t \in (t_0, t_0 + \epsilon)$.

The situation presented in Proposition 3 is the one normally encountered both in simulation and in experiment; higher order GDCM conditions are difficult to meet, since they require several state space quantities to be zero simultaneously.

Application to the Cuk converter



State variables:

$$
\begin{aligned}
x_1 &= i_{L1}\\
x_2 &= i_{L2}\\
x_3 &= v_{C1}\\
x_4 &= v_{C2}
\end{aligned}
$$

Complementary variables:

$$
\begin{aligned}
u_1 &= -v_D\\
u_2 &= i_S\\
y_1 &= i_D = x_1 - x_2 - u_2\\
y_2 &= v_S = x_3 - u_1
\end{aligned}
$$

$$
\begin{aligned}
&\dot{x} = Ax + Bu + EV\\
&y = Cx + Du + FV\\
&0 \leq y_1 \perp u_1 \geq 0\\
&y_2 \perp u_2
\end{aligned}
$$

For each position of S one can write the system equations in terms of the single complementarity pair y_1, u_1.

We will obtain the first order GDCMC for the two positions of the switch of the Čuk converter. Setting $v_S = 0$ or $i_S = 0$ one gets the following two LCS.

S closed ($v_S = 0$)

$$\begin{array}{rcl} L_1\dot{x}_1 & = & V, \\ L_2\dot{x}_2 & = & -x_3 - x_4, \\ C_1\dot{x}_3 & = & x_2 + i_D, \\ C_2\dot{x}_4 & = & x_2 - \dfrac{1}{R}x_4, \\ x_3 & = & -v_D. \end{array}$$

We apply Proposition 3 with $x = x_3$, $z = (x_1, x_2, x_4)$, $u = i_D$, $y = -v_D$, $f(x, z) = x_2/C_1$, $\alpha = 0$, $\beta = 1$, $a = 1/C_1$. The first order GDCMC is given by

$$x_3(t_0) = 0, \quad \gamma_1 = \frac{1}{C_1}x_2(t_0) < 0.$$

This implies $x_2(t_0) < 0$. Notice that it is a voltage which gets fixed to zero for a while.

On $x_3(t) = 0$ one can prove that u obeys

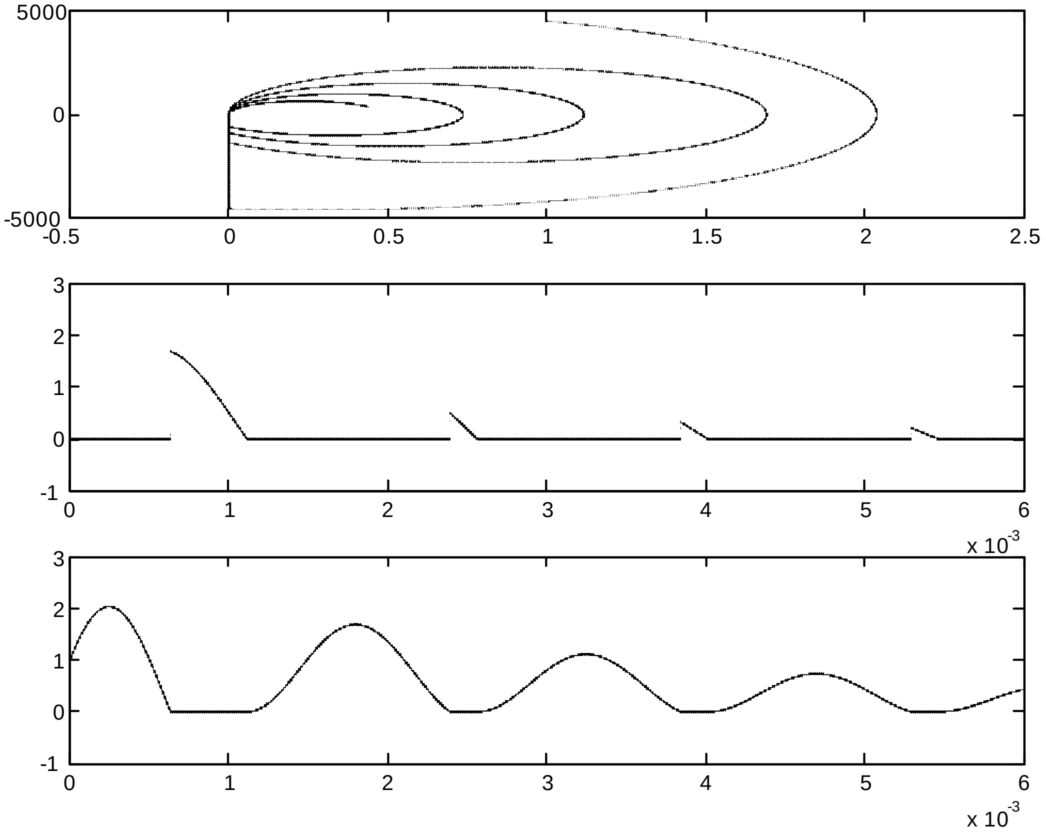
$$\ddot{u} + \frac{1}{RC_2}\dot{u} + \frac{1}{L_2C_2}u = 0$$

which has oscillating solutions for usual values of the parameters. Hence u reaches zero in finite time and the GDCM disappears.

Simulations can be done using a backwards Euler scheme in `Matlab`.

Parameter values (in SI units) : $L_1 = 750 \cdot 10^{-6}$, $L_2 = 800 \cdot 10^{-6}$, $C_1 = 220 \cdot 10^{-6}$, $C_2 = 130 \cdot 10^{-6}$, $R = 10$, $V = 24$.

Initial conditions: $x(0) = (2, 1, 1, 1)$.
Fixed integration step: $h = 10^{-6}$.



$$x_3(t_0) = 0$$

$$\gamma_1 = \frac{1}{C_1} x_2(t_0) < 0$$

GDCM for the Cuk converter with switch closed. Upper: x_3 on the horizontal axis and Γ (γ_1 at arbitrary time) on the vertical one. Middle: u as a function of time. Lower: y as a function of time. The GDCM has a finite duration, but it is re-entrant for the parameters and initial conditions chosen.

S open $(i_S = 0)$

$$
\begin{aligned}
L_1\dot{x}_1 &= -x_3 - v_D + V, \\
L_2\dot{x}_2 &= -x_4 + v_D, \\
C_1\dot{x}_3 &= x_1, \\
C_2\dot{x}_4 &= x_2 - \frac{1}{R}x_4, \\
i_D &= x_1 - x_2.
\end{aligned}
$$

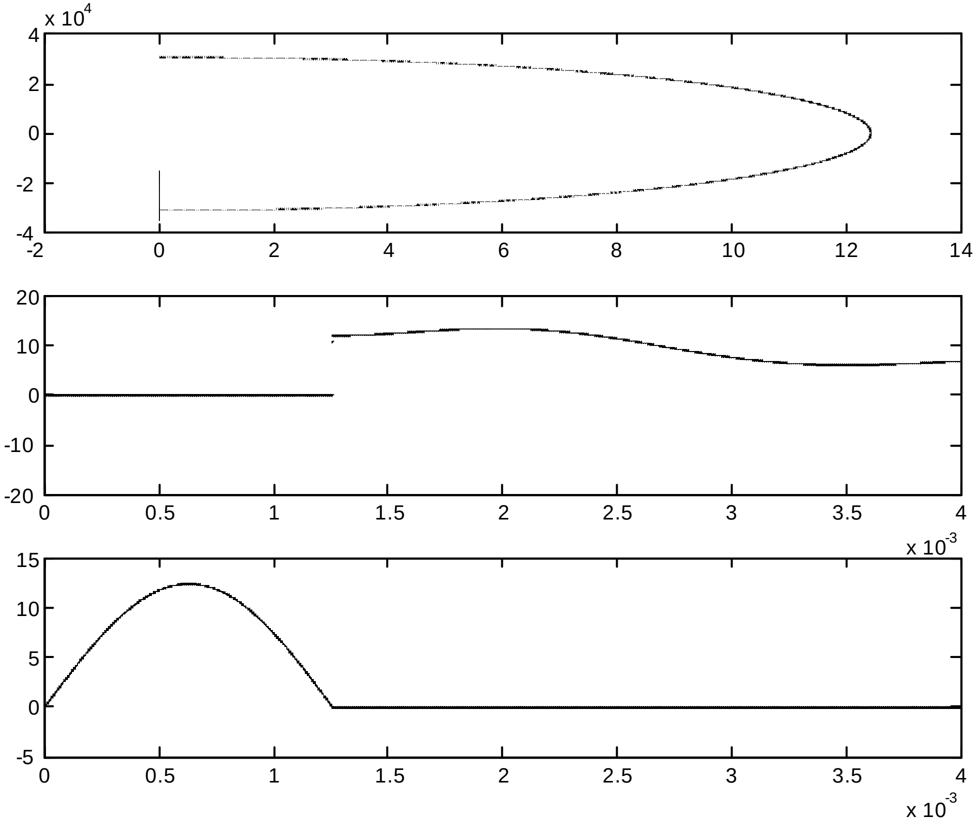
Notice that, if $x = x_1 - x_2$, then

$$\dot{x} = \frac{V}{L_1} - \frac{1}{L_1}x_3 + \frac{1}{L_2}x_4 - \left(\frac{1}{L_1} + \frac{1}{L_2}\right)v_D.$$

Thus, we can apply Proposition 3 with $x = x_1 - x_2$, $z = (x_1 + x_2, x_3, x_4)$, $u = -v_D$, $y = i_D$, $f(x, z) = V/L_1 - x_3/L_1 + x_4/L_2$, $\alpha = 0$, $\beta = 1$, $a = 1/L_1 + 1/L_2$.

The first order GDCMC is given by

$$x_1(t_0) = x_2(t_0), \quad \gamma_1 = \frac{V}{L_1} - \frac{x_3(t_0)}{L_1} + \frac{x_4(t_0)}{L_2} < 0.$$



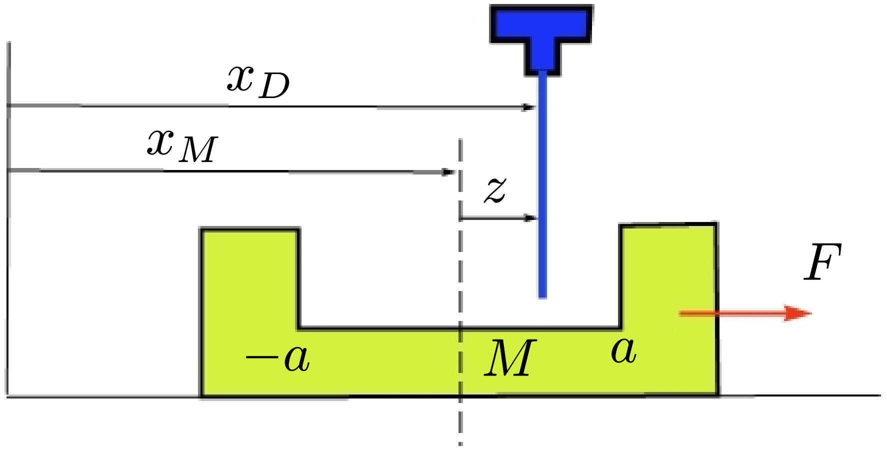
$$x_1(t_0) = x_2(t_0)$$

$$\gamma_1 = \frac{V}{L_1} - \frac{x_3(t_0)}{L_1} + \frac{x_4(t_0)}{L_2} < 0$$

GDCM for the Cuk converter with switch open. Upper: $x_1 - x_2$ on the horizontal axis and Γ on the vertical one. Middle: u as a function of time. Lower: y as a function of time. For the parameters and initial conditions used, the GDCM lasts indefinitely.

The case of several complementarity pairs: a mechanical example

A dead-zone nonlinearity:



$$M\ddot{x}_M = f_{DM}(z) + F(x_M, \dot{x}_M) \equiv v + F$$

The relation between z and v is given by the complementarity pairs

$$0 \leq u_1 \perp y_1 \geq 0, \qquad 0 \leq u_2 \perp y_2 \geq 0.$$

$$\begin{aligned} v &= u_1 - u_2 \\ y_1 &= \frac{a}{2} - \frac{z}{2} \\ y_2 &= \frac{a}{2} + \frac{z}{2} \end{aligned}$$

No contact force. $u_1 = 0$, $u_2 = 0$. Then $v = 0$, $y_1 \geq 0$, and hence $z \leq a$, and $y_2 \geq 0$, and hence $z \geq -a$, so $z \in [-a, a]$.

Left contact. $u_1 = 0$, $y_2 = 0$. Now $v = -u_2 \leq 0$, $z = -a$ and $y_1 = a > 0$.

Right contact. $u_2 = 0$, $y_1 = 0$. Then $v = u_1 \geq 0$, $z = a$ and $y_2 = a > 0$.

Forbidden. $y_1 = 0$, $y_2 = 0$. This implies $z = a$ and $z = -a$, which is impossible since $a > 0$.

Using the complementary variables, the dynamics can be written as

$$\begin{aligned}\dot{x} &= f(x) + Bu \equiv f + gu,\\ y &= Cx + E.\end{aligned}$$

$$f(x) = \begin{pmatrix} x_2 \\ \frac{1}{M}F(x) \end{pmatrix},\; B = \frac{1}{M}\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix},\; C = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix},\; E = \frac{1}{2}\begin{pmatrix} a - x_D \\ a + x_D \end{pmatrix}.$$

This is a uniform relative degree 2 system:

$$\begin{aligned}L_g y &= CB = 0,\\ L_f y &= C\begin{pmatrix} x_2 \\ \frac{1}{M}F(x) \end{pmatrix} = \frac{x_2}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}x,\\ L_g L_f y &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}B = \frac{1}{2M}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},\end{aligned}$$

DCP(0)

$$y^{(0)} = \frac{1}{2}\begin{pmatrix} x_1 + a - x_D \\ -x_1 + a + x_D \end{pmatrix}.$$

Case 1: $y_1 = 0$. Then $x_1 = x_D - a$, $y_2 = a > 0$, $u_2 = 0$, $u_1 \geq 0$ and $v \geq 0$.
The mode distribution is $J_0 = \{2\}$ and $K_0 = \{1\}$.
Contact force still not determined in $(t_0, t_0 + \epsilon)$.

Case 2: $y_2 = 0$. Then $x_1 = x_D + a$, $y_1 = a > 0$, $u_1 = 0$, $u_2 \geq 0$ and $v \leq 0$.
The mode distribution is $J_0 = \{1\}$ and $K_0 = \{2\}$.
Contact force still not determined in $(t_0, t_0 + \epsilon)$.

Case 3: $y_1 > 0$ and $y_2 > 0$. Then $u_1 = 0$, $u_2 = 0$, $v = 0$.
The mode distribution is $J_0 = \{1, 2\}$.
There is no contact for some $(t_0, t_0 + \epsilon)$.

DCP(1)

$$y^{(1)} = L_f y(t_0) + \partial_t y(t_0) = \frac{1}{2}\begin{pmatrix} x_2 - \dot{x}_D \\ -x_2 + \dot{x}_D \end{pmatrix}.$$

Case 3 is trivial, while case 2 is essentially the same as case 1, with complementary pairs interchanged. Hence, we only analyze case 1.

Case 1.1: $x_2 = \dot{x}_D$. Then $y_1^{(1)} = 0$, $y_2^{(1)} = 0$ and the mode distribution is unchanged: $J_1 = \{2\}$, $K_1 = \{1\}$.
Contact force still not determined in $(t_0, t_0 + \epsilon)$.

Case 1.2: $x_2 > \dot{x}_D$. Then $y_1^{(1)} > 0$, $y_2^{(1)} < 0$. This is no problem since $y_2 > 0$.
The mode distribution is $J_1 = \{1, 2\}$.
There is no contact for some $(t_0, t_0 + \epsilon)$.

Case 1.3: $x_2 < \dot{x}_D$. Then $y_1^{(1)} < 0$, $y_2^{(1)} > 0$.
This corresponds to a forbidden initial condition (the driving piece would penetrate the mass).

DCP(2)

$$y^{(2)} = \frac{1}{2}\begin{pmatrix} \frac{F}{M} - \ddot{x}_D \\ -\frac{F}{M} + \ddot{x}_D \end{pmatrix} + \frac{1}{2M}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} u. \quad (*)$$

The only interesting Case is 1.1 (and 2.1, which is symmetric).

Since in Case 1.1 we have $u_2 = 0$, $(*)$ reduces to

$$\begin{aligned} y_1^{(2)} &= \frac{1}{2}\left(\frac{F}{M} - \ddot{x}_D\right) + \frac{1}{2M}u_1, \quad (A1) \\ y_2^{(2)} &= \frac{1}{2}\left(-\frac{F}{M} + \ddot{x}_D\right) - \frac{1}{2M}u_1. \quad (A2) \end{aligned}$$

Equation $(A1)$ is an LCP for $(y_1^{(2)}, u_1)$, while $(A2)$ allows to compute the (irrelevant, since already $y_2 > 0$) value of $y_2^{(2)}$.

The solution to this LCP depends on the value of $F/M - \ddot{x}_D$:

Case 1.1.1: $F/M = \ddot{x}_D$. Then $y_1^{(2)} = 0$, $u_1 = 0$ and the mode distribution is unchanged: $J_2 = \{2\}$, $K_2 = \{1\}$.
Contact force still not determined in $(t_0, t_0 + \epsilon)$.

Case 1.1.2: $F/M - \ddot{x}_D > 0$. Then $u_1 = 0$, $y_1^{(2)} = \frac{1}{2}\left(F/M - \ddot{x}_D\right) > 0$.
The mode distribution is $J_2 = \{1, 2\}$.
There is no contact for some $(t_0, t_0 + \epsilon)$.

Case 1.1.3: $F/M - \ddot{x}_D < 0$. Then $y_1^{(2)} = 0$, $u_1 = -\left(F - M\ddot{x}_D\right) > 0$.
The mode distribution is $J_2 = \{2\}$, $I_2 = \{1\}$.
The contact force becomes active for some $(t_0, t_0 + \epsilon)$**.**

All this, which is rather trivial, makes sense physically.

The procedure can, in principle, be repeated indefinitely, and a branching of cases appears at each DCP(l) depending on the sign of $F^{(l-2)} - Mx_D^{(l)}$.

Notice however that unless $x_D(t)$ is quite special (carefully chosen depending on the initial conditions), it will be impossible to find an initial condition (x_1, x_2) satisfying $F^{(l-2)} - Mx_D^{(l)} = 0$ for arbitrary l (in fact for $l \geq 2$), and hence, the *normal* situation is that K_2 is empty.

Notice that we get a well defined problem at each step in spite of the decoupling matrix

$$D = \frac{1}{2M} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

not being positive definite.

This means that the van der Schaft-Schumacher paper states sufficient, but not necessary, conditions for the DCP algorithm to succeed.

Future work

- Several complementarity models of Coulomb-like friction. Description of stick-slip phenomena as the mechanical equivalent of discontinuous conduction modes.
  - Non solved problems with a singular decoupling matrix.
  - Analytical solution of low-order LCPs.
- Inclusion of impacts?
- Generalizations of van der Schaft-Schumacher theory?
  - In fact, for the linear case, there are many more results.
  - Inclusion of four-quadrant switches into the analysis.