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# Non-Existence, Uniqueness of Limit Cycles and Center Problem in a System That Includes Predator-Prey Systems and Generalized Lienard Equations

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**Abstract**

We study the number of periodic orbits of a family of planar differential equations that includes usual predator-prey systems and generalized Liénard equations. More explicitly, we give conditions for non-existence, uniqueness and hyperbolicity of periodic orbits and also for the existence of centers. The main tool used is a generalization of the Filippov transformation. We apply the results on non-existence of periodic orbits and on the existence of centers to some predator-prey systems found in the literature. The result on uniqueness is applied to some generalized Liénard equations.

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# 1 Introduction

This paper studies the number of limit cycles surrounding the origin of the system:

$$\begin{cases} \frac{dx}{dt} = b(x)\,(F(x) - \varphi(y))\,, \\ \frac{dy}{dt} = a(y)g(x). \end{cases} \tag{1}$$

As far as we know, this model was first introduced by Huang (see [19]) and, as we will see, it includes some well-known predator-prey systems. On the other hand, when $b(x) = 1$, $a(y) = 1$, and $\varphi(y) = y$, system (1) becomes a Liénard system (see [21]). This fact induces new applications since, as it is known, a wide class of predator-prey systems can be transformed into Liénard equations (see [13], [27], [28]). Related to this topic we also remark that another way to study systems of type (1) could be a transformation into a generalized Liénard equation by means of the change given by $Y = \int_0^y \frac{1}{a(u)}\,du$ and $\frac{dt}{d\tau} = \frac{1}{b(x)}$. Huang followed this way, using Zhang's theorem for generalized Liénard equations (see [35]) to study the uniqueness of limit cycles of (1).

Here we give both an uniqueness criterion, based on a theorem of Conti for Liénard systems, and results on non-existence of limit cycles and center problems. All these phenomena have special interest under a biological point of view because the appearance of anyone of them usually implies the coexistence of species. Furthermore, in case of having uniqueness, the hyperbolicity of the limit cycle is relevant. Without having this property, the limit cycle could bifurcate under small perturbations, and the dynamics of the population could be qualitatively modified. If a limit cycle is hyperbolic, it will persist under small $C^1$-perturbations. Hence, the hyperbolicity of a limit cycle for (1) implies the non-appearance of new periodic solutions near it and so, a similar behaviour for any $C^1$-close system, even if it is not of type (1).

In the next list, we provide some examples of predator-prey systems that can be written in the form (1):

I The Lotka-Volterra system (see [24], [33]), where $b(x) = x + x_0$, $F(x) = 0$, $\varphi(y) = \beta y$, $a(y) = y + y_0$, $g(x) = \gamma x$, and $(x_0, y_0)$ is the point $(\frac{\mu}{\gamma}, \frac{\alpha}{\beta})$.

II The Gause model (see [15]), where $b(x) = p(x + x_0)$, $F(x) = \frac{(x+x_0)\alpha}{p(x+x_0)} - y_0$, $\varphi(y) = y$, $a(y) = y + y_0$, $g(x) = -\gamma + q(x + x_0)$, and $(x_0, y_0)$ is the solution of the system $\alpha x - yp(x) = 0$, $-\gamma + q(x) = 0$ in the first quadrant.

III A generalization of (II), given by Kuang and Freedman (see [20]), where $b(x) = p(x + x_0)$, $F(x) = \frac{(x+x_0)h(x+x_0)}{p(x+x_0)}$, $\varphi(y) = \xi(y + y_0)$, $a(y) = \eta(y +$

$y_0)$, $g(x) = -\gamma + q(x + x_0)$, and $(x_0, y_0)$ is the solution of the system $xh(x) - \xi(y)p(x) = 0$, $-\gamma + q(x) = 0$ in the first quadrant.

Observe that in the above examples, the critical point (which is in the first quadrant in usual predator-prey systems) is at the origin. These expressions lead us to consider the following set of hypotheses for (1) in $D = \{(x,y) \in R^2 |\ x \in (x_1, x_2), y \in (y_1, y_2); x_1, y_1 \in R^- \cup \{-\infty\}; x_2, y_2 \in R^+ \cup \{+\infty\}\}$ which will be called *hypothesis (H1)* :

(a) $F(x)$, $g(x)$ and $b(x)$ are $C^1$-functions on the interval $(x_1, x_2)$. On this interval, $xg(x) \geq 0$ (zero only for $x = 0$) and $b(x) > 0$. Furthermore, $F(0) = 0$.

(b) $\varphi(y)$ and $a(y)$ are $C^1$-functions on the interval $(y_1, y_2)$. On this interval, $y\varphi(y) \geq 0$ (zero only for $y = 0$) and $a(y) > 0$.

The structure of this paper is as follows: in Section 2, we give the main statements which will be proved in Section 4. The most useful tool in the proofs is a generalization of the Filippov transformation, that will be studied in Section 3. Section 5 is devoted to apply the results to some examples.

## 2 Main Results

Define the function

$$G(x) := \int_0^x \frac{g(s)}{b(s)}\, ds. \tag{2}$$

The hypothesis (H1) implies that $G(x)$ is always positive, decreasing for $x \in (-\infty, 0)$ and increasing for $x \in (0, +\infty)$, as it is shown in Figure 1. Given a positive real number $z$, call $x_1(z) < 0 < x_2(z)$ the two solutions of $z = G(x)$, (see Figure 1).

We will say that $(w_1, w_2)$, $x_1 < w_1 < 0 < w_2 < x_2$, is a solution of $(F, G)$ if $F(w_1) = F(w_2)$ and $G(w_1) = G(w_2)$.

We also introduce

$$F_i(z) := F(x_i(z)), \quad \text{for } i = 1, 2. \tag{3}$$

With the above notations, we can state the first two results, which extend those given for Liénard equations (see [5], [12], [31]) to system (1).

**Theorem A.** (Non-existence of limit cycles). *Suppose that system (1) satisfies the hypothesis (H1). Then,*

*(a) System (1) has no periodic orbits in $D$ if $(F, G)$ has no solutions.*

*(b) System (1) has no periodic orbits entirely contained in the strip* $\{(x,y) : s_1 < x < s_2\} \cap D$, *if there exists a solution* $(s_1, s_2)$ *of* $(F,G)$ *such that for any other solution* $(w_1, w_2)$ *of* $(F,G)$, *we have* $w_1 < s_1 < 0 < s_2 < w_2$,

**Theorem B.** (Existence of centers). *Suppose that system (1) satisfies the hypotheses (H1) and (H2), (H3)*

*(H2)* $F_1(z) \equiv F_2(z)$ *for some positive neighbourhood of* $z = 0$, *where* $F_i(z)$, $i = 1,2$, *is defined as in (3);*

*(H3) There exists a neighbourhood* $U$ *of* $(0,0)$ *such that all the orbits in* $U$ *turn infinitely many times around the origin.*
*Then, the origin is a center for system (1).*

We will see in Section 4 that hypothesis (H2) is a generalization of the well-known condition of reversibility of Poincaré (see [29]).

We will also prove in that section that (H3) cannot be removed (see Lemma 4.1). Observe that when the origin is an elementary critical point for system (1), then (H3) holds if $F'(0)^2 - 4\frac{a(0)\varphi'(0)g'(0)}{b(0)} < 0$, while if this expression is positive, the hypothesis is not satisfied. In case that $F'(0)^2 - 4\frac{a(0)\varphi'(0)g'(0)}{b(0)} = 0$, we can decide whether (H3) holds by desingularising the critical point by means of the blowing-up technique (see [1], [2], [10]), assuming that the functions involved in (1) are sufficiently smooth (see again Lemma 4.1). In addition, when (1) is a Liénard equation, (H3) can be substituted by the simpler condition $|F_i(z)| < \sqrt{8z}$, $i = 1,2$, (see [12], [31]).

Some auxiliar functions are needed to give a clearer presentation of the following theorem. Set

$$\begin{aligned} L(x) &:= sgn(x)\sqrt{2\int_0^x \frac{g(\xi)}{b(\xi)}\,d\xi}, \\ H(y) &:= sgn(y)\sqrt{2\int_0^y \frac{\varphi(\xi)}{a(\xi)}\,d\xi}, \qquad \text{and} \\ \mathcal{G}_i(z,y) &:= 2a(y)H'(y)z - H(y)(F_i(z) - \varphi(y)), \quad \text{for } i = 1,2. \end{aligned}$$

**Theorem C.** (Uniqueness of limit cycles). *For system (1), consider the functions* $L(x)$, $H(y)$, $F_i(z)$ *and* $\mathcal{G}_i(z,y)$, *for* $i = 1,2$. *Suppose that this system satisfies the hypotheses (H1), and (H4), (H5)*

*(H4)* $\mathcal{G}_1(z,y)\mathcal{G}_2(z,y) \neq 0$, *for every pair* $(z,y) \neq (0,0)$, *and* $L$ *and* $H$ *are real* $C^1$*-functions with* $L'(0) \cdot H'(0) \neq 0$;

*(H5)* $(-1)^i k \frac{\partial}{\partial z}\left(\frac{F_i(z)}{\varphi(H^{-1}(k\sqrt{2z}))}\right) < 0$ $(> 0)$, *for* $i = 1,2$ *and any real number* $k \neq 0$.

*Then, system (1) has at most one periodic solution in $D$ and, if it exists, it is hyperbolic and stable (unstable).*

The effect of hypothesis (H4) is to force the periodic orbits to turn around the origin. However, for some special subclasses of system (1), it is not necessary that hypothesis (H4) holds for all $(z, y)$ in order to have uniqueness of limit cycles. See Remark 4.7, Example 5.4 and also [25].

Note that when $\varphi(y) = \int_0^y \frac{1}{a(u)}\, du$, then $H(y) = \varphi(y)$ and (H5) becomes H5') $(-1)^i \frac{\partial}{\partial z}\left(\frac{F_i(z)}{\sqrt{2z}}\right) > 0\ (< 0)$.
Therefore, when (1) is a Liénard equation, (H5) can be substituted by (H5') in Theorem C. This result improves ([6], Thm.III) because here we prove the hyperbolicity of the limit cycle.

The Filippov transformation has been used for generalized Liénard equations to get uniqueness of limit cycles in the previous papers [30], [34]. Another well-known result on uniqueness of limit cycles is Zhang's Criterion, (see [35]). In Example 5.5 we give a system for which , using our theorem, we can prove the uniqueness of limit cycles while the results obtained in the above papers do not work.

# 3 The Generalized Filippov Transformation

The transformation we use in this paper is a generalization of the Filippov transformation, which was originally designed for Liénard equations. The basic idea of the Filippov transformation is to take advantage of some symmetries, in order to compare solutions lying in different parts of the plane. The generalized Filippov transformation for system (1) is given by the change of variables

$$z = G(x) = \int_0^x \frac{g(s)}{b(s)}\, ds. \tag{F}$$

Then, system (1) splits into two $C^1$-systems:

$$\begin{cases} \frac{dz}{dt} = g_1(z)(F_1(z) - \varphi(y)), \\ \frac{dy}{dt} = g_1(z)a(y), \end{cases} \quad \text{and,} \tag{1.1'}$$

$$\begin{cases} \frac{dz}{dt} = g_2(z)(F_2(z) - \varphi(y)), \\ \frac{dy}{dt} = g_2(z)a(y), \end{cases} \tag{1.2'}$$

where $g_i(z) = g(x_i(z))$, and can be also considered as:

$$\frac{dz}{dy} = \frac{F_1(z) - \varphi(y)}{a(y)} =: H_1(z, y), \quad \text{and,} \tag{1.1}$$

$$\frac{dz}{dy} = \frac{F_2(z) - \varphi(y)}{a(y)} =: H_2(z, y). \tag{1.2}$$

From hypothesis (H1), this change of variables is well-defined between $\Omega^- = \{(x, y) : x < 0\}$ and $R = \{(z, y) : z > 0\}$, giving equation (1.1), and between $\Omega^+ = \{(x, y) : x > 0\}$ and $R$, giving equation (1.2) and both having uniqueness of solutions. The system has been split into two different ones according to a reflection of the part of the trajectories laying in $\Omega^-$ onto the right-hand side of the plane, plus a deformation related to the function $\frac{g(x)}{b(x)}$. So, it allows to compare both sides of any periodic orbit (see Figure 2).

**Remark 3.1** *$F_1(z^*) = F_2(z^*)$ if and only if there exist some $s_1 < 0 < s_2$ such that $F(s_1) = F(s_2)$ and $G(s_1) = G(s_2) = z^*$.*

The above result allows to distinguish Theorems A and B according to the relation between $F_1(z)$ and $F_2(z)$. Hence, the non-existence of periodic orbits is related to $F_1(z) \neq F_2(z)$ for all $z$, and the existence of centers to $F_1(z) \equiv F_2(z)$ in some neighbourhood of the origin. So, the existence of limit cycles needs the equality $F_1(z) = F_2(z)$ to be true for some $z > 0$.

Apart from the work of Filippov (see [12]) and Sansone and Conti (see [31]), the Filippov transformation has been successfully used also in [5], [30], [35] and [34], for instance. Other authors (see [7], [11]) have used a similar change of variables for Liénard equations, using of a symmetry with respect to the straight line $x = x_0$, in the case that there exists a unique $x_0$ such that $F'(x_0) = 0$ (a condition often satisfied in predator-prey systems of type (1)).

# 4 Proofs of the Theorems and related results

**Proof of Theorem A.**
(a) If $C(R^+) \cap C(R^-) = \emptyset$ then, from Remark 3.1, $F_1(z) \neq F_2(z)$, for all $z > 0$. Assume, for instance, that $F_1(z) < F_2(z)$.

Splitting (1) by (F), we obtain (1.1) and (1.2).

Suppose that there exists a periodic solution of (1), $\gamma \subset R^2$, and take $y_m = \min\{y : (x, y) \in \gamma\}$ and $y_M = \max\{y : (x, y) \in \gamma\}$. After applying the generalized Filippov transformation, see Figure 2, $\gamma$ is divided into $z_1(y)$ and $z_2(y)$ which are, respectively, solutions of the boundary value problems (1.1) and (1.2) with common boundary conditions $z_1(y_m) = z_2(y_m) = 0$ and $z_1(y_M) = z_2(y_M) = 0$.

Since, $H_1(z, y) < H_2(z, y)$, $z_1(y)$ is a solution of $\frac{dz}{dy} < H_2(z, y)$. Then, by the theory of differential inequalities (see [17], pp.30-32) , $z_1(y) \leq z_2(y)$ in $[y_m, y_M]$. If we prove that the inequality is strict, i.e. $z_1(y) < z_2(y)$ in $(y_m, y_M]$, we will

get a contradiction to the fact that $z_1(y)$ and $z_2(y)$ are solutions of the boundary varlue problems.

Assume that for some $y_0$, $y_0 \in (y_m, y_M)$, $z_1(y_0) = z_2(y_0)$. For all $y$ in a neighbourhood $(y_0 - \epsilon, y_0)$ of $y_0$, we have that

$$\frac{z_1(y) - z_1(y_0)}{y - y_0} > \frac{z_2(y) - z_2(y_0)}{y - y_0},$$

and so, $\frac{dz_1}{dy^-}(y_0) \geq \frac{dz_2}{dy^-}(y_0)$. Similarly, in the right-hand side, we obtain $\frac{dz_1}{dy^+}(y_0) \leq \frac{dz_2}{dy^+}(y_0)$. Hence, $z_1'(y_0) = z_2'(y_0)$. But, recalling that $z_1'(y_0) = H_1(z_1(y_0), y_0) < H_2(z_2(y_0), y_0) = z_2'(y_0)$ and $z_1(y_0) = z_2(y_0)$, a contradiction arises. Consequently,

$$z_1(y) < z_2(y), \qquad \text{for } y \in (y_m, y_M). \tag{4}$$

In order to show that $z_1(y_M) < z_2(y_M)$, take a point $y_1 \in (y_m, y_M)$. By (4), $z_1(y_1) < z_2(y_1)$. Consider the two following Cauchy problems:

$$\begin{cases} \frac{dz}{dy} = H_1(z, y), \\ z(y_1) = z_2(y_1); \end{cases} \qquad \begin{cases} \frac{dz}{dy} = H_2(z, y), \\ z(y_1) = z_2(y_1). \end{cases}$$

Let $\overline{z_1}(y), \overline{z_2}(y)$ be their respective solutions. In fact, $\overline{z_2}(y) = z_2(y)$. Again, by the theory of differential inequalities, $\overline{z_1}(y) < z_2(y)$ for all $y > y_1$. In particular, $\overline{z_1}(y_M) < z_2(y_M)$. On the other hand, $z_1(y_1) < z_2(y_1) = \overline{z_1}(y_1)$. Then, by the uniqueness of solutions of $\frac{dz}{dy} = H_1(z, y)$, we have that $z_1(y) < \overline{z_1}(y)$ for all $y \in [y_1, y_M]$ and, consequently, $z_1(y_M) < \overline{z_1}(y_M) < z_2(y_M)$.

Case (b) follows in a similar way.

**Proof of Theorem B.**
Using the Filippov transformation, system (1) is divided in systems (1.1') and (1.2'). From (H2), their solutions coincide although they may have different parameterizations. By (H3), the orbits of (1.i'), for $i = 1, 2$, join the positive and negative $y$-axes. Therefore, the theorem is proved.

We would like to stress the necessity of hypothesis (H3) in Theorem B. We give an example of system (1) satisfying (H2) but not (H3), such that its critical point is not a center.

**Lemma 4.1.** *Hypothesis (H3) cannot be removed from the hypotheses of Theorem B.*

**Proof.** Consider the Liénard system:

$$\begin{cases} \frac{dx}{dt} = F(x) - y & = ax^2 - y, \\ \frac{dy}{dt} = g(x) & = bx^3. \end{cases}$$

where $a$ and $b$ are positive real numbers.

It is easy to check that (H1) is fulfilled. The Filippov transformation, in this case, is given by $z = \frac{bx^4}{4}$ or, inversely, $x_i = (-1)^i \sqrt[4]{\frac{4z}{b}}$, for $i = 1, 2$. Therefore, $F_i(z) = F(x_i(z)) = a\sqrt{\frac{4z}{b}}$, for $i = 1, 2$, and so, $F_1(z) \equiv F_2(z)$. Hence, (H2) is satisfied.

In order to know whether (H3) is satisfied or not, we desingularise the critical point using the technique of quasi-homogeneous blowing-up (see [2], pp.182-190). Applying the change $x = r\cos\theta$, $y = r^2\sin\theta$, the Liénard system is transformed (after omitting the common factor $\frac{r}{1+\sin^2\theta}$) into:

$$\begin{cases} \frac{dr}{dt} = r\cos\theta((a + b\sin\theta)\cos^2\theta - \sin\theta), \\ \frac{d\theta}{dt} = b\cos^4\theta - 2a\cos^2\theta\sin\theta + 2\sin^2\theta. \end{cases}$$

According to the critical points on $r = 0$, the three following situations arise: if $a^2 - 2b < 0$, the above system has no critical points; if $a^2 - 2b = 0$, it has two critical points; and it has four critical points if $a^2 - 2b > 0$. All the critical points can be studied using [1], Sec.7 and Thm.65. Their phase portraits and those obtained without the blowing-up are given in Figure 3, where we can see that the origin is:

- an equilibrium point with an elliptic sector and a hyperbolic one, when $a^2 - 2b \geq 0$, and
- a focus or a center, when $a^2 - 2b < 0$.

(In this particular case, we could have obtained the same result from [1],Sec.22, but the method used here is much more general).

The most important aspect is that only in the case $a^2 - 2b < 0$ the orbits of the Liénard system turn around the origin and so, (H3) holds. Therefore, under this inequality, the above dilemma between focus or center is solved since, by Theorem B, we can conclude that the origin is a center. However, when $a^2 - 2b \geq 0$, the Liénard system satisfies (H1), (H2) and its origin is not a center. Hence, the lemma is proved.

On the other hand, hypothesis (H2) in Theorem B is strongly related to the concept of reversibility, see [36], as we pointed out in the previous section. This relation can be stated as follows.

**Lemma 4.2.** *Set $x_2 = x_2(x_1)$ the solution of the equation $G(x_1) = G(x_2)$ satisfying $x_1x_2 \leq 0$, see Figure 1. Suppose that $F_1(z) \equiv F_2(z)$. Then, if $(x_1(t), y(t))$ is a solution of (1), there exists a change of time $\tau = \tau(t)$, given by $\frac{d\tau}{dt} = \frac{g(x_1(t))}{g(x_2(t))}$,*

*such that* $(x_2(\tau), y(\tau))$ *is also a solution of (1).*

**Proof.** Since $G(x_2(x_1)) = G(x_1)$, we have that $x_2'(x_1) = \frac{b(x_2)}{b(x_1)}\frac{g(x_1)}{g(x_2)}$. Then,

$$\begin{aligned}\frac{dx_2(t)}{dt} &= x_2'(x_1(t))\frac{dx_1(t)}{dt} = \frac{b(x_2(t))}{b(x_1(t))}\frac{g(x_1(t))}{g(x_2(t))}b(x_1(t))\left(F(x_1(t)) - \varphi(y(t))\right)\\ &= \frac{b(x_2(t))g(x_1(t))}{g(x_2(t))}\left(F(x_1(t)) - \varphi(y(t))\right),\\ \frac{dy(t)}{dt} &= a(y(t))g(x_1(t)).\end{aligned}$$

Applying the above change of time, we have

$$\begin{aligned}\frac{dx_2(\tau)}{d\tau} &= b(x_2(\tau))\left(F(x_2(\tau)) - \varphi(y(\tau))\right),\\ \frac{dy(\tau)}{d\tau} &= a(y(\tau))g(x_2(\tau)),\end{aligned}$$

and so the lemma follows.

**Remark 4.3.** Assume that $g(x)$ is an odd function and $b(x)$ an even function. Therefore, from Lemma 3.1, if $(x(t), y(t))$ is a solution of (1), then $(-x(-t), y(-t))$ also is a solution. This is the condition of reversibility given by Poincaré to study center points (see [29]).

To demonstrate Theorem C we will use a general result on uniqueness and hyperbolicity of limit cycles which is already proved in ([16], Proposition 2). Here, we give a different proof.

**Proposition 4.4.** *Consider the differential equation*

$$\frac{d\rho}{d\theta} = S(\rho, \theta). \tag{5}$$

*on the cylinder* $\{(\rho, \theta) : \rho \in R^+, \theta \in R/[0, 2\pi]\}$, *for* $S$ *a real* $C^1$*-function. Assume that there exists a* $C^1$*-function,* $\omega : R^+ \to R^+$, *such that* $\frac{\partial}{\partial\rho}(\omega(\rho)S(\rho, \theta)) < 0 (> 0)$. *Then, there is at most one periodic orbit for system (5), which is hyperbolic and stable (resp., unstable) when it exists.*

**Proof.** Consider a new variable $\rho$ defined by $\frac{dr}{d\rho} = \omega(\rho)$. With this new variable, equation (5) becomes

$$\frac{dr}{d\theta} = \frac{dr}{d\rho}\frac{d\rho}{d\theta} = \omega(\rho(r))S(\rho(r), \theta). \tag{5'}$$

Let $\gamma$ be a periodic solution of (5), that is a solution $\rho(\theta)$ such that $\rho(0) = \rho(2\pi)$. After the change of variables, it will be transformed into a periodic solution $\gamma'$ of

(5'), which will have the same stability character as $\gamma$. Take $(\rho,\theta)=(x,0)$ the intersection point between $\gamma'$ and $\theta=0$. Let $h(x)$ be the Poincaré map associated to $\gamma'$. From [23], we know that the stability character of $\gamma'$ is given by the sign of $(h'(x)-1)$, where

$$h'(x)=\exp\int_0^{2\pi}\frac{\partial}{\partial r}\left(\omega(\rho(r(\theta,x)))S(\rho(r(\theta,x)),\theta)\right)\,d\theta.$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial r}\left(\omega(\rho(r))S(\rho(r),\theta)\right) &= \frac{\partial}{\partial\rho}\left(\omega(\rho)S(\rho,\theta)\right)\frac{\partial\rho}{\partial r}\\ &= \frac{\partial}{\partial\rho}\left(\omega(\rho)S(\rho,\theta)\right)\frac{1}{\omega(\rho)}.\end{aligned}$$

Since $\omega(\rho)>0$, the proposition is proved.

**Remark 4.5.** Proposition 4.4 also holds if the function $\frac{\partial}{\partial\rho}\left(\omega(\rho)S(\rho,\theta)\right)$ vanishes on some rays through the origin.

**Remark 4.6.** Applying Proposition 4.4 with $\omega(\rho)=\frac{1}{\rho}$, we can prove that the equation $\frac{d\rho}{d\theta}=\frac{a(\theta)\rho+f(\theta)\rho^n}{b(\theta)+g(\theta)\rho^{n-1}}$ has at most one limit cycle, which is hyperbolic, in the region where $\dot\theta$ does not vanish if $(f(\theta)b(\theta)-a(\theta)g(\theta))$ does not change sign. This result (without proving hyperbolicity) can be found in [3].

**Proof of Theorem C.** Consider the change of variables $u=L(x)$, $v=H(y)$. Its Jacobian is $L'(x)H'(y)$ and therefore, from hypothesis (H4), it is well-defined. In these variables, system (1) reads

$$\begin{cases}\frac{du}{dt}=\frac{\overline{g}(u)}{u}\left(\overline{F}(u)-\tilde\varphi(v)\right),\\ \frac{dv}{dt}=\frac{\tilde\varphi(v)}{v}\overline{g}(u),\end{cases}$$

where for any function $f$, $\overline{f}(u)=f(x(u))$ and $\tilde f(v)=f(y(v))$. Note that, again from hypothesis (H4), $\frac{\overline{g}(u)}{u}$ and $\frac{\tilde\varphi(v)}{v}$ are defined at zero because $\lim_{u\to0}\frac{\overline{g}(u)}{u}=\frac{g'(0)}{L'(0)}$ and $\lim_{v\to0}\frac{\tilde\varphi(v)}{v}=\frac{\varphi'(0)}{H'(0)}$.

In order to apply Proposition 4.4, we take polar coordinates $(\rho,\theta)$. The system is transformed into

$$\begin{cases}\frac{d\rho}{dt}=\frac{\overline{g}(u)\overline{F}(u)}{\rho},\\ \frac{d\theta}{dt}=\frac{\overline{g}(u)}{u}\left(\frac{\tilde\varphi(v)}{v}-\frac{\overline{F}(u)\sin\theta}{\rho}\right),\end{cases}\tag{6}$$

where $u=\rho\cos\theta$ and $v=\rho\sin\theta$.

Straightforward computations give that for $i = 1, 2$,

$$\mathcal{G}_i(z, y) = \rho^2 \left( \frac{\check{\varphi}(v)}{v} - \frac{\overline{F}(u) \sin\theta}{\rho} \right),$$

because $u^2 = 2z$ and $v = H(y)$. Hence, from hypothesis (H4), system (6) can be written as:

$$\frac{d\rho}{d\theta} = \rho \frac{\sin\theta \cos\theta \overline{F}(u)}{\check{\varphi}(v) - \overline{F}(u) \sin^2\theta} := S(\rho, \theta).$$

Note that

$$\frac{\partial}{\partial\rho} \left( \frac{S(\rho,\theta)}{\rho} \right) = \sin\theta \cos\theta \frac{\check{\varphi}(v) \frac{\partial \overline{F}(u)}{\partial\rho} - \overline{F}(u) \frac{\partial \check{\varphi}(v)}{\partial\rho}}{(\check{\varphi}(v) - \overline{F}(u) \sin^2\theta)^2}.$$

Then, using Proposition 4.4 with $\omega(\rho) = \frac{1}{\rho}$ and Remark 4.5, we have that system (1) has at most one limit cycle, which is hyperbolic in case it exists, if

$$\sin\theta \cos\theta \left( \check{\varphi}(v) \frac{\partial \overline{F}(u)}{\partial\rho} - \overline{F}(u) \frac{\partial \check{\varphi}(v)}{\partial\rho} \right) \tag{7}$$

does not change sign and vanishes only on some rays.

In order to provide a more intelligible condition we write this last expression using the Filippov variable $z = \frac{u^2}{2}$. Tedious but easy computations give that (7) is equivalent to

$$(-1)^i k \frac{\partial}{\partial z} \frac{F_i(z)}{\varphi(H^{-1}(k\sqrt{2z}))}.$$

where $k = (-1)^i \tan\theta$. Then, from hypothesis (H5), the theorem follows.

**Remark 4.7.** From the above proof, it is easy to show that when we remove the condition $\mathcal{G}_1(z, y)\mathcal{G}_2(z, y) \neq 0$ from Theorem C, we have the following result: Let $\overline{M_1}$ and $\overline{M_2}$ be connected components of $\{\mathcal{G}_1(z, y) \neq 0\} \subset \{(z, y) : z \geq 0\}$ and $\{\mathcal{G}_2(z, y) \neq 0\} \subset \{(z, y) : z \geq 0\}$, respectively, with intersection on the $y$-axis. Then there is at most one limit cycle of (1) entirely contained in $M = \{(x, y) : x = x_1(z), (z, y) \in \overline{M_1}\} \cup \{(x, y) : x = x_2(z), (z, y) \in \overline{M_2}\}$.

**Remark 4.8.** With the same notation as in the above remark, we have that if all the periodic orbits lie in one connected component of the set $M$, the uniqueness and hyperbolicity of limit cycles in all the phase plane follows.

# 5 Examples

**Example 5.1.** (Non-existence of limit cycles). Consider the system

$$\begin{cases} \dot{x} = \alpha x - \beta(x-k)y, \\ \dot{y} = \delta(x-k) - \gamma y, \end{cases} \tag{8}$$

where $x$ and $y$ are positive numbers representing, respectively, the prey and the predator populations, and $\alpha$, $\beta$, $\gamma$, $\delta$ and $k$ are positive constants.

This system (see [18]) describes a predator-prey model in which the prey has a refuge; that is, a fixed number $k$ of prey is not subject to predation. Here we give a proof of the non-existence of periodic orbits for system (7) different from Harrison's.

The system has a critical point $(x_0, y_0) = (k + \frac{\gamma}{\delta}, \frac{\alpha(\gamma+k\delta)}{\beta\gamma})$ in $(0,+\infty) \times (0,+\infty)$. Translating this point to the origin, system (7) becomes:

$$\begin{cases} \dot{x} = \beta(x+x_0)\left(\frac{\alpha(x+x_0+k)}{\beta(x+x_0)} - y_0 - y\right), \\ \dot{y} = (y+y_0)\delta x. \end{cases} \tag{9}$$

Clearly, this system is of type (1), defined in $D = (-x_0, +\infty) \times (-y_0, +\infty)$ and, moreover, it satisfies hypothesis (H1). Then, in order to apply Theorem A to ensure the non-existence of limit cycles, we only need to prove that there are no solutions $(w_1, w_2)$, $w_1 < 0 < w_2$, of the system $(F, G)$, where $F(x) = \frac{\alpha(x+x_0+k)}{\beta(x+x_0)} - y_0$ and $G(x) = \frac{\delta}{\beta}x - \frac{\gamma}{\delta}\ln\frac{\delta x+\gamma}{\gamma}$.

From $F(x) = F(y)$, we easily get that $w_1 = w_2$. Hence, system $(F, G)$ has no solutions with $w_1 < 0 < w_2$ and so, (8) has no limit cycles.

**Example 5.2.** (Non-existence of limit cycles). Consider the predator-prey system

$$\begin{cases} \dot{x} = \gamma\left(1 - \frac{x}{k}\right)x - f(x)y, \\ \dot{y} = y\left(cf(x) - m\right), \end{cases} \tag{10}$$

where $x$ and $y$ are positive numbers representing, respectively, the prey and the predator populations, and $\gamma$, $c$, $m$ and $k$ are positive constants. Assume also that $f(x)$ is an increasing $C^1$-function in $[0,+\infty)$.

There exists a unique critical point $(x_0, y_0)$ in $R^+ \times R^+$, where $x_0 = f^{-1}(\frac{m}{c})$ and $y_0 = \frac{\gamma c x_0(1-\frac{x_0}{k})}{m}$. Of course, the condition $x_0 < k$ must be satisfied in order to have

$y_0 > 0$. By a translation of this point to the origin, system (9) is transformed in:

$$\begin{cases} \dot{x} = f(x+x_0)\left(\left(\frac{\gamma(x+x_0)(1-\frac{x+x_0}{k})}{f(x+x_0)} - y_0\right) - y\right), \\ \dot{y} = (y+y_0)\,(cf(x+x_0) - m)\,. \end{cases} \tag{11}$$

This system is of type (1), defined in $D = (-x_0, +\infty) \times (-y_0, +\infty)$ and it satisfies hypothesis (H1). We obtain that

$$F(x) = \frac{\gamma(x+x_0)(1-\frac{x+x_0}{k})}{f(x+x_0)} - y_0, \quad \text{and}$$

$$G(x) = \int_0^x (c - \frac{m}{f(s+x_0)})ds = cx - m\int_0^x \frac{ds}{f(s+x_0)}.$$

From now on, we will consider several examples of predator-prey systems of type (9) studied in the current literature, changing in each case the function $f(x)$.

**Example 5.2.** (a). Take $f(x) = \frac{x^2}{a+x^2}$, with $a$ a positive constant; that is, Holling type III functional response (see also [8],[9],[19]).

The values $x_0$, $y_0$ are given by

$$x_0 = \sqrt{am/(c-m)} \qquad \text{and} \qquad y_0 = \frac{\gamma c}{m}\left(1 - \frac{x_0}{k}\right)x_0.$$

So, we assume that $c > m$. Here, system $(F, G)$ can be written as

$$\begin{cases} w_1 w_2 + (w_1 + w_2)x_0 = 0, \\ (w_1 + w_2) - k\frac{2m-c}{m} + 2\sqrt{\frac{m}{c-m}} = 0. \end{cases}$$

We are interested in the case that the origin is an attractor. According to the relations among the parameters, there are two possibilities to have an attractor:

(a) $0 < m < c < 2m$, $k \leq (2m\sqrt{am}) \,/\, \left((2m-c)\sqrt{c-m}\right)$; and,

(b) $2m \leq c$.

Starting from $(F, G)$, we get that

$$w_1 w_2 = 2\frac{am}{c-m} - \frac{k(2m-c)\sqrt{am}}{m\sqrt{c-m}}.$$

In both cases, (a) and (b), we arrive at the same contradiction: $w_1 w_2 \geq 0$. So, when $f(x) = \frac{x^2}{a+x^2}$ and the parameters satisfy either (a) or (b), system (10) has no periodic orbits in the first quadrant.

This predator-prey model has been widely studied in [8],[9], but without any

mention of the Filippov transformation.

**Example 5.2.** (b). Take $f(x) = \alpha x^\beta$, with $\alpha > 0$ and $\beta \geq 1$ real numbers (see [4]).

The values of $x_0$ and $y_0$ are in this case

$$x_0 = \sqrt[\beta]{\frac{m}{c\alpha}} \qquad \text{and} \qquad y_0 = \frac{\gamma c}{m}\left(1 - \frac{x_0}{k}\right) x_0.$$

System $(F, G)$ becomes

$$\begin{cases} (1 - \frac{w_1+x_0}{k})(w_1 + x_0)^{1-\beta} &= (1 - \frac{w_2+x_0}{k})(w_2 + x_0)^{1-\beta}, \\ w_1 - \frac{x_0^\beta}{1-\beta}(w_1 + x_0)^{1-\beta} &= w_2 - \frac{x_0^\beta}{1-\beta}(w_2 + x_0)^{1-\beta}. \end{cases}$$

This system can be transformed into the shorter form

$$\begin{cases} w_2 &= h(w_1), \\ w_1 &= h(w_2), \end{cases}$$

where $h(z) = k - x_0 - \frac{x_0^\beta}{(1-\beta)(z+x_0)^{1-\beta}}$.

Call $A = w_1 + x_0$, $B = w_2 + x_0$ and $\delta = \beta - 1$. From the above system, we get the following relation

$$k(A^\delta - B^\delta) = AB(A^{\delta-1} - B^{\delta-1}). \tag{12}$$

Suppose that there exists a solution $(w_1, w_2)$ of $(F, G)$ with $w_1 < 0 < w_2$. Consequently, $0 < B < x_0 < A$.

We divide the study into two cases:

If $\delta \in [0, 1]$, $k(A^\delta - B^\delta) > 0$ and $AB(A^{\delta-1} - B^{\delta-1}) \leq 0$. So, the equality (11) cannot hold.

If $\delta > 1$, we have that

$$\begin{aligned} & w_1 w_2 \\ &= (A - x_0)(B - x_0) = AB - (A + B)x_0 + x_0^2 \overset{\text{using (11)}}{=} \\ &= k\frac{A^\delta - B^\delta}{A^{\delta-1} - B^{\delta-1}} - (A + B)x_0 + x_0^2 \\ &= \frac{1}{A^{\delta-1} - B^{\delta-1}}\left(k(A^\delta - B^\delta) - (A + B)x_0(A^{\delta-1} - B^{\delta-1}) + x_0^2(A^{\delta-1} - B^{\delta-1})\right) \\ &= \frac{1}{(A^{\delta-1} - B^{\delta-1})}\left((k - x_0)(A^\delta - B^\delta) + x_0((A - x_0)B^{\delta-1} + (x_0 - B)A^{\delta-1})\right). \end{aligned}$$

Taking into account that $0 < B < x_0 < A$ and $x_0 < k$, we get that $w_1 w_2 > 0$. This is a contradiction to $w_1 < 0 < w_2$.

Hence, system $(F, G)$ has no solutions such that $w_1 < 0 < w_2$ if $\delta \geq 0$ or, equivalently, if $\beta \geq 1$. Therefore, by Theorem A, we have proved the non-existence of periodic orbits.

**Example 5.2.** (c). Take $f(x) = (1 - e^{-ax})$, with $a$ a positive constant, and $k = c = 1$ (see also [32]).

The condition $x_0 < k$ is equivalent to $0 < m < d(a) = 1 - e^{-a}$ and so, the origin is a critical point of system (10). In this case, the functions $F(x)$ and $G(x)$ are given by

$$F(x) = \frac{\gamma(1 - x - x_0)(x + x_0)}{1 - e^{-a(x+x_0)}} - y_0, \quad G(x) = x - \frac{m}{a} \ln \frac{1 - (1 - m)e^{-ax}}{me^{-ax}}.$$

Unfortunately, we have not been able to prove the non-existence of a solution of system $(F, G)$ such that $w_1 < 0 < w_2$. Anyway we have reduced the question of non-existence of limit cycles to a problem of solving a system of equations. In [14] we have studied the above system numerically for some values of the parameters.

In the following example, we give an easy application of Theorem B:

**Example 5.3.** The Lotka-Volterra system (see Introduction) has a center around the origin.

In the region $\{(x, y) : x > -\frac{\mu}{\gamma},\ y > -\frac{\alpha}{\beta}\}$ (in fact, the only one with biological meaning), this system satisfies the hypothesis (H1), while (H2) holds everywhere since $F_1(z) \equiv F_2(z) \equiv 0$. Moreover, since $F'(0)^2 - 4\frac{a(0)\varphi'(0)g'(0)}{b(0)} = -4\frac{\alpha\gamma^2}{\mu} < 0$, the origin is an elementary critical point of focus or center type. Hence, (H3) is satisfied and by Theorem B, the origin is a center for the Lotka-Volterra system.

**Example 5.4.** Consider the planar system

$$\begin{cases} \dot{x} = a_3 x^3 + a_1 x - (y + cy^3), \\ \dot{y} = x, \end{cases} \tag{13}$$

where $a_1$, $a_3$ and $c$ are real constants with $c \geq 0$.

The associated functions defined in Section 2 are, in this example: $L(x) = x$, $H(y) = y\sqrt{1 + c\frac{y^2}{2}}$ and $F_i(z) = (-1)^i\sqrt{2z}(a_1 + 2a_3 z)$. It can be easily checked that hypothesis (H1) is satisfied. Furthermore, $F_2(z) - F_1(z) = 2\sqrt{2z}(a_1 + 2a_3 z)$. Since $z$ is always positive, from Theorems A and B we have:

**Result 1.** *System (12) has no limit cycles if $a_1 a_3 \geq 0$ and has a center if*

$a_1 = a_3 = 0$.

On the other hand, to check hypothesis (H5), we compute

$$\frac{F_i(z)}{\varphi(H^{-1}(k\sqrt{2z}))} = \frac{(-1)^i\sqrt{2z}(a_1+2a_3z)}{\varphi\left(sgn(k)\sqrt{\frac{\sqrt{1+4ck^2z}-1}{c}}\right)} = \frac{(-1)^i\sqrt{2z}(a_1+2a_3z)}{sgn(k)\sqrt{\frac{\sqrt{1+4ck^2z}-1}{c}}\sqrt{1+4ck^2z}}.$$

Consider the change of variables $w = \sqrt{1+4ck^2z}$. Then,

$$\begin{aligned}
&(-1)^i k\frac{\partial}{\partial z}\left(\frac{F_i(z)}{\varphi(H^{-1}(k\sqrt{2z}))}\right)\\
&= k\frac{\partial}{\partial w}\left(\frac{\sqrt{\frac{w^2-1}{2ck^2}}(a_1+a_3\frac{w^2-1}{2ck^2})}{sgn(k)\sqrt{\frac{w-1}{c}}w}\right)\frac{\partial w}{\partial z}\\
&= \frac{\sqrt{2}}{2}\frac{\partial}{\partial w}\left(\frac{\sqrt{w+1}(a_1+a_3\frac{w^2-1}{2ck^2})}{w}\right)\frac{2ck^2}{w}\\
&= \frac{1}{2\sqrt{2}w^3\sqrt{w+1}}\left((2a_3-4a_1ck^2)+(a_3-2a_1ck^2)w+2a_3w^2+3a_3w^3\right),
\end{aligned}$$

and hence, if $a_1a_3 < 0$, (H5) is fulfilled.

Observe also that $L'(0)H'(0) = 1$. But, unfortunately, in this case the expression of $\mathcal{G}_i(z,y)$, for $i = 1,2$, is very complicated and we have not been able to prove condition $\mathcal{G}_1(z,y)\mathcal{G}_2(z,y) \neq 0$. Then, as we pointed out in the Remark 4.7, we can conclude:

**Result 2.** *If $a_1a_3 < 0$, then there is at most one limit cycle, which is hyperbolic, in each set of the form $M = \{(x,y) : x = x_1(z), (z,y) \in \overline{M_1}\} \cup \{(x,y) : x = x_2(z), (z,y) \in \overline{M_2}\}$, where $\overline{M_1}$ and $\overline{M_2}$ are connected components of $\{\mathcal{G}_1(z,y) \neq 0\}$ and $\{\mathcal{G}_2(z,y) \neq 0\}$, respectively, with intersection in the $y$-axis.*

In the case that $c = 0$, we can go further and ensure the uniqueness of limit cycles in all the phase space since we can study the components of type $M$. Applying the same changes of variables as in the proof of Theorem C and the Filippov transformation, we have that for $c = 0$,

$$\begin{cases} \frac{d\rho}{dt} = a_1\cos^2\theta\rho + a_3\cos^4\theta\rho^3, \\ \frac{d\theta}{dt} = (-1)^i - (a_1\cos\theta\sin\theta + a_3\cos^3\theta\sin\theta\rho^2), \qquad \text{for } i = 1,2. \end{cases} \tag{12.$i$}$$

Consider $\overline{K_i} = \{(z,y) : \mathcal{G}_i(z,y) = 0\}$, for $i = 1,2$, and define $K = \{(x,y) : x = x_1(z), (z,y) \in \overline{K_1}\} \cup \{(x,y) : x = x_2(z), (z,y) \in \overline{K_2}\}$. If we know $K$, we will also

have a description of its complementary (the union of the components of type $M$). Note that

$$\mathcal{G}_i(z,y)=0 \Leftrightarrow \frac{d\theta}{dt}=0 \Leftrightarrow \rho^2=\frac{(-1)^i-a_1\cos\theta\sin\theta}{a_3\cos^3\theta\sin\theta}.$$

Depending on the values of $a_1$ and $a_3$, $K$ can have the different shapes depicted in Figure 4. The most important feature of these sets is that the shadowed regions (the regions $\{(\rho,\theta):\dot{\theta}<0\}$) are positively or negatively invariant under the flow of (12) (see [3]). Hence, periodic orbits cannot cross them and must lie in $\{(\rho,\theta):\dot{\theta}>0\}$ (see also [3]).

Summing up these facts, from Theorem C and Remark 4.8, we obtain:

**Result 3.** *If $a_1a_3<0$ and $c=0$, then there is at most one limit cycle, which is hyperbolic, for system (12). Moreover, if $a_3<0$ ($>0$), then the limit cycle is stable (unstable).*

In fact, it is already known that system (12, $c=0$) has at most one limit cycle, see [22], and that there are cases in which the system actually has a limit cycle. However, here we have also proved its hyperbolicity. In Figure 5, we show some orbits of the phase portrait when $a_1=1$, $a_3=-1$, $c=0$, and also the limit cycle.

In the following example we show the independence of Theorem C compared to some other known results.

**Example 5.5.** Consider the following system, introduced by Conti in [6]:

$$\begin{cases}\dot{x}=x\frac{cx^2-1}{cx^2+1}-y,\\ \dot{y}=x,\end{cases}\tag{14}$$

where $c$ is a real positive constant.

It is clear that system (13) satisfies the hypothesis (H1). Using the same notation as in Theorem C, we have that $L(x)=x$, $H(y)=y$, $F_i(z)=(-1)^i\sqrt{2z}\frac{2cz-1}{2cz+1}$ and $\mathcal{G}_i(z,y)=2z-F_i(z)y+y^2$, for $i=1,2$. We notice that $\mathcal{G}_i(z,y)$ can be seen as a quadratic polynomial in the variable $y$. From this, it is easily seen that the hypothesis (H4) of Theorem C follows.

In addition,

$$\begin{aligned}(-1)^ik\frac{\partial}{\partial z}\left(\frac{F_i(z)}{\varphi(H^{-1}(k\sqrt{2z}))}\right) &= (-1)^ik\frac{\partial}{\partial z}\left(\frac{F_i(z)}{k\sqrt{2z}}\right)\\ &= \frac{\partial}{\partial z}\left(\frac{2cz-1}{2cz+1}\right)=\frac{4c}{(2cz+1)^2}>0,\end{aligned}$$

and so, hypothesis (H5) is fulfilled. We conclude, by Theorem C, that system (13) has at most one limit cycle and, if it exists, it is hyperbolic and unstable.

In fact, system (13) is a simple Liénard system, but it is a good example to compare Theorem C to other results on uniqueness of limit cycles for systems of Liénard type. For instance, to apply Zhang's theorem (see [35]) we would need to prove that $\frac{F'(x)}{g(x)} = \frac{cx^4+4c^2x^2-1}{x(cx^2+1)^2}$ is monotone; but, in this case it is not. Similarly, the hypothesis that $F_i(z)F_i'(z)$, is non decreasing $i = 1$ or 2, of Zeng's theorem (see [34]) does not hold here. Moreover, system (13) neither satisfies some hypotheses of Rychkov's theorem (see [30], Lemma 2); for instance, that $F_2'(z) < 0$.

# References

[1] A.A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maier, "Qualitative Theory of Dynamic Systems on the Plane", Israel Program for Scientific Translations, Jerusalem, 1973.

[2] H.W. Broer, F. Dumortier, S.J. van Strien and F. Takens, "Structures in Dynamics", Studies in Mathematical Physics, Vol.2. Elsevier Science Publishers, B.V. North-Holland, 1991.

[3] M. Carbonell and J. Llibre, Limit cycles of a class of polynomial systems , Proceedings of the Royal Society of Edinburgh. Sect.A 109(1988), 187–199.

[4] K.S. Cheng, S.B. Hsu and S.S. Lin, Some results on global stability of a predator-prey system, Journal of Mathematical Biology, 12(1981), 115–126.

[5] L.A. Cherkas, Estimation of the number of limit cycles of autonomous systems, Differential Equations, 13(1977), 529–547.

[6] R. Conti, Soluzioni periodiche dell'equazione di Liénard generalizatta. Esistenza ed unicità , Bolletino della Unione Matematica Italiana, 3(1952), 111–118 .

[7] W.A. Coppel, Some quadratic systems with at most one limit cycle, Dynamics Reported, John Wiley and Sons Ltd, 2(1988), 61-88.

[8] Ding Sunhong, On a kind of predator-prey system, SIAM Jour. Math. Anal., 20(1989), 1426–1435.

[9] Ding Sunhong, Global structure of a kind of predator-prey system, Applied Mathematics and Mechanics, Shangai(English edition), 9(1988), 999–1003.

[10] F. Dumortier, "Singularities of vector fields", Monografias de Matemática, 32. IMPA, 1978.

[11] F. Dumortier and C. Rousseau, Cubic Liénard equations with linear damping, Nonlinearity, 3(1990), 1015–1039 .

[12] A.F. Filippov, Sufficient conditions for the existence of stable limit cycles of second order equations, Mat. Sb., 72(1952), 171-180, (in Russian).

[13] A. Gasull, Differential equations that can be transformed into equations of Liénard type, Actas del XVII Colloquio Brasileiro de Matemática, IMPA, Rio de Janeiro, 1989.

[14] A. Gasull and A. Guillamon, Non-existence of limit cycles for some predator-prey systems, Proceedings of Equadiff'91. World Scientific, Singapore, 1993.

[15] G.F. Gause, The struggle for existence", Baltimore, Williams and Wilkins, 1934, new edition New York, Dover, 1971.

[16] K.P. Hadeler and K.R. Schneider, Periodic solutions of scalar first order differential equations , Preprint, Karl-Weierstrass-Institut Für Mathematik, Berlin 1991.

[17] J. Hale, "Ordinary Differential Equations", Pure and Applied Mathematics, 21, 1980.

[18] G.W. Harrison, Global stability of predator-prey interactions, J.Math.Biology, 8(1979), 159–171.

[19] Huang Xun-Cheng, Uniqueness of limit cycles of generalized Liénard systems and predator-prey systems, J. Phys. A: Math. Gen., 21(1988), L685–L691.

[20] Y. Kuang and H.I. Freedman, Uniqueness of limit cycles in Gause-type models of predator-prey systems, Math. Biosciences, 88(1988), 67–84.

[21] A. Liénard, Étude des oscillations entretenues, Revue générale de l'eléctric., 23(1928).

[22] A. Lins, W. de Melo and C.C. Pugh, On the Liénard's equation, Lecture Notes in Mathematics, Springer Verlag, 597(1977), 335–357.

[23] N.F. Lloyd, A note on the number of limit cycles in a certain two-dimensional systems, J. London Math. Soc., 20(1979), 277–286.

[24] A.J. Lotka, "Elements of physical biology", Baltimore, Williams and Wilkins, 1925, new edition, "Elements of mathematical biology", New York, Dover, 1956.

[25] J.L. Massera, Sur un théorème de G.Sansone sur l'équation de Liénard, Bolletino della Unione Matematica Italiana, 9(1954), 367–369.

[26] R.M. May, "Stability and complexity in model ecosystems", Princeton, Princeton University, 1973.

[27] H.N. Moreira, H.N. Unicidade de ciclos limite em sistemas presa-predador, Ph. D. Thesis, IMPA, Rio de Janeiro, 1989.

[28] H.N. Moreira, On Liénard's equation and the uniqueness of limit cycles in predator-prey systems, J.Math.Biology, 28(1990), 341–354.

[29] H. Poincaré, Mémoire sur les courbes définies par une équation differentielle, Œuvres d'Henri Poincaré, 1(1951), Paris, Gauthiers-Villars et Cie.

[30] G.S. Rychkov, A complete investigation of the number of limit cycles of the equation $(b_{10}x + y)dy = \sum_{i+j\geq 1}^{2} a_{ij}x^i y^j \, dx$, Differential Equations, 6(II) 1970, 1665–1670.

[31] G. Sansone and R. Conti, "Nonlinear differential equations", Macmillan, New York, 1964.

[32] W. Szlenk, "Matemáticas aplicadas a la Ecología", Centro de Invest. y Estud. Avanzad. del IPN, México, 1987.

[33] V. Volterra, Leçons sur la théorie mathématique de la lutte pour la vie, Paris, Gauthiers-Villars, 1931.

[34] Zeng Xianwu, Zhang Zhifen and Gao Suzhi, On the uniqueness of limit cycle of generalized Liénard equation, 1991, Preprint.

[35] Zhang Zhifen, Proof of the uniqueness theorem of limit cycles of generalized Liénard equations, Applicable Analysis, 23(1986), 63–76.

[36] H. Zoladek, The solution of the center-focus problem, 1992, Preprint.

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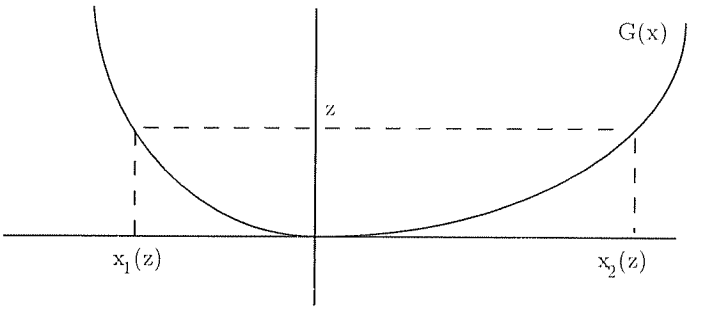



Figure 1 The graph of $G(x)$ and the definition of $x_i(z)$, for $i = 1, 2$

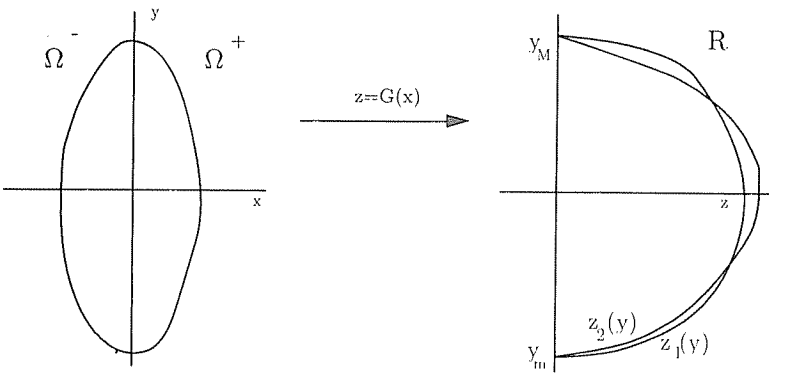



Figure 2 The Filippov transformation divides the periodic orbit in two orbits in the half-plane $z \geq 0$

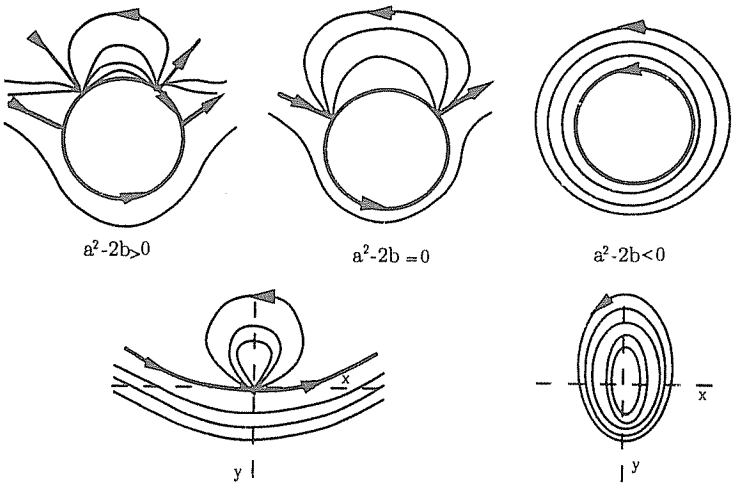



**Figure 3.** In the first row, the phase portraits obtained after applying the quasi-homogeneous blowing-up are shown. In the second row, we give those of the Liénard system when $a^2 - 2b \geq 0$ and $a^2 - 2b < 0$, respectively.

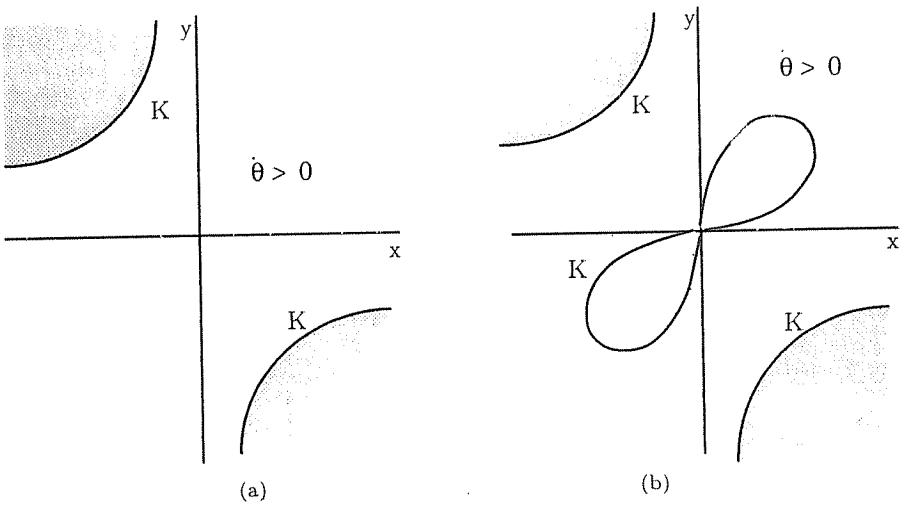



Figure 4. Region $\dot{\theta} > 0$ for system (12, $c = 0$), with $a_1 a_3 < 0$.
(a) $a_1 \in [0, 2]$; (b) $a_1 > 2$.

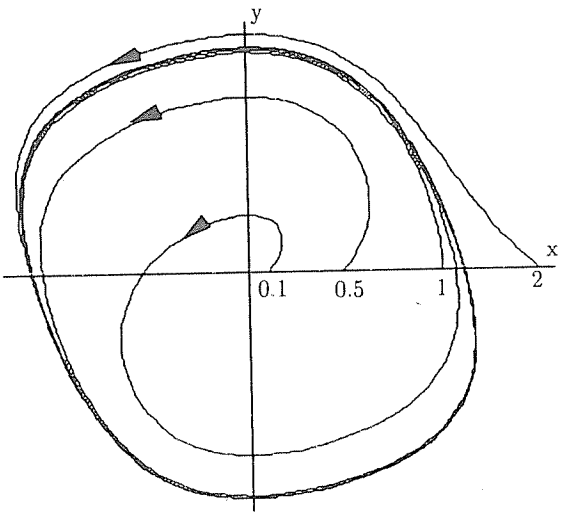



Figure 5.