On the other hand, one may think that the function $	ext{dir} \chi$ is not the only one that can control this kind of fact. Roughly speaking, what one needs to control in these problems is the tendency of the trajectories of the flow (1) to join $u_{\omega}$ around the equilibrium points, or the non-existence of periodic orbits, or such as the classification of critical points, the study of stability of the critical points, etc.

(1) \[ \frac{d}{dt} (\chi(t), x) = x \left( (x) \frac{\partial (x)}{\partial x} X = \frac{\partial}{\partial x} X \right) \]

Equations of this type appear in many problems related to systems of ordinary differential equations on the plane or in the real plane. We will present and show results.

I. Introduction and main results

In the first section of geometric quaternions, we present the first three derivatives of the flow at a point in the global manifold to the general case, and also provide a global study of the flow at a point on the manifold. Under this assumption, we classify the possible critical points and give a criterion for the stability of the flow. Given a vector field on the plane, we study the influence of the critical points.

Acknowledgments

Received 3rd December 1949

Address: GASLITAI, MALVA, TS.

Department of Mathematics, University of Barcelona, 80139, Barcelona.

A. GASLITAI and A. CUTTANON.

By R. A. GARIA

Geometric conditions for the stability of orbits in planar systems

Published in Geometric
A suitable candidate is found in the literature when one studies the stability of a periodic orbit of (1). In [12], the author gives a criterion (due to S. Diliberto), see also [3] (2) in which the stability is determined by means of the function \( K^-(x) \). The function \( K^-(x) \) that will be called \( \text{orthogonal curvature} \) associated to a point \( x \) of \( \mathbb{R}^2 \) seems to be a natural substitute of system (1)’s orthogonality because the orbits of (1) are orthogonal to those of (1), and so the share of these last ones indicates in some sense the tendency of the solutions of (1). Since the curvature is not defined at a critical point of (1), it will be more convenient to deal with the numerator of \( K^-(x) \), which will be denoted by \( K^-(x) \) and written

\[
K^-(x) = Q_x = P_x + Q_x \cdot P + Q_x + P_x Q_x,
\]

when we consider \( K^-(x) = (P_x + Q_x \cdot P + Q_x + P_x) 
\). In this paper, we will simplify expressions like \( K^-(x) \). By \( K^-(x,y) \) or \( K^-(x,y) \), we understand the orthogonal vector field \( \mathbf{X} \) at the point \( (x,y) \), respectively.

The concept of orthogonal curvature can be generalized to \( \mathbb{R}^2 \) endowed with a Riemannian metric \( g \). In this context, a vector field \( \mathbf{X} \) is described also by \( (x,y) \) to emphasize the orthogonal vector field \( \mathbf{X} \) in the metric \( g \).

In the proofs of our results, the geometrical meaning of the orthogonal curvature will be very helpful and some of the proofs will be reached by using the orthogonality arguments rather than analytical. This is an advantage of this operator compared to the divergence. Section 2 is devoted to the main geometrical tools to be used in the proofs of the principal results, and mainly concerns systems in which the orthogonal curvature is negative. From now on, they will be called path-convergent systems.

In Section 3, we discuss the nature of the critical points for path-convergent systems, paying more attention to the case of non-degenerate points. The main results of that section can be synthesized as:

**Theorem A.** Let \( E \) and \( H \) represent the number of elliptic and hyperbolic sectors, respectively, of an isolated critical point \( p \) of system (1). Then, if \( K^-(x,y) = 0 \) in a punctured neighbourhood of \( p \),

(a) \( E = 0 \) and \( H \neq 0 \) if \( (r')^2 < 0 \) for some \( r \). Moreover, there are examples having zero or two hyperbolic sectors.

(b) \( H = \text{in addition det} \left( DX(p) \right) > 0 \), then \( H = 0 \) and \( p \) is locally asymptotically stable (unstable) assuming \( K^-(x,y) < 0 \) (\( K^-(x,y) > 0 \)).

In Section 4, we study the stability of periodic orbits for system (1) and we see that it is governed by the following result:

**Theorem B.** Let \( \gamma \) be the \( q \)-arc-length parameterization of periodic orbit \( \gamma(t) \) in \( \mathbb{R}^2 \) endowed with a Riemannian metric \( g \). Then \( \gamma \) is hyperbolic if \( \gamma(t) \cdot \gamma'(t) \) is constant, otherwise \( \gamma \) is elliptic or parabolic. Furthermore, if \( \gamma \) is hyperbolic, then \( \gamma \) is hyperbolic (unstable) for a parameter \( q \).

For the Euclidean metric, we also give expressions for the second and third derivatives of the return map associated to \( \gamma \) in terms of the curvatures, see Theorems 4 and Appendix A.
where all the functions $A$, $V$, and $\varphi$ depend on $x$, $y$, $d$, and $d_0$. For $n \geq 6$, we obtain the formula:

$$
\overline{\partial A} + \overline{\partial V} + \sum_{n=1}^{\infty} \overline{\partial A}^{n+1} = \overline{\partial A}^{n+1}
$$

The corresponding systems of differential equations:

$$(1) \quad (d_0^* \partial A) (x) = x$$

$$(1') \quad (d_0^* \partial A) (x) = x$$

In this section, we introduce some Peano's notion and technical lemmas.

In what follows, we prove:

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space.

Theorem 1. Suppose that $\mathbb{R}^n$ is $n$-dimensional Euclidean space.

Then the origin is a global attractor of $\mathbb{R}^n$. To be a global attractor, $\mathbb{R}^n$ satisfies the following conditions:

1. For all $x, y \in \mathbb{R}^n$, there exists a constant $C > 0$ such that $d(x, y) < C$.

2. For all $x, y \in \mathbb{R}^n$, there exists a constant $C > 0$ such that $d(x, y) < C$.

3. For all $x, y \in \mathbb{R}^n$, there exists a constant $C > 0$ such that $d(x, y) < C$.

In particular, when $\Omega$ is a connected, $\Gamma$-connected, theorem C requires that $\Omega$ be a-connected, and the $\partial$-connection-unique $\Omega$-connected.

Note that in $\Omega$-connected, $\Gamma$-connected, and $\partial$-connection-unique $\Omega$-connected.

Zerstörung des in planaren $\Omega$-Kernen.
Theorem 2.1. Let \( f(z) \) be an oriented simple regular \( C^2 \)-curve on the plane. Suppose that for any point \( z \) on the curve, there exists a unique \( \xi \) such that \( \frac{df}{dz}(z) = \xi \). Then the next lemma extends this notion of tangent to any point on the curve as defined by the right-hand coordinate function \( f'(x) \).

### Lemma

The right-hand coordinate function \( f'(x) \) is defined for all \( x \in I \), with \( I \) being the interval of definition of \( f(x) \). The value of \( f'(x) \) at any point \( x \) on the curve is the tangent vector to the curve at that point.

### Proof

1. **Existence and Uniqueness**: For any point \( x \) on the curve, there exists a unique \( \xi \) such that \( \frac{df}{dz}(x) = \xi \).

2. **Continuity**: The function \( f'(x) \) is continuous on \( I \).

3. **Orientation**: The orientation of the curve is preserved under \( f'(x) \).

4. **Differentiability**: The function \( f'(x) \) is differentiable on \( I \).

When a curve on the plane is defined by the left-hand coordinate function \( f''(x) \), then the next lemma extends this notion of tangent to any point on the curve as defined by the left-hand coordinate function \( f''(x) \).
The construction is inspired by the proof of Lemma 1 in Section 5 and makes use of the construction employed in the following section of parallel-conformal systems in regions with critical points. It will be applied in the construction of the curve of the constant sign of the curvature. For instance, see [1, Lemma 2].

Finally, the last result of this section describes the qualitative motion of the curve.

We remark that when $\kappa$ is a closed orbit, the Lemma implies the well-known result:

So the Lemma follows.

Theorem: The only way for $\Gamma(x,y)$ to arise is through the formation of a region containing $\kappa$ for $\Gamma(x,y)$. Then the curvature $\kappa$ of all $\Gamma(x,y)$ is the same.

Since $\kappa(0) = 0$ for all $\Gamma(x,y)$ and $\kappa(0) = 0$ for all $\Gamma(x,y)$, we have $\kappa(0) = 0$ for all $\Gamma(x,y)$.

Since $\kappa(0) = 0$ for all $\Gamma(x,y)$, we have $\kappa(0) = 0$ for all $\Gamma(x,y)$.

Therefore, the only way for $\Gamma(x,y)$ to arise is through the formation of a region containing $\kappa$ for $\Gamma(x,y)$. Then the curvature $\kappa$ of all $\Gamma(x,y)$ is the same.

Since $\kappa(0) = 0$ for all $\Gamma(x,y)$ and $\kappa(0) = 0$ for all $\Gamma(x,y)$, we have $\kappa(0) = 0$ for all $\Gamma(x,y)$.

Since $\kappa(0) = 0$ for all $\Gamma(x,y)$, we have $\kappa(0) = 0$ for all $\Gamma(x,y)$.

Since $\kappa(0) = 0$ for all $\Gamma(x,y)$, we have $\kappa(0) = 0$ for all $\Gamma(x,y)$.

Therefore, the only way for $\Gamma(x,y)$ to arise is through the formation of a region containing $\kappa$ for $\Gamma(x,y)$. Then the curvature $\kappa$ of all $\Gamma(x,y)$ is the same.

Since $\kappa(0) = 0$ for all $\Gamma(x,y)$ and $\kappa(0) = 0$ for all $\Gamma(x,y)$, we have $\kappa(0) = 0$ for all $\Gamma(x,y)$.

Therefore, the only way for $\Gamma(x,y)$ to arise is through the formation of a region containing $\kappa$ for $\Gamma(x,y)$. Then the curvature $\kappa$ of all $\Gamma(x,y)$ is the same.

Since $\kappa(0) = 0$ for all $\Gamma(x,y)$ and $\kappa(0) = 0$ for all $\Gamma(x,y)$, we have $\kappa(0) = 0$ for all $\Gamma(x,y)$.

Therefore, the only way for $\Gamma(x,y)$ to arise is through the formation of a region containing $\kappa$ for $\Gamma(x,y)$. Then the curvature $\kappa$ of all $\Gamma(x,y)$ is the same.

Since $\kappa(0) = 0$ for all $\Gamma(x,y)$ and $\kappa(0) = 0$ for all $\Gamma(x,y)$, we have $\kappa(0) = 0$ for all $\Gamma(x,y)$.

Therefore, the only way for $\Gamma(x,y)$ to arise is through the formation of a region containing $\kappa$ for $\Gamma(x,y)$. Then the curvature $\kappa$ of all $\Gamma(x,y)$ is the same.
Theorem 3.1. Assume that \( \text{H}\) holds for solution (1). Let \( B\) and \( H\) represent the

path-connected systems.

The purpose of the next result is to describe the critical points that can appear in

3. Critical points for path-connected solutions

variables: the lemma follows.

\[
xp \frac{\| (\tilde{h}, x) \|}{\| (\tilde{h}, x) \|} - BP \frac{\| (\tilde{h}, x) \|}{\| (\tilde{h}, x) \|} \int
\]

where we have used that on \( \delta \) and the integral

\[
(x, \tilde{h})^{\frac{3}{2}} \Phi - (x, \tilde{h})^{\frac{3\alpha}{2}} = np \int \frac{\| (i, h, t) \|}{\| (i, h, t) \|} \int
\]

\[
\left( \begin{array}{c}
\Phi \text{ of} \delta \text{ on} \delta
\end{array} \right) \int \left( \begin{array}{c}
\Phi \text{ of} \delta \text{ on} \delta
\end{array} \right) \int
\]

\[
xp \frac{\| (\tilde{h}, x) \|}{\| (\tilde{h}, x) \|} - BP \frac{\| (\tilde{h}, x) \|}{\| (\tilde{h}, x) \|} \int
\]

\[
= np \int \frac{\| (\tilde{h}, x) \|}{\| (\tilde{h}, x) \|} \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]

\[
= np \int (\text{pr} (\tilde{h}, x) \| (\tilde{h}, x) \|) \int
\]
where \( \mathbb{C} \) denotes the difference of \( \mathbb{C} \).

Theorem. If \( x \neq 0 \) then the solutions of the equation for \( \gamma > 0 \) are unique and depend only on the coefficients of \( \gamma \).

Proof. Assume \( x = 0 \) and \( \gamma > 0 \). Then the solution \( \gamma \) is unique and depends only on the coefficients of \( \gamma \).

The solution \( \gamma \) is unique and depends only on the coefficients of \( \gamma \).

We can also write the solution as

\[
\gamma = \frac{1}{\alpha} + \frac{\beta}{\alpha^2} + \frac{\gamma}{\alpha^3}
\]

where \( \alpha > 0 \) and \( \beta > 0 \).

Theorem. Assume \( \gamma > 0 \). Then the solutions of the equation for \( \gamma > 0 \) are unique and depend only on the coefficients of \( \gamma \).

Proof. Assume \( x = 0 \) and \( \gamma > 0 \). Then the solution \( \gamma \) is unique and depends only on the coefficients of \( \gamma \).

The solution \( \gamma \) is unique and depends only on the coefficients of \( \gamma \).

We can also write the solution as

\[
\gamma = \frac{1}{\alpha} + \frac{\beta}{\alpha^2} + \frac{\gamma}{\alpha^3}
\]

where \( \alpha > 0 \) and \( \beta > 0 \).

Theorem. Assume \( \gamma > 0 \). Then the solutions of the equation for \( \gamma > 0 \) are unique and depend only on the coefficients of \( \gamma \).

Proof. Assume \( x = 0 \) and \( \gamma > 0 \). Then the solution \( \gamma \) is unique and depends only on the coefficients of \( \gamma \).

The solution \( \gamma \) is unique and depends only on the coefficients of \( \gamma \).

We can also write the solution as

\[
\gamma = \frac{1}{\alpha} + \frac{\beta}{\alpha^2} + \frac{\gamma}{\alpha^3}
\]

where \( \alpha > 0 \) and \( \beta > 0 \).
Second part: non-existence of hyperbolic sectors whose angles are equal to \( \pi/2 \).

We suppose that \( \theta \) and \( \phi \) are consecutive separatrices.

First, we notice that the flow of \( X \) is not normal to its separatrices, because we observe that the separatrices are not tangent to the integral curve of \( X \).

Second, we suppose that the flow of \( X \) is not normal to its separatrices, because we observe that the separatrices are not tangent to the integral curve of \( X \).

Finally, we suppose that the flow of \( X \) is not normal to its separatrices, because we observe that the separatrices are not tangent to the integral curve of \( X \).

Therefore, the property implies that the existence of at most four hyperbolic sectors with angles less than \( \pi/2 \).

Proof: non-existence of hyperbolic sectors with angles less than \( \pi/2 \).

Consider the case where the angles are contained under the flow of \( X \). Where the angles could be introduced to the flow in a straight line, \( \theta < \pi/2 \), and where this angle is given by the angle between the separatrices. First, we will prove that \( \theta \) cannot have hyperbolic sectors whose separatrices

\[
\theta < \pi/2.
\]

On the other hand, since \( \theta < \pi/2 \), the angle, as between and \( \phi \), satisfies

\[
\phi < \pi/2.
\]
vector $X$ forms an angle less than $\pi/2$ with $y$ and observe that, since $\theta$, the $(\mu, \nu, \lambda)$-plane and choose a point $b = \theta$ such that for all $0 \leq t \leq 1$ the possible line $0 \leq t \leq 1$ is a point $b = \theta$ and consider the solution of (1), that passes through this point.

Finally, we prove the stable separatrix, we finally get that $f = 0$. For $f = 0$, $2.3.4$

Analogously, it can be seen that $\gamma$ is a nest coordinate with $\gamma$ and anything in a similar condition. If $\gamma$ intersects $\gamma$, we can perform similar arguments.

In particular, we get a positively invariant region $U_1$ for the flow of $X$, and we reach the same situation as the one in the Lemma 2.1. Since $\gamma$ and $\nu$-invariant region of $X$, we can consider the region with boundary given by $\gamma$. We can choose a point $p$ on $\gamma$, sufficiently close to the origin, such that $p$ lies in $U_1$. If some point on $\gamma$ does not belong to $U_1$ then it lies either in $\gamma$ or in $\gamma^c$. If it lies in $\gamma^c$ then we will show that $\gamma$ must coincide with $\gamma$.

We want to prove that under hypotheses (H1) this configuration is impossible.

$\xi$ and $\zeta$, assuming that $\gamma = \gamma_0$ and $\zeta$. For $\gamma = \gamma_0$ and $\zeta$, $\zeta$ is the open quadrant bounded by $\gamma$. By $\gamma_0$ and $\zeta$, the open quadrant bounded by $\gamma$.

In this case, we can assume that we have four separatrices, two of them unstable.

In Fig. 6, the labeling of the configuration of four hyperbolic sectors.

\[ \text{Fig. 6: The labeling of the configuration of four hyperbolic sectors.} \]

\[ \text{(a) The region is associated to a hyperbolic sector.} \]

\[ \text{(b) The vertex flags (black arrows)} \]

\[ \text{(c) The vertex flags (black arrows)} \]

\[ \text{Orbits in planar systems} \]

\[ 207 \]
If the first possibly occurs, then the lengths of the triangle $x$ and $x'$ would

$$d = (x) f^{-1} (x) f^{-1}$$

Thus there are only two possibilities: either that $x$ is not 

Therefore $d$ is the length of the path of the region $x$ to the

We denote a map $x \mapsto (x)$ such that $x$ where $x = (x) f^{-1} (x)$

In such a way that $d$ lies on the right-hand side of $b$ and can be done.

Now consider the critical points of passing through $b$ and, for negative $b$, take a point $b \in (b) f^{-1} (b)$ which is on the right-hand side of $b$ and $x \in (x) f^{-1} (x)$.

Lemma 3. Suppose that a critical point lies on a rectangle around $x$.

Proof of Lemma 3.

As before, $q \in (q) f^{-1} (q)$.

Figure 8. The construction of Lemma 3.

Figure 7. The situation given by the existence of a saddle with separatrices at the line.
If the divergence in the problem of the stability of periodic orbits is described in detail, we justify why the orthogonality criteria play the same role.

In the proof of the theorem, the exact coordinates are not considered near the cycle.

**Theorem 1.** Stability of the cycles of a limit cycle can be known also as the theorem of 10.

**Generalization for Nonlinear systems.** which also includes theorem 2 of 11.

For the theorem in the previous section, we can develop it into a more general form. We present in theorem 1 an equivalent criterion in a more general form of the periodic as the periodic orbit of a periodic orbit.

A summary of the previous theorem is to determine the stability of a periodic orbit.

4. Stability of periodic orbits for path-connected systems.

Theorem 1 follows from the proof of theorems 3 and 2.

A stable focus which completes the proof.

Applying the Paley-Chebyshev lemma, it is not difficult to see that it can only be a saddle point with a focus. If the critical point is either a focus of a contour-focus, then

\[ 0 = \alpha + q \neq 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0) \quad \text{or} \quad 0 = \alpha - q \neq 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0) \quad \text{or} \quad 0 = \alpha - q \neq 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0)
\]

Hence it can only happen that a case in such a case

\[ 0 \neq p = 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0) \quad \text{or} \quad 0 \neq p = 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0)
\]

First of all, observe that if the point is stable, it is not difficult to see that it can only be a saddle point with a focus. We will explore all the possibilities to see that only saddle nodes and stable focus are compatible with the only possible point at the origin. We can write system (1) as:

\[ (h \cdot x) \theta + b_{x_0} + x_0 = 0 \]

Suppose that we have a critical point at the origin. We can write systems (2):

\[ (h \cdot x) \theta + b_{x_0} + x_0 = 0 \]

The critical points can only be nodes, focus, or critical-focus.

Theorem 2 of 13 in the previous section, we can develop it into a more general form. We present in theorem 3 an equivalent criterion in a more general form of the periodic as the periodic orbit of a periodic orbit.

A summary of the previous theorem is to determine the stability of a periodic orbit.

5. Stability of periodic orbits for path-connected systems.

Theorem 1 follows from the proof of theorems 3 and 2.

A stable focus which completes the proof.

Applying the Paley-Chebyshev lemma, it is not difficult to see that it can only be a saddle point with a focus. If the critical point is either a focus of a contour-focus, then

\[ 0 = \alpha + q \neq 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0) \quad \text{or} \quad 0 = \alpha - q \neq 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0) \quad \text{or} \quad 0 = \alpha - q \neq 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0)
\]

Hence it can only happen that a case in such a case

\[ 0 \neq p = 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0) \quad \text{or} \quad 0 \neq p = 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0)
\]

First of all, observe that if the point is stable, it is not difficult to see that it can only be a saddle point with a focus. We will explore all the possibilities to see that only saddle nodes and stable focus are compatible with the only possible point at the origin. We can write system (1) as:

\[ (h \cdot x) \theta + b_{x_0} + x_0 = 0 \]

Suppose that we have a critical point at the origin. We can write systems (2):

\[ (h \cdot x) \theta + b_{x_0} + x_0 = 0 \]

The critical points can only be nodes, focus, or critical-focus.

Theorem 2 of 13 in the previous section, we can develop it into a more general form. We present in theorem 3 an equivalent criterion in a more general form of the periodic as the periodic orbit of a periodic orbit.

A summary of the previous theorem is to determine the stability of a periodic orbit.

5. Stability of periodic orbits for path-connected systems.

Theorem 1 follows from the proof of theorems 3 and 2.

A stable focus which completes the proof.

Applying the Paley-Chebyshev lemma, it is not difficult to see that it can only be a saddle point with a focus. If the critical point is either a focus of a contour-focus, then

\[ 0 = \alpha + q \neq 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0) \quad \text{or} \quad 0 = \alpha - q \neq 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0) \quad \text{or} \quad 0 = \alpha - q \neq 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0)
\]

Hence it can only happen that a case in such a case

\[ 0 \neq p = 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0) \quad \text{or} \quad 0 \neq p = 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0)
\]

First of all, observe that if the point is stable, it is not difficult to see that it can only be a saddle point with a focus. We will explore all the possibilities to see that only saddle nodes and stable focus are compatible with the only possible point at the origin. We can write system (1) as:

\[ (h \cdot x) \theta + b_{x_0} + x_0 = 0 \]

Suppose that we have a critical point at the origin. We can write systems (2):

\[ (h \cdot x) \theta + b_{x_0} + x_0 = 0 \]

The critical points can only be nodes, focus, or critical-focus.

Theorem 2 of 13 in the previous section, we can develop it into a more general form. We present in theorem 3 an equivalent criterion in a more general form of the periodic as the periodic orbit of a periodic orbit.

A summary of the previous theorem is to determine the stability of a periodic orbit.

5. Stability of periodic orbits for path-connected systems.

Theorem 1 follows from the proof of theorems 3 and 2.

A stable focus which completes the proof.

Applying the Paley-Chebyshev lemma, it is not difficult to see that it can only be a saddle point with a focus. If the critical point is either a focus of a contour-focus, then

\[ 0 = \alpha + q \neq 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0) \quad \text{or} \quad 0 = \alpha - q \neq 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0) \quad \text{or} \quad 0 = \alpha - q \neq 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0)
\]

Hence it can only happen that a case in such a case

\[ 0 \neq p = 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0) \quad \text{or} \quad 0 \neq p = 0 \Rightarrow (z_0 + h_{x_0} + x_0)(0, 0)X(0, 0)
\]
Applying directly the Poincaré's criterion, we need to parametrize the hyperplanes and change signs on them. Hence, to the order hand, the derivative is given by

\[ \dot{x} + x \beta - x = f \]

Substituting in the hyperplane, we get

\[ x + x \beta - x = f \]

This gives rise to the hyperplane orbit given by the equation

\[ x \cdot 0 = 0 \]

whenever \( t \) is a periodic orbit of the system.

Proof: First, it can be easily seen that \( Y \) is a periodic orbit of the system since

\[ x + x \beta - x = f \]

is stable and hyperbolically stable orbit for the planar system.

We want to present two examples to show that sometimes Poincaré's criterion is easier to apply than theorems B and C, while others require the converse happens. Here are two examples of this phenomenon.

\[ (x)(X) = \left( \frac{\|x\|}{\|X\|} \right)^{\delta_{LP}} \]

The equality

\[ \text{with} \quad \|x\|^\gamma \text{ and \( \|X\|^\gamma \)} \]

and finally, we obtain that

\[ \frac{\|x\|^\gamma}{\|X\|^\gamma} = (x)(X) \]

using first with \( (x)(X) = \|x\|^\gamma \).

Theorem

We know that the sign of \( \text{det}(x)(X)(\gamma) \) is always the same as the period of the periodic orbit.”

\[ \text{Proof of Theorem B. First, we recall that \( f \) is a C^1 function, then} \]

\[ (x)(X) + \frac{(x)^f}{(x)(X)(\gamma)} A = \frac{(x)^f}{(x)(X)(\gamma)} A \]

R. A. CARAVIOLA, A. GASTIL AND A. GULLON
For a more geometric interpretation of the above theorem, see

\[ ((\lambda, \psi) X, d(\theta' x)) \]

Finally, set the change of coordinates

\[ ((\lambda, \psi) d, \mu) \]

and define \( a \) such that \((\lambda, \psi) X, d(\theta' x) = ((\lambda, \psi) X, a) \psi = (\theta' x) \).

\[ \text{For any point } (x, y) \text{ in a neighborhood of } (\theta' x) \text{ there exists some } s \text{ such that} \]

\[ \text{see} ((\lambda, \psi) X, a) \psi = (\theta' x) \]

Suppose that \( (x, y) \) is a periodic orbit of \((\lambda, \psi) X, a) \psi \).

For the sake of simplicity, we must compute the higher derivatives. In the following theorem, we consider the basis

\[ (1, 0) \]

necessary to introduce some notation.

Suppose the orbit \( \gamma \) is periodic. Let \( a \) be the projection of \((\lambda, \psi) X, a) \psi \) onto \((\theta' x) (x, y) \), and denote by \( (\lambda, \psi) X, a) \psi \) the solution of the equation \( \dot{x}(t) = (\lambda, \psi) X, a) \psi(x, y) \) in the neighborhood of \((\theta' x) (x, y) \), with initial conditions \((x, y), (\theta' x) (x, y) \).

The solution \( \gamma \) is given by the relation \( \gamma(t) = \psi(t) (x, y) \), where \( \psi(t) \) is the solution of the initial value problem

\[ \frac{d}{dt} \psi(t) = (\lambda, \psi) X, a) \psi(t) \]

with \( \psi(0) = (x, y) \).

From the proof of Theorem 1, it can be deduced that the first derivative of the solution map is given by the matrix

\[ \left[ \begin{array}{cc} \lambda & \psi \frac{d}{dt} \psi \end{array} \right] \]

which is invertible for \( \lambda > 0 \) and \( \psi \neq 0 \).

The equalities (1) and (2) make us think that, in the problems where they appear,
but we do not study this problem here.

\[ 0 \equiv \frac{\alpha - \beta \cdot \gamma}{\gamma - \alpha} \equiv \cdots \equiv \frac{\alpha}{\gamma} \equiv \gamma \]

when

\[ sp \left( \frac{\alpha - \beta \cdot \gamma}{\gamma - \alpha} \right) \]

the return map could be written as

From this result above it seems that a general expression for the nth derivative of

\[ sp \left( \frac{\alpha}{\gamma} \right)^n \]

0 = \( (0)_n \mu \) then

This assertion implies theorem 3 in [10]. If, moreover,

\[ sp \left( \frac{\alpha}{\gamma} \right)^n \]

then

0 = \( (0)_n \mu \) then

Observe that when \( 0 \equiv \gamma \), the orthogonal solution of the same problem

Orthogonal solution of the same problem

where \( Y(s) \) denotes the curvature of \( Y(s) \) and \( \gamma(\sigma) \), as usual, the curvature of the

\[ sp \left( \frac{\alpha}{\gamma} \right)^n \]

\[ sp \left( \frac{\alpha}{\gamma} \right)^n \]

\[ \gamma \]

\[ \gamma \]

\[ \gamma \]

Here, 

\[ \gamma \]

\[ \gamma \]

\[ \gamma \]

\[ \gamma \]
The scheme of the proof is based on proving that the basin of attraction of the origin in each of the phase planes is a union of periodic orbits, not graphs.

Theorem 1. Consider the system

\[ \dot{\mathbf{x}} = f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n \]

\[ f \in C^1(\mathbb{R}^n, \mathbb{R}^n) \]

We know from the properties of \( f \) that \( f = 0 \) has only one critical point \( x^* \). For \( x \neq x^* \), the vector field \( \mathbf{f}(x) \) is aligned with \( \mathbf{x} - x^* \), and the flow is given by

\[ \mathbf{x}(t) = \mathbf{x}(0) \exp(\int_0^t \mathbf{f}(\mathbf{x}(s)) \, ds) \]

for \( t \geq 0 \). The union of all forward orbits that escape to infinity is denoted by \( \mathcal{O}(x^*) \).

Theorem 1. Let \( I \) be a forward invariant set such that \( I \cap \mathcal{O}(x^*) = \emptyset \). Then \( \mathcal{O}(x^*) \) is a global attractor.

Proof. If \( \mathcal{O}(x^*) \) is not a global attractor, then there exists a sequence \( \{x_n\} \) such that \( x_n \to x^* \) and \( x_n \not\in \mathcal{O}(x^*) \) for all \( n \). Consider the sequence \( \{y_n\} = \{x_1, x_2, \ldots, x_n, \ldots\} \).

If \( y_n \to y^* \) then \( y^* \in \mathcal{O}(x^*) \). But since \( \mathcal{O}(x^*) \) is attracting, \( y^* \in \mathcal{O}(x^*) \) and hence \( y^* \not\in \mathcal{O}(x^*) \), which is a contradiction.

Therefore, \( \{y_n\} \) cannot converge to any point in \( \mathcal{O}(x^*) \). Thus, \( \mathcal{O}(x^*) \) is a global attractor.

\[ \Box \]
COROLLARY 6.1. Consider a graph \((G, A, \text{GEM})\) and a cutset (i) such that (H) holds in \(\mathbb{R}^2\) and suppose the point \(p^\ast\) is the unique terminal point and that \(p^\ast\) is locally combinatorially adequate. Then, it is also locally combinatorially adequate.

Proof: We get the same conclusion. If we observe carefully, the proof of the lemma follows from the observation that the point \(p^\ast\) is the only critical point not included in any compact set. So, if we assume that the only critical point not included in any compact set is the only critical point not included in any compact set, then the hypothesis (H) holds.

Finally, since \(f(x)\) goes to the origin and \(f(x)\) is not included in any compact set, the function \(f(x)\) tends to 0 as \(x\) tends to the origin. Then, \(f(x) = 0\) if and only if \(x = 0\). The result follows.

To show that \(B\) is also closed, let \((x_n)\) be a sequence in \(B\). Since \(B = \{x: f(x) = 0\}\), we have that \(f(x_n) = 0\).

For this purpose, we define

\[ f(x) = \begin{cases} 1 & \text{if } x \notin B \\ 0 & \text{if } x \in B \end{cases} \]

For all \(x\), we prove that \(f(x)\) is continuous. The proof follows since \(f(x)\) is continuous on \(\mathbb{R}^2\) and \(f(x) = 0\) for all \(x\) in \(B\).

To show that \(B\) is connected, let \(A, B\) be two open sets in \(\mathbb{R}^2\) that are disjoint and such that \(A \cup B = \mathbb{R}^2\). Then, \(A \cap B = \emptyset\).

Therefore, \(f(x) = 0\) for all \(x\) in \(B\). The result follows.

Theorem 1. The contradistancy situation in case that \(f(x) = 0\) and \(f(x) = 1\) is stable.
\[ \dot{\phi} = \phi, \quad \phi(0) = \phi_0 \]

where \( \phi \) is the phase variable. The system is periodic if \( \phi(t+T) = \phi(t) \) for some period \( T \). To find the period, we solve the differential equation

\[ \dot{\phi} = \sin(\phi) \]

with initial condition \( \phi(0) = \phi_0 \).

The period \( T \) is given by

\[ T = 2\pi \frac{1}{\sqrt{1 + \sin^2(\phi_0)}} \]

for \( \phi_0 \neq n\pi \), where \( n \) is an integer.

\[ \begin{align*}
\frac{d}{dt}(a \cdot \phi) &= a \dot{\phi} \\
\frac{d}{dt}(a \cdot \dot{\phi}) &= a \ddot{\phi}
\end{align*} \]

If we define

\[ \alpha = a \cdot \phi \]

we have

\[ \frac{d}{dt}(a \cdot \phi) = \alpha \ddot{\phi} \]

for \( \phi = \alpha \).

Consider the change of coordinates given in Section 4, where also

\[ X' = X, \quad Y' = Y - kX \]

with change of sign in \( k \).

We remark that in this case the linearized system is

\[ \begin{align*}
\dot{x} + \alpha x &= 0 \\
\dot{y} - k \dot{x} &= 0
\end{align*} \]

which has zeros at \( x = 0 \) and \( y = k \).

The system is stable if the real parts of the eigenvalues are negative.

The system is unstable if the real parts of the eigenvalues are positive.

The system is neutrally stable if the real parts of the eigenvalues are zero.

**Example:** Consider the system of differential equations in the plane:

\[ \begin{align*}
\dot{x} + \alpha x + \beta y &= 0 \\
\dot{y} - \gamma y - x &= 0
\end{align*} \]

which is equivalent to

\[ \begin{align*}
\dot{x} &= -\beta y \\
\dot{y} &= -x - \gamma y
\end{align*} \]

We can easily see that the origin is the unique critical point. Moreover, it is unstable.

It can be easily seen that the origin is the unique critical point. Moreover, it is not stable.

If \( \alpha = \beta = \gamma = 0 \), then the system becomes linear and can be analyzed using linearization techniques.

If \( \alpha = 0 \), the system is non-linear and cannot be analyzed using linearization techniques.

If \( \alpha = \beta = 0 \), the system is linear and can be analyzed using linearization techniques.

If \( \alpha = \beta = \gamma = 0 \), the system is non-linear and cannot be analyzed using linearization techniques.
\[ \beta (s) \cdot \chi (a \cdot s) \cdot \gamma (a \cdot s) \cdot \partial + (a \cdot s) \cdot \chi (a \cdot s) \cdot \partial = ((a \cdot s) \cdot \partial) \cdot \chi \]

Assume that

\[ \text{read the expression in } \text{the statement of the Theorem} \]

\[ \text{we will simplify denoted } \partial \text{ by } \gamma \text{ and } X \text{ and } \chi \text{ to which will be written the vector field } \vec{\chi} \text{ in terms of the frame } \{(a \cdot s) \cdot \gamma \} \text{ or } \{(a \cdot s) \cdot \chi \} \text{ which gives for geometrical meaning of } \gamma \text{ and } \chi \text{ to which red and } \{ \gamma = k \} = \gamma \text{ or } \{ \chi = s \} = \chi \text{ and } \gamma \text{ and } \chi \text{ to which red and } \} \text{ to which read and } \gamma \text{ and } \chi \text{ to which red and } \}

\[ \text{Then, if we take a transversal section } \chi \text{ to which read and } \gamma \text{ and } \chi \text{ to which red and } \}

\[ \text{so we can consider parameterization of } \chi \text{ as a result does not vanish in a neighborhood of } \]
\[ a \beta = \langle X^t (X \gamma - \gamma X) \rangle + \langle X (X \gamma - \gamma X) \rangle \]

and

\[ a \sigma = \langle X^t (X \gamma - \gamma X) \rangle + \langle X (X \gamma - \gamma X) \rangle \]

Differentiating at \( t = 0 \), we have

\[ (\sigma, Y) = (\sigma, Y) \]

Then, the required formulae give

\[ (\sigma, b) = (\sigma, b) \]

and

\[ (\sigma, d) = (\sigma, d) \]

Differentiating at \( t = 0 \), it follows that

\[ (\sigma, d) \]

From (6), we have

\[ (\sigma, b) \]

At this point, it remains only to understand the meaning of the terms \( \sigma \) and \( \sigma \). Then, substituting \( \sigma \) and \( \sigma \) and their derivatives and equalizing terms according to the powers on \( \sigma \) and \( \sigma \), we obtain

\[ (\sigma, b) \]

\[ (\sigma, d) \]

where

\[ (\sigma, b) = (\sigma, d) \]

From (8), we have

\[ (\sigma, b) = (\sigma, d) \]

The same arguments used with \( \sigma \) and \( \sigma \) give

\[ (\sigma, b) = (\sigma, d) \]

with

\[ \frac{\sigma}{\sigma} \]

and

\[ \frac{\sigma}{\sigma} \]

with

\[ \frac{\sigma}{\sigma} \]

with

\[ \frac{\sigma}{\sigma} \]

with

\[ \frac{\sigma}{\sigma} \]

with

\[ \frac{\sigma}{\sigma} \]
The proof ends by substituting the last expression in
\[ (s)_{\mathcal{Y}}(s)_{\mathcal{Y}} + \frac{\partial y}{\partial x} (s)_{\mathcal{Y}} + (s)_{\mathcal{Y}} = (s)_{\mathcal{Y}} \]
and
\[ (s)_{\mathcal{Y}}(s)_{\mathcal{Y}} + (s)_{\mathcal{Y}} = (s)_{\mathcal{Y}} \]
so that \( (s)_{\mathcal{Y}} = (s)_{\mathcal{Y}} \)
and
\[ (s)_{\mathcal{Y}} = (s)_{\mathcal{Y}} \]

Next, form in terms of curvatures.

As a consequence, the first term of the expansion of \( (s)_{\mathcal{Y}} \) and \( (s)_{\mathcal{Y}} \) present the

\[ (s)_{\mathcal{Y}}(s)_{\mathcal{Y}} - (s)_{\mathcal{Y}} = (s)_{\mathcal{Y}} \]

In a similar way, we could obtain further derivatives. Particularly,

\[ \frac{\partial y}{\partial x} = y \]

and

\[ z = \frac{\partial y}{\partial x} \]

Then, since \( \frac{\partial x}{\partial z} = (x)_{\mathcal{Y}} \), we have that

\[ (x)_{\mathcal{Y}} = (x)_{\mathcal{Y}}(x)_{\mathcal{Y}} = (x)_{\mathcal{Y}}(x)_{\mathcal{Y}}(x)_{\mathcal{Y}} \]

and so, restricting to \( z \), we obtain:

\[ 0 = (x)_{\mathcal{Y}}(x)_{\mathcal{Y}} + (x)_{\mathcal{Y}}(x)_{\mathcal{Y}} + (x)_{\mathcal{Y}}(x)_{\mathcal{Y}} \]

and

\[ 0 = ((x)_{\mathcal{Y}})(x)_{\mathcal{Y}} + ((x)_{\mathcal{Y}})(x)_{\mathcal{Y}} + ((x)_{\mathcal{Y}})(x)_{\mathcal{Y}} \]

To give a geometrical interpretation of these inner products, consider the

H. A. GARCIA, A. GASAUL, AND A. GELLON
REFERENCES

As we wanted to prove,

\[ \mathcal{H} = (t_\theta \cdot t_\phi) (g p) = (t_\theta \cdot \mathcal{T}) (g p) \]

\[ = (t_\theta \cdot \mathcal{D} \cdot t_\phi) (g p) = (t_\theta \cdot \mathcal{D} \cdot t_\phi) \partial = (t_\theta \cdot \mathcal{D} \cdot t_\phi) \partial = \mathcal{Y} \]

\[ \begin{align*}
\text{where } & \mathcal{Y} \text{ is the covariant derivative, } \partial \text{ is the differential of one-form } \alpha, \\
\text{and the expression for the differential of one-form } \alpha. \\
\text{On the other hand, the geodesic curvature in the direction } \mathcal{Y}_p \text{ is defined by the } \\
\text{expression } & \theta \wedge \mathcal{Y}_p = \mathcal{Y} p, \\
& \theta \wedge \mathcal{Y}_p = \mathcal{Y} p. \\
\text{In particular, } & \text{ the volume element } \mathcal{L} \text{ is the Lie derivative of } \mathcal{X} \text{ with respect to } \mathcal{Y} p,
\end{align*} \]

\[ O(\text{plane schemes}) \]