Graphs and matroids determined by their Tutte polynomials

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Introduction

It is usual to associate invariants to combinatorial objects, in particular to graphs and matroids, in order to study some of their properties. These invariants can be of various types: structural, numerical, algebraic, or polynomial. This work is devoted to one such invariant, the Tutte polynomial, which has received much attention over the years. The Tutte polynomial is a two-variable polynomial originally defined for graphs by Tutte and Whitney and later generalized to matroids by Crapo. It was first conceived as an extension of the chromatic polynomial, but nowadays it is known to have applications in many areas of combinatorics and other areas of mathematics. One of its striking features is that it contains a great deal of information about the underlying graph or matroid. For instance, from the knowledge of the Tutte polynomial one can deduce enumerative results on bases, flat cardinalities, colourings, and orientations, and also structural properties such as connectivity. It is then a natural question to what extent this information determines the matroid or graph up to isomorphism. That is, given two matroids with the same Tutte polynomial, are they necessarily isomorphic? It is well-known that the answer to this question is “no”. This thesis focuses precisely on graphs and matroids that do have this uniqueness property, which we call T-uniqueness.

A large part of this thesis consists of proving that several well-known families of graphs and matroids are T-unique. We study specific families and also try to provide methods that have a wide range of application. We are able to prove the T-uniqueness in many particular cases, but, unfortunately, it seems rather difficult to find general theorems that guarantee T-uniqueness from a few given properties. The closest we can get to this is finding general frameworks that provide strategies for proving T-uniqueness for some classes of graphs and matroids. In some other cases we show that, even if a given family is not T-unique, its members can be recognized from the knowledge of their Tutte polynomials.

Another contribution of this thesis is finding families of T-unique matroids whose growth is exponential in the rank. This is interesting since it has been proved that asymptotically almost no matroid is T-unique; essentially, this is due to the fact that asymptotically the number of matroids is much larger than the number of possible Tutte polynomials. In contrast, it is conjectured that most graphs are T-unique, although we seem still to be far from a solution.

T-uniqueness results give us information on the strength of the Tutte polynomial as an invariant and allow us to compare it to other polynomial invariants, such as the chromatic polynomial. Also, a proof of the T-uniqueness of a graph or a matroid provides a characterization of this object in terms of some basic invariants. Indeed, essential ingredients in proving that a given graph or matroid is T-unique are the numerical invariants that are contained in the Tutte polynomial. Although there are many of them, only a few turn out to be really useful for our purposes. In fact, for most properties it is not known whether they are deducible from the Tutte polynomial, and actually many of the ones we would like to have are not.

The structure of the dissertation is as follows. Chapter 1 starts by reviewing the basics of graph and matroid theory; then we introduce the Tutte polynomial and the concept of T-uniqueness, discuss their properties, and give a survey of known results and invariants. Chapter 2 proves a 30 year old conjecture by Brylawski that states that the Tutte polynomial of a connected matroid is irreducible. From this it follows that the decomposition into connected components of a matroid is in exact correspondence with the factorization of its Tutte polynomial. Chapters 3, 4 and 5 are devoted to proving that several families of graphs are T-unique. In the first we deal with well-known families of graphs, such as complete multipartite graphs, wheels, squares of cycles, ladders, and hypercubes. Chapter 4 is completely devoted to the proof of the T-uniqueness of the toroidal grid; the nature of this example leads us to develop a more involved argument that has as one of its main steps a classification theorem for graphs satisfying a local condition. In Chapter 5 we give a technique that can be applied to prove the T-uniqueness of several families of line graphs. More concretely, we show that under certain hypotheses a graph with the same Tutte polynomial as a line graph is itself a line graph. We then apply this technique to the line graphs of regular complete multipartite graphs.

Chapters 6, 7, and 8 focus on T-unique matroids. Chapter 6 introduces the concept of a chordal matroid, and gives a sufficient condition for a chordal matroid to be T-unique; this condition is in terms of the existence of a bijection with certain properties. We apply this result to truncations of complete graphs and to cycle matroids of complete bipartite graphs. Chapter 7 defines generalizations of wheels, whirls, and spikes. Properties of these generalizations are discussed, and their T-uniqueness is proved. Finally, in Chapter 8 we find large families of T-unique matroids. Two such families are given; one of them is graphic, and the other consists of matroids that are representable only over sufficiently large fields. The dissertation ends with conclusions and open problems, and with an appendix of properties that cannot be deduced from the Tutte polynomial.

Some parts of this thesis have already been published or accepted for publication. Chapter 2 corresponds entirely to [39]; most of Chapter 3 is in [40]; the contents of Chapter 4 can be found in [38]; reference [11] contains Section 6.1 and Chapter 7.

1

Graphs, matroids, and Tutte polynomials

The first two sections of this chapter are devoted to review the basic terminology and notation of graph and matroid theory. In the third section we introduce the Tutte polynomial and recall its basic properties. The essential concept of T-uniqueness is introduced in Section 1.4, where we also survey known results. The last section is intended as a summary of various invariants determined by the Tutte polynomial that are useful in the following chapters.

1.1 Graph theory

Given that terminology in graph theory is not uniform, we review here the notation we use and define several concepts that appear frequently in this thesis. For all undefined notions, we refer to any introductory book such as [29].

Throughout this work G denotes a graph without isolated vertices, but possibly with loops and multiple edges. $V(G)$ and $E(G)$ denote, respectively, the set of vertices and the set of edges of G; if there is no danger of confusion, we write V, E. An edge of G with ends v and w is denoted by $\{v, w\}$ or vw. If vw is an edge, we say that v and w are *adjacent* and that the vertex v and the edge vw are *incident* (we do not use the notion of adjacent edges since in Chapter 4 it has a meaning different from the usual one). We denote by $N(v)$ the set of neighbours of v, that is, the set of vertices adjacent to v. We denote by $d(v, w)$ the *distance* between the vertices v and w, that is, the number of edges of the shortest path that links v and w. For a subset X of V, we denote by $G - X$ the graph that has $V - X$ as set of vertices and whose edges are the edges of G having both ends in $V - X$. If A is a subset of E, we denote by $G|A$ the graph (V, A).

A graph G is *connected* if any two of its vertices are linked by a path. It is called *k-connected* if it has at least $k + 1$ vertices and for every set $X \subset V$ with $|X| < k$, $G - X$ is connected. The greatest integer k such that G is k-connected is called the *connectivity* of G. A graph is *l-edge-connected* if for every set $A \subset E$ with $|A| < l$, the graph $(V, E - A)$ is connected. The greatest integer l such that G is l-edge-connected is called the *edge-connectivity* of G and it is denoted by $\lambda(G)$.

For every subset $A \subseteq E$, its *rank* is $r(A) = n - k(G|A)$, where $n = |V|$ and $k(G|A)$ is the number of connected components of $G|A$. Equivalently, it can be shown that $r(A)$ is the maximum size of an acyclic subgraph of $G|A$. We write $r(G)$ instead of $r(E)$. The *nullity* of an edge-set is $n(A) = |A| - r(A)$.

A cycle of length n is called an n-cycle and denoted by C_n; for $n = 3, 4, 5$ we also refer to them as triangles, squares, and pentagons. The minimum n such that G has an n-cycle is called the *girth* of G, denoted by $g(G)$. A *chord* of a cycle C is an edge joining two nonconsecutive vertices of C. A *clique of order* d, or a d*-clique*, is a subgraph isomorphic to the complete graph K_d.

A *colouring* of a graph G with λ colours is a mapping $c : V \to \{1, 2, \ldots, \lambda\}$ such that if vw is an edge of G then $c(v) \neq c(w)$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum integer λ such that there exists a colouring of G with λ colours. The *chromatic polynomial* is a one-variable polynomial $\chi(G; \lambda)$ such that, for all integer positive values of λ,

$$\chi(G; \lambda) = \text{ number of colourings of } G \text{ with } \lambda \text{ colours.}$$

It is not difficult to show that $\chi(G; \lambda)$ is actually a polynomial. The chromatic number is then the smallest positive integer that is not a root of $\chi(G; \lambda)$.

Given a graph G, an orientation $\vec{E}$ of its edges, and an abelian group Γ, a *nowhere-zero* Γ*-flow* is a mapping $f : \vec{E} \to \Gamma - \{0\}$ such that Kirchhoff's law is satisfied at every vertex. It can be shown (see Theorem 1.9) that the number of nowhere-zero Γ-flows of G does not depend either on the orientation $\vec{E}$ or on the group Γ, but only on the order of Γ. We denote by $\phi(G)$ the minimum order of a group Γ for which there exists a nowhere-zero Γ-flow in G. It also makes sense to define the *flow polynomial* as

$$\phi(G; x) = \text{ number of nowhere-zero } \Gamma\text{-flows of } G \text{ for } |\Gamma| = x.$$

For planar graphs, colourings and flows are strongly related.

Theorem 1.1 *If G is planar and G^* is a geometric dual, then the number of λ-colourings of G equals the number of nowhere-zero $\mathbb{Z}_\lambda$-flows of G^* times $\lambda^{k(G)}$. In particular, $\chi(G) = \phi(G^*)$.*

The connectivity, the girth, and the chromatic and flow polynomials are examples of graph invariants. More generally, a function f defined on graphs is a *graph invariant* if $f(G) = f(H)$ whenever G and H are isomorphic.

1.2 Introduction to matroid theory

In this section we summarize the matroid theory used in this dissertation. We give the basic definitions and examine some of the ways of producing a matroid from given ones (minors, duality, relaxation, truncations, and direct sums). We conclude with Whitney's 2-isomorphism Theorem relating matroid isomorphism and graph 2-isomorphism. We refer to Oxley's book [43] for further details and proofs of the results in this section.

1.2.1 Basic definitions

Matroids were introduced by Whitney [60] in 1935 as an abstraction of both linear independence and the properties of cycles in graphs. Matroids can be defined in several dozens of ways. Here we define them in terms of independent sets; other equivalent definitions are given below.

A *matroid* is a pair $(S, \mathcal{I})$, where S is a (finite) set and $\mathcal{I}$ is a collection of subsets of S satisfying the following properties:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.

(I3) If $I, J \in \mathcal{I}$ and $|J| < |I|$, then there exists an element $x \in I - J$ such that $J \cup x \in \mathcal{I}$.

The set S is called the *ground set* of M, and the sets in $\mathcal{I}$ are called *independent sets*. When several matroids are under consideration, we use the notation $S(M)$, $\mathcal{I}(M)$. An *isomorphism* between the matroids M and N is a bijection $\varphi : S(M) \to S(N)$ such that for each subset $I \subseteq S(M)$, I is independent in M if and only if $\varphi(I)$ is independent in N.

Examples. Let $G = (V, E)$ be a graph and let $\mathcal{A}$ be the set of acyclic subsets of E. Then $(E, \mathcal{A})$ is a matroid. We denote it by $M(G)$ and we call it the *cycle matroid of* G . If a matroid is isomorphic to the cycle matroid of some graph we say that it is a *graphic* matroid.

Another important family of matroids are *uniform* matroids. The uniform matroid $U_{r,m}$ has as ground set the set $[m] = \{1, 2, \ldots, m\}$ and a subset of $[m]$ is independent if it has r or fewer elements. The uniform matroid $U_{r,r}$ is the *free matroid*.

Given a field F and a matrix A over F with columns $c_1, \ldots, c_m$, we associate to A the matroid $M[A]$ that has $\{c_1, \ldots, c_m\}$ as ground set and whose independent sets are the sets of columns of A that are linearly independent over F. We say that $M[A]$ is *representable over* F. Matroids representable over GF(2) are called *binary*.

The subsets of the ground set that are not independent are called *dependent*; minimal dependent sets are *circuits* and maximal independent sets are *bases*. We denote by $\mathcal{C}(M)$ and $\mathcal{B}(M)$ the sets of circuits and bases of M. As in the graphic case, the *girth* of a matroid is the number of elements of its smallest circuit, and it is denoted by $g(M)$. An element x such that $\{x\}$ is a circuit is called a *loop*; hence loops are in no basis. If an element belongs to every basis then it is called an *isthmus*. If a set contains a basis, it is called *spanning*.

From the independence axioms it follows that all bases have the same cardinality; this cardinality is denoted by $r(M)$, the rank of the matroid. For any set $A \subseteq S$, the *rank* of A is the size of the largest independent set in A, and it is denoted by $r(A)$ (or $r_M(A)$ if confusion might arise); as in the graphic case, the *nullity* of A is $n(A) = |A| - r(A)$. If $r(\{x\}) = r(\{y\}) = r(\{x, y\}) = 1$, we say that the elements x and y are *parallel*. A matroid without loops or parallel elements is called a *geometry* (or a *simple matroid*).

A set $F \subseteq S$ is a *flat* if $r(F \cup x) = r(F) + 1$ for all $x \notin F$. The *closure* of a set $A \subseteq S$ is the smallest flat containing A; it is denoted by $\mathrm{cl}_M(A)$ or simply $\mathrm{cl}(A)$. Flats of rank 1, 2, 3, and $r(M) - 1$ are called respectively *points*, *lines*, *planes*, and *hyperplanes*. A set that is both a circuit and a hyperplane is called a *circuit-hyperplane*.

The sets of bases, circuits, and flats, the rank function, and the closure operator are enough to determine the matroid up to isomorphism. For all of them there exist results analogous to the following theorem.

Theorem 1.2 *Let $\mathcal{C}$ be a collection of subsets of a set S such that:*

(C1) $\emptyset \notin \mathcal{C}$.

(C2) *If $C \in \mathcal{C}$ and B is a proper subset of C, then $B \notin \mathcal{C}$.*

(C3) *If $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, then there exists $C \in \mathcal{C}$ such that $x \notin C$ and $C \subseteq (C_1 \cup C_2) - x$.*

Then there exists a unique matroid M on the ground set S having $\mathcal{C}$ as its set of circuits. In particular, the independent sets of M are the subsets of S that contain no member of $\mathcal{C}$.

Property (C3) is called the *circuit elimination axiom*.

A matroid *invariant* is a function f defined on matroids such that $f(M) = f(N)$ whenever $M \cong N$. For instance, the rank, the number of bases, and the girth are matroid invariants.

1.2.2 Matroid constructions

Given a matroid M on the ground set S and an element $x \in S$, we can produce two matroids on the ground set $S - x$. The *deletion* $M\backslash x$ is the matroid on $S - x$ having as independent sets

$$\mathcal{I}(M\backslash x) = \{I : I \in \mathcal{I}(M), x \notin I\}.$$

In particular, the bases of $M\backslash x$ are the bases of M that do not contain x, provided that x is not an isthmus.

If x is not a loop of M, the *contraction* M/x is the matroid on $S - x$ having as bases

$$\mathcal{B}(M/x) = \{B - x : B \in \mathcal{B}(M), x \in B\}.$$

When x is a loop, the contraction M/x is defined as $M\backslash x$.

If a matroid N can be obtained from M by a sequence of deletions and contractions, N is called a *minor* of M. In particular, for $N = M\backslash(S - T)$, we write $N = M|T$ and we say that N is the *restriction of M to T*. If $\mathcal{M}$ is a class of matroids such that for every $M \in \mathcal{M}$ all minors of M belong to $\mathcal{M}$, then $\mathcal{M}$ is *minor-closed*. An *excluded minor* for $\mathcal{M}$ is a matroid M such that $M \notin \mathcal{M}$ but all its minors belong to $\mathcal{M}$. For instance, graphic matroids and matroids representable over a field F are minor-closed classes.

The *dual matroid* of M, denoted by M^*, is the matroid on the same ground set S as M having as set of bases

$$\mathcal{B}(M^*) = \{B : S - B \in \mathcal{B}(M)\}.$$

If G is a planar graph and G^* is a geometric dual of G, then $M(G)^* = M(G^*)$. It is easy to show from the definitions that $(M\backslash x)^* = M^*/x$. A matroid is called *self-dual* if it is isomorphic to its dual.

Let H be a circuit-hyperplane of a matroid M; then the set $\mathcal{B}(M) \cup H$ is the set of bases of a matroid M', which is called a *relaxation* of M.

The *truncation to rank s* of a matroid M is the matroid that has as independent sets the independent sets of M of rank at most s; it is denoted by $T^s(M)$. For example, if $s < r$, we have that $T^s(U_{r,m}) = U_{s,m}$.

So far we have seen operations that produce a matroid from a given one. The simplest operation in matroid theory that joins two matroids into one is direct sum. Given two matroids M_1, M_2 with disjoint ground sets S_1, S_2, the matroid $M_1 \oplus M_2$ has as ground set $S_1 \cup S_2$ and as independent sets

$$\mathcal{I}(M_1 \oplus M_2) = \{I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1), I_2 \in \mathcal{I}(M_2)\}.$$

The matroid $M_1 \oplus M_2$ is called the *direct sum* of M_1 and M_2. If $M_1 = M(G_1)$ and $M_2 = M(G_2)$, then $M_1 \oplus M_2$ is isomorphic to both $M(G_1 \cup G_2)$ and $M(G_1 \odot G_2)$, where $G_1 \odot G_2$ is the graph obtained by identifying any vertex of G_1 with any vertex of G_2.

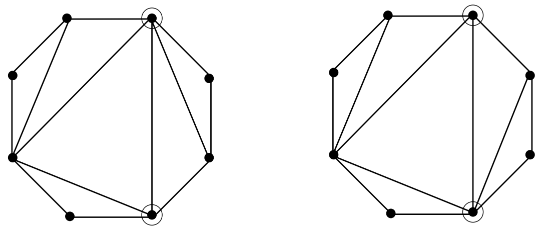


Figure 1.1 A Whitney twist about a 2-cut of a graph. The corresponding cycle matroids are isomorphic.

A matroid that cannot be expressed as the direct sum of two nonempty matroids is said to be *connected*. This is equivalent to the fact that for every pair of elements there is a circuit containing both. From this it follows that, for loopless G, the graphic matroid $M(G)$ is connected if and only if G is 2-connected.

1.2.3 Whitney's 2-isomorphism Theorem

Since graphic matroids only capture the structure of cycles of the underlying graph, it can be the case that two different graphs have isomorphic cycle matroids (recall the example above with direct sums). Whitney's 2-isomorphism Theorem [59] gives a necessary and sufficient condition for two graphic matroids to be isomorphic.

Two graphs G and H are said to be *2-isomorphic* if G can be transformed into H by means of the following two operations and their inverses:

(i) Identify two vertices in different connected components of G.

(ii) Suppose G is obtained from the disjoint graphs G_1 and G_2 by identifying the vertices u_1 of G_1 and u_2 of G_2, and v_1 of G_1 and v_2 of G_2. The *Whitney twist* of G is the graph obtained by identifying u_1 with v_2 and u_2 with v_1.

It should be clear that if two graphs are 2-isomorphic, then their cycle matroids are isomorphic, since the two operations above do not change the cycle structure (see Figure 1.1). Whitney's 2-isomorphism Theorem says that this is the only case where two graphic matroids can be isomorphic (see [43, Section 5.3] for a proof).

Theorem 1.3 *$M(G)$ and $M(H)$ are isomorphic if and only if G and H are 2-isomorphic. In particular, if G is 3-connected and $M(H)$ is isomorphic to $M(G)$, then H is isomorphic to G.*

1.3 The Tutte polynomial

The Tutte polynomial is one of the most studied polynomial invariants in combinatorics. As a graph invariant it was first defined by Tutte [54], although it turns out to be equivalent to the rank generating function of Whitney. It was extended to matroids by Crapo [27]. For a thorough survey on the properties and applications of the Tutte polynomial we refer to [24]. We also cite [58] for complexity issues and [8, Chapter X] as a reference for graphs that includes proofs of the basic facts.

The Tutte polynomial can be defined in several ways. For our purposes it is most interesting to view it as a reformulation of the generating function for rank and size. The *rank-size generating polynomial* of a matroid M with ground set S is defined as

$$F(M;x,y)=\sum_{A\subseteq S}x^{r(A)}y^{|A|}.$$

Hence the coefficient of x^iy^j in $F(M;x,y)$ counts the number of subsets of S with rank i and j elements. The *Tutte polynomial* is given by

$$t(M;x,y)=\sum_{A\subseteq S}(x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}.$$

The rank-size generating polynomial and the Tutte polynomial of a graph are defined analogously. In particular,

$$F(G;x,y)=F(M(G);x,y)\quad\text{and}\quad t(G;x,y)=t(M(G);x,y).$$

It should be clear that we can obtain either of $F(M;x,y)$ and $t(M;x,y)$ from the other one; actually, we have that

$$\begin{aligned}F(M;x,y)&=(xy)^{r(M)}t(M;\frac{xy+1}{xy},y+1),\\ t(M;x,y)&=(x-1)^{r(M)}F(M;\frac{1}{(x-1)(y-1)},y-1).\end{aligned}$$

Thus, both polynomials contain exactly the same information about M. But the Tutte polynomial has many properties not shared by the rank-size generating polynomial. Many of these properties translate combinatorial facts about a matroid or a graph into algebraic properties of their Tutte polynomials. We mention here the ones that are relevant to our work (see the references above for other interesting aspects as the deletion-contraction rule or the recipe theorem).

An easy to establish property is the following formula for the Tutte polynomial of a direct sum of matroids.

Theorem 1.4 *Let M be the direct sum of M_1 and M_2. Then*

$$t(M;x,y)=t(M_1;x,y)t(M_2;x,y).$$

In Chapter 2 we show that the converse also holds. A formula for the Tutte polynomial of the more complicated operation of generalized parallel connection has recently been found [12].

If $P(x,y)$ is a two-variable polynomial, we denote by $[x^iy^j]P(x,y)$ the coefficient of x^iy^j in $P(x,y)$. In particular, the coefficient of x^iy^j in the Tutte polynomial will be denoted by b_{ij} (the numbers b_{ij} are positive since they can be interpreted combinatorially as the number of certain types of bases, see [24, Section 6.6.A]). The following theorem summarizes some basic properties of the coefficients b_{ij}. In particular it says that we can read from the Tutte polynomial whether a matroid is connected and whether a graph is 2-connected.

Theorem 1.5 *The following hold for a matroid M with m elements and nullity $n(M)$.*

(i) $b_{00} = 0$ *if* $|S| \geq 1$*;*

(ii) $b_{10} \neq 0$ *if and only if* M *is connected;*

(iii) $x^k \,|\, t(M;x,y)$ *if and only if* M *has at least* k *isthmuses;*

(iv) $y^k \,|\, t(M;x,y)$ *if and only if* M *has at least* k *loops;*

(v) *if* $i \geq r(M)$ *or* $j \geq n(M)$*, then* $b_{ij} = 0$*, except if* $i = r(M)$ *and* $j = 0$*, or if* $i = 0$ *and* $j = n(M)$ *(in these two cases the coefficient* b_{ij} *is 1).*

The Tutte polynomial of the dual matroid M^* has a particularly simple expression in terms of $t(M;x,y)$.

Theorem 1.6 $t(M^*;x,y) = t(M;y,x)$.

In view of this theorem one might wonder whether self-dual matroids are the only matroids whose Tutte polynomial is symmetric in x and y. Qin [48] proved that this is not the case by constructing matroids that are not self-dual but whose Tutte polynomials are symmetric (another such example with graphic matroids can be found in the Appendix).

Besides its algebraic properties, one of the striking features of the Tutte polynomial $t(M;x,y)$ is that it contains a great deal of information about the matroid M. We say that a graph or matroid invariant is a *T-invariant* if its value can be deduced from the Tutte polynomial. For instance, the number of sets with given rank and size is one such invariant. We list some T-invariants here and in Section 1.5. For further invariants related to statistical physics, knot theory, coding theory, and other fields, see [24, 58].

Lemma 1.7 *The rank of a matroid* M *is the highest power of* x *in* $t(M;x,y)$*; the nullity of* M *is the highest power of* y*. The number of isthmuses (respectively, loops) of* M *is the maximum power of* x *(resp., of* y*) that divides* $t(M;x,y)$*.*

The following theorem contains results on the enumeration of independent and spanning sets, and of orientations of graphs.

Theorem 1.8 *For a matroid* M *and a connected graph* G*,*

(i) $t(M;1,1)$ *is the number of bases of* M *(spanning trees of* G *if* $M = M(G)$*);*

(ii) $t(M;2,1)$ *is the number of independent sets of* M *(forests in a graph); dually,* $t(M;1,2)$ *is the number of spanning sets of* M *(connected subgraphs in a graph);*

(iii) $t(G;2,0)$ *is the number of acyclic orientations of* G*;* $t(G;0,2)$ *is the number of totally cyclic orientations of* G*;*

(iv) $t(G;1,0)$ *is the number of acyclic orientations of* G *with a unique source at a fixed vertex.*

The original name of the Tutte polynomial, the dichromatic polynomial, reveals its strong relationship with colourings. Actually, the chromatic polynomial can be recovered from the Tutte polynomial by evaluating it at $y = 0$. Since for planar graphs colourings and flows behave dually, it is not surprising that the Tutte polynomial on the line $x = 0$ gives the flow polynomial.

Theorem 1.9 *The chromatic polynomial of a graph* G *is*

$$(-1)^{r(G)} x^{k(G)} t(G; 1-x, 0),$$

where $k(G)$ is the number of connected components of G.

The number of nowhere-zero Γ-flows of a graph G is

$$(-1)^{|E(G)|-r(G)}t(G;0,1-|\Gamma|).$$

1.4 Tutte uniqueness

In this section we define the concept of T-uniqueness and survey related results.

1.4.1 Definitions

We have just seen that the Tutte polynomial contains a great deal of combinatorial information. A question that arises naturally is whether a graph of matroid can be recovered up to isomorphism from its Tutte polynomial. This leads to the following definitions.

Definition 1.10 *Two matroids M and N are* T-equivalent *if $t(M;x,y) = t(N,x,y)$. A class of matroids $\mathcal{M}$ is* T-closed *if for every matroid $M \in \mathcal{M}$, all matroids T-equivalent to M belong to $\mathcal{M}$. A matroid M is* T-unique *if every matroid T-equivalent to M is isomorphic to M.*

We define analogous concepts for graphs. Recall that we assume that graphs have no isolated vertices.

Definition 1.11 *Two graphs G and H such that $t(G;x,y) = t(H,x,y)$ are* T-equivalent*. A class of graphs $\mathcal{H}$ is* T-closed *if for every graph $G \in \mathcal{H}$, all graphs T-equivalent to G belong to $\mathcal{H}$. A graph G is* T-unique *if every graph T-equivalent to G is isomorphic to G.*

In other words, a T-unique matroid (respectively, graph) is distinguished among all other matroids (resp., graphs) just by looking at its Tutte polynomial. For a T-closed class, we can decide membership to the class from the Tutte polynomial, but possibly some members of the class have the same Tutte polynomial.

T-uniqueness is the subject of this thesis. Most of our results consist of showing that several graphs and matroids are T-unique. We do this for individual families (Chapters 3, 4 and 7) and we also provide techniques that have a wider range of applicability (Chapters 5 and 6).

Before reviewing known results on T-uniqueness and T-equivalence, we discuss the relationship between the T-uniqueness of a graph and that of the associated graphic matroid. It should be clear that knowing that G is T-unique does not give much information about the T-uniqueness of $M(G)$, since in general one cannot deduce from the Tutte polynomial whether a matroid is graphic or not (see the examples in the Appendix). On the contrary, by Whitney's Theorem the T-uniqueness of the graphic matroid implies in most cases the T-uniqueness of the graph.

Corollary 1.12 *Let G be a 3-connected graph whose cycle matroid $M(G)$ is T-unique. Then G is T-unique.*

However, if the graph is not 3-connected, then we may possibly use either of the operations that define a 2-isomorphism to find another graph with the same cycle matroid, an hence with the same Tutte polynomial (for instance, the two graphs in Figure 1.1). Hence, we restrict our attention to 3-connected graphs.

Observe that we could have defined T-unique graphs as those graphs G such that if $M(G)$ and $M(H)$ are T-equivalent, then $M(G) \cong M(H)$. By Whitney's Theorem, we can replace the condition $M(G) \cong M(H)$ by "G and H are 2-isomorphic". Which definition one chooses does not make much difference, since they agree for 3-connected graphs; if the definition in terms of 2-isomorphism is chosen, then trees and forests are T-unique (since they are all 2-isomorphic). We have chosen to define T-uniqueness in terms of isomorphism rather than 2-isomorphism since the former is more natural for graphs.

1.4.2 Known results

We now turn to examine known results. The problem of finding graphs determined by polynomial invariants has been studied before for other polynomials, such as the characteristic polynomial (the polynomial having as roots the eigenvalues of the adjacency matrix, see [28] and the references there) and the matching polynomial (see [6]). The precedent that is most related to our problem is the study of chromatically unique graphs (χ-unique for short), that is, graphs determined up to isomorphism by their chromatic polynomials. Since the chromatic polynomial of a 2-connected graph can be recovered from its Tutte polynomial, we get that 2-connected chromatically unique graphs are also T-unique. References [34, 35] provide a thorough survey on chromatically unique graphs. These include in particular cycles, complete graphs, complete bipartite graphs, wheels with even rim, and several variations on wheels and complete graphs. One of the few nontrivial families of graphs that are known to be T-closed are series-parallel networks (see Brylawski [17] and Section 8.1).

There exist by now many results that imply the T-uniqueness of families of matroids. Most of these results arise from characterizations of matroids in terms of some invariants (mainly statistics on flats) that are determined by the Tutte polynomial. As an example, the following theorem of Bonin and Miller [15], together with results in the next section, implies the T-uniqueness of the affine geometry $AG(r-1,q)$ for $r \geq 4$.

Theorem 1.13 *Assume M is a rank-r geometry with q^{r-1} points in which lines have q points and planes have at least q^2 points. Then M is an affine geometry of order q.*

In the same paper [15] Bonin and Miller also characterize projective geometries $PG(r-1,q)$ in a similar way. Kung [37] recently proved an alternative characterization of projective and affine geometries in terms of very few of their Whitney numbers of the first and second kinds (the coefficients of the characteristic polynomial and the number of flats of each rank, respectively). Ankney and Bonin proved the T-uniqueness of the complement of a projective subgeometry inside of a projective geometry, that is, $PG(r-1,q)\backslash PG(k-1,q)$, for $r \geq 4$ and $1 \leq k \leq r-2$.

Another family that has received much attention are Dowling lattices $Q_r(\Gamma)$, where Γ is a (finite) group. They were introduced by Dowling [30] and, in some sense, they are to groups what projective geometries are to fields. Bonin and Miller [15] characterized $Q_r(\Gamma)$ in terms of the number of points, lines and planes of certain sizes; this characterization determines the order of Γ, but not the complete structure of the group. From this it follows that the class of Dowling lattices over a group of order g is T-closed, and if Γ is the only group with g elements (for instance, if Γ is $\mathbb{Z}_p$ with p prime), then $Q_r(\Gamma)$ is T-unique. This in particular implies that the cycle matroid of a complete graph $M(K_n)$ is T-unique, since $M(K_n)$ is isomorphic to $Q_{n-1}(\{1\})$, the Dowling lattice over the trivial group. In the same spirit, Sarmiento [50] and Qin [49] have proved the T-uniqueness of jointless Dowling lattices and that of complete principal truncations of Dowling lattices at modular flats (with the same restrictions on the group Γ as above).

A related question is that of finding the proportion of T-unique graphs among all graphs. Let $\mathcal{G}_n$ denote the set of graphs with n vertices, and $\mathcal{U}_n$ the set of T-unique graphs with n vertices. Bollobás, Pebody and Riordan [9] conjectured that

$$\lim_{n\to\infty} \frac{|\mathcal{U}_n|}{|\mathcal{G}_n|} = 1. \tag{1.1}$$

That is, that almost all graphs are T-unique. Actually, the same result is conjectured for chromatically unique graphs. There has been very little progress on either conjecture.

One way of proving that this conjecture is not true is finding many T-equivalent graphs; there are some known results along this line. The first nontrivial construction to find pairs of T-equivalent graphs was given by Tutte [55]; it gives rise to pairs of T-equivalent graphs with connectivities 3, 4, and 5. Brylawski [22] generalized these examples to arbitrarily high connectivity. A simpler construction was given by Bollobás, Pebody and Riordan [9]; this construction produces families of T-equivalent highly connected graphs that grow exponentially on the number of vertices. However, these results are not strong enough to give useful information about the limit (1.1).

The situation for matroids is completely different. It is known that most matroids are not T-unique. It follows from [24, Exercise 6.9] that the quotient of the number of possible Tutte polynomials for a matroid on m elements over the number of such matroids is at most

$$\frac{m!\ 2^{\frac{1}{4}(m+1)^3}}{2^{\frac{1}{2m}\binom{m}{\lfloor m/2 \rfloor}}},$$

that tends to zero as m tends to infinity. In spite of this negative result, in Chapter 8 we find exponentially many T-unique matroids.

There are other examples of families of matroids with the same Tutte polynomial besides the graphic families mentioned above. For instance, taking duals and direct sums of T-equivalent matroids gives other T-equivalent matroids. A less trivial construction was given by Bonin and Qin [16]; they use q-cones to give sets of T-equivalent matroids with arbitrarily high connectivity. Also, Bonin [10] uses the "complements" theorem of Brylawski [19, Proposition 5.9] and inequivalent representations of matroids over finite fields to produce many sequences of families of 3-connected T-equivalent representable matroids where the sizes of the families grow exponentially as a function of the rank.

We conclude this section with an application to reconstruction theory. The *deck of hyperplanes* of a matroid is the multiset of unlabeled hyperplanes; a matroid M is *hyperplane reconstructible* if any other matroid with the same deck of hyperplanes as M is isomorphic to M. Similarly, a matroid is *deletion reconstructible* if it can be recovered from the multiset of unlabeled single-element deletions. A graph G is *vertex reconstructible* if any other graph with the same multiset of vertex deletions is isomorphic to G. Miller [41] showed that if a matroid is hyperplane reconstructible then it is deletion reconstructible. Brylawski [21] showed that the Tutte polynomial of a matroid can be computed from its deck of hyperplanes, and Tutte proved that the Tutte polynomial of a graph can be computed from its deck of vertex deletions (see [56]). These observations lead to the following corollary.

Corollary 1.14 *T-unique matroids are hyperplane and deletion reconstructible. T-unique graphs are vertex and edge (deletion) reconstructible.*

1.5 Tutte invariants

The purpose of this section is to summarize the parameters that can be deduced from the Tutte polynomial and are used in the following chapters. We start with a list of matroidal T-invariants.

Theorem 1.15 *The following invariants of a connected matroid M on a set S can be deduced from its Tutte polynomial $t(M;x,y)$:*

(i) $r(M)$ *and* $|S|$;

(ii) *for each i with $0 \le i \le r(M)$, the number of independent sets of M of cardinality i;*

(iii) *the girth $g(M)$;*

(iv) *the number of circuits of M that have cardinality $g(M)$;*

(v) *for each i with $0 \le i \le r(M)$, the largest cardinality among all flats of M of rank i, and the number of rank-i flats of this cardinality.*

Whether M is a geometry can be deduced from $t(M;x,y)$. Furthermore, if M is a geometry, one can also deduce the following invariants from $t(M;x,y)$:

(vi) *for each integer j with $j \ge 2$, the number of lines of M that have cardinality j;*

(vii) *the number of 4-circuits of M.*

Proof. We only prove the less immediate statements. The girth $g(M)$ is the first integer l such that the coefficient of $x^{l-1}y^l$ in $F(M;x,y)$ is nonzero. The number of circuits of size $g(M)$ is this coefficient. A matroid is a geometry if its girth is at least three. To address assertion (vi), use (v) to deduce the number, say k, of lines of the largest cardinality, say t. Since we are assuming that M is a geometry, these k lines have no multiple points or loops. For i with $2 \le i \le t$, subtract $k\binom{t}{i}$ from the coefficient of x^2y^i in $F(M;x,y)$. From the resulting polynomial, one can deduce the second largest cardinality among lines of M, and the number of lines of M having this cardinality. Applying this idea recursively gives assertion (vi). For assertion (vii), note that the number of sets of size four and rank three is the coefficient of x^3y^4 in $F(M;x,y)$, and in a geometry such sets are of two types: 4-circuits and 4-sets that contain a unique 3-circuit. By assertion (vi), we know the number of 4-sets that contain a unique 3-circuit, so assertion (vii) follows. □

We remark here that the two matroids in Figure A.1 in the Appendix show that the simplicity assumption is necessary for assertions (vi) and (vii).

The ideas used to prove (vi) in the previous theorem can be extended to yield the following useful result. A stronger formulation was proved by Brylawski [20, Proposition 5.9].

Theorem 1.16 *For a rank-r matroid M and any integer i with $0 \le i \le r$, let c_i be the largest cardinality among rank-i flats of M. Then for each i with $1 \le i \le r$ and each j with $c_{i-1} < j \le c_i$, the number of flats of M having rank i and cardinality j can be deduced from the Tutte polynomial of M.*

We now turn to specific T-invariants for graphs. Even if the following theorem has some intersection with Theorem 1.15, we state it separately for the sake of completeness. Some of the invariants below are known to be χ-invariants, that is, they can be deduced from the knowledge of the chromatic polynomial.

Theorem 1.17 *Let G be a 2-connected graph. Then the following parameters of G are determined by its Tutte polynomial $t(G; x, y)$:*

(i) *the number of vertices and the number of edges;*

(ii) *the girth $g(G)$ and the number of cycles of shortest length $g(G)$;*

(iii) *for every k, the number of edges with multiplicity k; in particular, whether G is simple or not;*

(iv) *the edge-connectivity $\lambda(G)$; in particular, a lower bound for the minimum degree $\delta(G)$;*

(v) *the chromatic number $\chi(G)$.*

If G is simple, one can also deduce:

(vi) *the number of cliques of each order and the clique-number $\omega(G)$;*

(vii) *the number of cycles of length three, four and five, and the number of cycles of length four with exactly one chord.*

Proof. Assertion (i) is clear. The girth of a graph is the girth of the cycle matroid, hence (ii) follows from (iii)-(iv) in Theorem 1.15. Statement (iii) is proved using Theorem 1.16. To prove assertion (iv), note that the edge-connectivity of G is the size of a minimum edge-cut, and that edge-cuts are duals of circuits. Hence, $\lambda(G)$ is the girth of $M(G)^*$; by Theorems 1.6 and 1.15, $\lambda(G)$ is determined by $t(G; x, y)$. The well-known inequality $\lambda(G) \leq \delta(G)$ implies that every vertex in G has degree at least $\lambda(G)$. Assertion (v) follows immediately from Theorem 1.9.

Assume now that G is simple. In this case the cliques of order d are the only subgraphs of G with rank $d-1$ and size $\binom{d}{2}$; from this (vi) follows. In particular we get the number of 3-cycles. The number of 4-cycles follows from (vii) in Theorem 1.15; the number of 4-cycles with only one chord is the number of subgraphs with rank 3 and size 5. Finally, to get 5-cycles consider which subsets of $E(G)$ contribute to $[x^4y^5]F(G; x, y)$. There are three types:

(1) Cycles of length five.

(2) Cycles of length four plus one edge that is not a chord of the cycle.

(3) Cycles of length three plus two edges such that they do not form a 4-cycle with a chord.

Since the cardinalities of the last two classes can be deduced from the Tutte polynomial, assertion (vii) follows. □

For most invariants it is not known whether they are T-invariants or not. In the Appendix we show several counterexamples to properties that cannot be deduced from the Tutte polynomial.

2

Factorization of the Tutte polynomial

Recall that one of the basic properties of $t(M;x,y)$ is that, if M is the direct sum of two matroids M_1 and M_2, then

$$t(M;x,y) = t(M_1;x,y)\,t(M_2;x,y).$$

In particular, this implies that if M is disconnected, then $t(M;x,y)$ has a nontrivial factor in $\mathbb{Z}[x,y]$. Brylawski [18] conjectured that the converse also holds; in this chapter we give a proof of this conjecture.

Theorem 2.1 *If M is a connected matroid, then $t(M;x,y)$ is irreducible in $\mathbb{Z}[x,y]$.*

Hence, the nontrivial factors of $t(M;x,y)$ only arise from the connected components of M.

Corollary 2.2 *If a matroid M has c connected components $M_1,\ldots,M_c$, then the factorization of $t(M;x,y)$ in $\mathbb{Z}[x,y]$ is*

$$t(M;x,y) = t(M_1;x,y)\,\cdots\,t(M_c;x,y).$$

Theorem 2.1 implies that the T-uniqueness of a matroid reduces to the T-uniqueness of its connected components.

Corollary 2.3 *A matroid is T-unique if and only if each of its connected components is T-unique.*

Proof. Let $M = M_1 \oplus \cdots \oplus M_c$ be a matroid with c connected components. Assume first that M is T-unique and that there is at least one of its connected components, say M_1, that is not T-unique; let M_1' be T-equivalent to M_1 but not isomorphic to M_1. The matroid $M' = M_1' \oplus M_2 \oplus \cdots \oplus M_c$ is not isomorphic to M, but by Theorem 1.4, the matroids M and M' are T-equivalent.

For the converse, assume that M_i is T-unique for all i with $1 \le i \le c$ and let $N = N_1 \oplus \cdots \oplus N_s$ be a matroid T-equivalent to M, with $N_1,\ldots,N_s$ its connected components. Since M and N are T-equivalent, we have that

$$t(M_1;x,y)\,\cdots\,t(M_c;x,y) = t(N_1;x,y)\,\cdots\,t(N_s;x,y).$$

By Corollary 2.2, both sides of the equation are the factorization of $t(M;x,y)$. Hence, M and N have the same number of connected components and there exists a bijection $\varphi : \{1,\ldots,c\} \to$

$\{1,\ldots,c\}$ such that $t(M_i;x,y)=t(N_{\varphi(i)};x,y)$ for all i with $1\le i\le c$. Since M_i is T-unique, we have that $M_i\cong N_{\varphi(i)}$, and hence $M\cong N$; thus, M is T-unique. □

We now turn to the proof of Theorem 2.1. The main tool is the following set of linear equations (B_k) that are satisfied by the coefficients of the Tutte polynomial of any matroid and that were proved by Brylawski in [18].

Lemma 2.4 *Let $t(M;x,y)=\sum b_{ij}x^iy^j$ be the Tutte polynomial of a matroid M and let m be the number of elements in M. Then*

$$\sum_{s=0}^{k}\sum_{t=0}^{k-s}(-1)^t\binom{k-s}{t}b_{st}=0, \qquad (B_k)$$

for $k=0,1,\ldots,m-1$.

Suppose now that M is a connected matroid on a set of m elements, and that there is a nontrivial factorization

$$t(M;x,y)=\sum b_{ij}x^iy^j=A(x,y)\,C(x,y), \qquad (2.1)$$

where $A(x,y)=\sum a_{ij}x^iy^j$ and $C(x,y)=\sum c_{ij}x^iy^j$.

Since $b_{00}=0$, either a_{00} or c_{00} is zero; we may assume $a_{00}=0$. By Theorem 1.5, b_{10} is nonzero; this and the equality

$$b_{10}=a_{00}c_{10}+a_{10}c_{00}$$

imply that $c_{00}\neq 0$. The aim of the proof is to obtain a contradiction by showing that $c_{00}=0$. We start by introducing some notation.

Since M is connected, by Theorem 1.5 neither x nor y are factors of $A(x,y)$ or $C(x,y)$. If $P(x,y)=\sum p_{ij}x^iy^j$ is a polynomial not divisible by x and y, define

$$r_x(P)=\max\{i\,:\,p_{i0}\neq 0\},\qquad r_y(P)=\max\{j\,:\,p_{0j}\neq 0\},$$

and let

$$m(P)=r_x(P)+r_y(P).$$

Clearly, from (2.1), $m=m(t)=m(A)+m(C)$. Since we are assuming that the factorization is nontrivial, it follows that $r_x(A),r_y(A)<m(A)<m(t)$. We next show that the coefficients of $A(x,y)$ behave as the coefficients of the Tutte polynomial, in the sense that they satisfy a property analogous to statement (v) in Theorem 1.5.

Lemma 2.5 *Let M be a connected matroid and let $t(M;x,y)$ be its Tutte polynomial with a factorization as in (2.1). If i,j are integers with either $i\ge r_x(A)$ or $j\ge r_y(A)$, then $a_{ij}=0$, except if $i=r_x(A)$ and $j=0$, or if $i=0$ and $j=r_y(A)$.*

Proof. Let $\alpha=\max\{i\,:\,a_{ij}\neq 0$ for some $j\}$ and $\beta=\max\{j\,:\,a_{\alpha j}\neq 0\}$; define analogously α' and β' for the polynomial $C(x,y)$. The monomial $a_{\alpha\beta}c_{\alpha'\beta'}x^{\alpha+\alpha'}y^{\beta+\beta'}$ appears in $t(M;x,y)$, as it cannot be cancelled, and it is the term with maximum degree of x in $t(M;x,y)$. Using property (v) in Theorem 1.5 we see that $\alpha+\alpha'=r(M)$ and $\beta+\beta'=0$, so $\beta=0$ and $\alpha=r_x(A)$. Thus, the highest power of x that appears in $A(x,y)$ is $x^{r_x(A)}$. A similar argument shows that the highest power of y in $A(x,y)$ is $y^{r_y(A)}$. □

We next prove two lemmas that together imply Theorem 2.1.

Let (A_k) be the same equation as (B_k), but with a_{st} replacing b_{st}; that is,

$$\sum_{s=0}^{k}\sum_{t=0}^{k-s}(-1)^t\binom{k-s}{t}a_{st}=0. \tag{A_k}$$

Note that we do *not* assume $A(x,y)$ to be the Tutte polynomial of a matroid, hence we do not know whether equations (A_k) hold or not. In fact, we have the following result.

Lemma 2.6 *With hypotheses as in Lemma 2.5, there is at least one equation* (A_k) *with* $r_x(A)\le k\le m(A)$ *that does not hold.*

Proof. First, for k with $r_x(A)\le k\le m(A)$ and $i\ge 0$ we define the auxiliary equation $(A_{k,i})$ as follows:

$$\sum_{s=0}^{k}\sum_{t=0}^{k-s}(-1)^{t+i}\binom{k-s}{t}a_{s,t+i}=0. \tag{$A_{k,i}$}$$

Note that $(A_{k,0})$ is the same equation as (A_k). Now we prove a recurrence relation involving these equations.

Observe that for $i>0$ the left-hand side of equation $(A_{k,i-1})$ is

$$\sum_{s=0}^{k}\sum_{t=0}^{k-s}(-1)^{t+i-1}\binom{k-s}{t}a_{s,t+i-1}. \tag{2.2}$$

Using the fact that $\binom{k-s}{t}=\binom{k-s-1}{t}+\binom{k-s-1}{t-1}$ if $k-s>0$, and assuming $\binom{a}{b}=0$ if $b<0$ or $a<b$, we can rewrite (2.2) in the following way:

$$\sum_{s=0}^{k-1}\sum_{t=0}^{k-s}(-1)^{t+i-1}\left[\binom{k-1-s}{t}+\binom{k-1-s}{t-1}\right]a_{s,t+i-1}+(-1)^{i-1}a_{k,i-1}$$

$$=\sum_{s=0}^{k-1}\sum_{t=0}^{k-1-s}(-1)^{t+i-1}\binom{k-1-s}{t}a_{s,t+i-1} \tag{2.3}$$

$$+\sum_{s=0}^{k-1}\sum_{t=1}^{k-s}(-1)^{t+i-1}\binom{k-1-s}{t-1}a_{s,t+i-1}+(-1)^{i-1}a_{k,i-1}. \tag{2.4}$$

The last term appears because $\binom{0}{0}$ cannot be decomposed into two binomial coefficients. The two double summands in (2.3) and (2.4) are, respectively, the left-hand sides of equations $(A_{k-1,i-1})$ and $(A_{k-1,i})$. Note that if $k>r_x(A)$, by Lemma 2.5 the term $(-1)^{i-1}a_{k,i-1}$ is zero. Hence we can write symbolically:

$$(A_{k,i-1})=(A_{k-1,i})+(A_{k-1,i-1})$$

or

$$(A_{k-1,i})=(A_{k,i-1})-(A_{k-1,i-1}) \tag{2.5}$$

for k,i with $r_x(A)<k\le m(A)$ and $i>0$.

Let us suppose now that all equations (A_k) hold for k with $r_x(A) \leq k \leq m(A)$. In order to find a contradiction, consider equation $(A_{r_x(A),r_y(A)})$. By Lemma 2.5, the only term a_{ij} involved in this equation that is not zero is $a_{0,r_y(A)}$. The left-hand side of $(A_{r_x(A),r_y(A)})$ reduces then to $(-1)^{r_y(A)} a_{0,r_y(A)}$, which is nonzero. On the other hand, using Equation (2.5) repeatedly $r_y(A)$ times, we can express this nonzero term as a sum of the left-hand sides of equations $(A_{k,0})$ for k with $r_x(A) \leq k \leq m(A) = r_x(A) + r_y(A)$. Since equations $(A_{k,0})$ and (A_k) are the same, and we are assuming that all equations (A_k) hold, we obtain a contradiction; we conclude that for some k with $r_x(A) \leq k \leq m(A)$ equation (A_k) does not hold. □

Lemma 2.7 *If not all equations (A_k) hold for $k \leq m(A)$, then $c_{00} = 0$.*

Proof. Let (A_j) be the first equation that does not hold. Equation (B_j) holds because $j \leq m(A) < m$. First, we rewrite this equation taking into account that

$$b_{st} = \sum_{h \leq s} \sum_{l \leq t} c_{hl} a_{s-h,t-l}.$$

Then we have the following equalities for the left-hand side of (B_j):

$$\begin{aligned}
\sum_{s=0}^{j} \sum_{t=0}^{j-s} (-1)^t \binom{j-s}{t} b_{st} &= \sum_{s=0}^{j} \sum_{t=0}^{j-s} (-1)^t \binom{j-s}{t} \sum_{h \leq s} \sum_{l \leq t} c_{hl} a_{s-h,t-l} \\
&= c_{00} \sum_{s=0}^{j} \sum_{t=0}^{j-s} (-1)^t \binom{j-s}{t} a_{st} \\
&+ \sum_{0 < h+l \leq j} c_{hl} \left[\sum_{s=h}^{j} \sum_{t=l}^{j-s} (-1)^t \binom{j-s}{t} a_{s-h,t-l} \right]. \qquad (2.6)
\end{aligned}$$

Note that each c_{hl} has as coefficient an expression similar to the left hand side of equation (A_j); in particular, for c_{00} this coefficient is exactly the left-hand side of (A_j). We introduce a new set of equations $(A'_{n,i})$ for $i \leq n$ as

$$\sum_{s=0}^{n-i} \sum_{t=0}^{n-i-s} (-1)^{t+i} \binom{n-s}{t+i} a_{st} = 0. \qquad (A'_{n,i})$$

Observe that the left-hand side of equation $(A'_{j-h,l})$ is the coefficient of c_{hl} in (2.6) above: change indices $s \leftarrow s+h, t \leftarrow t+l$, and remark that for $s > j-l$ and $t \geq l$ the binomial $\binom{j-s}{t}$ is zero. Also note that $(A'_{n,0})$ is precisely equation (A_n), which we are assuming holds for all n with $0 \leq n < j$. Now we prove that $(A'_{n,i})$ holds for n, i with $1 \leq n \leq j$ and $1 \leq i \leq n$ using induction on n.

For $n = 1$ we only have to consider equation $(A'_{1,1})$, that reduces to $a_{00} = 0$, which was supposed from the beginning. Assuming the result for all indices less than n, we use again a formula for the binomial coefficients to decompose the left-hand side of equation $(A'_{n,i})$ into a sum of previous equations:

$$\sum_{s=0}^{n-i} \sum_{t=0}^{n-i-s} (-1)^{t+i} \left[\binom{n-s-1}{t+i-1} + \binom{n-s-2}{t+i-1} + \cdots + \binom{t+i-1}{t+i-1} \right] a_{st}.$$

Each binomial coefficient $\binom{n-s}{t+i}$ is partitioned into exactly $n-s-t-i+1$ terms, so that the last expression equals

$$\sum_{s=0}^{n-1-(i-1)} \sum_{t=0}^{n-1-(i-1)-s} (-1)^{t+i}\binom{n-1-s}{t+i-1} a_{st}$$
$$+ \sum_{s=0}^{n-2-(i-1)} \sum_{t=0}^{n-2-(i-1)-s} (-1)^{t+i}\binom{n-2-s}{t+i-1} a_{st}$$
$$+ \cdots + \sum_{s=0}^{0}\sum_{t=0}^{0}(-1)^{t+i}\binom{t+i-1}{t+i-1} a_{st}.$$

Now it is easy to check that the p-th term in the last sum is, up to a sign, the left hand side of equation $(A'_{n-p,i-1})$, for p with $1 \leq p \leq n-i+1$. Thus we obtain the following relation:

$$(A'_{n,i}) = -\left((A'_{n-1,i-1}) + (A'_{n-2,i-1}) + \cdots + (A'_{i-1,i-1})\right).$$

We use this recurrence relation to show that equation $(A'_{n,i})$ holds for $n \leq j$ and all i with $1 \leq i \leq n$. If $i = 1$, the equations on the right are $(A_{n-1}),\ldots,(A_0)$, all of which hold because $n-1 < j$. If $i > 1$, equations $(A'_{n-1,i-1}),\ldots,\ (A'_{i-1,i-1})$ hold by inductive hypothesis. In both cases $(A'_{n,i})$ holds, and this concludes the induction.

Using this result we see that the coefficient of c_{hl} in (2.6) is zero unless $h = l = 0$. Hence equation (B_j) holds if and only if $c_{00} = 0$ or (A_j) holds. Since (A_j) does not hold, c_{00} must be zero and the lemma is proved. □

Lemmas 2.6 and 2.7 show that c_{00} is 0. Since from the factorization (2.1) it follows that $c_{00} \neq 0$, this contradiction implies that $t(M; x, y)$ is irreducible if M is connected.

Observe that the proof makes no use of the fact that the coefficients a_{ij} and c_{ij} are integers; thus, the Tutte polynomial of a connected matroid is irreducible in $\mathbb{C}[x, y]$ or over any field of characteristic zero. However, for other characteristics the Tutte polynomial might have nontrivial factors that do not arise from the connected components of the matroid. For example,

$$\begin{aligned} t(M(K_4); x, y) &= 2x + 2y + 3x^2 + 4xy + 3y^2 + x^3 + y^3 \\ &= (x+y)(x+y+x^2+xy+y^2) \mod 2, \end{aligned}$$

whereas $M(K_4)$ is a connected matroid. The example above also shows that one of the key points in the proof, the fact that b_{10} is nonzero for a connected matroid, does not remain true for arbitrary characteristic.

3

T-uniqueness of some families of graphs

This is the first of three chapters on T-unique graphs. As we mentioned in Section 1.4, T-unique graphs can be viewed as a generalization of chromatically unique graphs. In this section we study the T-uniqueness of several well-known families of graphs whose χ-uniqueness has not been settled yet. It is not surprising that many graphs that are not known to be χ-unique can be proved to be T-unique. These include complete multipartite graphs, wheels, squares of cycles, ladders, and hypercubes. Although all the techniques used in the proofs have a similar flavour, each of the families under consideration requires a special treatment. Even if it is conjectured that most graphs are T-unique, we can only provide proofs for graphs that are highly symmetric.

3.1 Complete multipartite graphs

In this section we show that complete multipartite graphs are T-unique. The chromatic uniqueness of complete multipartite graphs is far from being settled. A proof of the χ-uniqueness of $K_{p,q}$ for $p \geq q \geq 2$ can be found in [33] (recall that $K_{1,p}$ is a tree, and therefore it shares the same chromatic and Tutte polynomials with all other trees with $p+1$ vertices). Not much is known about complete graphs with three stable sets. For instance, $K_{1,p,q}$ is not chromatically unique; but $K_{p_1,\ldots,p_r}$ is if $|p_i - p_j| \leq 1$ for all i, j with $1 \leq i < j \leq r$, because in this case it is a Turán graph [34, Theorem 19].

Observe that Theorem 3.1 below gives $p(n)$ nonisomorphic T-unique graphs of order n, where $p(n)$ is the number of partitions of the integer n. It is known [1] that $p(n)$ behaves asymptotically as $\frac{A}{n}e^{B\sqrt{n}}$ for some positive constants A and B.

Theorem 3.1 *For every set of positive integers $p_1, p_2, \ldots, p_r$, the complete multipartite graph $K_{p_1,p_2,\ldots,p_r}$ is T-unique, with the only exception of $K_{1,p}$.*

Proof. Since complete graphs K_n and bipartite graphs $K_{p,q}$ with $p, q \geq 2$ are χ-unique, they are T-unique. So let H be a graph T-equivalent to $K_{p_1,p_2,\ldots,p_r}$ for some $r \geq 3$. By Theorem 1.5 we have that H is 2-connected, and hence we can apply Theorem 1.17 to show that H has $n = \sum p_i$ vertices and $\sum_{i<j} p_i p_j$ edges. Also, from Theorem 1.9 we have that H has chromatic number r, and hence is r-partite.

Let K_{r+1}^- denote the complete graph K_{r+1} with one edge deleted, and let us consider the set

$$\mathcal{K}(H) = \{(A, B) \,:\, A \subseteq B \subseteq E(H),\ A \cong K_r,\ B \cong K_{r+1}^-\}.$$

If $\alpha(H)$ is the number of subgraphs of H isomorphic to K_{r+1}^-, then

$$|\mathcal{K}(H)| = 2\alpha(H),$$

since every K_{r+1}^- contains exactly two subgraphs isomorphic to K_r. Since H and $K_{p_1,p_2,\dots,p_r}$ are T-equivalent, they contain the same number $p_1 \cdots p_r$ of r-cliques. Let $W_1(H), \dots, W_{p_1 \cdots p_r}(H)$ be the r-cliques of H, and let $k_i(H)$ be the number of subgraphs of H isomorphic to K_{r+1}^- that contain $W_i(H)$. Then

$$|\mathcal{K}(H)| = k_1(H) + \cdots + k_{p_1 \cdots p_r}(H) \le p_1 \cdots p_r(n-r), \tag{3.1}$$

the last inequality because every r-clique in H can be extended in at most $n-r$ ways (the number of remaining vertices) to a K_{r+1}^-.

If we define $\mathcal{K}$, α, $W_1, \dots, W_{p_1 \cdots p_r}$, and $k_1, \dots, k_{p_1 \cdots p_r}$ analogously for the graph $K_{p_1,\dots,p_r}$, then we have

$$|\mathcal{K}| = 2\alpha = k_1 + \cdots + k_{p_1 \cdots p_r} = p_1 \cdots p_r(n-r),$$

since now every r-clique in $K_{p_1,\dots,p_r}$ is contained in exactly $n-r$ subgraphs isomorphic to K_{r+1}^-. But $\alpha(H) = \alpha$, since these quantities correspond precisely to the number of edge-sets, respectively in H and in $K_{p_1,\dots,p_r}$, of rank r and cardinality $\binom{r+1}{2} - 1$; this quantity is determined by the Tutte polynomial for simple graphs. We deduce that the inequality in (3.1) is in fact an equality and that $k_i(H) = n - r$ for every i.

We can now prove that H is a complete r-partite graph. Let $u, v \in V(H)$ be in different parts of the r-partition of H and let $\{w_1, \dots, w_r\}$ be an r-clique of H, where u and w_1, and also v and w_2, are in the same part. Since $k_1(H) = n - r$, necessarily $\{u, w_2, \dots, w_r\}$ is a clique in H; by the same reason, $\{u, v, w_3, \dots, w_r\}$ must be a clique and hence u and v are adjacent.

We are not finished yet, since we only have $H \cong K_{q_1,\dots,q_r}$ for some set of integers $q_1, q_2, \dots, q_r$. By counting the number of cliques of every possible size from 1 to r (all of them determined by the Tutte polynomial) we obtain:

$$\begin{aligned}
\sum_i p_i &= \sum_i q_i \\
\sum_{i<j} p_i p_j &= \sum_{i<j} q_i q_j \\
\cdots &= \cdots \\
p_1 \cdots p_r &= q_1 \cdots q_r.
\end{aligned}$$

Since two sets of r numbers having the same elementary symmetric functions must be equal, we have that $H \cong K_{p_1,\dots,p_r}$. □

3.2 Wheels and squares of cycles

The *wheel* $\mathcal{W}_n$ is the graph obtained from a cycle C_n by adding a new vertex (the *hub*) adjacent to all the vertices in the cycle. The edges joining the hub to the other vertices are called *spokes*. The wheels $\mathcal{W}_3$ and $\mathcal{W}_9$, and all wheels $\mathcal{W}_{2n}$ with an even number of spokes are χ-unique. But $\mathcal{W}_5$ and $\mathcal{W}_7$ are not, and the question of whether $\mathcal{W}_{2n+1}$ is χ-unique is still open (see [34]).

We prove in Section 7.1 that the cycle matroid of a wheel is T-unique; since wheels are 3-connected, by Corollary 1.12 the graph $\mathcal{W}_n$ is T-unique. Wheels can also be proved to be T-unique as graphs directly (see [40] for details). Instead of giving that proof, we give a very similar one showing that squares of cycles are T-unique.

The *square of cycle* C_n^2 is the graph obtained from the cycle C_n by adding all the edges between vertices at distance two. All graphs C_n^2 for $n \leq 9$ are known to be χ-unique; for larger values of n the question remains open, although they are conjectured to be χ-unique [34].

Theorem 3.2 *For every* $n \geq 3$*, the graph* C_n^2 *is T-unique.*

Proof. For $n = 3, 4, 5$, the graph C_n^2 is isomorphic to K_n, which we know is T-unique. For $n = 6, 7, 8$, the table of χ-unique graphs in [34] ensures that C_6^2, C_7^2 and C_8^2 are also T-unique. So we restrict to the general case $n \geq 9$.

Let H be a graph T-equivalent to C_n^2 for some $n \geq 9$. Applying Theorems 1.5 and 1.17 to H, we obtain the following information: H is a 2-connected simple graph, $|V(H)| = n$, $|E(H)| = 2n$, H has n triangles, H has neither K_4 nor any cycle of length four without chords, and H has n cycles of length four with exactly one chord. (Note that if $n = 6$, H has twelve 4-cycles with one chord, and if $n = 7, 8$, H has some 4-cycles without chords). By Theorem 1.17 again, we know that the minimum degree in H is at least four. Since the sum of all degrees must be $4n$ (twice the number of edges), and as there are exactly n vertices, we conclude that H is 4-regular.

Claim 1 There is no edge in $E(H)$ belonging to three or more triangles.

Proof. Let us show that it is possible to obtain from $t(H; x, y) = t(C_n^2; x, y)$ the number of subsets of $E(H)$ consisting of three triangles meeting at a single edge, or, which is the same, subgraphs $K_{2,3}$ with an extra edge joining the two vertices in the partite set of cardinatlity two: call these subsets $K_{2,3}^+$. All of them contribute to the coefficient of x^4y^7 in the rank-size generating polynomial $F(H; x, y)$. We study first which other subsets of $E(H)$ contribute to this coefficient. There are three possibilities (see Figure 3.1):

(1) Cycles of length five with two chords meeting at a common vertex (that is, a triangulated pentagon).

(2) Subgraphs $K_{2,3}$ with an extra edge joining two vertices in the partite set of order three (this is the same as a cycle of length 5 with two disjoint chords).

(3) Complete subgraphs K_4 plus one edge (either vertex-disjoint or not).

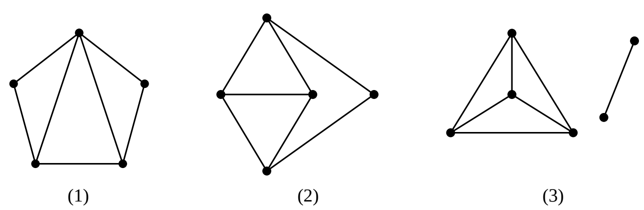


Figure 3.1 All possible subgraphs in H with rank 4 and size 7 in addition to $K_{2,3}^+$.

We already know that the situation described in (3) is impossible because there are no K_4 in H. Using the fact that each cycle of length four must have one and only one chord it is easy to prove that situation (2) is also impossible. Observe that this implies that a cycle of length five cannot have three or more chords.

Therefore, the only contributions to $[x^4y^7]F(G;x,y)$ are $K_{2,3}^+$ and triangulated pentagons, so

$$[x^4y^7]F(G;x,y) = t + c, \tag{3.2}$$

where c is the number of cycles of length five as in (1) above and t is the number of $K_{2,3}^+$.

To see that $t = 0$ we need another equation involving t and c. In order to obtain this second equation, we analyse the coefficient of x^4y^6 in $F(H;x,y)$, that is, subgraphs in H with rank 4 and size 6. The possibilities are the following (see Figure 3.2):

(1) Two edge-disjoint triangles.

(2) A cycle of length four with its chord and any other edge.

(3) A cycle of length five with one chord.

(4) A subgraph $K_{2,3}$.

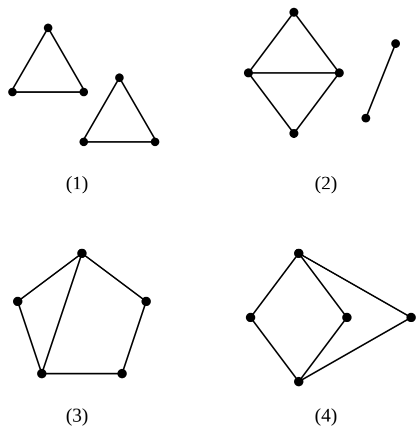


Figure 3.2 All possible subgraphs in H with rank 4 and size 6.

The contributions of 1 and 2 are, respectively, $\binom{n}{2} - n$ and $n(2n-5)$. Note that each cycle of length five with one chord must be triangulated, for if not there would be a cycle of length four without chords. Since the maximum number of chords in a cycle of length five is two, we deduce that the number of cycles of length five with one chord equals twice the number of triangulated pentagons. On the other hand, in $K_{2,3}$ there are three cycles of length four without chords; as we have seen previously the chord must join the two vertices in the 2-stable set. This means that in H the number of $K_{2,3}$ equals the number of $K_{2,3}^+$. Hence

$$[x^4y^2]F(G;x,y) = t + 2c + \binom{n}{2} - n + n(2n-5) \tag{3.3}$$

Equations (3.2) and (3.3) imply that it is possible to obtain t and c from the coefficients of $t(H;x,y) = t(C_n^2;x,y)$. As in C_n^2 there is no edge belonging to three triangles, we deduce that $t = 0$. Actually both t and c are zero; this implies that in H there is no subgraph with rank 4 and size 7. □

We call an edge $e \in E(H)$ a *diagonal edge* if it is the chord of a cycle of length four; if e is the chord of the 4-cycle C, then we write $e = d(C)$. If e is not the chord of any cycle of length 4, we call it a nondiagonal edge.

Claim 2 In H there are n diagonal edges and n nondiagonal edges. A diagonal edge is in exactly two triangles and a nondiagonal edge is in exactly one.

Proof. Let $\mathcal{U} = \{(e, C) : C$ is a 4-cycle and $e = d(C)\}$. Define ρ_i as the number of cycles of length four having the edge e_i as a chord. By Claim 1 we know that $0 \leq \rho_i \leq 1$. The cardinality of $\mathcal{U}$ is n because there are exactly n cycles of length four with a chord. But also $|\mathcal{U}| = \rho_1 + \cdots + \rho_{2n}$, so there must be n edges with $\rho_i = 1$ and n edges with $\rho_i = 0$. From now on we assume that the diagonal edges are $e_1, \ldots, e_n$; that is, $\rho_1 = \cdots = \rho_n = 1$ and $\rho_{n+1} = \cdots = \rho_{2n} = 0$.

To prove the second assertion, consider the set

$$\mathcal{T} = \{(e, t) : e \text{ is an edge of the triangle } t\}.$$

Define τ_i as the number of triangles that contain e_i. We know that $\tau_i = 2$ for i with $1 \leq i \leq n$ and that $\tau_i \leq 1$ for i with $n + 1 \leq i \leq 2n$. Summing over all triangles in H, we deduce that $|\mathcal{T}| = 3n$, and summing over all edges, we obtain $|\mathcal{T}| = \tau_1 + \cdots + \tau_{2n} = 2n + \tau_{n+1} + \cdots + \tau_{2n} \leq 3n$. So $\tau_i = 1$ for i with $n + 1 \leq i \leq 2n$ and the claim is proved. □

Claim 3 There is no subgraph in H isomorphic to the 4-wheel $\mathcal{W}_4$.

Proof. Suppose H has a subgraph isomorphic to $\mathcal{W}_4$. By deleting any edge of a 4-wheel, we obtain a graph with rank 4 and 7 elements. As observed in the proof of Claim 1, H cannot contain such a subgraph. Hence there is no wheel $\mathcal{W}_4$ in H. □

Claim 4 Every vertex $v \in V(H)$ is incident with two diagonal and two nondiagonal edges.

Proof. Let δ'_i be the nondiagonal degree of the vertex v_i, $1 \leq i \leq n$, that is, the number of nondiagonal edges incident with v_i. If δ'_i were zero for some i, then the four diagonal edges incident with v_i would be the spokes of a 4-wheel, contradicting Claim 3. It is easy to check that δ'_i cannot be either 1 or 3. Since $\delta'_1 + \delta'_2 + \cdots + \delta'_n = 2n$, if some δ'_i were 4, then this would force some vertex to have nondiagonal degree zero, which we just saw is impossible. Therefore $\delta'_i = 2$ for all i, $1 \leq i \leq n$. □

We know now the local structure around every vertex $v \in V(H)$. It is easy to prove that the four edges incident with v must be joined as depicted in Figure 3.3, where d_1, d_2 are diagonal edges and f_1, f_2 are nondiagonal edges.

The last step in the proof consists of determining the global structure of H using this local information.

Claim 5 The subgraph H_1 induced by the diagonal edges in H is a cycle.

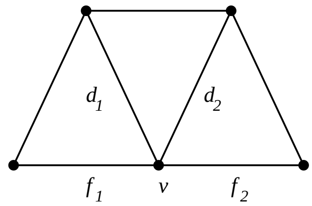


Figure 3.3 Local structure around every vertex in H.

Proof. We already know that H_1 is 2-regular, so each of its connected components is a cycle. Suppose now that H_1 is disconnected. Then there is a nondiagonal edge e joining two vertices $v_1 \in V(\Gamma_1)$ and $v_2 \in V(\Gamma_2)$ in two different components Γ_1 and Γ_2 of H_1. Let x_1 and y_1 (respectively, x_2 and y_2) be the neighbours of v_1 (resp. v_2) in Γ_1 (resp. Γ_2). As $x_1v_1, y_1v_1, x_2v_2, y_2v_2$ are all diagonal edges, the local structure of H implies that $x_1y_1, x_2y_2 \in E(H)$, and also that either x_1v_2 or y_1v_2 is an edge; assume $x_1v_2 \in E(H)$. Observe that the vertices v_1, v_2 and x_1 form a triangle supported by the nondiagonal edge v_1v_2. Now v_1 must be adjacent to either x_2 or y_2, but in both cases we obtain a second triangle containing v_1v_2, which is impossible. Therefore, H_1 is a 2-regular connected graph, hence a cycle. □

To finish the proof of the theorem, consider the cycle induced by the diagonal edges. If two vertices are at distance two in this cycle, from the local structure of H it follows that they are joined by an edge; since this edge is not in the cycle, it must be a nondiagonal edge. This implies that $H \cong C_n^2$. □

3.3 Ladders and Möbius ladders

The graph $C_n \times K_2$ is called the *ladder* L_n . The *Möbius ladder* M_n is constructed from an even cycle C_{2n} by joining every pair of vertices at distance n. These graphs are only known to be χ-unique for very few values of n. We now show that they are T-unique.

Theorem 3.3 *For every $n \geq 3$, the ladders L_n are T-unique.*

Proof. In [34] the ladders L_3, L_4, and L_5 are shown to be χ-unique, hence they are T-unique. Let now H be a graph T-equivalent to L_n for some $n \geq 6$. From Theorems 1.5 and 1.17 we deduce the following information: H is a 2-connected simple graph with $2n$ vertices and $3n$ edges, in H there is no cycle of length three or five, and there are n cycles of length four, all of them without chords. Since the Tutte polynomial specializes to the chromatic polynomial, it is possible to determine the chromatic number $\chi(H)$ from $t(H; x, y)$: $\chi(H) = 2$ if n is even and $\chi(H) = 3$ if n is odd. We also know from Theorem 1.17 that all the vertices in H have degree at least three, and since the sum of the degrees of all the vertices equals $6n$, we conclude that H is 3-regular.

Some other parameters, such as cycles of length six or bipartite subgraphs $K_{2,3}$, are not T-invariants. But we can use the information above to determine them in this particular case. We say that two squares are *adjacent* if they have one common edge, or, which is the same, if their union is a cycle of length six with a chord. The two following claims tell us how many pairs of adjacent squares there are.

Claim 1 If $t(H, x, y) = t(L_n, x, y)$, then

(1) in H there is no subgraph isomorphic to $K_{2,3}$ (Figure 3.4.1);
(2) in H there is no cycle of length six with two chords (Figure 3.4.2);
(3) in H there are n cycles of length six with exactly one chord (Figure 3.4.3);
(4) in H there is no subgraph isomorphic to the graph in Figure 3.4.4; therefore, if a square is adjacent to two other squares, the only possible configuration is that of Figure 3.4.5.

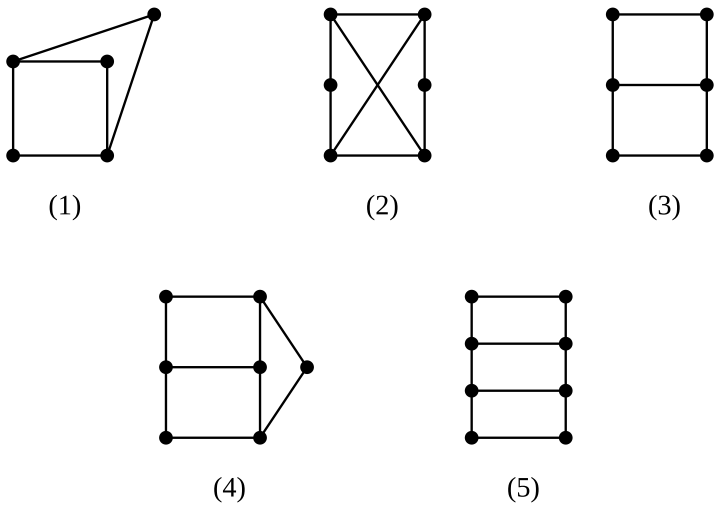


Figure 3.4 Illustrating Claim 1.

Proof. All of them follow by studying which subsets $A \subseteq E(H)$ are counted by suitable coefficients of $F(H, x, y)$, and using the properties of H proved previously. Note that the only chords allowed in a cycle of length six are those joining two vertices at distance three, otherwise H would contain a triangle. □

Claim 2 Every square in H is adjacent to two other squares.

Proof. Let c_i be the number of squares adjacent to the i-th square, $1 \leq i \leq n$. Observe first that in Figure 3.4.5 there is no place for a third square adjacent to the central one, so $c_i \leq 2$ for all $1 \leq i \leq n$. Let $\mathcal{C}_4$ be the set of all squares in H, and let $\mathcal{H}$ be the set of all cycles of length six with one chord; now define $\mathcal{S} = \{(c, h) \in \mathcal{C}_4 \times \mathcal{H} : c \subseteq h\}$. The cardinality of $\mathcal{S}$ is clearly $2n$, since there are n elements in $\mathcal{H}$ and each of them contains two squares. But we also have that $|\mathcal{S}| = c_1 + \cdots + c_n \leq 2n$. So equality must hold and $c_i = 2$ for every i. □

Finally, given a square s_1 there is a second square s_2 adjacent to s_1, and a third one s_3 adjacent to s_2. We add a new square in this way repeatedly until we return to s_1. The union of these squares $s_1, \ldots, s_k$ is a connected component in H, because all the vertices already have three neighbours. Since H is 2-connected, $k = n$ and H is isomorphic to either L_n or M_n. But we can distinguish L_n from M_n using the chromatic number ($\chi(M_n) = 3$ if n is even and $\chi(M_n) = 2$ if n is odd), so $H \cong L_n$. □

The same proof gives the following.

Theorem 3.4 *The Möbius ladders M_n are T-unique for all $n \geq 2$.*

Proof. Let H' be a graph T-equivalent to M_n for some n. If $n \geq 5$, all the facts proved for H in the above theorem are still valid for H'; when we arrive at the end of the proof, the inversion of the chromatic numbers leads us to $H' \cong M_n$.

For the particular cases remaining, we have that $M_2 \cong K_4$ and $M_3 \cong K_{3,3}$ are both covered by Theorem 3.1, and M_4 can be treated directly without difficulty. □

3.4 Hypercubes

The *n-cube* Q^n is defined as the product of n copies of K_2. They are conjectured to be χ-unique [34], but this has been proved only for $n = 2, 3$. Cubes have been extensively studied and have been characterized in several ways. In this section we use one such characterization to prove their T-uniqueness. We need the following result due to Mulder (see [4]).

Lemma 3.5 *A connected n-regular graph is isomorphic to the n-cube if and only if it has 2^n vertices and every pair of vertices at distance 2 have precisely two common neighbours.*

We can now prove that cubes are T-unique.

Theorem 3.6 *The n-cube is T-unique for every $n \geq 2$.*

Proof. Let H be T-equivalent to the n-cube Q^n. Then H is bipartite and 2-connected, has 2^n vertices and $n2^{n-1}$ edges. Since the n-cube has edge-connectivity equal to n, so does H. Hence H has minimum degree at least n, and by counting incidences between vertices and edges, we deduce that H is in fact n-regular.

In view of the previous lemma, we only need to check that every pair of vertices at distance 2 in H have precisely two common neighbours. Observe that two vertices at distance 2 have a common neighbour by definition, and that two common neighbours define a cycle C_4 (all C_4's are chordless since H is bipartite); conversely, every C_4 gives rise to two pairs of vertices with two common neighbours. Since Q^n and H are T-equivalent, they have the same number $2^{n-2}\binom{n}{2}$ of cycles C_4.

For every pair of vertices x and y at distance 2 in H, let $n(x, y)$ be number of common neighbours they have. We have that $n(x, y) \leq 2$, since if x and y have a, b, c as common neighbours, then $\{x, y, a, b, c\}$ induces a subgraph in H isomorphic to $K_{3,2}$, which has size 6 and rank 4; but such a subgraph does not exist in Q^n, hence neither in H.

Now count in two ways the pairs $(\{x, y\}, z)$, where z is a common neighbour of x and y. Since H is n-regular, we have

$$\sum_{d(x,y)=2} n(x, y) = 2^n \binom{n}{2}.$$

Counting C_4's, there are at least $2 \cdot 2^{n-2}\binom{n}{2}$ different pairs $\{x, y\}$ at distance 2 with $n(x, y) = 2$. Since equality holds, all pairs of vertices at distance 2 have two common neighbours, as was to be proved. □

4

Locally grid graphs

The goal of this chapter is to prove that the toroidal grid $C_p \times C_q$ is T-unique. As we shall see, the proof requires the introduction of new techniques far beyond the ones used in the previous chapter. We first need to define a family of graphs, locally grid graphs, whose local structure resembles that of a toroidal grid. Roughly speaking, locally grid graphs are those where around each vertex there is a 3×3 grid $\boxplus$ (the precise definition is slightly more technical and is given in Section 4.1). We then prove that the Tutte polynomial captures the locally grid condition; in other words, locally grid graphs are T-closed. The next step consists of finding all possible locally grid graphs. In particular, we prove that locally grid graphs fall into three families, each of them embeddable in either the torus or the Klein bottle. To complete the proof of the T-uniqueness of $C_p \times C_q$ we show that no locally grid graph has the same Tutte polynomial as a toroidal grid by carefully counting edge-sets of certain rank and size.

In order to avoid this chapter being too long, in some of the proofs we will only give the main ideas; the missing details can be completed without much effort.

4.1 The local condition

In this section we introduce a condition that abstracts the local structure of a toroidal grid, and we prove that this condition is captured by the Tutte polynomial.

Let $N(x)$ be the set of neighbours of a vertex x. We say that a 4-regular connected graph G is a *locally grid graph* if for every vertex $x \in V(G)$ there exists an ordering x_1, x_2, x_3, x_4 of $N(x)$ and four different vertices y_1, y_2, y_3, y_4, such that, taking the indices modulo 4,

$$\begin{aligned} N(x_i) \cap N(x_{i+1}) &= \{x, y_i\}, \\ N(x_i) \cap N(x_{i+2}) &= \{x\}, \end{aligned}$$

and there are no more adjacencies among $\{x, x_1, \ldots, x_4, y_1, \ldots, y_4\}$ than those required by this condition. In particular this implies that a locally grid graph is simple and triangle-free, and that every vertex belongs to exactly four squares. Note that this definition excludes the toroidal grids $C_4 \times C_q$, since with the above notation either x_1, x_3 or x_2, x_4 would have a common neighbour different from x. This restriction is due to some technical reasons that appear in the proof of Theorem 4.2.

It is easy to show that locally grid graphs are 2-connected. Their edge-connectivity is at most 4, the minimum degree; we prove that actually equality holds.

Lemma 4.1 *Locally grid graphs are 4-edge-connected.*

Proof. We show that any edge-cut contains at least 4 edges. Let C be an edge-cut of a locally grid graph G. Take an edge $e \in C$. This edge belongs to exactly two squares, and their intersection is e. Since a cut and a cycle cannot intersect in just one edge, in addition to e, each of these squares must have at least another one edge in C. Hence C contains at least 3 edges, say $\{e, f_1, f_2\} \subseteq C$. Now consider f_1; this edge is contained in a square that is disjoint from e and f_2. As before, this square intersects C in at least one other edge besides f_1; hence $|C| \geq 4$. □

We prove now that the fact of being a locally grid graph can be deduced from the Tutte polynomial.

Theorem 4.2 *Suppose H is a graph T-equivalent to a locally grid graph G that does not contain cycles of length 5. Then H is locally grid and contains no cycle of length 5.*

Proof. By Theorems 1.5 and 1.17 we deduce that H is simple, has $n = |V(G)|$ vertices and $2n$ edges, and edge-connectivity 4. This implies that H has minimum degree at least 4 and that, in fact, it is 4-regular. We also have that H has exactly n squares but no cycles of length 3 or 5.

Claim 1 H contains no subgraph isomorphic to $K_{2,3}$.

Proof. We show that in fact H does not contain any subgraph with rank 4 and size 6. Observe that $K_{2,3}$ is the only triangle-free graph with such rank and size. Since in a locally grid graph every two squares meet in at most one edge, G does not contain any $K_{2,3}$ as a subgraph, and hence no subgraph with rank 4 and size 6. Since G and H are T-equivalent, the same holds for H. □

Claim 2 H contains exactly $2n$ cycles C_6 with a *long* chord, that is, a chord joining opposite vertices.

Proof. We consider subgraphs of rank 5 and size 7. Besides C_6 with a long chord, there are two possibilities for a triangle-free graph with these rank and size: $K_{2,3}$ plus one edge sharing at most one endpoint with the edges in $K_{2,3}$; and a cycle C_5 plus one vertex joined to two vertices of the cycle. By Claim 1 and the fact that H has no C_5, these two possibilities are excluded in H. Hence the number of C_6 with a long chord in H is the same as in G, namely $2n$. □

Claim 3 Every edge of H is in exactly two squares.

Proof. Since H is 4-regular and contains no $K_{2,3}$, an edge can be in at most 3 squares. For $i = 0, 1, 2, 3$, let n_i be the number of edges contained in exactly i squares. Then, by double counting of the number of pairs (e, s) such that s is a square containing the edge e, we obtain

$$4n = 0n_0 + 1n_1 + 2n_2 + 3n_3.$$

By Claim 2 and double counting of the number of pairs (e, h) such that h is a C_6 having e as a long chord, we obtain

$$2n = 0n_0 + 0n_1 + 1n_2 + 3n_3.$$

From the above two equations it follows

$$n_1 + n_2 = 2n.$$

But since $n_0 + n_1 + n_2 + n_3 = 2n$ we have $n_0 = n_3 = 0$ and, consequently, $n_2 = 2n$ and $n_1 = 0$, as was to be proved. □

Observe that the union of the two squares in the previous claim must form a C_6 having the given edge as a long chord, since the other two possibilities imply the existence of either a $K_{2,3}$ or a double edge.

We now check the locally grid condition. Let $x \in V(H)$, and let y be a neighbour of x. From the above claim it follows that the edge xy is in two squares $xyy'x'$ and $xyy''x''$. Let z be the fourth vertex adjacent to x. Then we have the situation depicted in Figure 4.1.

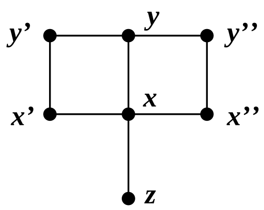


Figure 4.1 Local structure around vertex x.

Consider now a square containing xz. It must contain a second edge incident with x, and it cannot be xy since then xy would belong to three squares. Hence, the two squares containing xz must be $zxx'u$ and $zxx''v$ for some u and v. Note that $\{u, v\} \cap \{y', y''\} = \emptyset$, otherwise xy would be in three squares. Also $u \neq v$, since $u = v$ would force xx' to be in three squares.

In order to finish the proof we must show two more things. First, that there is no other edge among the vertices $\{x, x', x'', y, y', y'', z, u, v\}$; this is clear since otherwise we have a triangle, a C_5, or an edge contained in more than two squares. Secondly, that y and z do not have any common neighbour besides x, and that the same holds for x' and x''; this is because otherwise there would be edges contained in 3 squares. □

4.2 Classification

We have just proved that if a graph is T-equivalent to $C_p \times C_q$ with $p, q \geq 6$ then it is locally grid. In this section we temporarily put the Tutte polynomial aside and focus on the classification of locally grid graphs. We first define three families of locally grid graphs, each of them having a natural embedding in the torus or the Klein bottle, with the four squares around each vertex being faces of the embedding. We then prove that these families are the complete list of locally grid graphs.

Let $H = P_p \times P_q$ be the $p \times q$ grid, where P_l is a path with l vertices. Label the vertices of H with the elements of the abelian group $\mathbb{Z}_p \times \mathbb{Z}_q$ in the natural way. Notice that the vertices of degree four already have the locally grid property. In order to obtain a locally grid graph we add new edges suitably among the vertices of degree two and three. If we add the edges

$$\{(j, 0), (j, q-1) \mid 0 \leq j \leq p-1\} \cup \{(0, j), (p-1, j) \mid 0 \leq j \leq q-1\},$$

the result is the toroidal grid $C_p \times C_q$. Other ways to do this give the following three families of graphs.

The torus $T^{\delta}_{p,q}$. The graph $T^{\delta}_{p,q}$ is built just as the graph $C_p \times C_q$, but moving the adjacencies in the first direction δ vertices to the right. That is,

$$\begin{aligned} E(T^{\delta}_{p,q}) &= E(H) \cup \{\{(i,0),(i+\delta,q-1)\}, 0 \le i \le p-1\} \\ &\cup \{\{(0,j),(p-1,j)\}, 0 \le j \le q-1\}\,. \end{aligned}$$

Note that we can assume $\delta \le p/2$. For $\delta = 0$ we obtain the toroidal grid $C_p \times C_q$; in this case we simply write $T_{p,q}$.

All these graphs are embeddable in the topological torus. See Figure 4.2 for an embedding of $T^2_{7,5}$; in this figure, as in the next two, the vertices of the graph are represented by dots, and two points with the same label correspond to points that are identified in the surface.

Although we have not mentioned this explicitly, the values of p and q must be large enough so that the locally grid condition holds. For example, for $\delta = 0$, p and q must be at least 5, but for $\delta = 1$ we can take $q = 4$ and $p = 5$. Similar observations apply to the next two families. The exact bounds on p and q are stated in Theorem 4.3.

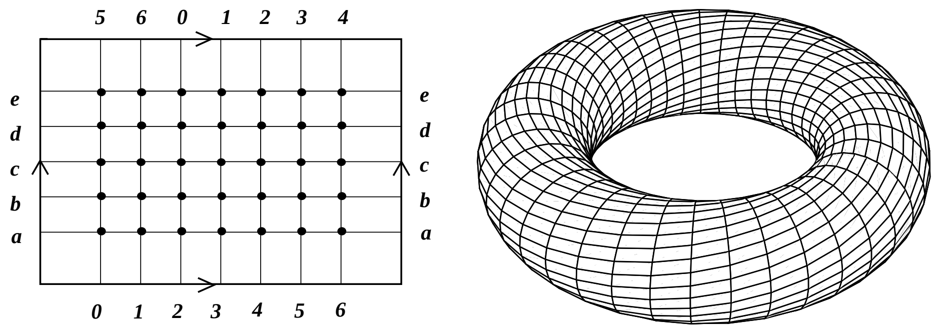


Figure 4.2 Left: the torus $T^2_{7,5}$. Right: an illustration of a "twisted torus" in $\mathbb{R}^3$.

The Klein bottles $K^{i}_{p,q}$. For $i \in \{0,1,2\}$, $i \equiv p \bmod 2$, define the graph $K^i_{p,q}$ as follows. Keep the adjacencies of the second direction untouched and reverse the ones in the first direction, thus obtaining graphs embeddable in the Klein bottle. If p is odd, the edges are as follows:

$$\begin{aligned} E(K^{1}_{p,q}) &= E(H) \cup \{\{(0,j),(p-1,j)\}, 0 \le j \le q-1\} \\ &\cup \{\{(j,0),(p-j-1,q-1)\}, 0 \le j \le p-1\}. \end{aligned}$$

The superscript 1 means that there is only one adjacency made in the usual way, namely $\{((p-1)/2,0),((p-1)/2,q-1)\}$.

If p is even, there are two cases:

$$\begin{aligned} E(K^{0}_{p,q}) &= E(H) \cup \{\{(0,j),(p-1,j)\}, 0 \le j \le q-1\} \\ &\cup \{\{(j,0),(p-j-1,q-1)\}, 0 \le j \le p-1\} \end{aligned}$$

and

$$\begin{aligned} E(K^{2}_{p,q}) &= E(H) \cup \{\{(0,j),(p-1,j)\}, 0 \le j \le q-1\} \\ &\cup \{\{(j,0),(p-j,q-1)\}, 0 \le j \le p-1\}. \end{aligned}$$

In the first case there are no adjacencies of the kind $\{(c,0)(c,q-1)\}$, and in the second one there are two of them.

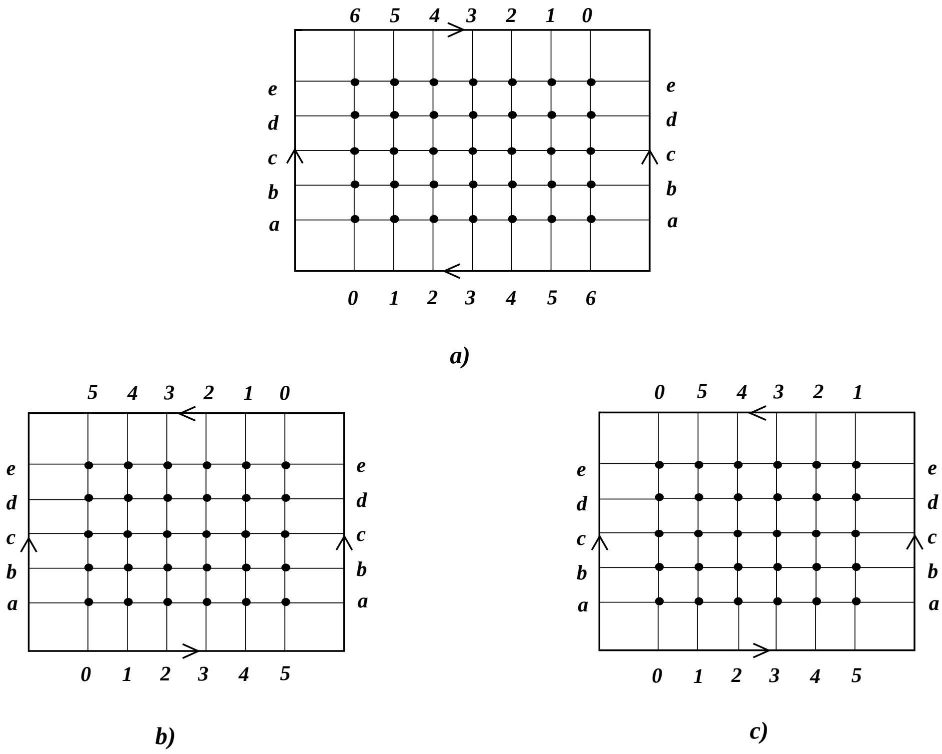


Figure 4.3 Three examples of Klein bottles: a) $K^1_{7,5}$, b) $K^0_{6,5}$, c) $K^2_{6,5}$.

In Figure 4.3 we show embeddings of these three kinds of graphs in the Klein bottle.

The graphs $S_{p,q}$. Define the graph $S_{p,q}$ (S is for "strange") for $p \leq q$ as follows:

$$\begin{aligned} E(S_{p,q}) = E(H) \quad & \cup \quad \{\{(j,0),(p-1,q-p+j)\}, 0 \leq j \leq p-1\} \\ & \cup \quad \{\{(0,i),(i,q-1)\}, 0 \leq i \leq p-1\} \\ & \cup \quad \{\{(0,i),(p-1,i-p)\}, p \leq i \leq q-1\}. \end{aligned}$$

For $q \leq p$, the edges of $S_{p,q}$ are as follows:

$$\begin{aligned} E(S_{p,q}) = E(H) \quad & \cup \quad \{\{(j,0),(0,q-1-j)\}, 0 \leq j \leq q-1\} \\ & \cup \quad \{\{(p-1,i),(p-1-i,q-1)\}, 0 \leq i \leq q-1\} \\ & \cup \quad \{\{(i,q-1),(i+q,0)\}, 0 \leq i \leq p-q-1\}. \end{aligned}$$

Note that when $p = q$ both definitions agree. Figure 4.4 shows embeddings of the two kinds of "strange" graphs in the Klein bottle. Notice that in the second one we use a different model for the Klein bottle.

It is straightforward to verify that all the graphs we have defined are locally grid and have pq vertices. We later show that, except in one case, they are pairwise nonisomorphic. We now turn to the proof that these families exhaust all the cases.

Theorem 4.3 *If G is a locally grid graph with N vertices, then one and only one of the following holds:*

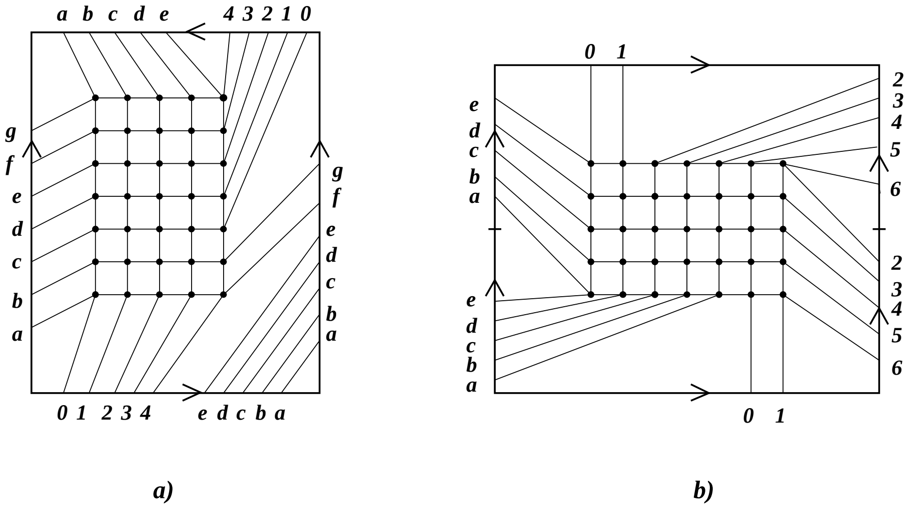


Figure 4.4 Two “strange” graphs: a) $S_{5,7}$, b) $S_{7,5}$.

(a) $G \cong T^{\delta}_{p,q}$ *with* $pq = N$, $p \geq 5$, $\delta \leq p/2$, *and*

$$\begin{cases} \delta + q \geq 5, & \text{if } q \geq 4; \\ \delta + q \geq 6, & \text{if } q = 2, 3; \\ 4 \leq \delta < p/2, \delta \neq p/3, p/4, & \text{if } q = 1. \end{cases}$$

(b) $G \cong K^{i}_{p,q}$ *with* $pq = N$, $p \geq 5$, $i \equiv p$ *mod* 2 *for* $i \in \{0, 1, 2\}$, *and* $q \geq 4 + \lceil i/2 \rceil$.

(c) $G \cong S_{p,q}$ *with* $pq = N$, $p \geq 3$, *and* $q \geq 6$.

Before proving Theorem 4.3 we mention here a paper by C. Thomassen [53]. In this paper the author defines a (4, 4)-tiling of the torus or the Klein bottle in a way that, except for minor technical details, is equivalent to our definition of a locally grid graph. Theorem 4.1 [53] is the analogue of our Theorem 4.3. After carefully comparing both theorems, we realized that the family $S_{p,q}$ with $q < p$ does not appear in the list given in Theorem 4.1 [53]. However, it does appear implicitly in the proof, which is very similar in spirit to ours. Since a good deal of the proof of Theorem 4.1 [53] is left to the reader, and since having an exact and complete list of locally grid graphs is essential for the remaining sections of this chapter, we think it is appropriate to give here a detailed proof of Theorem 4.3.

Let $G = (V, E)$ be a locally grid graph. Two edges e_1, e_2 incident with a vertex v are called *adjacent* if there is a square containing both; otherwise, they are called *opposite*. The four edges incident at v are thus classified into two pairs of opposite edges.

Given $v_1 \in V$ and $e_1 \in E$, e_1 incident with v_1, we define the walk $W(v_1, e_1)$ as the sequence $v_1 e_1 v_2 e_2 \ldots v_l e_l v_{l+1}$ such that e_i, e_{i+1} are opposite edges at v_{i+1} and v_{l+1} is the first vertex equal to v_i for some i with $1 \leq i < l$. In other words, we visit the vertices of the graph starting from e and following opposite edges until we repeat some vertex. Notice that $W(v_1, e_1)$ is a cycle only if $v_{l+1} = v_1$; in this case, e_1 and e_l may be opposite (this is always the case in torus and Klein bottles), or adjacent (this can only happen in “strange” graphs).

We call a path $v_1 e_1 \ldots v_l e_l v_{l+1}$ an *opposite path* if for every $i < l$, e_i and e_{i+1} are opposite edges. Similarly we call a cycle $v_1 e_1 \ldots v_l e_l v_1$ an *opposite cycle* if every pair of consecutive edges in the cycle is opposite. If we are not interested in the edges of the cycle or path, we simply write $v_1 \dots v_l v_1$. Two opposite cycles $v_1 \dots v_l v_1$ and $w_1 \dots w_l w_1$ of the same length are called *parallel* if $v_i w_i \in E$ and $v_i w_i w_{i+1} v_{i+1}$ is a square for all i with $1 \leq i \leq l$ (for example, those in Figure 4.5.a).

We need some easy, but essential, properties about opposite paths and cycles. All omitted proofs are straightforward.

Lemma 4.4 *If $v_1 v_2 v_3$ and $w_1 w_2 w_3$ are consecutive vertices in two opposite paths, and $v_2 w_2 \in E$ is adjacent to $v_1 v_2$, then either $v_1 w_1 \in E$ or $v_1 w_3 \in E$.*

Lemma 4.5 *If $v_1 v_2 v_3$ and $w_1 w_2 w_3$ are consecutive vertices in two opposite paths, and $v_1 w_1, v_2 w_2 \in E$ with $v_1 w_1 w_2 v_2$ a square, then $v_3 w_3 \in E$ and $v_2 w_2 w_3 v_3$ is a square.*

Lemma 4.6 *Let $e, f \in E$ be a pair of opposite edges at the vertex v. Then every square that contains v contains either e or f but not both.*

Lemma 4.7 *Every opposite cycle has length at least five.*

Lemma 4.8 *Let $v_1 \dots v_l$ be an opposite path with $v_i \neq v_j$ for $i \neq j$. If $v_1 v_k \in E$ for some $k < l$, and $v_1 v_k$ is adjacent to $v_1 v_2$, then necessarily $v_2 v_{k+1} \in E$.*

Proof. By Lemma 4.4, either $v_2 v_{k-1} \in E$ or $v_2 v_{k+1} \in E$. In the first case we apply repeatedly Lemma 4.5 and we get $v_3 v_{k-2}, v_4 v_{k-3}, \dots, v_i v_{k-i+1}, \dots \in E$. For $i = \lfloor k/2 \rfloor$ we obtain either a triangle, which is impossible in a locally grid graph, or a contradiction to Lemma 4.6. Thus we conclude that $v_2 v_{k+1} \in E$. □

The proof of Theorem 4.3 considers different cases, depending on the nature of the walks $W(v, e)$. We choose a distinguished walk and then use it to recover the locally grid graph.

Case 1 All walks $W(v, e)$ in G are opposite cycles.

Suppose first that there exists an opposite cycle $v_1 \dots v_n v_1$ in G that contains all vertices. Since G is 4-regular, the vertex v_1 is joined to some other vertex v_{k+1}, for some k with $4 \leq k \leq n/2$. By Lemma 4.8 and repeated application of Lemma 4.5, we obtain that v_i is adjacent to v_{i+k} and v_{i-k}, where all the subindices are read modulo n. This determines all the adjacencies between vertices of G. Note that if n is even and $k = n/2$ the resulting graph is not locally grid. Therefore, G is isomorphic to $T^k_{n,1}$ for some k with $4 \leq k < n/2$, and with $k \neq n/3, n/4$ because in these cases there are triangles or every vertex is contained in five squares.

From now on, we can assume that not all vertices of G are contained in an opposite cycle. Let $C = v_1 \dots v_l v_1$ be a walk $W(v, e)$ of shortest length. We claim that every vertex in C has only two neighbours among $V(C)$. If not, we can assume that $v_1 v_{k+1} \in E$ for some k, $k \geq 4$. By Lemma 4.8 and repeated application of Lemma 4.5 we get that $v_{k+1} v_{2k+1} \in E$, and in general that $v_{ik+1} v_{(i+1)k+1} \in E$. Note that these edges have to form an opposite cycle C', because of the 4-regularity of G. Since G is connected and there are vertices not in C, the cycle C' cannot include all the vertices in C. Therefore, it is shorter than C, a contradiction.

Take now a vertex $w_1 \in V - C$ such that $v_1 w_1 \in E$. Let f be the edge incident with w_1 that shares a square with $v_1 v_2$, and consider the walk $W(w_1, f)$, which is an opposite cycle of length at least l, $w_1 w_2 \dots w_r$, with $r \geq l$. It is an immediate consequence of Lemma 4.5 and the choice of f that $v_2 w_2, \dots, v_l w_l \in E$ and, more generally, $w_s v_{s'} \in E$, with $s' \equiv s \bmod l$. By Lemma 4.6 it

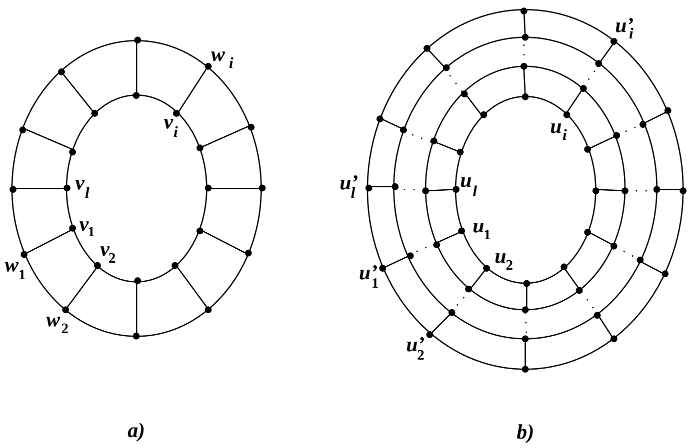


Figure 4.5 a) The cycle C and the walk $W(w_1, f)$ of length l; b) The same structure after s steps.

follows that $w_1, \ldots, w_r \notin C$. If $r = l$, then C and $W(w_1, f)$ are parallel cycles (see Figure 4.5.a); otherwise, the structure is as in Figure 4.6.a. We study each case separately.

Case 1.1 $r = l$

Suppose we have a set of opposite cycles $C_1, \ldots, C_s$ for s with $s \geq 2$, and such that C_i and C_{i+1} are parallel for all i with $1 \leq i < s$, and $C = C_j$ for some j (like those in Figure 4.5.b and with vertices named as in this figure). Let us denote by $C_{1\ldots s}$ a set of vertices and edges like this. Note that all vertices have degree four, except the vertices in C_1 and C_s, which have degree three. We show next that if it is not possible to add a new opposite cycle C_{s+1} parallel to C_s or to C_1, then the vertices in C_1 and C_s must be joined to complete a locally grid graph.

Assume there is a pair of adjacent vertices $v \in C_{1\ldots s}$ and $w \notin C_{1\ldots s}$; we can suppose $v \in C_s$. As we did before, let f be an edge incident with w and adjacent to vw, and consider the opposite cycle $W(w, f)$. By Lemmas 4.4 and 4.5 the cycles $W(w, f)$ and C_s are parallel (in this case, the length of the walk $W(w, f)$ must be necessarily l). Since all the vertices in $C_{1\ldots s}$ have degree either three or four, none of the vertices in $W(w, f)$ belongs to $C_{1\ldots s}$. We can thus define $C_{s+1} = W(w, f)$.

If it is not possible to add a new opposite cycle to $C_{1\ldots s}$, then the fourth neighbour of any vertex of C_1 belongs to either C_1 or C_s. Let w be the fourth neighbour of $u_1 \in C_1$. If $w \in C_1$, we apply Lemmas 4.8 and 4.5 and deduce that l must be even and that $w = u_{l/2+1}$, since we can add only one more edge to each vertex. Thus u_i is joined to $u_{i+l/2}$. The vertices in C_s must be joined in the same way. Since all the vertex degrees are already four, our graph is completed. In this case G is a Klein bottle $K^0_{2s,l/2}$. We note here that this situation is possible unless $s = 2$; in this case we have an opposite cycle of length 4, a contradiction to Lemma 4.7.

If the neighbour w of u_1 belongs to C_s, $w = u'_k$, there are two possibilities, depending on whether u_2 is adjacent to u'_{k+1} or to u'_{k-1}. In the first case the resulting graph is a Torus $T^{k-1}_{l,s}$, and in the second one a Klein bottle $K^i_{l,s}$, where i depends on k and the parity of l. As in the previous paragraph, for small values of s not all ranges of k are allowed. Straightforward checking shows that for $s = 2, 3, 4$, the minimum values of k are $5, 4, 1$, respectively; the parameter l must be at least 5.

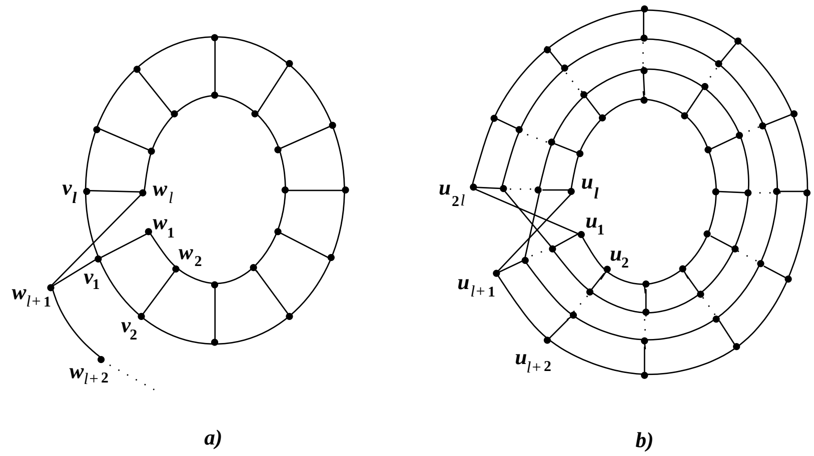


Figure 4.6 a) The cycle C and the walk $W(w_1, f)$ with length $l > r$; b) The same structure after s steps.

Case 1.2 $r > l$

We study now the case in which the length r of the walk $W(w_1, f)$ exceeds l, the length of C; the notation is as in Figure 4.6.a. By Lemma 4.5, $v_i w_{i+l} \in E$ for all i such that $l + i \leq r$. If $r < 2l$ then, again by Lemma 4.5, we obtain that $v_1 w_{2l-r+1} \in E$. This leads to a contradiction because the four neighbours of v_1 are v_2, v_l, w_1 and w_{l+1}, all of them different from w_{2l-r+1}. Therefore $r \geq 2l$. If $r > 2l$, then $v_1 w_{2l+1} \in E$, which is again a contradiction. We conclude that $r = 2l$ and that every vertex in C is adjacent to two vertices of $W(w_1, f)$.

Suppose that we have s cycles $C_1, \ldots, C_s$, $s \geq 2$, such that $C_1 = C$, $C_2 = W(w_1, f)$, and C_i is parallel to C_{i+1}, for all i, $2 \leq i \leq s-1$; note that this implies that $C_2, \ldots, C_s$ have length $2l$. We call this structure $M_{1\ldots s}$ (see Figure 4.6.b for notation). We prove next that either we can complete a locally grid graph or add another cycle of length $2l$ parallel to C_s.

If there is no vertex in $M_{1\ldots s}$ whose fourth neighbour lies outside $M_{1\ldots s}$, the neighbour of a vertex in the last cycle C_s belongs to this cycle. Let u_k be such that $u_1 u_k \in E$. By Lemmas 4.8 and 4.5, we obtain that $u_{2l-k+2} u_1 \in E$. Since u_1 had already degree three, the only choice is $k = l + 1$. It is easy to see that this graph is a Klein bottle $K^1_{2s-1,l}$, where l, s satisfy $l \geq 5$, $2s - 1 \geq 5$.

Suppose now that there exists $u'_1 \in V - M_{1\ldots s}$ such that $u_1 u'_1 \in E$. Take an edge f containing u'_1 and adjacent to $u_1 u'_1$, and consider the walk $W(u'_1, f) = u'_1 \ldots u'_r u'_1$. This walk is an opposite cycle of length $r \geq l$. If $r > l$, then we can show as before that $r = 2l$ and we can add a cycle C_{s+1} parallel to C_s. If $r = l$, by Lemma 4.5 $u_i u'_i, u_{l+i} u'_i \in E$ for $i \leq l$. All the vertices have degree four and thus we have completed a locally grid graph. In this case we obtain a Klein bottle $K^2_{2s,l}$. Note that this situation is not possible for $s = 2$, because in this case we would have opposite cycles of length four ($u_1 u'_1 u_{l+1} v_1$, for instance), contradicting Lemma 4.7.

We have proved that every locally grid graph such that all the walks $W(v, e)$ are opposite cycles is a torus or a Klein bottle. The family $S_{p,q}$ arises when considering the case in which there is a walk that is not an opposite cycle.

Case 2 Not all the walks $W(v,e)$ are opposite cycles.

Choose a vertex v_1 and an edge e_1 such that $W(v_1,e_1) = v_1e_1v_2\dots e_lv_1$ is a cycle, e_l and e_1 are adjacent edges, and such that $W(v_1,e_1)$ is the shortest walk with this property; call it C.

Since e_1 and e_l are adjacent at v_1, then v_2 and v_l have a second common neighbour w_1. By Lemma 4.6, $w_1 \notin C$. Now w_1 and v_3 have a second common neighbour, w_2, which also does not belong to C; note that if $w_2 = v_1$, then there would be a triangle in G. Suppose now that $w_1,\dots,w_{r-1} \in V - C$ for $r < l$ are such that w_i is a common neighbour of v_{i+1} and w_{i-1} (see Figure 4.7.a). Vertices w_{r-1} and v_{r+1} also have a second common neighbour w_r. This vertex w_r cannot be any of the v_i for i with $2 \le i \le r$, since these vertices have degree three. By Lemma 4.6, w_r is different from v_i for i with $r+2 \le i \le l$. In the next claim we prove that w_r is also different from v_1.

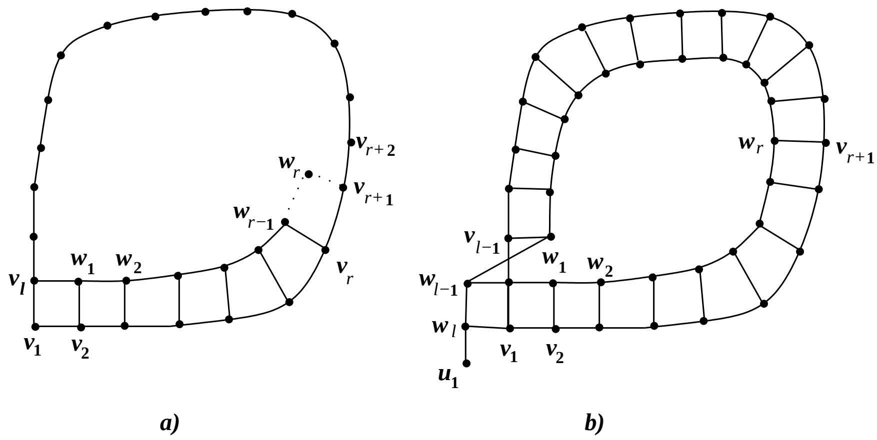


Figure 4.7 Two intermediate steps in Case 2.

Claim 1 The vertex w_r is different from v_1.

Proof. Suppose this is not the case. Then v_1 and v_{r+2} must have a common neighbour besides v_{r+1}; since v_1 already has degree four, this neighbour is either v_2 or v_l. If it is v_l, apply Lemma 4.8 to the opposite path $v_lv_{l-1}\dots$ and the edge v_lv_{r+2}, and deduce that v_{l-1} and v_{r+1} are adjacent; then $v_1v_lv_{l-1}v_{r+1}$ is a square that contradicts Lemma 4.6. Hence v_{r+2} is adjacent to v_2. Now apply repeatedly Lemma 4.5 and get that $v_{l-r+1}v_1 \in E$. Since the neighbours of v_1 are v_2, v_l, v_{r+1} and w_{r-1}, we obtain a contradiction unless $r = l/2$. In this case, the neighbours of v_1 are $v_2, v_l, v_{l/2+1}$ and $w_{l/2-1}$, and the edges v_1v_l and $v_1v_{l/2+1}$ are opposite. But the vertex $v_{l/2}$ is a common neighbour of v_l and $v_{l/2+1}$, and this contradicts the locally grid property at v_1. □

We define recursively w_i as the common neighbour of v_{i+1} and w_{i-1} for i with $i \le l-2$. By Claim 1 and the remarks before it, each w_i is a new vertex. Observe that if $w_{l-2}w_1$ was an edge it would contradict the minimality of C. Therefore the second common neighbour of w_{l-2} and v_l is a new vertex w_{l-1}; so it is w_l, the other common neighbour of v_1 and w_{l-1} (see Figure 4.7.b). Note that $w_lv_1v_2$ is an opposite path since $w_lv_1v_lw_{l-1}$ is a square.

Let us focus now on the vertex w_l. Its four neighbours are v_1, w_{l-1}, u_1 and u', where the edge w_lu_1 is opposite to $w_{l-1}w_l$, and the edge w_lu' is opposite to w_lv_1. We show that u_1 is different from all the vertices that have appeared previously.

Claim 2 The vertex u_1 is different from w_i for all i, $1 \leq i \leq l$, and from v_j for all j, $1 \leq j \leq l$.

Proof. The only cases that must be checked carefully are the following.

(1) Suppose that $u_1 = w_i$ for some i with $2 \leq i \leq l-3$. Then the cycle $w_i w_{i+1} \ldots w_l u_1$ contradicts the choice of C.

(2) Suppose $u_1 = v_j$ for some $3 \leq j \leq l-2$. Applying Lemma 4.8 to the opposite path $w_l v_1 v_2 \ldots$ and to the edge $w_l v_j$, we deduce $v_1 v_{j+1} \in E$. Now apply Lemma 4.5 $l - j$ times and obtain $v_{l-j} v_l \in E$. This is a contradiction since the neighbours of v_l are $v_1, v_{l-1}, w_1, w_{l-1}$.

□

Let u_2 be the second common neighbour of u_1 and v_1, let u_3 be the second common neighbour of u_2 and v_2, and so on. By an argument similar to that of Claim 1, all the vertices u_i for $1 \leq i \leq l$ are new vertices, and it is immediate that $u_i v_{i-1} \in E$ and $u_l w_1 \in E$ (see Figure 4.8a).

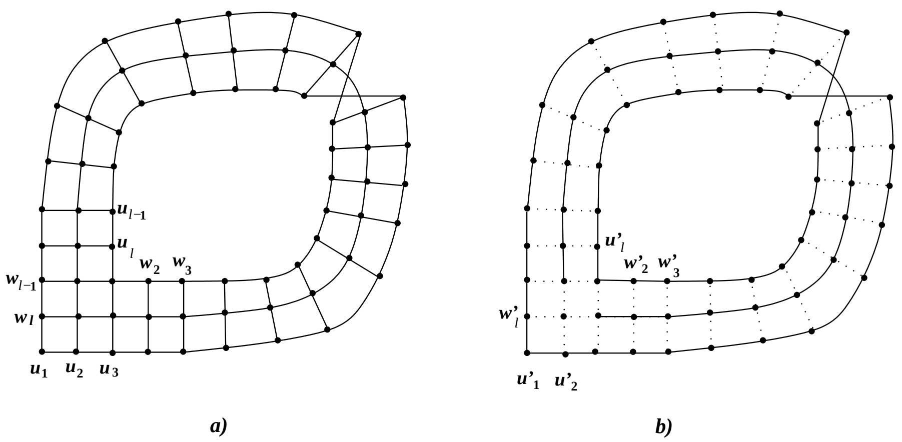


Figure 4.8 a) The first layer; b) The same structure after s steps.

Now suppose that we have built s layers like this from C and the vertices, adjacencies and notation are as in Figure 4.8.b. Call this structure R_s. The vertices w'_2 and u'_l have a second common neighbour, which is either a new vertex or u'_1 (because all the other vertices in R_s have degree at least three).

Assume first that u'_1 is a common neighbour of w'_2 and u'_l. By Lemma 4.4, u'_2 has to be adjacent to a neighbour of w'_2; the only choice is w'_3, since the other neighbours have already degree four. By repeated application of Lemma 4.5 we obtain that $u'_i w'_{i+1} \in E$ for all $i < l$. This settles all the adjacencies; the resulting graph is $S_{2s+1,l}$. This might not seem immediate from our definition of "strange" graphs. To see that G is indeed one of $S_{p,q}$, we have to find a walk in $S_{p,q}$ that plays the role of C, that is, we have to find a non-opposite walk of shortest length. There are several walks that satisfy this property; using the same notation as in the definition of $S_{p,q}$, one possibility is $C = W((p-3, q-3), \{(p-3, q-3), (p-2, q-3)\})$ if $p \leq q$ and $C = W((p-2, 1), \{(p-2, 1), (p-1, 1)\})$ if $q < p$. Notice that whether we have a "strange" graph of the first or second kind depends on the values of s and l, although all the pictures here correspond to the case $p \leq q$ for simplicity.

Let us treat now the case in which the second common neighbour z_1 of u'_l and w'_2 is not u'_1. The vertex z_1 is clearly different from all the vertices in R_s. As we did in the case of the first layer, let z_i be the second common neighbour of z_{i-1} and w'_{i+1} for $3 \leq i \leq l-1$, and let z_l be the second common neighbour of z_{l-1} and u'_1. With a reasoning analogous to that of Claim 1, it can be proved that none of the vertices z_i belongs to R_s, and thus we arrive to the situation depicted in Figure 4.9. If z'_1 is one of the previous vertices, by an argument similar to that in Claim 2 we see that the only possibilities are $z'_1 = z_1$ or $z'_1 = u'_{l-1}$. If $z'_1 = z_1$, by Lemma 4.5 we obtain $u'_{i-1}z_i \in E$ for all i, $1 \leq i \leq l$; the locally grid property is satisfied at each vertex and the resulting graph is $S_{2s+2,l}$. The other case is impossible, because if $z'_1 = u'_{l-1}$, then Lemma 4.8 applied to the opposite path $z_l u'_1 u'_2 \ldots$ implies that u'_1 is adjacent to u'_l, a vertex that already has degree 4.

The only case left is when z'_1 is a new vertex. Define z'_2 as the second common neighbour of z'_1 and u'_1, and recursively z'_j as the second common neighbour of z'_{j-1} and u'_{j-1}, for j with $3 \leq j \leq l$. Again as in Claim 1 all the vertices z'_j are new, $z'_j u'_{j-1} \in E$ and z'_l is joined to z_1. We have thus added a new layer to R_s.

This concludes the proof of the classification theorem of locally grid graphs.

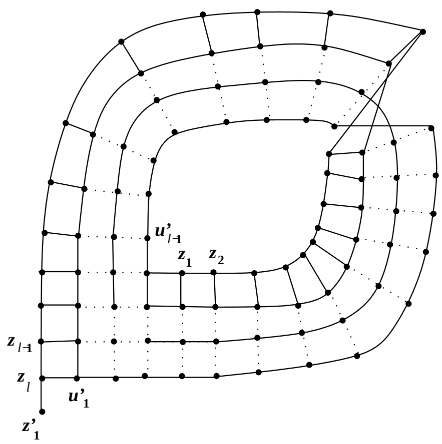


Figure 4.9 In the middle of a new layer.

We can also use the walks $W(v, e)$ to prove that almost all our graphs are nonisomorphic. It is easy to see that the structure of the walks $W(v, e)$ of a locally grid graph (how many of them there are, whether they are opposite or not, and how long they are) is invariant under isomorphism. Using this fact one sees that among all the graphs we have defined, and with a fixed number of vertices pq, the only pair that are isomorphic are $T^{\delta}_{p,q}$ and $T^{\delta'}_{p',q'}$ with $pq = p'q'$, $(p, \delta) = q'$ and $(p', \delta') = q$. The structure of the walks $W(v, e)$ can also be used to show that the only vertex-transitive locally grid graphs are the torus $T^{\delta}_{p,q}$ for every p, q, δ satisfying the conditions in Theorem 4.3. The Klein bottles $K^{0}_{p,q}$ might be thought to be vertex-transitive, but note that among the opposite cycles of length $2q$, there are only two of them that have chords, while the remaining ones are chordless.

4.3 Codifying and counting edge-sets

This and the following section are devoted to the proof of the T-uniqueness of $C_p \times C_q$. We show that for every locally grid graph G different from $C_p \times C_q$ and with pq vertices there is at least one coefficient $a_{i,j}$ of the rank-size generating polynomial $F(G;x,y)$ in which $C_p \times C_q$ and G differ. This coefficient is related to the topological structure of locally grid graphs.

Let $\mathcal{M}$ be the surface in which a locally grid graph G is embedded naturally, that is, $\mathcal{M}$ is the torus if $G \cong T^{\delta}_{p,q}$ and the Klein bottle if $G \cong K^{i}_{p,q}, S_{p,q}$, and consider the embeddings defined in the previous section. We say that a cycle C in G is *contractible* if $\mathcal{M} - C$ has two connected components, one of them contractible to a point. Otherwise we call C an *essential cycle*. In other words, a contractible cycle determines a simply connected region, whereas an essential cycle does not.

Let l_G be the shortest length of an essential cycle in G; for example,

$$l_{C_p \times C_q} = \min\{p,q\} \quad \text{and} \quad l_{K^0_{p,q}} = \min\{p,q+1\}.$$

The number of essential cycles of length l_G contributes to the coefficient a_{l_G-1,l_G} of $F(G;x,y)$, which counts the number of edge-sets with rank $l_G - 1$ and size l_G; but clearly there are other subgraphs that also contribute to this coefficient. We prove that if G and G' are locally grid graphs of the same order, and $m \leq \min\{l_G, l_{G'}\}$, then G and G' have the same number of edge-sets with rank $m-1$ and size m that do not contain essential cycles. Therefore, if $l_G < l_{G'}$, then the coefficients of $x^{l_G-1}y^{l_G}$ in $F(G;x,y)$ and $F(G';x,y)$ are different, and thus G and G' are not T-equivalent.

The aim of this section is to give the basic counting tool in order to prove Lemma 4.9. In the next section we find the quantities l_G for all locally grid graphs and using Corollary 4.16 we show that $C_p \times C_q$ is a T-unique graph.

Lemma 4.9 *Fix $m > 0$. The number of edge-sets with rank $m-1$ and size m that do not contain an essential cycle is the same for all locally grid graphs G with pq vertices and such that $m \leq l_G$.*

We call an edge-set $A \subseteq E(G)$ a *normal edge-set* if it does not contain any essential cycle. We first prove the lemma for connected normal edge-sets and then generalize to the case of several connected components. The proof goes as follows. First we show that locally grid graphs are *locally orientable*; we then use this fact to establish a canonical way to represent edge-sets with words over a given alphabet. Using this representation we count the number of edge-sets described above and show that it does not depend on the graph.

Let G be a locally grid graph. An *orientation at a vertex v* consists of labeling the four edges incident with v bijectively with the labels N, S, E, W in such a way that the edges labelled N and S are opposite, and so are the ones labelled E and W (note that the cyclic clockwise order around v need not be N, E, S, W). By the orientation (v, e, f) we mean that v is the origin vertex, e is labelled E and f is labelled N. For $\alpha \in \{N, S, E, W\}$, we denote by α^{-1} the label opposite to α. If w is a vertex adjacent to v, then the orientation at v induces an orientation at w in the natural way: if the edge vw was labelled α from v, it is labelled α^{-1} from w, and if $xvwy$ is a square, and vx was labelled β from v, wy is also labelled β from w. In the same way, if P is a path beginning at v, we can translate the orientation to all the vertices in P. If P and P' are paths joining v and v' the orientation at v' induced by P could be different from that induced by P'. This does not happen if the union of P and P' is a contractible cycle. Indeed, in this case the union of P and P' determines a simply connected region. This allows us to transform one path into the other one through the simply connected region by means of the following two elementary transitions and their inverses: if e, f, g, h are the four edges of a square ordered cyclically, we can change e, f, g in a path by h, or e, f by g, h, and these operations do not change the orientation at the endpoint.

Therefore, if we fix an orientation at a vertex $v \in V(A)$ of a connected normal edge-set A, then all the vertices in $V(A)$ are unambigously oriented. With this orientation fixed, every path in A can be described by a sequence of the labels $\{N, S, E, W\}$ (see Figure 4.10.a). This enables us to assign coordinates to every vertex in $V(A)$: the vertex v has coordinates $(0, 0)$ and the vertex $w \in V(A)$ has coordinates (i, j) if in one (and thus in every) path in A joining v to w, the number of labels E minus the number of labels W equals i and the number of labels N minus the number of labels S equals j. This procedure gives coordinates unambigously to every vertex of the normal edge-set A. We have to guarantee that there are no two vertices that are given the same coordinates.

Lemma 4.10 *If A is a normal edge-set of a locally grid graph G and $|A| \leq l_G + 2$, then the procedure described above does not assign the same coordinates to two different vertices of A.*

Proof. It is enough to prove that no vertex, except the origin of the orientation, can have coordinates $(0, 0)$. Suppose that x is a vertex of A that has coordinates $(0, 0)$ with orientation (v, e, f). By definition, in A there exists a path P from v to x having as many N edges as S edges, and as many E edges as W edges. The proof is by induction on the length of this path. The result is true for a path of length 4. Now assume it is true for lengths at most n and take P with length $n + 1$. If we view P as a sequence of the symbols $\{N, S, E, W\}$, in P there is a subsequence of the form $\alpha\beta^k\alpha^{-1}$, where $\beta \in \{N, S, E, W\} - \{\alpha, \alpha^{-1}\}$. Now we use induction on k. If k equals one, we can change the subsequence $\alpha\beta\alpha^{-1}$ by β. Doing this changes A to another normal edge-set A', but the coordinates at the vertices of $P \cap A'$, in particular at the endpoints of P, have not changed. Since the path P has been shortened, by induction hypothesis no vertex can have coordinates $(0, 0)$. Now assume the result is true for $k-1$. In this case we prove that we can change the sequence $\alpha\beta^k\alpha^{-1}$ to either $\alpha\beta^{k-1}\alpha^{-1}\beta$ or $\beta\alpha\beta^{k-1}\alpha^{-1}$; to guarantee that when modifying P we still get a normal edge-set, we have to make use of the hypothesis $|A| \leq l_G + 2$. □

In fact we know of no normal edge-set of any size where two vertices would be given the same coordinates, so it might be the case that the size restriction in the previous lemma is not necessary. Since for our purposes we do not need to deal with arbitrarily large edge-sets, we do not develop this further.

Now we are ready to codify the edge-sets of a locally grid graph. In order to do this, we define a set Γ of words over the alphabet $\{N, S, E, W\}$ that represents all possible connected normal edge-sets. The definition of Γ is given in the infinite square lattice L^∞ and we use the previous discussion on orientations to assign a unique word to every connected normal edge-set. Having done this, it is quite simple to evaluate the quantities defined in Lemma 4.9.

Let γ be a word over the alphabet $\{N, S, E, W\}$. Given γ and a locally grid graph G, we can produce from γ a connected edge-set of $E(G)$. Take a vertex $v \in V(G)$ and two edges $e, f \in E(G)$ adjacent at v. The set $A^G(\gamma, v, e, f)$ is produced by following the code given by γ from the vertex v with orientation (v, e, f); this means that if the i-th element of γ is α, at the i-th stage we add the edge that is labelled α from the endpoint of the edge added in the $(i-1)$-st stage. We call the set $A^G(\gamma, v, e, f)$ an *instance of γ in G* and the triple (v, e, f) is the *anchor* of the instance. We also want to produce instances of γ in the infinite plane square lattice L^∞, that is, the infinite graph having as vertices $\mathbb{Z} \times \mathbb{Z}$ and in which (i, j) is joined to

$(i-1,j), (i+1,j), (i,j-1)$, and $(i,j+1)$. The graph $B(\gamma)$ is the subgraph of L^∞ obtained from γ by starting at $(0,0)$ with $\{(0,0),(1,0)\}$ as E edge and $\{(0,0),(0,1)\}$ as N edge. Observe that when following γ we can cover a given edge more than once (see the example in Figure 4.10). The next lemma gives a sufficient condition for the instance of a word to be normal.

Lemma 4.11 *If $r(B(\gamma)) \leq l_G - 1$ and $|B(\gamma)| \leq l_G$, then $A^G(\gamma, v, e, f)$ is normal and has the same rank and size as $B(\gamma)$.*

Proof. The proof is by induction on the length of γ. If γ has length one, then the instance is an edge and the conclusion of the lemma is clear. Assume now that γ' is the word γ with the last element removed. By induction hypothesis $A^G(\gamma', v, e, f)$ is normal and has the same rank and size as $B(\gamma')$. The effect on $B(\gamma')$ when we add the last element of γ can be of three types: adding an isthmus, adding an edge joining two existing vertices, or repeating an edge. The proof consists of showing that the effect on $A^G(\gamma', v, e, f)$ is the same, and hence the rank and size of $A^G(\gamma, v, e, f)$ are the same as those of $B(\gamma)$. We only focus on the second case; the other two are simpler and use similar ideas.

Suppose that the last element α of γ adds a new edge between the vertices x and y of $B(\gamma')$. Let x' be the last vertex of $A^G(\gamma', v, e, f)$ covered by γ'. The edge e to be added is the one that is labelled α from x' following the orientation (v, e, f) (since $A^G(\gamma', v, e, f)$ is normal) . If $A^G(\gamma, v, e, f)$ is not normal, it is because adding e creates an essential cycle. Because of the bound on the size of $B(\gamma)$, the only possibility is that $A^G(\gamma, v, e, f)$ is an essential cycle. Since $B(\gamma)$ is obtained from $B(\gamma')$ by joining two already existing vertices, $B(\gamma)$ contains at least one cycle. This means that γ has a subword of the kind $\alpha\omega\alpha^{-1}$, where ω is a word over the alphabet $\{\beta, \beta^{-1}\} = \{N, S, E, W\} - \{\alpha, \alpha^{-1}\}$ with l elements β and $l-k$ elements β^{-1}. Then change this piece of γ by β^k and obtain a word γ^*. The instance of γ^* is normal by induction hypothesis, and contains a cycle. Consider the simply connected region determined by the instance of γ^*; when adding to it a "row of square cells" we obtain a region determined by the instance of γ. Since this instance $A^G(\gamma, v, e, f)$ is an essential cycle, at least one of the vertices that were added in the "row of square cells" must belong to $A^G(\gamma^*, v, e, f)$. But then it easy to show that $A^G(\gamma, v, e, f)$ contains at least two cycles, and hence cannot be an essential cycle. To prove the assertions on the rank and size of $A^G(\gamma, v, e, f)$, we use Lemma 4.10 to show that the edge added to $A^G(\gamma', v, e, f)$ joins two existing independent vertices. □

Note that taking different anchors we might obtain the same instance of γ (see Figure 4.11 for an example). If an instance of γ is normal, the number of anchors that lead to that same edge-set depends only on the symmetries of $B(\gamma)$ and we call it $\mathrm{sym}(\gamma)$. As we take arbitrary words γ and produce their instances, we obtain the same edge-set several times. What we do next is to select a set of words that represent normal edge-sets bijectively.

Let $\mathcal{S}$ be the group of the graph automorphisms of L^∞ (that is, the group generated by the translations, symmetries and rotations of the plane that map vertices to vertices). Given B_1 and B_2 finite connected edge-sets in L^∞, we say that $B_1 \sim B_2$ if there is $\sigma \in \mathcal{S}$ such that $\sigma(B_1) = B_2$. The relation $\sim$ is an equivalence relation. Let $\Sigma(L^\infty)$ be the set of all finite connected edge-sets of L^∞ and choose $\mathcal{B}$ to be a set of representatives of $\Sigma(L^\infty)/\sim$ such that every $B \in \mathcal{B}$ contains the vertex $(0,0)$. Intuitively, this set of representatives covers all the possible "shapes" that a normal edge-set can have. Now assign to each $B \in \mathcal{B}$ a sequence $\gamma_B = \alpha_1\alpha_2\ldots\alpha_{n(B)}$ over the alphabet $\{N, S, E, W\}$ in such a way that beginning at the origin with $\{(0,0),(1,0)\}$ as E edge and $\{(0,0),(0,1)\}$ as N edge, and following the instructions given by γ_B, the edges covered are exactly those of B (of course there are several choices for the sequence γ_B, but among all possibilities we choose one at random; see Figure 4.10.b for an example). We will refer to γ_B as the *word* of B and $\Gamma = \{\gamma_B : B \in \mathcal{B}\}$ is the set of all possible such words.

The next step consists of assigning one word from Γ to each normal connected edge-set of a locally grid graph. In particular we prove that every connected normal edge-set $A \subseteq E(G)$ is the instance of a unique word $\gamma \in \Gamma$. This is done through the following pair of lemmas.

Lemma 4.12 *Given a connected normal edge-set $A \subseteq E(G)$ with $|A| \leq l_G+2$ and an orientation (v, e, f), $v \in V(A)$, there exists a unique edge-set $B' \subseteq E(L^\infty)$ and a unique graph isomorphism $\varphi : V(A) \to V(B')$ such that:*

(i) $\varphi(v) = (0, 0)$ *and*

(ii) *if the edge $xy \in A$ is labelled α from x, then $\overline{\varphi}(xy)$ is labelled α from $\varphi(x)$ according to the orientation $\big((0, 0), \{(0, 0), (1, 0)\}, \{(0, 0), (0, 1)\}\big)$, where $\overline{\varphi}$ is the morphism induced on edges by φ.*

Proof. The sketch of the proof is as follows. Assign coordinates to the vertices of A according to the orientation (v, e, f), as explained before (note that we need A to be normal, connected, and with $|A| \leq l_G + 2$ to assign coordinates unambigously). Define $\varphi : V(A) \to V(L^\infty)$ by $\varphi(x) = (i, j)$ if the coordinates of x are (i, j), and consequently $\overline{\varphi} : A \to E(L^\infty)$ is defined by $\overline{\varphi}(xy) = \varphi(x)\varphi(y)$. Take $B' = \overline{\varphi}(A)$; the pair φ, B' satisfies the conclusion of the lemma. The uniqueness of B' is proved by induction on $\max\{d(v, x), x \in V(A)\}$. □

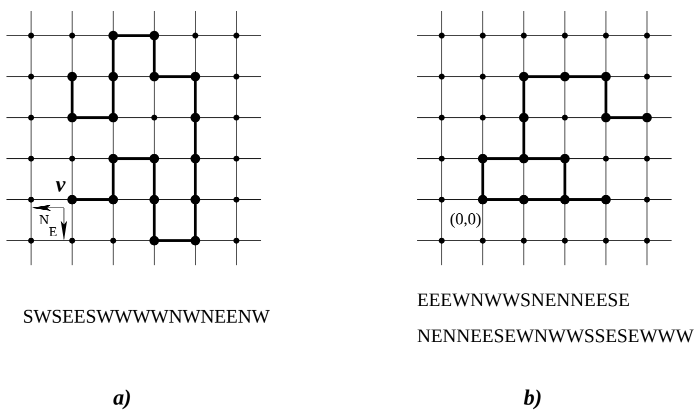
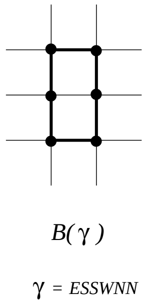


Figure 4.10 a) A path described using a sequence over $\{N, S, E, W\}$;
b) A connected subgraph and two different sequences codifying it.

Lemma 4.13 *Given a connected normal edge-set $A \subseteq E(G)$ with $|A| \leq l_G + 2$, there exists a unique word $\gamma(A) \in \Gamma$ and a (not necessarily unique) anchor (v, e, f) such that $A = A^G(\gamma(A), v, e, f)$.*

Proof. Fix $v' \in V(A)$ and a pair e', f' of adjacent edges at v'. Apply the previous lemma to A with orientation (v', e', f') and obtain B' and φ. Let $B \in \mathcal{B}$ be the representative of the equivalence class of B' and let $\sigma \in \mathcal{S}$ be the automorphism of L^∞ that maps B to B'. Take v to be the vertex $\varphi^{-1}\sigma((0, 0))$. Now assume $\sigma(\{(0, 0), (1, 0)\})$ is labeled α from $\sigma((0, 0))$. Then take as e the edge labeled α from v according to the orientation (v', e', f') (note that this edge does not necessarily belong to A). Find f analogously. Then $A = A^G(\gamma(A), v, e, f)$. To prove the uniqueness, suppose that there exists $B'' \in \mathcal{B}$ such that $A = A^G(\gamma_{B''}, v'', e'', f'')$. Then, by the definition of instance, we can find $\sigma' \in \mathcal{S}$ such that $\sigma'(B'')$ satisfies the conclusion of Lemma 4.12 when applied to A with orientation (v', e', f'). Hence B' and B'' are in the same equivalence class, and thus $B'' = B$. □



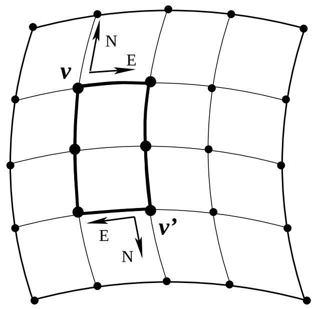


Figure 4.11 Various anchors of the same word that lead to the same instance in a locally grid graph.

Lemma 4.13 enables us to use words to count edge-sets and to prove the following weak version of Lemma 4.9.

Lemma 4.14 *For fixed $m > 0$, the number of connected normal edge-sets with rank $m - 1$ and size m is the same for all locally grid graphs G with pq vertices and such that $m \leq l_G$.*

Proof. The preceding lemma shows that every normal connected edge-set A is the instance of one and only one word $\gamma(A)$; also, by the proof of Lemma 4.12 the set A is isomorphic to $B(\gamma(A))$. By Lemma 4.11, the number of normal edge-sets of G with rank $m - 1$ and size m equals the number of distinct instances of words corresponding to edge-sets in $\mathcal{B}$ with rank $m - 1$ and size m. For fix γ, we can choose $8pq$ different anchors from which we obtain $8pq$ instances. The number of anchors that give rise to the same instance is $\mathrm{sym}(\gamma)$, that depends only on the word and not on the graph in which we produce the instance. We denote by $\Gamma^{r,s}$ the set of all words $\gamma \in \Gamma$ such that $r(B(\gamma)) = r$ and $|B(\gamma)| = s$. Then the number of connected normal edge-sets with rank $m - 1$ and size m is

$$\sum_{\gamma \in \Gamma^{m-1,m}} \frac{8pq}{\mathrm{sym}(\gamma)},$$

which does not depend on the graph G. □

Our next aim is to prove the non-connected version of this lemma.

Lemma 4.15 *For fixed $m > 0$ and $n > 1$, the number of normal edge-sets with rank $m - 1$, size m, and n connected components is the same for all locally grid graphs G with pq vertices and such that $m \leq l_G$.*

Proof. Let $A_1, \dots, A_n$ be the connected components of a normal edge-set $A \subseteq E(G)$ and denote by γ_i the word corresponding to A_i. Denote by $\mathbb{M}(\Gamma)$ the family of all multisets of Γ and define

$$\Gamma_n^{r,s} = \Big\{ \{\gamma_1, \dots, \gamma_n\} \in \mathbb{M}(\Gamma) : \sum_1^n r(B(\gamma_i)) = r, \sum_1^n |B(\gamma_i)| = s \Big\}.$$

We choose an ordering for every multiset $\widetilde{\gamma}$ and view it as an ordered tuple when necessary. We assign to A the multiset from $\Gamma_n^{m-1,m}$ that corresponds to the words of the connected components of A. An *anchor* for $\widetilde{\gamma} \in \Gamma_n^{m-1,m}$ is a tuple $x = (v_1, e_1, f_1, \dots, v_n, e_n, f_n)$ such that $v_i \in V(G)$ and e_i and f_i are incident with v_i (note that the triples (v_i, e_i, f_i) may not be pairwise different). The *instance* of $\widetilde{\gamma}$ with anchor x in a locally grid graph G is the edge-set $\bigcup A^G(\gamma_i, v_i, e_i, f_i)$. We say that an instance of $\widetilde{\gamma}$ is an *overlapping instance at* (i,j) if $A^G(\gamma_i, v_i, e_i, f_i) \cup A^G(\gamma_j, v_j, e_j, f_j)$ is connected; note that an instance overlapping at (i,j) might overlap at other pairs too. With these definitions it should be clear that the number of normal edge-sets as in the statement of the lemma is the sum over all words $\widetilde{\gamma} \in \Gamma_n^{m-1,m}$ of the number of distinct instances of $\widetilde{\gamma}$ nonoverlapping at any pair. The only thing that remains to be proved is that this last quantity, which we denote by $C_{\widetilde{\gamma}}^G$, does not really depend on G.

Let $\mathcal{X}^G$ be the set of all possible anchors in G for $\widetilde{\gamma}$ (with an abuse of notation we omit the reference to $\widetilde{\gamma}$ in $\mathcal{X}^G$) . Define $\mathcal{X}_{ij}^G, 1 \le i < j \le n$, as the set of all anchors in $\mathcal{X}^G$ that give rise to an instance overlapping at (i,j). As an application of the Principle of Inclusion and Exclusion we obtain the following expression for $C_{\widetilde{\gamma}}^G$:

$$C_{\widetilde{\gamma}}^G = \frac{\sum_{I \subseteq \{(i,j):1 \le i<j \le n\}} (-1)^{|I|} |\mathcal{X}_I^G|}{\mathrm{sym}(\gamma_1) \cdots \mathrm{sym}(\gamma_n) \mathrm{sym}(\widetilde{\gamma})},$$

where $\mathcal{X}_I^G = \bigcap_{(i,j) \in I} \mathcal{X}_{ij}^G$, $\mathcal{X}_\emptyset^G = \mathcal{X}^G$, and $\mathrm{sym}(\widetilde{\gamma})$ is the number of permutations $\pi \in \mathcal{S}_n$ such that $\pi(\widetilde{\gamma}) = \widetilde{\gamma}$. The lemma follows now from the next claim.

Claim $|\mathcal{X}_I^G|$ does not depend on G.

Proof. We can view I as the set of edges of a graph H_I with vertex set $\{1, \dots, n\}$. We prove first the case in which H_I is connected. This means that all the instances of $\widetilde{\gamma}$ with anchor in $\mathcal{X}_I^G$ are connected edge-sets in G.

Fix an orientation (v,e,f) in G and define $\mathcal{X}_I^{(v,e,f)}$ as the subset of $\mathcal{X}_I^G$ consisting of all the anchors that begin with (v,e,f). Define now the sets of anchors in the infinite plane lattice $\mathcal{X}^\infty, \mathcal{X}_{ij}^\infty$ and $\mathcal{X}_I^\infty$ analogously to $\mathcal{X}^G, \mathcal{X}_{ij}^G$ and $\mathcal{X}_I^G$. Define also $\mathcal{X}_I^{\text{origin}}$ as the subset of $\mathcal{X}_I^\infty$ in which anchors begin by $\big((0,0), \{(0,0),(1,0)\}, \{(0,0),(0,1)\}\big)$. Since $\widetilde{\gamma}$ consists of n words adding up to rank $m-1$ and size m, for $m \le l_G$, the instance of one of the components γ_i of $\widetilde{\gamma}$ contains a cycle of length less than l_G. By Lemma 4.11 this implies that the instances of $\widetilde{\gamma}$ with anchor in $\mathcal{X}_I^{(v,e,f)}$ do not contain essential cycles, and hence are normal connected edge-sets. Thus we can apply Lemma 4.12 to prove that there exists a bijection between $\mathcal{X}_I^{(v,e,f)}$ and $\mathcal{X}_I^{\text{origin}}$. It is easy to see that $|\mathcal{X}_I^G| = 8pq|\mathcal{X}_I^{(v,e,f)}|$. Therefore, if H_I is connected then $|\mathcal{X}_I^G| = 8pq|\mathcal{X}_I^{\text{origin}}|$ for every locally grid graph G with pq vertices.

If H_I is not connected, let $V_1, \dots, V_s \subseteq \{1, \dots, n\}$ be the vertices of its connected components and let $I_1, \dots, I_s \subseteq I$ be the edge-sets of these components (note that some I_k might be empty). Then

$$|\mathcal{X}_I^G| = |\overline{\mathcal{X}}_{I_1}^G| \cdots |\overline{\mathcal{X}}_{I_s}^G|,$$

where the anchors in $\overline{\mathcal{X}}^G_{I_k}$ refer only to the words $\gamma_i \in \widetilde{\gamma}$ for $i \in V_k$ and not to the whole of $\widetilde{\gamma}$. Since all the H_{I_k} are now connected, the argument above shows that none of $|\overline{\mathcal{X}}^G_{I_k}|$ depends on G, and therefore neither does $|\mathcal{X}^G_I|$, and the claim is proved. □

This claim finishes the proof of the lemma. □

Lemmas 4.14 and 4.15 together imply Lemma 4.9 and the following corollary.

Corollary 4.16 *Let G, G' be a pair of locally grid graphs with pq vertices. If $l_G \neq l_{G'}$, then $t(G; x, y) \neq t(G'; x, y)$. If $l_G = l_{G'}$, but G and G' do not have the same number of shortest essential cycles, then also $t(G; x, y) \neq t(G'; x, y)$.*

Proof. Suppose that $l_G < l_{G'}$. By Lemma 4.9, the number of normal edge-sets of rank $l_G - 1$ and size l_G is the same in G and G'. Since there are essential cycles of length l_G in G, but not in G', the coefficient of $x^{l_G-1}y^{l_G}$ in the rank-size generating polynomial is greater in G than in G', and thus their Tutte polynomials are different. The second statement follows in a similar way. □

A careful revision of the proof of Lemma 4.9 shows that in some special cases it is also possible to count the number of normal edge-sets with size greater than l_G. Denote by $N(\gamma, G, r, m)$ the number of normal edge-sets in G with rank r and size m, and such that have $B(\gamma)$ as a subgraph. We say that an edge-set $A \subseteq E(G)$ is a *forbidden edge-set for γ* in G if it contains an essential cycle and a subset $B \subseteq A$ isomorphic to $B(\gamma)$. The following corollary, which will be very useful in the next section, says that $N(\gamma, G, r, m)$ is the same in all locally grid graphs that do not have forbidden edge-sets for γ.

Corollary 4.17 *Let $\gamma \in \Gamma$ be such that $B(\gamma)$ contains at least one cycle. Then the quantity $N(\gamma, G, r, m)$ is the same for all locally grid graphs G with pq vertices, no forbidden edge-set for γ of size m, and such that $m \leq l_G + 2$.*

4.4 Tutte uniqueness

In the previous sections we have assembled all the necessary machinery in order to prove the T-uniqueness of toroidal grids. In the light of Corollary 4.16, we first examine the length and number of the shortest essential cycles in each type of locally grid graph.

Lemma 4.18 *If G is a locally grid graph, then the length l_G of the shortest essential cycle and the number cycles of length l_G, or a lower bound on this number, are given in the following table.*

G	l_G	number of essential cycles	
$C_p \times C_q$	$\min\{p, q\}$	q	if $p < q$
		$2p$	if $p = q$
		p	if $q < p$
$T^{\delta}_{p,q}$	$\min\{p, q + \delta\}$	q	if $p < q + \delta$
		$q + p\binom{q+\delta-1}{\delta}$	if $p = q + \delta$
		$p\binom{q+\delta-1}{\delta}$	if $q + \delta < p$
$K^0_{p,q}$	$\min\{p, q + 1\}$	q	if $p < q + 1$
		$5q$	if $p = q + 1$
		$4q$	if $q + 1 < p$
$K^1_{p,q}$	$\min\{p, q\}$	q	if $p < q$
		$q + 1$	if $p = q$
		1	if $q < p$
$K^2_{p,q}$	$\min\{p, q\}$	q	if $p < q$
		$q + 2$	if $p = q$
		2	if $q < p$
$S_{p,q}$	$\min\{2p, q\}$	at least 2^{q-1}	if $p \leq q \leq 2p$
		at least q	if $2p \leq q$
		2^q	if $q \leq p$

Proof. All locally grid graphs can be obtained by adding some edges to a (p, q)-grid H. Let us call any of these edges an *exterior edge*. Clearly every essential cycle must contain at least one exterior edge. The essential cycles obtained by joining the two ends of an exterior edge by a path contained in H are candidates to be the shortest essential cycles. We first study these cycles, and then prove that any essential cycle containing two or more exterior edges is actually longer (with one exception). We repeatedly use the fact that the length of a shortest path between the points $(0, 0)$ and (a, b) in a grid is $|a| + |b|$, and that there are $\binom{|a|+|b|}{|a|}$ such paths.

1. $T^{\delta}_{p,q}$.

The q edges of the form $\{(0, j), (p - 1, j)\}$ determine one essential cycle of length p each. The p edges of the form $\{(i, 0), (i + \delta, q - 1)\}$ determine $\binom{q+\delta-1}{\delta}$ essential cycles of length $q + \delta$. If $p = q$, then we have both kinds of cycles, that is, $2p$ cycles of length p.

2. $K^i_{p,q}$.

As in the previous case, the q edges of the form $\{(0, j), (p - 1, j)\}$ give rise to one essential cycle of length p each.

If $i = 0$, among the edges of the form $\{(j, 0), (p - j - 1, q - 1)\}$, the shortest essential cycle is determined by joining by a path the two ends of any of the following four edges:

$$\{(0, 0), (p - 1, q - 1)\}, \{(p - 1, 0), (0, q - 1)\},$$

$$\{(p/2, 0), (p/2 - 1, q - 1)\}, \{(p/2 - 1, 0), (p/2, q - 1)\}.$$

Any of these gives rise to q essential cycles of length $q+1$; therefore, we have a total of $4q$ essential cycles of length $q+1$.

If $i=1$, the edge $\{((p-1)/2,0),((p-1)/2,q-1)\}$ is, among the "twisted" edges, the one that determines a shortest cycle; in this case there is one cycle of length q. Similarly, if $i=2$ there are two essential cycles of length q.

3. $S_{p,q}$.

We treat first the case $p \leq q$. Each of the $q-p$ exterior edges of the form $\{(0,i),(p-1,i-p)\}$ determines $\binom{2p-1}{p}$ cycles of length $2p$. If $2p \leq q$, then $(q-p)\binom{2p-1}{p} > q$, and the bound in the table follows. There are other essential cycles of length $2p$; one of them is

$$\begin{aligned}&(0,0),(0,q-1),(1,q-1),(2,q-1),\dots,(p-2,q-1),\\&\quad(p-1,q-1),(p-1,0),(p-2,0),\dots,(1,0),(0,0).\end{aligned} \tag{4.1}$$

Observe that this cycle contains two exterior edges: $\{(0,0),(0,q-1)\}$ and $\{(p-1,q-1),(p-1,0)\}$.

The edges of the form $\{(0,i),(i,q-1)\}$ or $\{(i,0),(p-1,q-p+i)\}$ give rise to $\binom{q-1}{i}$ cycles of length q each. We have thus to evaluate the quantity

$$2\sum_{i=0}^{p-1}\binom{q-1}{i}=\sum_{i=0}^{p-1}\binom{q-1}{i}+\sum_{i=0}^{p-1}\binom{q-1}{q-1-i}.$$

If $q \leq 2p$, then $q-p \leq p$ and hence the expression above contains at least once all the binomials of the form $\binom{q-1}{j}$ for j with $0 \leq j \leq q-1$. Therefore the number of cycles of length q is at least 2^{q-1}.

Let us study now the case $q < p$. Each of the q edges of the form $\{(j,0),(0,q-1-j)\}$ determines $\binom{q-1}{j}$ cycles of length q. These quantities add up to 2^{q-1} cycles. Since the edges of the form $\{(p-1-i,q-1),(p-1,i)\}$ behave in the same way, we have a total of 2^q essential cycles of length q. The essential cycles determined by the edges $\{(i,q-1),(i+q,0)\}$ have length $2q$, and thus they are never the shortest ones.

To complete the proof we have to show that we cannot improve these numbers by considering essential cycles containing two or more exterior edges. The proof is long since it deals with several cases, but it has no difficulty and hence we omit it. The key idea is the following observation. Let C be an essential cycle and let Q be a cycle of length four with $|C \cap Q| \geq 2$. Then the set $(C-(C\cap Q)) \cup (Q-(C\cap Q))$ contains an essential cycle of length at most $|C|$. We use this result to prove that, with the exception of the cycle (4.1) above, any essential cycle that contains at least two exterior edges can be shortened. □

Now we can finally prove the T-uniqueness of the toroidal grid $C_p \times C_q$.

Theorem 4.19 *The graph $C_p \times C_q$ is T-unique for $p,q \geq 6$.*

Proof. Let $p,q \geq 6$ be fixed integers, and let G be a graph T-equivalent to $C_p \times C_q$. By Theorems 4.2 and 4.3 we know that G is one of $T^{\delta}_{p',q'}, K^{i}_{p',q'}$, or $S_{p',q'}$, for some p',q' with $p'q'=pq$.

In order to show that G is necessarily $C_p \times C_q$, we make use of Corollary 4.16 and Lemma 4.18. Thus, it only remains to distinguish the cases in which the length and number of shortest essential cycles in G agree with the length and number of shortest essential cycles in $T_{p,q}$. We assume that $p \leq q$.

Case I $G = T^{\delta}_{p,q}$, with $\delta > 0$, $p < q$.

Our aim is to show that $T_{p,q}$ has more edge-sets with rank $q-1$ and size q than $T^{\delta}_{p,q}$. Recall that H is the $p \times q$ grid used to define locally grid graphs. For every r with $0 \le r \le q-1$, denote by E_r the set of edges that join a vertex at height r in H with a vertex at height $r+1$ (the *height* of a vertex is its second coordinate). Let A be an edge-set with rank $q-1$ and size q in either $T_{p,q}$ or $T^{\delta}_{p,q}$. If there exists some r such that A does not contain any edge in E_r, define $s(A)$ as

$$s(A) = \min\{r | A \cap E_r = \emptyset\}.$$

Observe that if $A \subset E(T^{\delta}_{p,q})$ the minimum $s(A)$ always exists, whereas there are some essential cycles of length q in $T_{p,q}$ that contain one edge of each set E_r. To prove that $T_{p,q}$ has more edge-sets with rank $q-1$ and size q than $T^{\delta}_{p,q}$ it is enough to find a bijection φ_r between $\{A \subseteq E(T_{p,q}) | r(A) = q-1, |A| = q, s(A) = r\}$ and $\{A \subseteq E(T^{\delta}_{p,q}) | r(A) = q-1, |A| = q, s(A) = r\}$.

Define $\overline{\varphi}_r$ from $E(T_{p,q}) - E_r$ to $E(T^{\delta}_{p,q}) - E_r$ as

$$\{(x,y),(x',y')\} \longrightarrow \begin{cases} \{(x,y),(x',y')\}, & \text{if } r+1 \le y, y' \le q-1; \\ \{(x,y),(x'-\delta,y')\}, & \text{if } y = q-1, y' = 0; \\ \{(x-\delta,y),(x',y')\}, & \text{if } y = 0, y' = q-1; \\ \{(x-\delta,y),(x'-\delta,y')\}, & \text{otherwise.} \end{cases}$$

It is straightforward to show that $\overline{\varphi}_r$ is a bijection. To obtain the desired bijection φ_r define $\varphi_r(A)$ as $\cup_{e \in A} \overline{\varphi}_r(e)$.

Case II $G = K^{i}_{p,q}$, with $p < q$.

This case is solved in a way similar to that of Case I.

Case III $K^{0}_{p',q'}$, with $p = q'+1$, $q = 4q'$, $p' = 4(q'+1)$.

In this case we prove that in $T_{p,q}$ there are more edge-sets with rank $p+1$ and size $p+2$ than in $K^{0}_{p',q'}$. These edge-sets can be classified into three groups:

(i) normal edge-sets;

(ii) sets containing an essential cycle of length p and two other edges;

(iii) essential cycles of length $p+2$.

We show that the number of edge-sets in each of these groups is greater in $T_{p,q}$ than in $K^{0}_{p',q'}$, therefore proving that $F(T_{p,q}; x, y) \neq F(K^{0}_{p',q'}; x, y)$.

(i) We apply Corollary 4.17 to count normal edge-sets. The only possible forbidden edge-sets of size $p+2$ and containing a contractible cycle are the ones in $K^{0}_{p',q'}$ shown in Figure 4.12.a. Therefore, the number of normal edge-sets having rank $p+1$ and size $p+2$, and containing a cycle of length at least six, is the same in $K^{0}_{p',q'}$ as in $T_{p,q}$. It only remains to prove that the number of normal edge-sets having rank $p+1$ and size $p+2$, and containing a cycle of length four, is smaller in $K^{0}_{p',q'}$ than in $T_{p,q}$.

Again by Corollary 4.17, the number of edge sets with rank p and size $p+1$ and containing a square is the same in both graphs, call it s_p. Add one edge to each of these sets in order to obtain edge-sets with size $p+2$ and rank at most $p+1$. There are three possibilities depending on which edge we are adding. The resulting set can be of one of the following types.

(A) A normal edge-set with rank $p+1$.
(B) A normal edge-set containing two contractible cycles and hence having rank p.
(C) An edge-set containing an essential cycle of length p and a contractible cycle of length four.

Call $\mathcal{A}(G)$, $\mathcal{B}(G)$ and $\mathcal{C}(G)$, $G \in \{T_{p,q}, K^0_{p',q'}\}$, the families of all edge-sets in G that belong to the groups indexed by (A), (B), (C), respectively. We have to prove that $|\mathcal{A}(T_{p,q})| > |\mathcal{A}(K^0_{p',q'})|$. We have the following equality.

$$s_p(2pq - p - 1) = |\mathcal{A}(G)|(p-2) + \sum_{B \in \mathcal{B}(G)} (p + 2 - \delta(B)) + |\mathcal{C}(G)|(p-2),$$

where by $\delta(B)$ we denote the number of edges of B that belong to all cycles of length four in B.

Note that $\mathcal{C}(T_{p,q})$ is empty whereas the sets in Figure 4.12.a belong to $\mathcal{C}(K^0_{p',q'})$. Applying Corollary 4.17 several times we get that

$$\sum_{B \in \mathcal{B}(T_{p,q})} (p + 2 - \delta(B)) = \sum_{B \in \mathcal{B}(K^0_{p',q'})} (p + 2 - \delta(B)).$$

Therefore $|\mathcal{A}(T_{p,q})|$ must be greater than $|\mathcal{A}(K^0_{p',q'})|$.

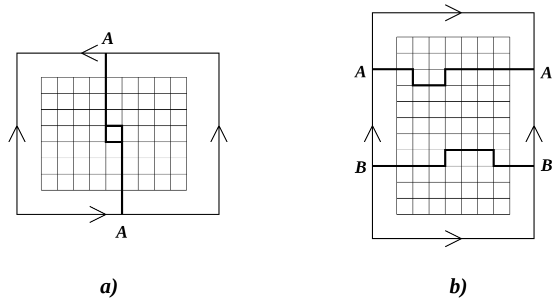


Figure 4.12 a) A set of rank $q'+1$ and size $q'+3$ in $K^0_{p',q'}$ containing an essential cycle; b) Essential cycles of length $p+2$ in $T_{p,q}$.

(ii) In $T_{p,q}$ every essential cycle of length p plus two edges has rank $p+1$. This is not true for all essential cycles in $K^0_{p',q'}$ (see Figure 4.12.a). Since by hypothesis the number of essential cycles is the same, the number of edge-sets in this case is greater in $T_{p,q}$ than in $K^0_{p',q'}$.

(iii) There is one single possibility in $T_{p,q}$ for essential cycles of length $p+2$ (see Figure 4.12.b) and we have $2q\binom{p}{2}$ of them. The two possibilities in $K^0_{p',q'}$ correspond to paths depicted in Figure 4.13. Since we are assuming that $p' = 4(q'+1) = 4p$, and $p \geq 6$, in both cases there are four different exterior edges that can be used to produce an essential cycle of length $p+2$. The total number of such cycles is then $4(p-1)\binom{p-1}{2} + 4\binom{p+1}{3}$, where the factor 4 stands for the possible exterior edges, and the remaining factors correspond to the possible choices of the vertical steps. We again see that the first quantity is greater than the second.

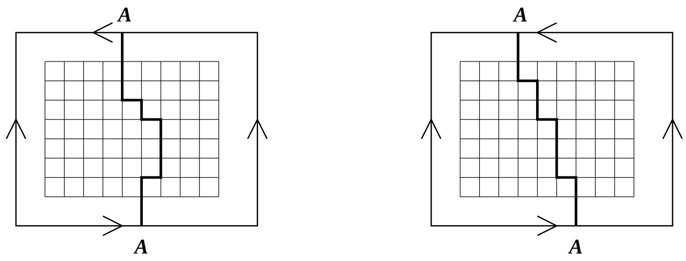


Figure 4.13 Essential cycles of length $q'+3$ in $K^0_{p',q'}$.

Case IV $G = S_{p',q'}$ with $p' = q = 2^p$, $q' = p$.

In this case we prove that $T_{p,q}$ has more edge-sets with rank $q'+1$ and size $q'+2$ that $S_{p',q'}$. The proof uses the same ideas as Case III and we omit it for the sake of brevity. □

We have shown that $C_p \times C_q$ is T-unique for $p, q \geq 6$. For the cases with $p \leq 5$ our techniques do not apply, since either the graphs are not locally grid, or they contain cycles of length 5; proving their T-uniqueness would require specific arguments. Also, it appears that all locally grid graphs are T-unique; proving this would need a careful cross-checking of any two of them as in Theorem 4.19.

5
Line graphs

In this chapter we focus on line graphs. In Section 5.1 we prove that a graph T-equivalent to the line graph of a d-regular d-edge-connected graph is the line graph of a d-regular graph. Thus, in contrast with the last two chapters, where we built specific arguments for each family of graphs, we find a general method for proving the T-uniqueness of d-regular line graphs. In Section 5.2 we apply this method to prove the T-uniqueness of the line graphs of regular complete t-partite graphs. We also prove that the line graph of any complete bipartite graph is T-unique. We conclude the chapter by showing how to construct pairs of nonisomorphic T-equivalent line graphs.

We mention here the related problem of characterizing line graphs by means of the characteristic polynomial (that is, the polynomial that has as roots the eigenvalues of the adjacency matrix); this problem has received much attention in the literature (see [7, Chapter 3] and [28]). In particular, it is known that $L(K_n)$ is determined by its characteristic polynomial except for $n = 8$; in this case there are three graphs with the same spectrum as $L(K_8)$. Similarly, $L(K_{n,n})$ is also determined by its characteristic polynomial except for $n = 4$; in this case there is one exceptional graph cospectral but not isomorphic to $L(K_{4,4})$.

5.1 Main result

All graphs in this chapter are simple. We start by recalling the definition of the *line graph* $L(G)$ of a graph G: it has as vertices the edges of G and two vertices are adjacent if the corresponding edges in G are incident with the same vertex. The basic property we use of $L(G)$ is that it decomposes into edge-disjoint cliques, each of them corresponding to the set of edges incident with a vertex. In fact, we have the following characterization of line graphs ([32, Chapter 8]).

Theorem 5.1 *A graph G is a line graph if and only if the edges of G can be partitioned into cliques such that no vertex of G lies in more than two of the cliques. Furthermore, if $C_1, \dots, C_t$ are cliques of G such that every vertex belongs to exactly two of them (some of the cliques may be trivial), then G is the line graph of G_0, where G_0 has as vertices $\{c_1, \dots, c_t\}$ and an edge between c_i and c_j if C_i meets C_j.*

We need the following result from [25] relating the connectivity of a graph and that of its line graph.

Lemma 5.2 *If a graph G is n-edge-connected, then its line-graph $L(G)$ is n-connected and $(2n-2)$-edge-connected.*

Next we prove that, under certain hypotheses, a graph T-equivalent to the line graph of a d-regular d-edge-connected graph is the line graph of a d-regular graph.

Theorem 5.3 *Let G be a d-regular d-edge-connected graph on n vertices and assume that $d \geq 3$ and, if $d = 3$, then G is triangle-free. If a graph H is T-equivalent to $L(G)$, then $H = L(G_0)$, where G_0 is a d-regular connected graph on n vertices.*

Proof. By Lemma 5.2, $L(G)$ is d-connected and $(2d-2)$-edge-connected. Hence, by Theorem 1.5, the graph H is 2-connected; by Theorem 1.17, it has $nd/2$ vertices and $nd(d-1)/2$ edges, and it is $(2d-2)$-edge-connected. Then the minimum degree of H is at least $2d-2$ and, since the sum of the degrees of all vertices must be $nd(d-1)$, it follows that H is $(2d-2)$-regular.

Under the hypotheses, $L(G)$ has clique number d and has exactly n cliques of order d (we have excluded triangles in G when $d = 3$ since they also give rise to triangles in $L(G)$). Since H is T-equivalent to $L(G)$, we deduce that H has exactly n cliques of size d; let us denote them by $C_1, C_2, \ldots, C_n$. The key ingredient of the proof is the following claim.

Claim For $i \neq j$, the cliques C_i and C_j meet in at most one vertex of H.

Proof. In $L(G)$ every edge belongs to either $d-1$ or $d-2$ triangles; we prove first that this also holds for H. For an edge $e \in E(L(G))$, let $t(e)$ be the number of triangles of $L(G)$ that contain e; similarly, for an edge $f \in E(H)$, denote by $t'(f)$ the number of triangles of H that contain f. By double counting the cardinalities of the sets $\{(t, e) : e \text{ is an edge of the triangle } t\}$ and $\{(c, e) : e \text{ is a chord of the 4-cycle } c\}$ in both H and $L(G)$ we have

$$\sum_{e \in E(L(G))} t(e) = \sum_{f \in E(H)} t'(f), \qquad \sum_{e \in E(L(G))} \binom{t(e)}{2} = \sum_{f \in E(H)} \binom{t'(f)}{2}.$$

If there exist two edges $f_1, f_2 \in E(H)$ such that $|t'(f_1) - t'(f_2)| > 1$, then

$$\sum_{e \in E(L(G))} \binom{t(e)}{2} < \sum_{f \in E(H)} \binom{t'(f)}{2}.$$

Therefore $t'(f)$ is either $d-1$ or $d-2$ for all $f \in E(H)$.

Suppose now that two cliques C_i and C_j meet in a complete subgraph $K \cong K_p$ for some p with $2 \leq p \leq d-1$. If f is an edge of K, then $t'(f) = 2d - p - 2$; since $t'(f) \leq d - 1$, we deduce that $p = d-1$. Hence $C_i \cup C_j$ induces a subgraph in H isomorphic to K_{d+1}^-, the complete graph K_{d+1} minus an edge. Observe that $L(G)$ contains no such subgraph. Since K_{d+1}^- is the only simple graph with rank d and $\binom{d+1}{2} - 1$ edges, we can deduce from the knowledge of the Tutte polynomial that H contains no subgraph isomorphic to K_{d+1}^-. Therefore $C_i \cap C_j$ is either empty or one vertex. □

Since H has $n\binom{d}{2}$ edges, the previous claim implies that each edge of H belongs to exactly one of the d-cliques; since H is $(2d-2)$-regular, every vertex of H belongs to at most two of the cliques. Actually, since there are nd cliques of order d and H has $nd/2$ vertices, each clique must intersect exactly d other cliques, and hence every vertex is in exactly two of the cliques. By Theorem 5.1, H is the line graph of a graph G_0 on n vertices, which is the intersection graph of the cliques $\{C_1, \ldots, C_n\}$. The graph G_0 is clearly d-regular, and it is connected since the line graph of a disconnected graph is not 2-connected. □

We can in fact obtain more information about the graph G_0 in the above theorem.

Theorem 5.4 *With the notation as in Theorem 5.3, G_0 has the same number of triangles as G.*

Proof. We show how to deduce the number of triangles in a d-regular graph G' with n vertices from the knowledge of the Tutte polynomial of its line graph $L(G')$. Let $\tau(G')$ be the number of triangles in G'. Since triangles in $L(G')$ arise either from triangles in G' or from three incident edges in G', we have

$$\tau(L(G')) = \tau(G') + n\binom{d}{3}.$$

As we have seen in the previous proof, n and d can be deduced from $t(L(G'); x, y)$. By Theorem 1.17, we can also deduce the value of $\tau(L(G'))$, and hence we know $\tau(G')$. □

5.2 Line graphs of complete multipartite graphs

Using the previous results we prove that line graphs of regular complete multipartite graphs are T-unique. We begin with a simple application of Theorem 5.3 that covers the case of a complete graph.

Corollary 5.5 *The graph $L(K_p)$ is T-unique for $p \geq 3$.*

Proof. Let H be a graph T-equivalent to $L(K_p)$. If $p \geq 5$, the hypotheses of Theorem 5.3 hold, and therefore we know that H is the line graph of a $(p-1)$-regular graph with p vertices. Since the only such graph is K_p, the result follows.

The two cases remaining have to be treated separately. For $p = 3$, we have that $L(K_3) = K_3$, which is clearly T-unique. If $p = 4$, the line graph of K_4 is $K_{2,2,2}$; its T-uniqueness follows from Theorem 3.1. □

In the same spirit we could prove the T-uniqueness of $L(K_{2n} - nK_2)$, the line graph of a complete graph minus a perfect matching.

Now take $G = K(p,t)$, the complete t-partite graph with parts of size $p \geq 2$, and suppose H is a graph T-equivalent to $L(G)$. The graph $K(p,t)$ is $(tp-p)$-regular and $(tp-p)$-edge-connected. If $p = t = 2$, $L(K(2,2)) = C_4$, a cycle of length four, which is T-unique. For all the remaining cases the hypotheses of Theorem 5.3 hold. Then we know that $H = L(G_0)$, where G_0 is a $(tp-p)$-regular graph with tp vertices. By Theorem 5.4, we also know that G_0 and $G = K(p,t)$ have the same number of triangles, namely $p^3\binom{t}{3}$. Then it is enough to prove the following extremal result.

Lemma 5.6 *If G is a $(tp-p)$-regular graph with tp vertices, then G has at least $p^3\binom{t}{3}$ triangles. Moreover, equality holds if and only if $G \cong K(p,t)$.*

Proof. Let $N(x)$ denote the set of vertices adjacent to a vertex x. For every edge $e \in E(G)$, label its ends arbitrarily as x and y, and define the following quantities:

$$\begin{array}{rcl} \alpha(e) & = & |N(x) \cap (V(G)\backslash N(y))|, \\ \beta(e) & = & |N(x) \cap N(y)|, \\ \gamma(e) & = & |(V(G)\backslash N(x)) \cap (V(G)\backslash N(y))|. \end{array}$$

Then, from

$$\begin{array}{l} \alpha(e) + \beta(e) = d_G(x) = tp - p \\ \alpha(e) + \gamma(e) = tp - d_G(y) = p \end{array}$$

it follows that $\beta(e) = tp - 2p + \gamma(e)$. Since $\beta(e)$ equals the number of triangles that contain e, the total number of triangles in G is

$$\frac{1}{3} \sum_{e \in E(G)} (tp - 2p + \gamma(e)) \;\geq\; \frac{1}{3}|E(G)|(tp - 2p) = p^3 \binom{t}{3}.$$

This proves the first part of the claim.

If we have an equality, then $\gamma(e) = 0$ for all $e \in E(G)$. This means that given any edge xy and a third vertex z, either x or y is adjacent to z.

Let s be the chromatic number of G, and let $V = V_1 \cup \cdots \cup V_s$ be a partition of the vertex set of G into s stable subsets. From the regularity of G it follows that there are at least $tp-p$ vertices outside each part; therefore, $|V_i| \leq p$ for all i, and $s \geq t$.

Let us prove that G is complete s-partite. By definition of the chromatic number, for any two different stable subsets V_i and V_j there exists an edge xy with $x \in V_i$ and $y \in V_j$. Let z be any vertex in $V_j \backslash y$; since $\gamma(xy) = 0$ and y and z are in the same part, it follows that xz is an edge. Repeated application of this argument shows that any pair of vertices in different stable subsets are adjacent.

Since G is complete s-partite, $(tp - p)$-regular, and has tp vertices, it follows that $s = t$ and $G \cong K(p, t)$. □

Corollary 5.7 *The graph $L(K(p, t))$ is T-unique for $t \geq 2$.*

Our next goal is to prove the T-uniqueness of $L(K_{p,q})$, a case not covered by Theorem 5.3 since $K_{p,q}$ is not regular for $p \neq q$. For this we need a simple combinatorial lemma.

Lemma 5.8 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of a positive integer n with $2 \leq k < n$ and $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$. Let μ be the conjugate partition of λ, that is, $\mu_j = |\{i \colon \lambda_i \geq j\}|$. Then*

$$\sum_i \binom{\lambda_i}{2} + \sum_j \binom{\mu_j}{2} \leq \binom{n-1}{2} + 1.$$

Proof. By induction on n, starting with the trivial case $n = 2$. Assume the claim holds for n, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of $n + 1$ into at least two parts and at most n, and let $\mu = (\mu_1, \ldots, \mu_l)$ be the conjugate partition of λ.

Define an auxiliary partition of n as $\lambda' = (\lambda_1, \lambda_2, \ldots, \lambda_k - 1)$, where the last term is omitted if $\lambda_k = 1$ (if $\lambda = (n, 1)$ then λ' would have only one part, but in this case the theorem can be checked directly). Letting $h = \lambda_k$, the conjugate partition of λ' is $\mu' = (\mu_1, \ldots, \mu_{h-1}, \mu_h - 1, \mu_{h+1}, \ldots, \mu_l)$. Then, by induction hypothesis,

$$\sum_i \binom{\lambda_i}{2} + \sum_j \binom{\mu_j}{2} = \sum_i \binom{\lambda'_i}{2} + \sum_j \binom{\mu'_j}{2} + (\lambda_k - 1) + (\mu_h - 1) \le \binom{n-1}{2} + \lambda_k + \mu_h - 1 \le \binom{n}{2} + 1,$$

the last inequality because $\lambda_k + \mu_h \le \lambda_k + k \le n + 1$. □

Theorem 5.9 *The graph $L(K_{p,q})$ is T-unique for all $p \le q$.*

Proof. The case $p = q$ is covered by Corollary 5.7, so let us assume $p < q$. Suppose H is T-equivalent to $L(K_{p,q})$. Then, by Theorems 1.5 and 1.17, H is 2-connected with pq vertices and $p\binom{q}{2} + q\binom{p}{2}$ edges. Since $L(K_{p,q})$ is $(p+q-2)$-edge-connected, so is H. Then the minimum degree of H is at least $p+q-2$ and H is in fact $(p+q-2)$-regular.

Since $L(K_{p,q})$ has clique number q and has exactly p cliques of size q, so does H. Let $C_1, \ldots, C_p$ be the cliques of size q in H. We claim that these cliques are vertex disjoint.

It cannot be that $V(C_i) \cap V(C_j) = \{x\}$, since then the degree of x in H would be at least $2q - 2 > p + q - 2$. Suppose $|V(C_i) \cap V(C_j)| > 1$ and let $x \in V(C_j) \backslash V(C_i)$. Then the vertex set $V(C_i) \cup \{x\}$ induces a subgraph in H of rank q and size at least $\binom{q}{2} + 2$. We now prove that such a subgraph does not exist in $L(K_{p,q})$. This will give a contradiction because H and $L(K_{p,q})$ are T-equivalent.

Let $A \subseteq E(L(K_{p,q}))$ have rank q and let us bound the size of A. The graph $L(K_{p,q})$ can be thought of as a $p \times q$ grid in which two vertices in the same row or column are adjacent. Let $R_1, \ldots, R_p$ be the vertex sets corresponding to the rows, and let S_i be the vertices in R_i incident to an edge of A (some S_i might be empty). In order to maximize $|A|$ we may assume that the vertices of S_i are left-justified, that is, they form a consecutive set of vertices in R_i starting in the first column, and thus $|S_1| + \cdots + |S_p| = q + 1$.

Assume without loss of generality that $|S_1| \ge |S_2| \ge \cdots \ge |S_p|$ (note that $L(K_{p,q})$ is not affected by permuting the rows). Since $|R_i| = q$ and $|A| = q + 1$, the set S_2 is nonempty. If we set $\lambda_i = |S_i|$, then $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of $q + 1$, where k is the largest index for which S_i is not empty; clearly, $k \le p < q$, and hence the hypotheses of Lemma 5.8 hold. If μ is the conjugate partition of λ, then the size of A is equal to

$$\sum_i \binom{\lambda_i}{2} + \sum_j \binom{\mu_j}{2}.$$

By the previous lemma this quantity is at most $\binom{q}{2} + 1$, thus proving that there are no subgraphs in $L(K_{p,q})$ with rank q and size $\binom{q}{2} + 2$.

Since the cliques C_i are disjoint, $V(C_1) \cup \cdots \cup V(C_p)$ accounts for all the pq vertices in H. Each vertex is adjacent to the $q - 1$ vertices in its own clique; since the degree is $p + q - 2$, it must be adjacent to other $p - 1$ vertices. If a vertex $x \in V(C_i)$ were adjacent to $y, z \in V(C_j)$ with $j \ne i$, then $C_j \cup \{x\}$ would induce a subgraph of rank q and size $\binom{q}{2} + 2$, which we just proved is not possible. Summarizing, each vertex is adjacent to exactly one vertex in each clique different from its own. This means that the vertices of H can be partitioned into q sets of size p, each of them containing one vertex from each of the cliques $C_1, \ldots, C_p$.

Finally, let us consider the number of p-cliques in H. It must be the same as in $L(K_{p,q})$, that is, $p\binom{q}{p} + q$. The number of p-cliques contained in the q-cliques C_i is $p\binom{q}{p}$; hence there must exist q additional ones. The fact that there are no edge-sets with rank q and size $\binom{q}{2} + 2$ implies that each p-clique intersects each q-clique in one vertex. This combined with the conclusion of the last paragraph shows that H has the structure of a $p \times q$ grid as defined above, and this finishes the proof. □

5.3 T-equivalent regular line graphs

The results above might lead us to conjecture that if two line graphs $L(G_1)$ and $L(G_2)$ have the same Tutte polynomial, then $L(G_1) \cong L(G_2)$ (this is equivalent to $G_1 \cong G_2$, except if G_1 is a triangle and G_2 a star $K_{1,3}$ [32, Theorem 8.3]). However, using a construction of Tutte [55] we can provide examples of pairs of nonisomorphic T-equivalent line graphs, which can even be chosen to arise from d-regular graphs.

A graph R is called a *rotor of order* n if there is a subset of the vertices $\{x_1, \ldots, x_n\} \subseteq V(R)$ and an automorphism φ of R such that $\varphi(x_i) = x_{i+1}$ for all i, where the indices are modulo n. The set $\{x_1, \ldots, x_n\}$ is called the *border* of R. We have the following theorem from [55].

Theorem 5.10 *Let R be a rotor of order n with $3 \le n \le 5$, and let S be a graph with n selected vertices $\{y_1, \ldots, y_n\} \subseteq V(S)$. Let G be the graph formed from $R \cup S$ by identifying x_i with y_i for all $1 \le i \le n$. Similarly, let H be the graph formed from $R \cup S$ by identifying x_{n-i+1} with y_i for all $1 \le i \le n$. Then G and H have the same Tutte polynomial.*

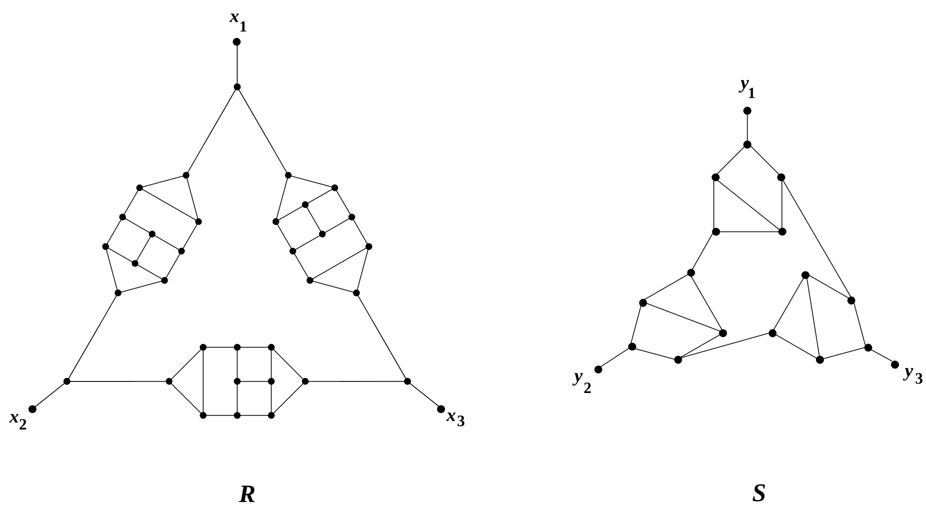


Figure 5.1 The line graphs of R and S are used by the rotor construction to produce T-equivalent line graphs.

We show next how to use this construction to produce pairs of nonisomorphic T-equivalent line graphs. Take as R any rotor of order n with $3 \le n \le 5$ such that the vertices $\{x_1, \ldots, x_n\}$ have degree one; denote by e_i the only edge incident with x_i. Let R' be the line graph $L(R)$;

R' is a rotor of the same order as R whose border are the vertices corresponding to the edges $\{e_1, \ldots, e_n\}$; moreover, each of these vertices belongs to only one nontrivial clique of $L(R)$. Similarly, take as S any graph with a selected set of vertices $\{y_1, y_2, \ldots, y_n\}$ such that y_i is incident with only one edge f_i. Denote by S' the line graph of S. The vertices of S' that correspond to the edges $\{f_1, \ldots, f_n\}$ also belong to only one clique in S'. By Theorem 5.1, the graphs G and H that are obtained from Theorem 5.10 with the rotors R' and S' are line graphs, since their edges can be partitioned into cliques and every vertex belongs to at most two of the cliques. By choosing conveniently the rotor R and the graph S so that G and H are not isomorphic, we obtain a pair of nonisomorphic T-equivalent line graphs.

Furthermore, if we choose R and S such that all vertices except $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ have degree d, then G and H are the line graphs of a pair of d-regular graphs (see Figure 5.1 for an example of such R and S). We have thus proved the following corollary.

Corollary 5.11 *There exist pairs of nonisomorphic T-equivalent line graphs $L(G)$ and $L(H)$ such that G and H are d-regular.*

6

k-Chordal matroids

So far, the examples of T-unique matroids known in the literature are obtained by giving characterizations of matroids by numerical invariants; it turns out that these invariants can be deduced from the Tutte polynomial, and hence the T-uniqueness follows. In this chapter we present a technique that can be applied to prove the T-uniqueness of a wider class of matroids, namely what we call r-chordal matroids. Like the theory we developed in Chapter 5 for line graphs, we have a tool that helps proving T-uniqueness, but one still has to check several conditions for each individual matroid and, as we shall see, this is usually nontrivial.

In the first section we introduce the notion of $[k_1, k_2]$-chordal matroid and that of r-chordal matroid, and we give sufficient conditions for a matroid to be isomorphic to a given r-chordal matroid. These conditions consist of the existence of a bijection that preserves circuits of the smallest size, together with agreement on some numerical parameters. These parameters can be computed from the Tutte polynomial, so that to prove T-uniqueness one only needs to find the bijection.

In the second part of the chapter we apply this tool to truncations of cycle matroids of complete graphs and to cycle matroids of complete bipartite graphs. As a previous step to T-uniqueness, we obtain characterizations of these matroids in terms of some numerical invariants. Another application of the methods developed here is given in the next chapter.

6.1 Characterization of $[k_1, k_2]$-chordal matroids

A graph is said to be chordal if every cycle of length at least 4 has a chord ([29]). Chordal graphs have many interesting properties (for instance they give rise to the supersolvable graphic matroids [52]) and they are the topic of much research in graph theory. The notion of a chordal graph has been extended to matroids in [5], where a binary matroid M is said to be chordal if each circuit C of M that has four or more elements has a chord; that is, an element $e \notin C$ such that for some partition of C into two blocks C_1 and C_2 with $|C_1|, |C_2| \geq 2$, both $C_1 \cup e$ and $C_2 \cup e$ are circuits of M. Here we are interested in a slightly different extension of the concept of a chordal graph.

Definition 6.1 *A circuit C of a matroid M is* chordal *if there are circuits C_1 and C_2 of M and an element x in $C_1 \cap C_2$ such that $|C_1|, |C_2| < |C|$ and $C = (C_1 \cup C_2) - x$.*

Let $[k_1, k_2]$ be an interval of integers with $k_1 > g(M)$ and $k_2 \leq r(M) + 1$. A matroid M is $[k_1, k_2]$-chordal *if each circuit C of M with $k_1 \leq |C| \leq k_2$ is chordal.*

A matroid M is k-chordal *if it is $[g(M) + 1, k]$-chordal.*

Note that only circuits with four or more elements can be chordal. Thus, only geometries (i.e. simple matroids) can be k-chordal. We will be most interested in geometries M that are $r(M)$-chordal; when no confusion arises, we refer to them as r-chordal.

Examples. If a matroid is chordal in the sense of [5] and has girth 3, then it is $\big(r(M)+1\big)$-chordal. This includes in particular the cycle matroids of all chordal graphs. However, the cycle matroid of $K_{m,n}$ is not chordal, but it is $(m+n)$-chordal. An easy argument based on linear combinations shows that any projective geometry M is $\big(r(M)+1\big)$-chordal; likewise, any affine geometry M over a field of characteristic not 2 is $\big(r(M)+1\big)$-chordal (note that 4-circuits are not chordal for characteristic is 2).

The following theorem gives necessary conditions to guarantee that a matroid is isomorphic to a given chordal matroid.

Theorem 6.2 *Assume M and M' are matroids on the ground sets S and S', respectively, and M is r-chordal. Assume $\phi : S \to S'$ is a bijection such that for every circuit C of M with $|C| = g(M)$, its image $\phi(C)$ is a circuit of M'. If either of the following two conditions holds, then M and M' are isomorphic and ϕ is an isomorphism.*

(a) *The matroids M and M' have the same girth, the same number of circuits of size $g(M)$, and, for each integer i with $g(M)+1 \le i \le r(M)$, the same number of independent sets of cardinality i.*

(b) $t(M;x,y) = t(M';x,y)$.

Proof. By Theorem 1.15, condition (b) implies condition (a), so we focus on (a). By Theorem 1.2 it is enough to show that a subset C of S is a circuit of M if and only if $\phi(C)$ is a circuit of M'. Since the spanning circuits are precisely the sets of size $r(M)+1$ that do not contain smaller circuits, it suffices to prove this statement in the case that $|C| \le r(M)$. The proof is by induction on $|C|$.

The base case is $|C| = g(M)$. By hypothesis, for every circuit C of M with $|C| = g(M)$, its image $\phi(C)$ is a circuit of M'. The converse follows since M and M' have the same number of circuits of size $g(M)$.

Assume C is a circuit of M with $|C| = i$ and $g(M) < i \le r(M)$. Since M is $r(M)$-chordal, there are circuits C_1 and C_2 of M and an element x in $C_1 \cap C_2$ such that $|C_1|, |C_2| < |C|$ and $C = (C_1 \cup C_2) - x$. Now $\phi(C) = \big(\phi(C_1) \cup \phi(C_2)\big) - \phi(x)$. By the inductive assumption, both $\phi(C_1)$ and $\phi(C_2)$ are circuits of M', so by circuit elimination, $\phi(C)$ contains a circuit of M', say C'. If C' were properly contained in $\phi(C)$, then by induction hypothesis $\phi^{-1}(C')$ would be a circuit properly contained in the circuit C, which is impossible. Thus, $\phi(C)$ is a circuit of M'.

For the converse, it suffices to show that M and M' have the same number of i-circuits. Since ϕ is a bijection, $|S| = |S'|$, so S and S' have the same number of i-subsets. By assumption, M and M' have the same number of independent sets of cardinality i, and therefore the same number of dependent sets of cardinality i. Dependent sets either are circuits or they properly contain a circuit. By the inductive assumption the mapping ϕ provides a bijection between i-sets that properly contain a circuit, hence M and M' have the same number of i-circuits, thereby completing the proof. □

The next theorem addresses the more general case in which M is k-chordal for some $k \le r(M)$. For the matroids M and M' in Theorem 6.3 it is only possible to guarantee that they correspond up to rank k. Observe that if M is k-chordal, its truncation $T^h(M)$ to rank h is $\min(k, h+1)$-chordal.

Theorem 6.3 *Assume M and M' are matroids on the ground sets S and S', respectively, and M is k-chordal. Assume $\phi : S \to S'$ is a bijection such that for every circuit C of M with $|C| = g(M)$, its image $\phi(C)$ is a circuit of M'. If either of the following two conditions holds, then M' is k-chordal.*

(a) *The matroids M and M' have the same girth, the same number of circuits of size $g(M)$, and, for each integer i with $g(M)+1 \leq i \leq k+1$, the same number of independent sets of cardinality i.*

(b) $t(M; x, y) = t(M'; x, y)$.

Furthermore, ϕ is an isomorphism of the truncations of M and M' to rank k. The map ϕ is a bijection between the chordal $(k+1)$-circuits of M and the chordal $(k+1)$-circuits of M'. Also, M and M' have the same number of nonchordal circuits of size $k+1$.

Proof. Again we focus on condition (a). Note that the induction argument used in the proof of Theorem 6.2 works for i with $g(M) \leq i \leq k$; this means that the map ϕ is a bijection from the set of circuits of M of size at most k onto the set of circuits in M' of size at most k. From this we deduce that M' is k-chordal. The assertion about truncations follows by applying Theorem 6.2 to the matroids $T^k(M)$ and $T^k(M')$.

The argument in the proof of Theorem 6.2 also shows that for each chordal circuit C of M with $|C| = k+1$, its image $\phi(C)$ is a chordal circuit of M'. Note that ϕ^{-1} satisfies the hypotheses of the theorem. From this it follows that if $\phi(C)$ is a chordal circuit of M' of size $k+1$, then C is a chordal circuit of M. Therefore, ϕ is a bijection between the chordal $(k+1)$-circuits of M and M'.

As in the proof of Theorem 6.2, ϕ gives a bijection between the sets of size $k+1$ that properly contain circuits. Also, M and M' have the same number of dependent sets of size $k+1$, and thus the same number of $(k+1)$-circuits. This, together with the conclusion of the last paragraph, shows that M and M' have the same number of nonchordal circuits of size $k+1$. □

6.2 Truncations of $M(K_n)$

Cycle matroids of complete graphs have been characterized by numerical invariants by Bonin and Miller [15]. In this section we combine their proof and Theorem 6.2 to provide a characterization of the truncations $T^s(M(K_n))$ for $s \geq 5$. The numerical characterization of $M(K_n)$ given in [15] is based on treating $M(K_n)$ as a Dowling lattice over the trivial group. The sketch of the proof is the following. First, take a geometry M sharing some basic statistics about low-rank flats with $M(K_n)$ and deduce structural properties of M. Then use these properties to show that M satisfies the axioms of a Dowling lattice over the trivial group. To characterize $T^s(M(K_n))$ we modify this argument slightly; we start by taking a geometry with the same statistics about low-rank flats as $T^s(M(K_n))$, and use these statistics to find a bijection between the ground sets of M and $T^s(M(K_n))$ satisfying the conditions of Theorem 6.2. The key point is that $M(K_n)$ and $T^s(M(K_n))$ have the same statistics about low-rank flats if s is not too small. Therefore, the first part of the proof of Bonin and Miller is still valid. In particular, the following lemma is a consequence of their proof of [15, Theorem 3.2].

Lemma 6.4 *Assume that a rank-s geometry M has, for some integer $n > s$,*

(i) $\binom{n}{2}$ *points;*
(ii) $\binom{n}{3}$ *lines with three points;*
(iii) *no five-point planes,* $\binom{n}{4}$ *planes with six points, no planes with more than six points; and*
(iv) *no rank-4 flats with more than 10 points.*

Then,

(1) *all six-point planes in M are isomorphic to $M(K_4)$;*
(2) *for every point x in M, there exist exactly $n-2$ three-point lines $\ell_1, \dots, \ell_{n-2}$ through x;*
(3) *for every point x in M, there are two disjoint sets $\{x_1, \dots, x_{n-2}\}$ and $\{y_1, \dots, y_{n-2}\}$ such that $\ell_i = \{x, x_i, y_i\}$ for all i; also, $\{x_i, x_j, z_{ij}\}$ and $\{y_i, y_j, z_{ij}\}$ are three-point lines, where z_{ij} is the only point in $\mathrm{cl}(\ell_i \cup \ell_j) - (\ell_i \cup \ell_j)$, for all i, j with $i < j$.*

Note that $T^s(M(K_n))$ satisfies assumptions (i)–(iv) of Lemma 6.4 if $5 \le s < n$. Now we use the previous lemma to establish the bijection needed in Theorem 6.2.

Theorem 6.5 *Assume that a rank-s geometry M has, for some integer $n > s \ge 5$,*

(i) $\binom{n}{2}$ *points;*
(ii) $\binom{n}{3}$ *lines with three points;*
(iii) *no five-point planes,* $\binom{n}{4}$ *planes with six points, no planes with more than six points;*
(iv) *no rank-4 flats with more than 10 points;*
(v) *the same number of rank-i independent sets as $T^s(M(K_n))$ for all i with $4 \le i \le s$.*

Then $M \cong T^s(M(K_n))$.

Proof. The conclusions of Lemma 6.4 follow; we fix a point $x \in S(M)$ and use the notation given in that lemma. Since

$$\left|\{x, x_1, \dots, x_{n-2}, y_1, \dots, y_{n-2}\} \cup \{z_{ij} : 1 \le i < j \le n-2\}\right| = \binom{n}{2},$$

the points given in Lemma 6.4 are all the points of M. Now we define the following bijection from the ground set of $T^s(M(K_n))$, which we take to be $\{(i,j) : 1 \le i < j \le n\}$, to the ground set S of M.

$$\begin{cases} \phi\left((n-1,n)\right) = x; & \\ \phi\left((i,n-1)\right) = x_i, & \text{if } 1 \le i \le n-2; \\ \phi\left((i,n)\right) = y_i, & \text{if } 1 \le i \le n-2; \\ \phi\left((i,j)\right) = z_{ij}, & \text{if } 1 \le i < j \le n-2. \end{cases}$$

It only remains to show that the image under ϕ of a 3-circuit of $T^s(M(K_n))$ is a 3-circuit of M, that is, that the image of $\{(i_1,i_2),(i_2,i_3),(i_1,i_3)\}$ is a 3-circuit of M for all i_1, i_2, i_3 with $1 \le i_1 < i_2 < i_3 \le n$. By Lemma 6.4, this is equivalent to showing that the sets $\{x, x_i, y_i\}, \{x_i, x_j, z_{ij}\}, \{y_i, y_j, z_{ij}\}$ and $\{z_{i_1 i_2}, z_{i_2 i_3}, z_{i_1 i_3}\}$ are circuits of M. The first three follow by statement (3) in that lemma, so it only remains to prove that $\{z_{i_1 i_2}, z_{i_2 i_3}, z_{i_1 i_3}\}$ is a circuit of M for all i_1, i_2, i_3 with $1 \le i_1 < i_2 < i_3 \le n$.

To prove this consider the lines $\{x_{i_1}, x_{i_2}, z_{i_1 i_2}\}$ and $\{x_{i_2}, x_{i_3}, z_{i_2 i_3}\}$ and apply the circuit elimination axiom to get that $\{x_{i_1}, x_{i_3}, z_{i_1 i_2}, z_{i_2 i_3}\}$ is a 4-circuit. Its closure is a six-point plane

$\Pi = \{x_{i_1}, x_{i_2}, x_{i_3}, z_{i_1i_2}, z_{i_2i_3}, z_{i_1i_3}\}$. By Lemma 6.4, this plane is isomorphic to $M(K_4)$; we get that $\{z_{i_1i_2}, z_{i_2i_3}, z_{i_1i_3}\}$ is a three-point line by studying the possible lines in Π. □

Since conditions (i)–(v) of Theorem 6.5 can be deduced from the Tutte polynomial, we have the following corollary.

Corollary 6.6 *$T^s(M(K_n))$ is a T-unique matroid for all n, s with $5 \leq s < n$.*

6.3 The cycle matroid of $K_{m,n}$

Our next aim is to give a numerical characterization of the cycle matroid of a complete bipartite graph $M(K_{m,n})$. This section is devoted to the proof of Theorem 6.7. For the matroid M in this theorem, let $\mathcal{F}^{i,j}$ denote the set of all rank-i flats with j elements, and let $f^{i,j}$ be $|\mathcal{F}^{i,j}|$.

Theorem 6.7 *Assume M is a geometry with ground set S and let m, n be integers with $n \geq m \geq 2$ such that:*

(i) $r(M) = m + n - 1$;

(ii) $|S| = mn$;

(iii) *the girth $g(M)$ of M is 4;*

(iv) *M has $\binom{m}{2}\binom{n}{2}$ 4-point planes, and no planes with 5 or more points;*

(v) *M has $\binom{m}{2}\binom{n}{2}(nm-4)$ rank-4 sets with 5 points, M has $\binom{m}{3}\binom{n}{2} + \binom{m}{2}\binom{n}{3}$ rank-4 flats with 6 points, and no rank-4 flats with more than 6 points;*

(vi) *M has the following statistics on rank-5 flats:*

$$\begin{aligned} f^{5,7} &= (m-3)(n-2)\binom{m}{3}\binom{n}{2} + (m-2)(n-3)\binom{m}{2}\binom{n}{3}, \\ f^{5,8} &= \binom{m}{4}\binom{n}{2} + \binom{m}{2}\binom{n}{4}, \\ f^{5,9} &= \binom{m}{3}\binom{n}{3}, \end{aligned}$$

and $f^{5,k} = 0$ for all $k \geq 10$;

(vii) *the maximum size of a rank-s flat F is given by*

$$|F| = \begin{cases} k(k+1), & \text{if } s \leq 2m-1 \text{ and } s = 2k; \\ (k+1)^2, & \text{if } s \leq 2m-1 \text{ and } s = 2k+1; \\ m(s-m+1), & \text{if } s \geq 2m; \end{cases}$$

(viii) *for each i with $4 \leq i \leq m+n-1$, the geometries M and $M(K_{m,n})$ have the same number of independent sets of cardinality i.*

Then $M \cong M(K_{m,n})$.

Since conditions (i)–(viii) of Theorem 6.7 can be deduced from the Tutte polynomial and the T-uniqueness of $M(K_{1,n})$ is immediate, we have the following corollary.

Corollary 6.8 *The geometry $M(K_{m,n})$ is T-unique for all integers m, n with $1 \leq m \leq n$.*

From here on, let M denote a geometry satisfying conditions (i)–(viii) in Theorem 6.7 for some fixed integers m, n with $2 \leq m \leq n$. To prove that $M \cong M(K_{m,n})$ we find a bijection of the ground set of $M(K_{m,n})$ with S, the ground set of M, such that the image of a 4-circuit of $M(K_{m,n})$ is a 4-circuit of M. The background needed to define this bijection is given in the following series of propositions. For the sake of simplicity, in what remains of this section a set $A \subseteq S$ will be called a *plane* if it is a nontrivial plane, that is, a 4-point plane.

Proposition 6.9 *All sets in M of rank four with five elements consist of a plane Π and an element not in the closure of Π. Therefore, there are no 5-circuits in M.*

Proof. By assumption (v), M has $\binom{m}{2}\binom{n}{2}(nm-4)$ subsets with rank 4 and size 5. Note that, by assumption (iv), if Π is a 4-point plane and x is a point outside Π, the set $\Pi \cup \{x\}$ has rank 4 and size 5. These sets account for all the $\binom{m}{2}\binom{n}{2}(nm-4)$ subsets with rank 4 and size 5, and therefore M has no circuit of length 5. □

As a consequence we obtain the following proposition; it can be proved easily using the circuit elimination axiom.

Proposition 6.10 *If two planes $\Pi_1 = \{a, b, c_1, d_1\}$ and $\Pi_2 = \{a, b, c_2, d_2\}$ meet in exactly two points, then $\{c_1, c_2, d_1, d_2\}$ is a plane. Every $A \subseteq S$ in $\mathcal{F}^{4,6}$ is isomorphic to $M(K_{2,3})$.*

From here on, the sets in $\mathcal{F}^{4,6}$ will be called *prisms*.

Proposition 6.11 *If Π and Π' are two planes meeting at a unique point, then the closure $\mathrm{cl}(\Pi \cup \Pi')$ is isomorphic to $M(K_{3,3})$.*

Proof. Define the set

$$\mathcal{A} = \{(A, x) : A \in \mathcal{F}^{4,6}, x \notin A\}.$$

By assumptions (ii) and (v), $|\mathcal{A}| = \left(\binom{m}{3}\binom{n}{2} + \binom{m}{2}\binom{n}{3}\right)(nm-6)$.

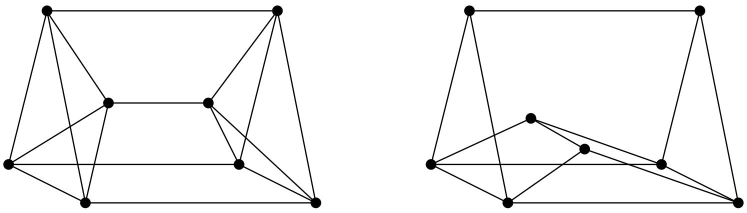
Note that by assumptions (v) and (vi) the closure $\mathrm{cl}(A \cup x)$ for $(A, x) \in \mathcal{A}$ belongs to one of $\mathcal{F}^{5,7}$, $\mathcal{F}^{5,8}$ or $\mathcal{F}^{5,9}$. We abuse notation and say that a set B contains the pair (A, x) if $A \cup x \subseteq B$. In the following three claims we count the number of pairs of $\mathcal{A}$ that are contained in flats belonging to $\mathcal{F}^{5,7}$, $\mathcal{F}^{5,8}$ or $\mathcal{F}^{5,9}$.

> **Claim 1** Every flat $B \in \mathcal{F}^{5,7}$ contains at most one pair belonging to $\mathcal{A}$, with equality holding if and only if $M|B \cong M(K_{2,3}) \oplus y$, where y is an isthmus of $M|B$.

Proof. Suppose that $x \in B$ is such that $B - x \in \mathcal{F}^{4,6}$; since the rank has gone down by one, x is an isthmus of $M|B$. If two or more subsets of B belong to $\mathcal{F}^{4,6}$, then $M|B$ has at least two isthmuses, say x and y. Then $M|B - \{x, y\}$ has rank three and size 5, thus contradicting assumption (iv). The case of equality follows easily. □

Note that in Claim 1 we do not need B to be a flat; hence, the result applies as well to all sets with rank 5 and size 7.

> **Claim 2** Every flat $B \in \mathcal{F}^{5,8}$ contains at most 8 pairs belonging to $\mathcal{A}$, with equality holding if and only if $M|B \cong M(K_{2,4})$.



a) b)

Figure 6.1 The truncation to rank 4 of: a) $M(K_{2,4})$; b)$M(K^-_{3,3})$.

Proof. Since M has no rank-4 set with more than 6 points, for every $x \in B \in \mathcal{F}^{5,8}$ we have that $B - x$ has rank 5 and size 7. By the remark after Claim 1, $B - x$ contains at most one pair in $\mathcal{A}$; since $|B| = 8$, the upper bound follows.

Suppose now that $B \in \mathcal{F}^{5,8}$ contains exactly 8 pairs; this implies that there are exactly four 6-subsets of B that are prisms. Since B has 8 points, it follows that any two prisms T_1 and T_2 in B are such that $M|(T_1 \cap T_2)$ is a plane. It is easy to prove now that $M|(T_1 \cup T_2)$ is either isomorphic to $M(K_{2,4})$ or to $M(K^-_{3,3})$, the cycle matroid of a complete graph $K_{3,3}$ minus an edge (see Figure 6.1). Since this last matroid contains only two prisms, we deduce that $M|B \cong M(K_{2,4})$. □

Claim 3 Every flat $B \in \mathcal{F}^{5,9}$ contains at most 18 pairs that belong to $\mathcal{A}$, with equality holding if and only if $M|B \cong M(K_{3,3})$.

Proof. If every point $x \in B$ is such that $B - x$ contains at most two prisms, we have at most 18 pairs. So assume that for some $x \in B$, $B - x$ contains at least three prisms.

As in the proof of the case of equality in Claim 2, any two prisms $T_1, T_2 \subseteq B - x$ are such that $M|(T_1 \cap T_2)$ is a plane; we deduce that $M|(B - x) \cong M(K_{2,4})$. The closure in $M|(B - x)$ of any five points contains at least one flat with rank 4 and size 6. Hence the point x cannot belong to any prism and therefore the number of pairs as in $\mathcal{A}$ contained in B is 12.

Suppose now that B contains exactly 18 pairs in $\mathcal{A}$. This means that the deletion of any point of B contains two prisms, and hence is isomorphic to $M(K^-_{3,3})$. It is easy to show now that $B \cong M(K_{3,3})$. □

The previous three claims imply that

$$\left(\binom{m}{3}\binom{n}{2} + \binom{m}{2}\binom{n}{3}\right)(nm - 6) = |\mathcal{A}| \leq f^{5,7} + 8f^{5,8} + 18f^{5,9}. \tag{6.1}$$

Using the values of $f^{5,7}, f^{5,8}$ and $f^{5,9}$ given in assumption (vi), we see that equality must hold in (6.1), and thus in Claims 1, 2 and 3.

To prove that $\text{cl}(\Pi \cup \Pi')$ is isomorphic to $M(K_{3,3})$, note that it has rank 5 and size at least 7. Since $M(K_{2,3}) \oplus y$ and $M(K_{2,4})$ contain no pair of planes intersecting in only one point, the closure must be isomorphic to $M(K_{3,3})$ (see Figure 6.2.a). □

We say that the planes $\Pi_1, \ldots, \Pi_t$ are *x-intersecting* if $\Pi_i \cap \Pi_j = \{x\}$ for all i, j with $1 \leq i < j \leq t$. Let $\Pi_1 = \{x, a_1, b_1, c_1\}$ and $\Pi_2 = \{x, a_2, b_2, c_2\}$ be a pair of x-intersecting planes. By

Proposition 6.11, the closure $\mathrm{cl}(\Pi_1 \cup \Pi_2)$ is isomorphic to $M(K_{3,3})$. Let y and z be the two points in $\mathrm{cl}(\Pi_1 \cup \Pi_2) - (\Pi_1 \cup \Pi_2)$. We can choose the notation such that the planes in $\mathrm{cl}(\Pi_1 \cup \Pi_2)$ are, as shown in Figure 6.2.b,

$$\{x, a_1, b_1, c_1\}, \{x, a_2, b_2, c_2\}, \{x, a_1, b_2, y\},$$
$$\{x, a_2, b_1, z\}, \{c_1, c_2, y, z\}, \{a_1, a_2, c_2, y\},$$
$$\{a_1, a_2, c_1, z\}, \{b_1, b_2, c_1, y\} \text{ and } \{b_1, b_2, c_2, z\}.$$

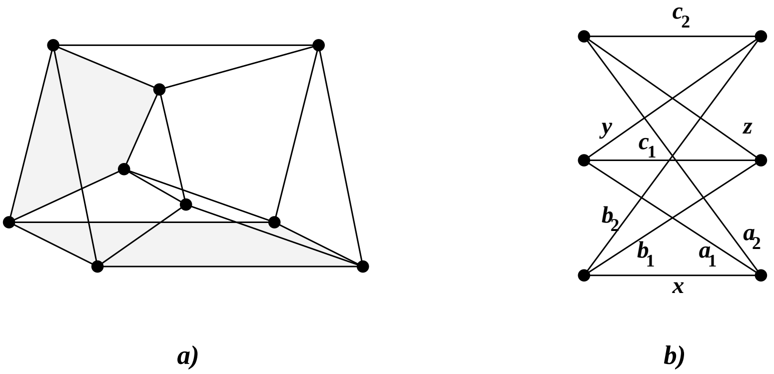


Figure 6.2 a) Truncation to rank 4 of $M(K_{3,3})$, the closure of two planes meeting in a single point; b) a graph representing the closure of $\Pi_1 = \{x, a_1, b_1, c_1\}$ and $\Pi_2 = \{x, a_2, b_2, c_2\}$.

The following propositions prove several properties of x-intersecting planes.

Proposition 6.12 *Let Π_1, Π_2 be a pair of x-intersecting planes. Let P be the only plane in $\mathrm{cl}(\Pi_1 \cup \Pi_2)$ that contains both of the points in $\mathrm{cl}(\Pi_1 \cup \Pi_2) - (\Pi_1 \cup \Pi_2)$; then for every point $p \in P$, there is exactly one plane of M containing x and p.*

Proof. We use the notation introduced in Figure 6.2.b. The plane P is $\{c_1, c_2, y, z\}$. We have to prove that the only plane that contains x and y is $\{x, y, a_1, b_2\}$, and similarly that $\{x, a_1, b_1, c_1\}$ is the only plane containing x and c_1 (the cases of z and c_2 follow analogously).

Suppose that Π is a plane containing x and y; then Π_1, Π_2 and Π are x-intersecting. Since $\mathrm{cl}(\Pi_1 \cup \Pi_2)$ contains no plane x-intersecting with Π_1, Π_2, the plane Π is of the form $\{x, y, a, b\}$ with $a, b \notin \mathrm{cl}(\Pi_1 \cup \Pi_2)$. The closure $\mathrm{cl}(\Pi_1 \cup \Pi_2 \cup \Pi)$ has rank 6; we prove that it contains at least 13 points, thus contradicting assumption (vii).

Since $\{x, y, a_1, b_2\}$ is a plane, b_2 belongs to $\mathrm{cl}(\Pi_1 \cup \Pi)$ and a_1 belongs to $\mathrm{cl}(\Pi_2 \cup \Pi)$. Let c and d be the only points in $\mathrm{cl}(\Pi_1 \cup \Pi) - (\Pi_1 \cup \Pi \cup b_2)$ and $\mathrm{cl}(\Pi_2 \cup \Pi) - (\Pi_2 \cup \Pi \cup a_1)$, respectively. We prove that c and d are two new points. It is easy to check that if c was any of z, a_2 or c_2, then $\mathrm{cl}(\Pi_1 \cup \Pi)$ would have more than nine points; similarly, d is different from z, b_1 and c_1. To see that c and d are different, note that $\mathrm{cl}(\Pi_1 \cup \Pi)$ is isomorphic to $M(K_{3,3})$ and therefore there exists a 4-circuit containing c, b_2, one of $\{a, b, y\}$ and one of $\{b_1, c_1\}$. If $c = d$, then $\mathrm{cl}(\Pi_2 \cup \Pi)$ would contain one of $\{b_1, c_1\}$ and hence it would have at least 10 points. Therefore, c and d are different and $|\mathrm{cl}(\Pi_1 \cup \Pi_2 \cup \Pi)| \geq 13$, which is a contradiction.

For the case of c_1, note that c_1 plays the role of y if we take the planes $\{x, y, a_1, b_2\}$ and $\{x, z, a_2, b_1\}$ as the initial pair of x-intersecting planes. $\square$

The following is an easy consequence of Proposition 6.12.

Proposition 6.13 *Let $\Pi_i = \{x, a_i, b_i, c_i\}$, $1 \le i \le t$, be x-intersecting planes, and let y_{ij} and z_{ij} be the points in $\mathrm{cl}(\Pi_i \cup \Pi_j) - (\Pi_i \cup \Pi_j)$ for $1 \le i < j \le t$. Then all the points y_{ij} and z_{ij} are different.*

Corollary 6.14 *Any set of x-intersecting planes contains at most $m - 1$ planes.*

Proof. Let $\Pi_1, \dots, \Pi_t$ be a collection of x-intersecting planes. The rank of $\mathrm{cl}(\Pi_1 \cup \cdots \cup \Pi_t)$ is at most $2t+1$ and, by Proposition 6.13, we have that $|\mathrm{cl}(\Pi_1 \cup \cdots \cup \Pi_t)| \ge 1 + 3t + 2\binom{t}{2} = (t+1)^2$. We first prove that the rank s of $\mathrm{cl}(\Pi_1 \cup \cdots \cup \Pi_t)$ is in fact $2t+1$. Assume the contrary, that is, $s \le 2t$. By examining the possible values of the size of a rank-s flat given in assumption (vii) it follows that the number of elements in a flat of rank at most $2t$ is strictly less than $(t+1)^2$; hence, the rank of $\mathrm{cl}(\Pi_1 \cup \cdots \cup \Pi_t)$ is $2t + 1$.

To complete the proof suppose $t > m-1$. This implies that $1+2t \ge 2m$, and hence the third case in assumption (vii) applies. Hence the size of the flat $\mathrm{cl}(\Pi_1 \cup \cdots \cup \Pi_t)$ is at most $m(2t - m + 2)$, which can be shown to be strictly less than $(t+1)^2$. Therefore, t is at most $m - 1$. □

Recall that by Proposition 6.13, given a collection of x-intersecting planes, $\{\Pi_i = \{x, a_i, b_i, c_i\} : 1 \le i \le t\}$ with $t \le m - 1$, there are points y_{ij} and z_{ij} for i, j with $1 \le i < j \le t$, all of them different and such that $\mathrm{cl}(\Pi_i \cup \Pi_j) = \Pi_i \cup \Pi_j \cup \{y_{ij}, z_{ij}\}$. By Proposition 6.12 and the fact that the closure of two x-intersecting planes is isomorphic to $M(K_{3,3})$, we know that only one of a_i, b_i, c_i is contained in exactly one plane with x. We choose the notation such that c_i is that point for every plane Π_i. The structure of planes of $M(K_{3,3})$ implies now that $\{c_i, c_j, y_{ij}, z_{ij}\}$ is a plane for all i, j with $1 \le i < j \le t$.

Proposition 6.15 *Let Π_1, Π_2 and Π_3 be three x-intersecting planes as above. Suppose the notation is chosen so that $\{x, a_1, b_2, y_{12}\}$ and $\{x, a_1, b_3, y_{13}\}$ are planes. Then either $\{x, a_2, b_3, y_{23}\}$ or $\{x, a_2, b_3, z_{23}\}$ is a plane.*

Proof. Since $\mathrm{cl}(\Pi_2 \cup \Pi_3)$ is isomorphic to $M(K_{3,3})$, if neither $\{x, a_2, b_3, y_{23}\}$ nor $\{x, a_2, b_3, z_{23}\}$ is a plane, then either $\{x, b_2, b_3, y_{23}\}$ or $\{x, b_2, b_3, z_{23}\}$ is a plane; assume $\{x, b_2, b_3, y_{23}\}$ is a plane. Proposition 6.10 applied to the planes $\{x, a_1, b_2, y_{12}\}$ and $\{x, a_1, b_3, y_{13}\}$ implies that $\{b_2, b_3, y_{12}, y_{13}\}$ is a plane. Applying Proposition 6.10 to this plane and to $\{x, b_2, b_3, y_{23}\}$ we obtain that $\{x, y_{13}, y_{12}, y_{23}\}$ is a plane, thus contradicting Proposition 6.12. □

Let $\{\Pi_i = \{x, a_i, b_i, c_i\} : 1 \le i \le t\}$ be a collection of x-intersecting planes as above. Proposition 6.15 implies that we can choose the notation such that the nine planes of $\mathrm{cl}(\Pi_i \cup \Pi_j)$, $1 \le i < j \le t$, are the following:

$$\begin{aligned}\Pi_{ij} = \{&\{x, a_i, b_i, c_i\}, \{x, a_j, b_j, c_j\}, \{x, a_i, b_j, y_{ij}\},\\ &\{x, a_j, b_i, z_{ij}\}, \{c_i, c_j, y_{ij}, z_{ij}\}, \{a_i, a_j, c_j, y_{ij}\},\\ &\{a_i, a_j, c_i, z_{ij}\}, \{b_i, b_j, c_i, y_{ij}\}, \{b_i, b_j, c_j, z_{ij}\}\}.\end{aligned}$$

Proposition 6.16 *Through any point $x \in S$ there are exactly $(n-1)(m-1)$ planes. Among these planes we can choose $(m-1)$ x-intersecting planes $\Pi_1, \dots, \Pi_{m-1}$, with $\Pi_i = \{x, a_i, b_i, c_i\}$, and $n - m$ planes $P_1, \dots, P_{n-m}$, with $P_k = \{x, a_1, d_k, e_k\}$, such that $M = \mathrm{cl}(\Pi_1 \cup \cdots \cup \Pi_{m-1} \cup P_1 \cup \cdots \cup P_{n-m})$.*

Proof. The average number of planes through a point is $(m-1)(n-1)$. Choose a point $x \in S$ such that the number of planes through x is at least $(m-1)(n-1)$. Let t be the maximum size of a collection of x-intersecting planes; by Corollary 6.14, $t \leq m-1$. Let $\{\Pi_i = \{x, a_i, b_i, c_i\} : 1 \leq i \leq t\}$ be such a collection of x-intersecting planes, with the same notation as above.

Consider the set spanned by these planes, $\mathcal{P} = \mathrm{cl}(\Pi_1 \cup \cdots \cup \Pi_t)$. As in Corollary 6.14, we have that $r(\mathcal{P}) = 1+2t$ and $|\mathcal{P}| \geq (t+1)^2$; since $t \leq m-1$, assumption (vii) implies that $|\mathcal{P}| = (t+1)^2$ and thus $\mathcal{P} = \Pi_1 \cup \cdots \cup \Pi_t \cup \{y_{ij}, z_{ij} : 1 \leq i < j \leq t\}$. Therefore the number of planes through x that are contained in $\mathcal{P}$ is $t + 2\binom{t}{2} = t^2$. We now distinguish two cases.

Case 1 The number of planes through x is t^2.

Since $t \leq m-1 \leq n-1$ and $t^2 \geq (m-1)(n-1)$, we necessarily have that $t = m-1 = n-1$ and that there are $(m-1)^2$ planes through any point. Since $|\mathcal{P}| = |\mathrm{cl}(\Pi_1 \cup \cdots \cup \Pi_{m-1})| = m^2 = |S|$, the geometry M is spanned by $\bigcup_{1 \leq i \leq m-1} \Pi_i$.

Case 2 The number of planes through x is at least $t^2 + 1$.

In this case, there is at least one more plane P_1 through x; this plane must intersect one of $\Pi_1, \ldots, \Pi_t$ in two points. Assume $P_1 \cap \Pi_1 = \{x, a_1\}$ and $P_1 = \{x, a_1, d_1, e_1\}$, with e_1 such that x and e_1 lie in exactly one common plane. Since $\{x, a_1, b_j, y_{1j}\}$ is a plane for all $2 \leq j \leq t$, each of the closures $\mathrm{cl}(\Pi_j \cup P_1)$ with $2 \leq j \leq t$ contains only one new point, w_{j1}, and a new plane through x, $\{x, a_j, d_1, w_{j1}\}$. Therefore, we have now $t^2 + 1 + (t-1) = t^2 + t$ planes through x.

Suppose now that the number of planes through x exceeds $t^2 + t$. This means that there is at least one more plane P_2 through x; as before, this plane must intersect one of $\Pi_1, \ldots, \Pi_t$ in two points. Note that if $\Pi_1 \cap P_2 = \{x, b_1\}$, then $P_1, P_2, \Pi_2, \ldots, \Pi_t$ would be a collection of $t+1$ x-intersecting planes, thus contradicting the choice of $\Pi_1, \ldots, \Pi_t$. It is easy to prove that we can choose this second plane P_2 such that $P_2 \cap \Pi_1 = \{x, a_1\}$ (if P_2 does not intersect Π_1, we can find a plane in $\mathrm{cl}(\Pi_1 \cup P_2)$ that contains x and a_1). As before, the closure of P_2 and Π_j provides $(t-1)$ new planes through x.

We repeat this process until we reach the number of planes through x, which is at least $(m-1)(n-1)$. Suppose we have added l new planes $P_1, \ldots, P_l$, with $P_k = \{x, a_1, d_k, e_k\}$ for $1 \leq k \leq l$. The number of planes through x is then $t^2 + lt$, and in $\mathrm{cl}(\Pi_1 \cup \cdots \cup \Pi_t \cup P_1 \cup \cdots \cup P_l)$ there are at least $(t+1)^2 + l(t+1)$ points. We have the following two inequalities.

$$(t+1)^2 + l(t+1) \leq nm \tag{6.2}$$

$$(n-1)(m-1) \leq t^2 + lt \tag{6.3}$$

Combining these two inequalities we get $l \leq m + n - 2t - 2$. Putting this in (6.3) we obtain $(m-1)(n-1) \leq -t^2 + t(m+n-2)$. If $t < m-1$, the right-hand side of this last expression is strictly less than $(m-1)(n-1)$, a contradiction. Therefore, $t = m-1$ and, in order to satisfy (6.3), $l = n - m$. The total number of planes through x is $(m-1)(n-1)$, thus the same for every point of M. We now have equality in (6.2) and from this we get that the set $\left(\bigcup_{1 \leq i \leq m-1} \Pi_i\right) \cup \left(\bigcup_{1 \leq k \leq n-m} P_k\right)$ spans M. □

Before proving Theorem 6.7 we need the list of all the planes in M. In $\mathrm{cl}(\Pi_1 \cup P_k)$, only the plane $\{b_1, c_1, e_k, d_k\}$ appears as new. For i, k with $2 \leq i \leq m-1$ and $1 \leq k \leq n-m$, let w_{ik} be such that $\mathrm{cl}(\Pi_i \cup P_k) = \Pi_i \cup P_k \cup \{y_{1i}, w_{ik}\}$; then the planes in $\mathrm{cl}(\Pi_i \cup P_k)$ that are not in

$\mathrm{cl}(\Pi_1 \cup \Pi_i)$ are the following:

$$P_{ik} = \{\{x, a_1, d_k, e_k\}, \{x, a_i, d_k, w_{ik}\}, \{c_i, e_k, y_{1i}, w_{ik}\},$$
$$\{a_1, a_i, e_k, w_{ik}\}, \{c_i, b_i, d_k, w_{ik}\}, \{b_i, e_k, d_k, y_{1i}\}\}.$$

Starting with the planes listed in Π_{ij} and in P_{ik} and repeatedly applying Proposition 6.10, we obtain the list of planes displayed in the following corollary. Since they account for $\binom{m}{2}\binom{n}{2}$ planes, by statement (iv) of Theorem 6.7 it is the complete list of planes of M.

Corollary 6.17 *The planes in M are the following:*

(i) *the planes in Π_{ij} for $1 \le i < j \le m-1$, those in P_{ik} for $2 \le i \le m-1$ and $1 \le k \le n-m$, and the plane $\{b_1, c_1, d_k, e_k\}$ for $1 \le k \le n-m$;*

(ii)
$$\{a_{i_1}, a_{i_2}, y_{i_1 i_3}, y_{i_2 i_3}\}, \{a_{i_1}, a_{i_3}, y_{i_1 i_2}, z_{i_2 i_3}\}, \{a_{i_2}, a_{i_3}, z_{i_1 i_2}, z_{i_1 i_3}\},$$
$$\{b_{i_1}, b_{i_2}, z_{i_1 i_3}, z_{i_2 i_3}\}, \{b_{i_1}, b_{i_3}, z_{i_1 i_2}, y_{i_2 i_3}\}, \{b_{i_2}, b_{i_3}, y_{i_1 i_2}, y_{i_1 i_3}\},$$
$$\{c_{i_1}, y_{i_1 i_3}, y_{i_2 i_3}, z_{i_1 i_2}\}, \{c_{i_1}, y_{i_1 i_2}, z_{i_1 i_3}, z_{i_2 i_3}\}, \{c_{i_2}, y_{i_1 i_2}, y_{i_1 i_3}, y_{i_2 i_3}\},$$
$$\{c_{i_2}, z_{i_1 i_2}, z_{i_1 i_3}, z_{i_2 i_3}\}, \{c_{i_3}, z_{i_1 i_2}, z_{i_1 i_3}, y_{i_2 i_3}\}, \{c_{i_3}, y_{i_1 i_2}, y_{i_1 i_3}, z_{i_2 i_3}\},$$

for $1 \le i_1 < i_2 < i_3 \le m-1$;

(iii)
$$\{y_{i_1 i_3}, y_{i_2 i_3}, y_{i_1 i_4}, y_{i_2 i_4}\}, \{y_{i_1 i_2}, y_{i_1 i_4}, y_{i_3 i_4}, z_{i_2 i_3}\}, \{y_{i_1 i_2}, y_{i_1 i_3}, z_{i_2 i_4}, z_{i_3 i_4}\},$$
$$\{z_{i_1 i_2}, z_{i_1 i_3}, y_{i_2 i_4}, y_{i_3 i_4}\}, \{z_{i_1 i_2}, z_{i_1 i_4}, z_{i_3 i_4}, y_{i_2 i_3}\}, \{z_{i_1 i_3}, z_{i_1 i_4}, z_{i_2 i_3}, z_{i_2 i_4}\},$$

for $1 \le i_1 < i_2 < i_3 < i_4 \le m-1$;

(iv) $\{c_1, e_k, z_{1i}, w_{ik}\}, \{b_1, d_k, z_{1i}, w_{ik}\}$, *for $2 \le i \le m-1$ and $1 \le k \le n-m$;*

(v) $\{d_{k_1}, d_{k_2}, e_{k_1}, e_{k_2}\}$, *for $1 \le k_1 < k_2 \le n-m$;*

(vi) $\{d_{k_1}, d_{k_2}, w_{ik_1}, w_{ik_2}\}, \{e_{k_1}, e_{k_2}, w_{ik_1}, w_{ik_2}\}$, *for $2 \le i \le m-1$ and $1 \le k_1 < k_2 \le n-m$;*

(vii)
$$\{a_{i_1}, a_{i_2}, w_{i_1 k}, w_{i_2 k}\}, \{c_{i_1}, z_{i_1 i_2}, w_{i_1 k}, w_{i_2 k}\},$$
$$\{c_{i_2}, y_{i_1 i_2}, w_{i_1 k}, w_{i_2 k}\}, \{b_{i_2}, d_k, y_{i_1 i_2}, w_{i_1 k}\},$$
$$\{e_k, y_{1 i_2}, y_{i_1 i_2}, w_{i_1 k}\}, \{e_k, y_{1 i_1}, z_{i_1 i_2}, w_{i_2 k}\}, \{b_{i_1}, d_k, z_{i_1 i_2}, w_{i_2 k}\},$$

for $2 \le i_1 < i_2 \le m-1$ and $1 \le k \le n-m$;

(viii) $\{w_{i_1 k_1}, w_{i_1 k_2}, w_{i_2 k_1}, w_{i_2 k_2}\}$, *for $2 \le i_1 < i_2 \le m-1$ and $1 \le k_1 < k_2 \le n-m$;*

(ix)
$$\{y_{i_1 i_3}, y_{i_2 i_3}, w_{i_1 k}, w_{i_2 k}\}, \{y_{i_1 i_2}, z_{i_2 i_3}, w_{i_1 k}, w_{i_3 k}\},$$

for $2 \le i_1 < i_2 < i_3 \le m-1$ and $1 \le k \le n-m$;

(x) $\{z_{i_1 i_2}, z_{i_1 i_3}, w_{i_2 k}, w_{i_3 k}\}$, *for $1 \le i_1 < i_2 < i_3 \le m-1$ and $1 \le k \le n-m$.*

Now we are ready to complete the proof of Theorem 6.7.

Proof of Theorem 6.7. In order to apply Theorem 6.2, we have to define a bijection $\phi : S \to S'$ from the ground set of $M(K_{m,n})$ to the ground set of M such that the image under ϕ of a 4-circuit of $M(K_{m,n})$ is a 4-circuit of M. Take the set $\{(i,j) : 1 \le i \le m, 1 \le j \le n\}$ as the ground set of $M(K_{m,n})$. Let x be a point in M, and let $\Pi_i = \{x, a_i, b_i, c_i\}$ with $1 \le i \le m-1$

and $P_k = \{x, a_1, d_k, e_k\}$ with $1 \leq k \leq n - m$ be as in Proposition 6.16 and Corollary 6.17. Define ϕ as follows.

$$\begin{cases} \phi((m,m)) = x; & \\ \phi((i,m)) = a_i, & \text{if } 1 \leq i \leq m-1; \\ \phi((m,i)) = b_i, & \text{if } 1 \leq i \leq m-1; \\ \phi((i,i)) = c_i, & \text{if } 1 \leq i \leq m-1; \\ \phi((i,j)) = y_{ij}, & \text{if } 1 \leq i < j \leq m-1; \\ \phi((j,i)) = z_{ij}, & \text{if } 1 \leq i < j \leq m-1; \\ \phi((m,m+k)) = d_k, & \text{if } 1 \leq k \leq m-n; \\ \phi((1,m+k)) = e_k, & \text{if } 1 \leq k \leq m-n; \\ \phi((i,m+k)) = w_{ik}, & \text{if } \ 2 \leq i \leq m-1 \text{ and } 1 \leq k \leq m-n. \end{cases}$$

Using the list of planes given in Corollary 6.17 it is easy to check that the image of the set $\{(i_1, i_2), (i_3, i_2), (i_3, i_4), (i_1, i_4)\}$ is a 4-circuit for all i_1, i_2, i_3, i_4 with $1 \leq i_1, i_3 \leq m$ and $1 \leq i_2, i_4 \leq n$. Since M satisfies condition (a) in Theorem 6.2, we conclude that $M \cong M(K_{m,n})$. $\square$

7

Generalizations of wheels, whirls, and spikes

Wheels, whirls, and spikes are matroids that arise frequently in matroid structure theory [43, 44]. In this chapter we generalize these matroids by freely adding points to their lines. By means of the tools developed in Section 6.1 we give a characterization of generalized wheels and whirls by numerical invariants, and use this characterization to prove their T-uniqueness. In the second section we study generalized spikes; we give a necessary and sufficient condition for two generalized spikes to have the same Tutte polynomial and prove that certain classes are T-closed.

7.1 Generalizations of wheels and whirls

In this section we apply Theorems 6.2 and 6.3 to give characterizations of the matroids obtained from wheels and whirls by adding points freely to nontrivial lines. We begin by giving the background on principal extensions and parallel connections that is needed in this section.

7.1.1 Principal extensions and parallel connections

Principal extensions are a special case of the so-called single-element extensions. The matroid M^+ on the ground set $S \cup e$ is a *single-element extension* of the matroid M on the ground set S if $M = M^+ \backslash e$. Single-element extensions were completely understood by Crapo [26, 43]. Here we only need to define principal extensions.

Let F be a flat of a matroid M on the ground set S. The *principal extension* of M with respect to F is the matroid $M +_F e$ on the ground set $S \cup e$ whose closure operator is defined as:

$$\mathrm{cl}(X) = \begin{cases} X \cup e, & \text{if } F \subseteq \mathrm{cl}_M(X); \\ X, & \text{otherwise;} \end{cases}$$

$$\mathrm{cl}(X \cup e) = \begin{cases} \mathrm{cl}_M(X \cup F) \cup e, & \text{if } r_M(X \cup F) = r_M(X) + 1; \\ \mathrm{cl}_M(X) \cup e, & \text{otherwise;} \end{cases}$$

for every subset $X \subseteq S$. Hence a principal extension consists of adding a point to the flat F as freely as possible. If $N = M +_F e$, we say that the matroid N is obtained from the matroid M by adding the element e freely to the flat F.

We now turn to the basic results we need about parallel connections of matroids [17, 43].

Assume that M and N are loopless matroids with ground sets S and T, respectively, and that $S \cap T = \{p\}$. The *parallel connection of* M *and* N *with respect to the basepoint* p is the matroid $P(M, N)$ whose collection of circuits is

$$\mathcal{C}(M) \cup \mathcal{C}(N) \cup \{(C \cup C') - p \mid C \in \mathcal{C}(M), C' \in \mathcal{C}(N), \text{ and } p \in C \cap C'\},$$

where $\mathcal{C}(M)$ and $\mathcal{C}(N)$ are the collections of circuits of M and N, respectively.

The following theorem is part (ii) of [43, Theorem 7.1.16].

Theorem 7.1 *Assume that* M *is a connected matroid and that* p *is in the ground set of* M*. If* $M/p = M_1 \oplus M_2$*, where* M_1 *and* M_2 *have ground sets* S_1 *and* S_2*, respectively, then* $M = P\big(M|(S_1 \cup p), M|(S_2 \cup p)\big)$.

We will use the following corollary of Theorem 7.1.

Corollary 7.2 *Assume that* M *is a rank-*n *geometry and that the ground set of* M *is the union of* $n-1$ *lines,* $\ell_1, \ell_2, \ldots, \ell_{n-1}$*, where*

$$\ell_1 \cap \ell_2 = \{p_1\}, \ell_2 \cap \ell_3 = \{p_2\}, \ldots, \ell_{n-2} \cap \ell_{n-1} = \{p_{n-2}\},$$

and the $n-2$ *points* $p_1, p_2, \ldots, p_{n-2}$ *are distinct. Then* M *is formed by taking the parallel connection of* ℓ_1 *and* ℓ_2 *with respect to* p_1*, and then the parallel connection of the resulting matroid and* ℓ_3 *with respect to* p_2*, and so on.*

Proof. A rank calculation shows that M/p_{n-2} is the direct sum of the restrictions $(M/p_{n-2})|\big((\ell_1 \cup \ell_2 \cup \cdots \cup \ell_{n-2}) - p_{n-2}\big)$ and $(M/p_{n-2})|(\ell_{n-1} - p_{n-2})$. The result now follows from Theorem 7.1 by induction on n. □

7.1.2 Definition of generalized wheels and whirls

Recall that the *rank-*n *wheel* $\mathcal{W}_n$ is the graph that consists of an n-cycle, the *rim*, and one additional vertex, the *hub*, that is adjacent to each vertex on the rim. Label the edges that are incident with the hub as $b_0, b_1, \ldots, b_{n-1}$; these edges are the *spokes*. Label the rim edges as $a_0, a_1, \ldots, a_{n-1}$ so that for each i, the edges b_i, a_i, b_{i+1} form a 3-circuit; here and below, subscripts are interpreted modulo n. The rim edges form a circuit-hyperplane of the cycle matroid $M(\mathcal{W}_n)$ of $\mathcal{W}_n$. The matroid obtained by relaxing this circuit-hyperplane is the *rank-n whirl*, denoted $\mathcal{W}^n$. Wheels and whirls play a major role in matroid structure theory (see [43, 44]).

It is easy to see that the circuits of the n-wheel $\mathcal{W}_n$ are of two types:

(i) $\{a_0, a_1, \ldots, a_{n-1}\}$, and

(ii) $\{b_i, a_i, a_{i+1}, \ldots, a_{j-1}, b_j\}$ for any distinct integers i and j in $\{0, 1, \ldots, n-1\}$.

It follows that the circuits of the n-whirl $\mathcal{W}^n$ are also of two types:

(i′) $\{b_i, a_0, a_1, \ldots, a_{n-1}\}$ for any i in $\{0, 1, \ldots, n-1\}$, and

(ii′) $\{b_i, a_i, a_{i+1}, \ldots, a_{j-1}, b_j\}$ for any distinct integers i and j in $\{0, 1, \ldots, n-1\}$.

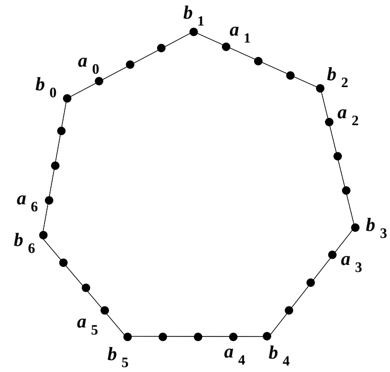


Figure 7.1 The truncation to rank 3 of the $(7, 5)$-wheel; the set $\{a_0, a_1, \ldots, a_6\}$ was a circuit before the truncation.

We now define recursively the (n, t)-wheel, $\mathcal{W}_{n,t}$, and the (n, t)-whirl, $\mathcal{W}^{n,t}$, for any integer $t \geq 3$. They are both rank-n matroids with $n(t-1)$ points; roughly speaking, they are like usual wheels and whirls but with t points in each line instead of 3. The matroids $\mathcal{W}_{n,3}$ and $\mathcal{W}^{n,3}$ are $M(\mathcal{W}_n)$ and $\mathcal{W}^n$, respectively. We obtain the (n, t)-wheel by adding one point to every nontrivial line of the $(n, t-1)$-wheel. More precisely, let M_0 be the matroid obtained from $\mathcal{W}_{n,t-1}$ by adding the point $x_{0,t-3}$ freely to the flat $\mathrm{cl}(\{b_0, b_1\})$ of $\mathcal{W}_{n,t-1}$. Now extend M_0 to M_1 by adding the point $x_{1,t-3}$ freely to the flat $\mathrm{cl}_{M_0}(\{b_1, b_2\})$ of M_0. The points $x_{2,t-3}, \ldots, x_{n-1,t-3}$ are added in a similar way. The resulting matroid is the (n, t)-wheel $\mathcal{W}_{n,t}$. The line spanned by the points b_i, b_{i+1} is then $\{b_i, a_i, x_{i,1}, \ldots, x_{i,t-3}, b_{i+1}\}$. The (n, t)-whirl $\mathcal{W}^{n,t}$ is defined in an analogous manner, starting with the n-whirl $\mathcal{W}^n$. In principle, our definition of (n, t)-wheels and (n, t)-whirls depends on the order the points are added to the lines. It follows from Lemma 7.3 that the collection of circuits of an (n, t)-wheel and an (n, t)-whirl is independent of this order, and hence we can refer to the (n, t)-wheel and the (n, t)-whirl. Figure 7.1 shows a geometric representation of the truncation to rank 3 of the $(7, 5)$-wheel.

The $(n, 4)$-whirl $\mathcal{W}^{n,4}$ is also known as the *swirl*. Swirls were first defined in [46] and they play a key role in [10], where their inequivalent representations over $\mathrm{GF}(q)$ are used to produce many T-equivalent matroids.

We shall see that (n, t)-wheels and (n, t)-whirls are k-chordal matroids. Towards this end we need to understand their circuits.

Lemma 7.3 *The circuits of the (n, t)-wheel $\mathcal{W}_{n,t}$ are the sets C that satisfy one of the following four properties.*

(I) *C consists of any 3 points of the line $\mathrm{cl}(\{b_j, b_{j+1}\})$ for some j.*

(II) *For some s and k with $0 \leq s \leq n-1$ and $1 < k < n$, the set C consists precisely of the following points:*

  (1) *any two points from $\mathrm{cl}(\{b_s, b_{s+1}\}) - b_{s+1}$;*
  (2) *any two points from $\mathrm{cl}(\{b_{s+k-1}, b_{s+k}\}) - b_{s+k-1}$; and*
  (3) *for each j with $1 \leq j \leq k-2$, one point from $\mathrm{cl}(\{b_{s+j}, b_{s+j+1}\}) - \{b_{s+j}, b_{s+j+1}\}$.*

(III) $C = \{a_0, a_1, \ldots, a_{n-1}\}$.

(IV) (1) $|C| = n + 1$;

(2) *for all i we have* $|C \cap \{a_i, x_{i,1}, x_{i,2}, \ldots, x_{i,t-3}\}| \geq 1$; *and*
(3) $\{a_0, a_1, \ldots, a_{n-1}\} \not\subseteq C$.

The circuits of the (n,t)*-whirl* $\mathcal{W}^{n,t}$ *are the sets* C *that satisfy* (I), (II), *or*

(III′) $|C| = n+1$ *and for all* i, *we have* $|C \cap \{a_i, x_{i,1}, x_{i,2}, \ldots, x_{i,t-3}\}| \geq 1$.

In particular, no circuit of the (n,t)*-whirl* $\mathcal{W}^{n,t}$ *contains exactly one point from each line* $\mathrm{cl}(\{b_i, b_{i+1}\})$.

Proof. By Corollary 7.2, the deletion $\mathcal{W}_{n,t} \backslash \{a_i, x_{i,1}, x_{i,2}, \ldots, x_{i,t-3}\}$ is the parallel connection of the $n-1$ lines, $\mathrm{cl}(\{b_{i+1}, b_{i+2}\}), \mathrm{cl}(\{b_{i+2}, b_{i+3}\}), \ldots, \mathrm{cl}(\{b_{i-1}, b_i\})$, with respect to the basepoints $b_{i+2}, b_{i+3}, \ldots, b_{i-1}$. From the structure of the circuits in parallel connections of lines, it follows that the circuits of $\mathcal{W}_{n,t}$ that, for some i, do not contain any of the points $a_i, x_{i,1}, x_{i,2}, \ldots, x_{i,t-3}$ are the sets C that satisfy either property (I) or (II). All other circuits contain at least one point from each set $\{a_i, x_{i,1}, x_{i,2}, \ldots, x_{i,t-3}\}$. Since $\mathcal{W}_{n,t}$ is an extension of $M(\mathcal{W}_n)$, the circuits of $\mathcal{W}_{n,t}$ that do not contain any of the new points $x_{i,j}$ are the same as those of $M(\mathcal{W}_n)$. Therefore the set C in (III) is a circuit of $\mathcal{W}_{n,t}$. We claim that (III) gives the only n-circuit that contains one element from each of the sets $\{a_i, x_{i,1}, x_{i,2}, \ldots, x_{i,t-3}\}$. Indeed, assume that C is an n-circuit that contains one element from each of these sets, and that $x_{i,j}$ is the last element in C that is added in the sequence of single-element extensions that yield $\mathcal{W}_{n,t}$. Both $(C - x_{i,j}) \cup b_i$ and $(C - x_{i,j}) \cup b_{i+1}$ are bases of the parallel connection $\mathcal{W}_{n,t} \backslash \{a_i, x_{i,1}, x_{i,2}, \ldots, x_{i,t-3}\}$ of $n-1$ lines, so the closure of $C - x_{i,j}$ does not contain the flat $\mathrm{cl}(b_i, b_{i+1})$ and by definition of principal extension, the point $x_{i,j}$ is not in the closure of $C - x_{i,j}$ in $\mathcal{W}_{n,t}$; this is a contradiction since we assumed C was a circuit. It follows that the remaining circuits are all spanning circuits and satisfy property (IV). The structure of the circuits of the (n,t)-whirl $\mathcal{W}^{n,t}$ can be deduced in a similar way. □

Note that all circuits that satisfy properties (IV) and (III′) and some of the circuits that satisfy property (II) are spanning circuits. Also, a circuit that satisfies property (IV) or (III′) contains at most one point b_i among $b_0, b_1, \ldots, b_{n-1}$.

To apply Theorems 6.2 and 6.3 to $\mathcal{W}_{n,t}$ and $\mathcal{W}^{n,t}$, we need the following proposition.

Proposition 7.4 *The* (n,t)*-wheel* $\mathcal{W}_{n,t}$ *is* $(n-1)$*-chordal. The* (n,t)*-whirl* $\mathcal{W}^{n,t}$ *is* $(n+1)$*-chordal.*

Proof. The result follows immediately from the analysis of the circuits of $\mathcal{W}_{n,t}$ and $\mathcal{W}^{n,t}$ in Lemma 7.3. Note that the circuit $\{a_0, a_1, \ldots, a_{n-1}\}$ of the (n,t)-wheel is not chordal. □

7.1.3 T-uniqueness

We give a characterization of (n,t)-wheels and (n,t)-whirls by numerical invariants using the results on chordal matroids of the previous chapter; this characterization implies that (n,t)-wheels and (n,t)-whirls are T-unique. This combined with Corollary 1.12 gives the T-uniqueness of the graph $\mathcal{W}_n$.

Note that $\mathcal{W}_{3,3}$ and $\mathcal{W}^{3,3}$ have particularly simple characterizations: it is easy to check that $\mathcal{W}_{3,3}$ and $\mathcal{W}^{3,3}$ are the only rank-3 geometries on six points for which the number of 3-point lines is, respectively, four and three. We omit these cases in Theorem 7.5 since condition (iii) of Theorem 7.5 does not hold in the case of $\mathcal{W}_{3,3}$.

Theorem 7.5 *Assume that n and t are integers with $n, t \geq 3$ and either $n > 3$ or $t > 3$. Assume that M is a geometry on the ground set S that satisfies the following properties:*

(i) $r(M) = n$;

(ii) $|S| = (t-1)n$;

(iii) *there are exactly n lines $\ell_1, \ell_2, \ldots, \ell_n$ with $|\ell_i| = t$;*

(iv) *for s with $2 \leq s \leq n-1$, flats of rank s have at most $(s-1)(t-1)+1$ points; and*

(v) *for each s with $3 \leq s \leq n$, the geometry M has the same number of independent sets of size s as $\mathcal{W}_{n,t}$.*

Then M is isomorphic to the (n,t)-wheel $\mathcal{W}_{n,t}$.

Assume that M is a geometry on the ground set S that satisfies properties (i)–(iv) *and*

(v′) *for each s with $3 \leq s \leq n$, the geometry M has the same number of independent sets of size s as $\mathcal{W}^{n,t}$.*

Then M is isomorphic to the (n,t)-whirl $\mathcal{W}^{n,t}$.

In particular, (n,t)-wheels and (n,t)-whirls are T-unique.

Proof. We first show that M is a *ring of t-point lines* in the following sense: M has a basis $p_0, p_1, \ldots, p_{n-1}$ such that each of the lines $\mathrm{cl}(\{p_i, p_{i+1}\})$, for i with $0 \leq i \leq n-1$, has t points and these lines contain all points of M. Towards this end, we introduce several more definitions. A sequence $\ell'_1, \ell'_2, \ldots, \ell'_k$ of t-point lines *intersects well* if for each i with $1 < i \leq k$, there is a j such that $j < i$ and $\ell'_j \cap \ell'_i \neq \emptyset$. An *ordered component of M* is a maximal sequence of t-point lines that intersects well. Our interest in ordered components is more in the collections of lines rather than in the ordering. Note that each maximal component with at least two lines has more than one ordering with respect to which it intersects well. We say that M *has a unique ordered component* if all ordered components of M use all t-point lines of M.

To show that M is a ring of t-point lines, we prove several properties about sequences that intersect well.

> **Claim 1** Assume $1 \leq k \leq n-2$. Every sequence $\ell'_1, \ell'_2, \ldots, \ell'_k$ of t-point lines that intersects well has rank $k+1$ and contains $(t-1)k+1$ points. Furthermore, $\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_k$ is a flat and $\ell'_1, \ell'_2, \ldots, \ell'_k$ are the only nontrivial lines of $M|(\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_k)$.

Proof. We prove the following four statements by induction on k:

(R_k) $r(\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_k) = k+1$;

(S_k) $|\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_k| = (t-1)k+1$;

(C_k) $\mathrm{cl}(\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_k) = \ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_k$;

(L_k) $\ell'_1, \ell'_2, \ldots, \ell'_k$ are the only nontrivial lines of $M|(\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_k)$.

The case $k = 1$ is trivial. For $k > 1$, assume that $\ell'_1, \ell'_2, \ldots, \ell'_{k-1}, \ell'_k$ is a sequence of t-point lines that intersects well. Note that $\ell'_1, \ell'_2, \ldots, \ell'_{k-1}$ is also a sequence of t-point lines that intersects well. Statements (C_{k-1}), (L_{k-1}), and the definition of intersecting well imply that ℓ'_k intersects $\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_{k-1}$ in exactly one point. Statements (R_k) and (S_k) follow now immediately from (R_{k-1}), (C_{k-1}), and (L_{k-1}). From (R_k), (S_k), and assumption (iv), we get (C_k). Since ℓ'_k intersects $\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_{k-1}$ in exactly one point, it follows from this and statements (L_{k-1}) and (C_k) that the only nontrivial line of $M|(\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_k)$ other than $\ell'_1, \ell'_2, \ldots, \ell'_{k-1}$ is ℓ'_k, as asserted in (L_k). □

Claim 2 A sequence $\ell'_1, \ell'_2, \ldots, \ell'_{n-1}$ of t-point lines that intersects well has rank n and the remaining t-point line ℓ'_n must intersect ℓ'_{n-1} in a point that is not in $\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_{n-2}$. Furthermore, ℓ'_n must intersect one line from among $\ell'_1, \ell'_2, \ldots, \ell'_{n-2}$. Thus, $\ell'_1, \ell'_2, \ldots, \ell'_{n-1}, \ell'_n$ intersects well.

Proof. The rank assertion follows as in the proof of Claim 1. By comparing the size of S with that of $\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_{n-1}$ and ℓ'_n, it follows that ℓ'_n intersects $\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_{n-1}$ in at least two points. Since the distinct lines ℓ'_n and ℓ'_{n-1} can intersect in at most one point and since $\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_{n-2}$ is a flat of M whose only nontrivial lines are $\ell'_1, \ell'_2, \ldots, \ell'_{n-2}$, it follows that ℓ'_n intersects $\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_{n-1}$ in exactly two points and that ℓ'_n intersects ℓ'_{n-1} in one point that is not in $\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_{n-2}$. □

Claim 3 The geometry M has a unique ordered component.

Proof. Assume M has h ordered components; let c_j be the number of t-point lines in the j-th ordered component. Thus, $\sum_{j=1}^{h} c_j = n$. If $h > 1$, then by Claim 2 each c_j is less than $n-1$, so the conclusion of Claim 1 applies to each ordered component. Thus, the number of points in M is at least

$$\sum_{j=1}^{h}\big((t-1)c_j + 1\big) = (t-1)\left(\sum_{j=1}^{h} c_j\right) + h = (t-1)n + h,$$

which exceeds $(t-1)n$. This contradiction shows that $h = 1$, as claimed. □

Claim 4 The geometry M is a ring of t-point lines.

Proof. From Claim 3, we know that M has a unique ordered component; assume that the sequence $\ell_1, \ell_2, \ldots, \ell_n$ of t-point lines intersects well. We first claim that we may assume that the lines $\ell_1, \ell_2, \ldots, \ell_n$ are ordered so that for i with $1 \leq i < n$, each intersection $\ell_i \cap \ell_{i+1}$ is nonempty and these points of intersection are distinct. We show this by induction. Since the sequence intersects well, it follows that the intersection $\ell_1 \cap \ell_2$ is a point p_1. Assume that $j < n-1$ and that the sequence $\ell_1, \ell_2, \ldots, \ell_j$ has the property that for each i with $i < j$, the intersection $\ell_i \cap \ell_{i+1}$ consists of one point p_i distinct from $p_1, p_2, \ldots, p_{i-1}$. We claim that there is a line ℓ_k with $k > j$ with the intersection $\ell_j \cap \ell_k$ being a point other than p_{j-1}. Since ℓ_j intersects $\ell_1 \cup \ell_2 \cup \cdots \cup \ell_{j-1}$ in exactly one point, if there were no such ℓ_k, then the sequence $\ell_1, \ell_2, \ldots, \ell_{j-1}, \ell_{j+1}, \ldots, \ell_n, \ell_j$ would intersect well and ℓ_j would contain only one point from $\bigcup_{i \neq j} \ell_i$, contrary to Claim 2. Thus we may assume that $\ell_j \cap \ell_{j+1}$ is a point p_j different from p_{j-1}. By Claim 2, ℓ_n intersects ℓ_{n-1} in a point p_{n-1} with $p_{n-1} \neq p_{n-2}$.

Note that the sequence $\ell_{n-1}, \ell_{n-2}, \ldots, \ell_1, \ell_n$ intersects well. It therefore follows from Claim 2 that ℓ_n intersects ℓ_1 in a point p_0 with $p_0 \neq p_1$.

Since $p_0, p_1, \ldots, p_{n-1}$ span the lines $\ell_1, \ell_2, \ldots, \ell_n$ and, by assumption (ii), these lines contain all points of M, it follows that $\{p_0, p_1, \ldots, p_{n-1}\}$ is a basis and M is a ring of t-point lines. □

Let $\{p_0, p_1, \ldots, p_{n-1}\}$ be a basis that shows that M is a ring of t-point lines. From Corollary 7.2, it follows that $M\backslash\big(\mathrm{cl}(\{p_i, p_{i+1}\}) - \{p_i, p_{i+1}\}\big)$ is a parallel connection of t-point lines. Thus for $n > 3$, the 3-circuits of M are precisely the 3-element subsets of the lines $\mathrm{cl}(\{p_j, p_{j+1}\})$.

Now assume condition (v′) holds. Since M is a geometry, in the case $n = 3$, this condition implies that M has the same number of 3-circuits as $\mathcal{W}^{3,t}$. This conclusion, along with the fact that all 3-element subsets of the lines $\mathrm{cl}(\{p_i, p_{i+1}\})$ are 3-circuits, implies that the 3-circuits of M are precisely the 3-element subsets of the lines $\mathrm{cl}(\{p_i, p_{i+1}\})$. Therefore, we can strengthen the conclusion of the last paragraph: for any $n \geq 3$, the 3-circuits of M are precisely the 3-element subsets of the lines $\mathrm{cl}(\{p_j, p_{j+1}\})$. Thus, M and the (n, t)-whirl $\mathcal{W}^{n,t}$ have the same number of 3-circuits, and any bijection of the ground set of $\mathcal{W}^{n,t}$ with the ground set S of M that maps b_i in $\mathcal{W}^{n,t}$ to p_i in M and that maps the line $\mathrm{cl}(\{b_i, b_{i+1}\})$ of $\mathcal{W}^{n,t}$ to the line $\mathrm{cl}_M(\{p_i, p_{i+1}\})$ of M gives a bijection between the 3-circuits of $\mathcal{W}^{n,t}$ and the 3-circuits of M. This observation, Proposition 7.4, and Theorem 6.2 complete the proof in the case of the (n, t)-whirl $\mathcal{W}^{n,t}$.

Now assume condition (v) holds and assume $n > 3$. The same argument as above shows that any bijection of the ground set of the (n, t)-wheel $\mathcal{W}_{n,t}$ with the ground set S of M that maps b_i in $\mathcal{W}_{n,t}$ to p_i in M and that maps the line $\mathrm{cl}(\{b_i, b_{i+1}\})$ of $\mathcal{W}_{n,t}$ to the line $\mathrm{cl}_M(\{p_i, p_{i+1}\})$ of M gives a bijection between the 3-circuits of $\mathcal{W}_{n,t}$ and the 3-circuits of M. By Theorem 6.3 and Proposition 7.4, it follows that M, like $\mathcal{W}_{n,t}$, has precisely one nonchordal n-circuit. That $M\backslash\big(\mathrm{cl}(\{p_i, p_{i+1}\}) - \{p_i, p_{i+1}\}\big)$ is a parallel connection of t-point lines allows us to conclude that this nonchordal n-circuit of M must contain precisely one point in each set $\mathrm{cl}(\{p_i, p_{i+1}\}) - \{p_i, p_{i+1}\}$. Since the bijection ϕ of Theorem 6.3 can be chosen to map the circuit $a_0, a_1, \ldots, a_{n-1}$ of $\mathcal{W}_{n,t}$ to this nonchordal n-circuit of M, the map ϕ gives a bijection between all nonspanning circuits of $\mathcal{W}_{n,t}$ and those of M, which suffices to complete the proof in the case of the (n, t)-wheel $\mathcal{W}_{n,t}$ for $n > 3$.

The case of $n = 3$ and $t \geq 4$ follows using similar ideas. In particular, from condition (v), it follows that there is exactly one 3-circuit in addition to those arising from the three t-point lines. From this, it is easy to construct a bijection of the ground set of $\mathcal{W}_{3,t}$ with the ground set of M that gives a bijection between the nonspanning circuits.

Finally, note that, by Theorem 1.15, conditions (i)–(v) and (v′) can be deduced from the Tutte polynomial. □

7.2 Generalized spikes

In this section, we generalize the notion of a spike as defined in [44] and we prove a number of properties about these matroids. In particular, we show that a large class of these generalized spikes are distinguished from all other matroids by their Tutte polynomials and we show that binary spikes and the generalizations of free spikes are T-unique. We start by defining this more general notion of a spike.

Definition 7.6 *Assume n, s, and t are integers with $n \geq 3$, $s \geq n-1$, and $t \geq 3$. An (n, s, t)-*spike with tip *a is a rank-n geometry whose ground set is the union of s lines $\ell_1, \ell_2, \ldots, \ell_s$ for which the following properties hold:*

(S1) *$a \in \ell_i$ for $1 \leq i \leq s$;*

(S2) *$|\ell_i| = t$ for $1 \leq i \leq s$;*

(S3) *for any* $k < n$ *and any integers* $1 \leq i_1 < i_2 < \cdots < i_k \leq s$, *we have* $r(\ell_{i_1} \cup \ell_{i_2} \cup \cdots \cup \ell_{i_k}) = k + 1$.

Thus, an (n, s, t)-spike consists of s lines with t points each, that intersect at a common point and lie in rank n. Consistent with [44], an $(n, n, 3)$-spike will be called an *n-spike*. The following $n \times (2n + 1)$ matrix D over the field GF(2) represents an n-spike; the last column corresponds to the tip.

$$D = \left(\begin{array}{ccccc|ccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 0 & 1 \end{array}\right) \tag{7.1}$$

Throughout this section, if M is an (n, s, t)-spike, we let $\ell_1, \ell_2, \ldots, \ell_s$ denote the lines through the tip a of M. It follows from condition (S3) that the restriction $M|(\ell'_1 \cup \ell'_2 \cup \cdots \cup \ell'_{n-1})$ of an (n, s, t)-spike to any $n-1$ of the lines through the tip is the parallel connection of $\ell'_1, \ell'_2, \ldots, \ell'_{n-1}$ with respect to the common basepoint a. Hence, if we want to distinguish among (n, s, t)-spikes, we have to choose n of the lines and consider sets that contain one point from each of them. More concretely, we define the set $\mathscr{Z}_n$ as

$$\mathscr{Z}_n = \{X \mid X \subseteq S - a, |X| = n, \text{ and } |X \cap \ell_i| \leq 1 \text{ for } 1 \leq i \leq s\}. \tag{7.2}$$

Thus, $\mathscr{Z}_n$ is the collection of all n-sets X such that for some $i_1, i_2, \ldots, i_n$ with $1 \leq i_1 < i_2 < \cdots < i_n \leq s$, the set X consists of precisely one point in each of the sets $\ell_{i_1} - a, \ell_{i_2} - a, \ldots, \ell_{i_n} - a$.

It follows from condition (S3) of Definition 7.6 that each set X in $\mathscr{Z}_n$ is either a basis or a circuit of M. If $s = n$, the circuits in $\mathscr{Z}_n$ are necessarily circuit-hyperplanes of M, but if $s > n$, a circuit in $\mathscr{Z}_n$ might not be a flat.

Condition (S3) of Definition 7.6 also implies that there are only three types of nonspanning circuits of an (n, s, t)-spike, namely, the circuits in $\mathscr{Z}_n$, all 3-subsets of the lines ℓ_i, and, if $n > 3$, all sets of the form $A \cup B$ where A and B are 2-subsets of any two distinct sets $\ell_i - a$ and $\ell_j - a$. The *free* (n, s, t)*-spike* is the (n, s, t)-spike in which there are no circuits in $\mathscr{Z}_n$; for each triple n, s, t, there is precisely one free (n, s, t)-spike. Free n-spikes play a role in [46] where they are proved to have many inequivalent representations over sufficiently large finite fields.

We start by giving a necessary and sufficient condition for two (n, s, t)-spikes to have the same Tutte polynomial; in particular, we show that all (n, n, t)-spikes with the same number of circuit-hyperplanes share the same Tutte polynomial. Figure 7.2 shows two nonisomorphic 4-spikes, each having two circuit-hyperplanes and so, according to Corollary 7.9, having the same Tutte polynomial.

The criterion in Theorem 7.7 for (n, s, t)-spikes to have the same Tutte polynomial is based on hyperplanes that do not contain the tip. It follows from condition (S3) of Definition 7.6 that the restriction of an (n, s, t)-spike M to a hyperplane that does not contain the tip a is isomorphic to a uniform matroid $U_{n-1,h}$ for some h with $n - 1 \leq h \leq s$. For such h, let c_h^M denote the number of hyperplanes of M that do not contain the tip and for which the corresponding restrictions of M are isomorphic to $U_{n-1,h}$. In particular, the number of circuit-hyperplanes of M is given by c_n^M. For the free (n, s, t)-spike M we have $c_{n-1}^M = \binom{s}{n-1}(t-1)^{n-1}$ and $c_h^M = 0$ for $n \leq h \leq s$. In

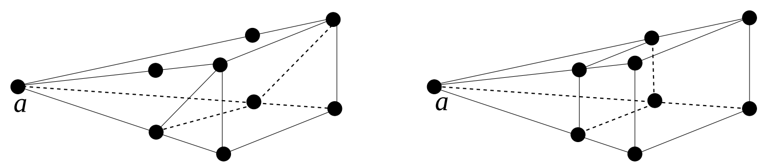


Figure 7.2 Nonisomorphic 4-spikes with the same Tutte polynomial.

general, since any $n-1$ points, not including the tip, chosen from distinct lines span a unique hyperplane that does not contain the tip, we have

$$\binom{s}{n-1}(t-1)^{n-1} = \sum_{h=n-1}^{s} c_h^M \binom{h}{n-1}.$$

Thus, any $s-n+1$ of the numbers $c_{n-1}^M, c_n^M, \ldots, c_s^M$ determine the other number in this sequence. In the following theorem we prove that any $s-n+1$ of the numbers $c_{n-1}^M, c_n^M, \ldots, c_s^M$ determine the Tutte polynomial of an (n,s,t)-spike M, and conversely.

Theorem 7.7 *Two (n,s,t)-spikes M and N have the same Tutte polynomial if and only if $c_h^M = c_h^N$ for any $s-n+1$ integers h with $n-1 \le h \le s$.*

Proof. By the remark above, $c_h^M = c_h^N$ for all h with $n-1 \le h \le s$ if and only if $c_h^M = c_h^N$ for any $s-n+1$ integers h with $n-1 \le h \le s$. We focus on the first of these conditions.

We first show that we can determine the number of subsets with a given rank and cardinality from the numbers c_h^M and the conditions that define an (n,s,t)-spike. Let A be a subset of S. First assume that A contains the tip a or that A contains two or more points from some line ℓ_i. If A contains at least one point other than the tip from j of the lines $\ell_1, \ell_2, \ldots, \ell_s$, then it follows from condition (S3) of Definition 7.6 that the rank of A is given as follows:

$$r(A) = \begin{cases} j+1, & \text{if } j < n; \\ n, & \text{otherwise.} \end{cases}$$

Now assume that A does not contain the tip and that A contains at most one point from each line $\ell_1, \ell_2, \ldots, \ell_s$. Note that if $|A| < n$, then $r(A) = |A|$ since we have $r(A \cup a) = |A| + 1$ from condition (S3). All sets not yet considered have cardinality k, for some $k \ge n$, and rank $n-1$ or n. The number of such subsets having k points is $\binom{s}{k}(t-1)^k$, and among these, exactly

$$\sum_{h=k}^{s} c_h^M \binom{h}{k}$$

have rank $n-1$.

For the converse, it follows in the same way that if two (n,s,t)-spikes M and N have the same Tutte polynomial, then for all k with $n-1 \le k \le s$ we have

$$\sum_{h=k}^{s} c_h^M \binom{h}{k} = \sum_{h=k}^{s} c_h^N \binom{h}{k}. \tag{7.3}$$

Let a_k be the sum in equation (7.3). The matrix with rows and columns indexed by $n-1, n, \ldots, s$ whose k,h entry is $\binom{h}{k}$ is upper triangular with all 1s on the diagonal, so the system of linear equations $\sum_{h=k}^{s} x_h \binom{h}{k} = a_k$, with $n-1 \leq k \leq s$, has a unique solution. Thus, from equation (7.3), we conclude that $c_h^M = c_h^N$ for all h with $n-1 \leq h \leq s$. □

Since the number of circuits of cardinality n is given by $\sum_{h=n}^{s} c_h^M \binom{h}{n}$, the following corollary is immediate.

Corollary 7.8 *If two (n,s,t)-spikes have the same Tutte polynomial, then they have the same number of circuits of cardinality n. In particular, two (n,n,t)-spikes have the same Tutte polynomial if and only if they have the same number of circuit-hyperplanes.*

The following extremal property of (n,n,t)-spikes will be useful.

Theorem 7.9 *Assume $n \geq 4$. An (n,n,t)-spike has at most $(t-1)^{n-1}$ circuit-hyperplanes; in particular, an n-spike has at most 2^{n-1} circuit-hyperplanes. The only n-spikes with 2^{n-1} circuit-hyperplanes are binary, and all binary n-spikes are isomorphic.*

Proof. As observed before, a circuit-hyperplane of an (n,s,t)-spike must be a member of $\mathcal{Z}_n$, that is, it contains a point from each of the sets $\ell_1 - a, \ell_2 - a, \ldots, \ell_n - a$. Consider any $(n-1)$-set A containing one element from each of $\ell_1 - a, \ell_2 - a, \ldots, \ell_{n-1} - a$; if A belongs to a circuit-hyperplane, then from the $t-1$ elements of $\ell_n - a$, there is only one that adjoined to A gives a circuit-hyperplane (the other $t-2$ give bases). From this the general bound follows.

Assume now that M is an n-spike with 2^{n-1} circuit-hyperplanes. Since $|\mathcal{Z}_n| = 2^n$ and the elements of $\mathcal{Z}_n$ are either bases or circuit-hyperplanes, from the last paragraph it follows that M has 2^{n-1} bases. Assume the lines of the n-spike M are $\ell_i = \{a, x_i, y_i\}$. By the argument of the first paragraph, we have that if B is a basis and $x \in B$ belongs to $\ell_n - a$, then the set $B - x \cup (\ell_n - a - x)$ is a circuit-hyperplane. This idea can be generalized to yield the following claim.

> **Claim** For a circuit-hyperplane C of M and any integers $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, we have that the symmetric difference
>
> $$C \triangle \{x_{i_1}, y_{i_1}\} \triangle \{x_{i_2}, y_{i_2}\} \triangle \cdots \triangle \{x_{i_k}, y_{i_k}\}$$
>
> is a circuit-hyperplane if and only if k is even; otherwise this symmetric difference is a basis.

Now using the claim one can easily construct an isomorphism between any two n-spikes with 2^{n-1} circuit-hyperplanes. Using this and the fact that the n-spike represented by matrix D in equation (7.1) has 2^{n-1} circuit-hyperplanes, it follows that any n-spike with 2^{n-1} circuit-hyperplanes is binary and isomorphic to D. To finish the proof we have to show that every binary spike has 2^{n-1} circuit-hyperplanes. By the Scum Theorem [43, Proposition 3.3.7], a binary matroid has the property that no coline (i.e. flat of rank $r(M)-2$) is covered by 4 or more hyperplanes. If M is a binary n-spike with fewer than 2^{n-1} circuit-hyperplanes, then there is an $(n-1)$-set B with one element from each of $\ell_i - a$, for i with $1 \leq i \leq n-1$, such that both $B \cup x_n$ and $B \cup y_n$ are bases. Then it is easy to check that any subset of B with $n-2$ elements is a coline that is covered by at least 4 hyperplanes, contradicting the fact that M is binary. □

The situation is very different if we consider representations over GF(q); Wu [61] shows that the number of nonisomorphic $(n,n,3)$-spikes that are representable over GF(q) grows rapidly as n and q increase.

We now show that certain classes of spikes can be detected by a few numerical invariants that can be determined from the Tutte polynomial; condition (vi) of Theorem 7.10 is what mildly limits the scope of this result.

Theorem 7.10 *Assume n, s, and t are integers with $n \geq 5$, $s \geq n-1$, and $t \geq 3$. Assume that M is a rank-n geometry that has:*

(i) $s(t-1)+1$ *points;*

(ii) s *lines that each have exactly t points;*

(iii) $s\binom{t}{3}$ *circuits with three elements;*

(iv) $\binom{s}{2}\binom{t-1}{2}^2$ *circuits with four elements;*

(v) $\binom{s}{2}$ *planes with $2t-1$ points; and*

(vi) *for each j with $j \geq n-1$, no hyperplane with $j(t-1)+1$ points.*

Then M is an (n,s,t)-spike. If in addition there are no n-circuits, then M is the free (n,s,t)-spike. If $s=n$, $t=3$, and M has 2^{n-1} circuits with n elements, then M is the binary n-spike.

Proof. Assumptions (ii) and (iii) imply that the t-point lines are the only nontrivial lines. Note that $\binom{t-1}{2}^2$ is the minimum number of 4-circuits in a plane that has $2t-1$ points in which each line has either 2 or t points; furthermore, the only such plane that has $\binom{t-1}{2}^2$ circuits with four elements is the parallel connection of two t-point lines. Since M has $\binom{s}{2}$ planes with $2t-1$ points and $\binom{s}{2}\binom{t-1}{2}^2$ circuits with four elements, it follows that each $(2t-1)$-point plane of M is a parallel connection of two t-point lines. Furthermore, since there are $\binom{s}{2}$ planes of M that have $2t-1$ points, each of the $\binom{s}{2}$ pairs of t-point lines spans one of these planes and therefore has nonempty intersection. Since $n \geq 5$, this implies that all t-point lines contain some common point, say a. Thus, conditions (S1) and (S2) in Definition 7.6 hold.

From assumption (vi) it follows that each hyperplane of M that contains the point a can contain at most $n-2$ of the t-point lines. This implies that, for $i \leq n-1$, any rank-i flat of M that contains a can contain at most $i-1$ of the t-point lines. Thus, condition (S3) in Definition 7.6 holds, so M is an (n,s,t)-spike.

The assertion about free (n,s,t)-spikes follows immediately; that about binary n-spikes follows from Theorem 7.9. □

Let $\mathcal{S}^k_{n,s,t}$ be the set of all (n,s,t)-spikes that satisfy condition (vi) in Theorem 7.10 for which the number of n-circuits is exactly k. Matroids in $\mathcal{S}^k_{n,s,t}$ are distinguished from all other matroids by their Tutte polynomials.

Corollary 7.11 *Assume n, s, and t are integers with $n \geq 5$, $s \geq n-1$, and $t \geq 3$. Then $\mathcal{S}^k_{n,s,t}$ is T-closed. In particular, if N is the only (n,s,t)-spike in $\mathcal{S}^k_{n,s,t}$, then N is T-unique.*

Proof. Let M be an (n,s,t)-spike in $\mathcal{S}^k_{n,s,t}$ and let N be T-equivalent to M. From Theorems 1.15 and 1.16, it follows that N satisfies conditions (i)–(vi) in Theorem 7.10. Thus, N is an (n,s,t)-spike. That k is the number of n-circuits in N follows from Corollary 7.8 or, alternatively, from Theorem 6.3 and the fact that N is $(n-1)$-chordal. □

Note that condition (vi) in Theorem 7.10 is automatically satisfied by any (n, s, t)-spike with $s < (n-1)(t-1)+1$. In particular, the first assertion in Corollary 7.11 applies to all (n, n, t)-spikes. From Corollary 7.11, we also get the T-uniqueness of some families of (n, s, t)-spikes.

Corollary 7.12 *For integers n, s, and t with $n \geq 5$, $s \geq n-1$, and $t \geq 3$, the following matroids are T-unique:*

(1) *the free (n, s, t)-spike;*

(2) *the binary n-spike;*

(3) *the (n, s, t)-spike M where, for some integer h with $n \leq h \leq s$ and h not of the form $j(t-1)+1$ for $j \geq n-1$, we have*

$$c_k^M = \begin{cases} 1, & \text{if } k = h; \\ 0, & \text{if } n \leq k \leq s \text{ and } k \neq h. \end{cases}$$

(that is, there is only one nontrivial hyperplane that does not contain the tip).

That the hyperplanes of a $(4, s, t)$-spike isomorphic to $U_{3,h}$, for $h \geq 4$, contain 4-circuits makes the argument about the structure of $(2t-1)$-point planes in the proof of Theorem 7.10 fail in general for $n = 4$. However, the same ideas as appear in the proofs of Theorem 7.10 and Corollary 7.11 give the following result.

Theorem 7.13 *The free $(4, s, t)$-spike is T-unique.*

8

Large families of T-unique matroids

As mentioned in Section 1.4, there exist several constructions that give large families of T-equivalent graphs and matroids [9, 10, 22]. In this chapter we consider a related question, namely that of finding large families of T-unique matroids. By a "large family" of matroids we mean that the number of rank-r members of the family grows exponentially as a function of r. Recall that asymptotically almost no matroid is T-unique.

We present two such families. The matroids in the first one are graphic; they are obtained by taking a subclass of maximal outerplanar graphs which is large enough to include many matroids, but concrete enough to allow a proof of T-uniqueness. The second family consists of matroids that are only respresentable over sufficiently large fields. These matroids are transversal matroids defined in terms of lattice paths. To our knowledge, these two are the first known large families of T-unique matroids.

8.1 Maximal outerplanar graphs

Cycle matroids of series-parallel networks are one of the few known T-closed classes of graphic matroids. As we will see later, in general they are not T-unique. We construct our family of T-unique graphic matroids by successively restricting the class of series-parallel networks to smaller T-closed classes, until we get to a class whose members are T-unique.

Let C_2 denote the cycle of length 2. A graph is a *series-parallel network* if either it has one edge or it can be obtained from C_2 by a sequence of operations each of which is either a series or a parallel extension (see [43, Section 5.4] for definitions and results). It is well-known that series-parallel networks are planar and 2-connected, but not 3-connected. Although series-parallel networks may have multiple edges, in this section we only consider simple graphs.

Brylawski [17] proved that cycle matroids of series-parallel networks can be recognized by the Tutte polynomial. Recall that the *beta invariant* $\beta(M)$ of a matroid M is the coefficient of x in $t(M; x, y)$.

Theorem 8.1 *A matroid M is the cycle matroid of a series-parallel network if and only if $\beta(M) = 1$.*

A simple series-parallel network G with n vertices has at most $2n - 3$ edges; if G has precisely this number of edges, then it is called a *2-tree*. To build a 2-tree, start with a triangle and repeatedly add a new vertex adjacent to the two ends of an existing edge. The class of cycle matroids of 2-trees is T-closed.

Lemma 8.2 *If G is a 2-tree and M is T-equivalent to $M(G)$, then $M \cong M(H)$ for some 2-tree H.*

Proof. By Theorem 8.1, M is the cycle matroid of a series-parallel network H. Since the number of vertices and edges are T-invariants, H is a 2-tree. □

A graph is *outerplanar* if it is planar and can be drawn in the plane in such a way that all the vertices are in the outer face. It is *maximal outerplanar* if, in addition, all the internal faces are triangles. It is easy to show that a maximal outerplanar graph G on n vertices has $2n-3$ edges and $n-2$ internal faces; in fact, maximal outerplanar graphs correspond to outerplanar 2-trees.

Lemma 8.3 *Let G be a maximal outerplanar graph. If M is T-equivalent to $M(G)$, then M is the cycle matroid of a maximal outerplanar graph H.*

Proof. By Lemma 8.2, H is a 2-tree. Thus, it only remains to show that H is outerplanar. The following claim is easily proved by induction on n.

> **Claim** A 2-tree with n vertices has at least $n-3$ cycles of length four with one chord, with equality holding if and only if the 2-tree is outerplanar.

By Theorem 1.17, the number of C_4 with a chord is determined by the Tutte polynomial; therefore H is outerplanar. □

Since series-parallel networks are not 3-connected, by Whitney's Theorem 1.3 two maximal outerplanar graphs can have the same cycle matroid (see Figure 1.1). This means that we need a way to represent cycle matroids of maximal outerplanar graphs that avoids this ambiguity. To achieve this, we define the *dual tree* of a maximal outerplanar graph G as the graph obtained from the geometric dual G^* by removing the vertex corresponding to the outer face in an outerplanar drawing of G. We denote the dual tree by $\tau(G)$; note that the maximum degree of $\tau(G)$ is at most 3.

Lemma 8.4 *Two maximal outerplanar graphs G and H have isomorphic cycle matroids if and only their dual trees $\tau(G)$ and $\tau(H)$ are isomorphic.*

Proof. If $M(G) \cong M(H)$, by Whitney's Theorem 1.3, G and H are 2-isomorphic. It is easy to see that a Whitney twist in a maximal outerplanar graph has to be performed on the two extremes of a chord, that is, an edge not on the boundary of the outer face. But then it is clear that such a twist in G does not change the isomorphism type of $\tau(G)$.

To prove the converse, if $\tau(G) \cong \tau(H)$, then clearly $G^* \cong H^*$. Then $M(G)^* \cong M(H)^*$ and so $M(G) \cong M(H)$. □

We introduce labels on the vertices of $\tau(G)$. A vertex is called of type E (end) if it has degree 1 in $\tau(G)$; type S (series) if it has degree 2; and type B (branching) if it has degree 3. Alternatively, we refer to them as E-vertices, S-vertices and B-vertices. Denote by $e(G), s(G)$ and $b(G)$, the number of vertices in $\tau(G)$ which are of type E, S and B, respectively.

The following relations are easy to establish:

$$e(G) = b(G) + 2, \qquad e(G) + s(G) + b(G) = |V(G)| - 2. \tag{8.1}$$

In our way to constructing a family of T-unique matroids, we consider the following family $\mathcal{O}$ of maximal outerplanar graphs satisfying the next two conditions (see Figure 8.1):

(1) the dual tree is a *caterpillar*, namely, it is such that removing all the E-vertices what results is a path, called the *spine*;

(2) the two extremes of the spine are adjacent to two E-vertices in the dual tree.

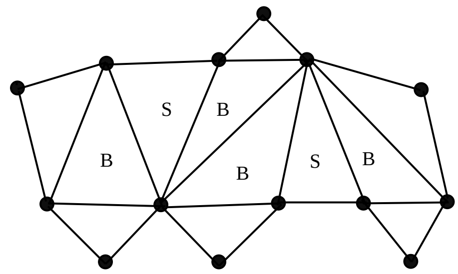


Figure 8.1 A graph in the family $\mathcal{O}$; internal faces are labeled S or B, according to their type in the dual tree.

To prove that the class of cycle matroids of graphs in $\mathcal{O}$ is T-closed, we need the following simple result, whose proof is omitted.

Lemma 8.5 *A maximal outerplanar graph G is in $\mathcal{O}$ if and only if every B-vertex in $\tau(G)$ is adjacent to exactly one E-vertex, except for two B-vertices (the extremes of the spine), which are adjacent to exactly two E-vertices.*

We next prove that the class $\mathcal{O}$ can be characterized in terms of Tutte polynomials.

Lemma 8.6 *If G is a graph in $\mathcal{O}$ and M is a matroid T-equivalent to $M(G)$, then $M \cong M(H)$ for some graph H in $\mathcal{O}$.*

Proof. By Lemma 8.3 we know that $M \cong M(H)$ for some maximal outerplanar graph H, so it only remains to show that H is in $\mathcal{O}$.

The number of E-vertices in the dual tree of a maximal outerplanar graph is a Tutte invariant, since they correspond to circuits of size two in the dual graph. Thus $\tau(G)$ and $\tau(H)$ have the same number of E-vertices; using equations (8.1) we see that they also have the same number of vertices of type S and B.

The goal now is to count the number of adjacencies in $\tau(H)$ involving vertices of type E and B, and then apply Lemma 8.5.

Suppose first that xy is an edge of $\tau(H)$ with x of type E and y of type S. If w is the vertex corresponding to the outer face of H^*, then $\{x, y, w\}$ induces a subgraph in H^* of size 4 and rank 2. The only subgraphs which also contribute to $[x^2y^4]F(H^*; x, y)$ are those induced by two E vertices and w (see Figure 8.2). Of these, H^* and G^* contain the same number, namely $\binom{e(G)}{2} = \binom{e(H)}{2}$. Since in G^* there is no adjacency between vertices of type E and S, we deduce that the same happens in H^*. Hence all E-vertices in $\tau(H)$ are adjacent only to B-vertices.

Suppose next xy and yz are edges in $\tau(H)$ with x, z of type E and y of type B. If w is as before, then $\{x, y, z, w\}$ induces a subgraph in H^* of size 6 and rank 3. The only subgraphs which also contribute to $[x^3y^6]F(H^*; x, y)$ are those shown in Figure 8.3. Of the second kind, G^* and H^*

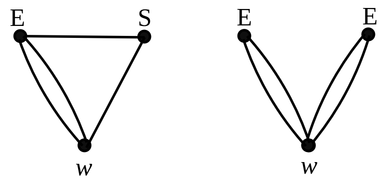


Figure 8.2 Subgraphs of rank 2 and size 4 in H^*.

contain the same number, namely $\binom{e(G)}{3} = \binom{e(H)}{3}$. Of the third and fourth kinds, we proved in the previous paragraph that neither G^* nor H^* contain any. Since in $\tau(G)$ there are exactly two B-vertices adjacent to precisely two E-vertices, we deduce that his holds in $\tau(H)$ too. We conclude that H satisfies the hypothesis in Lemma 8.5 and hence belongs to $\mathcal{O}$. □

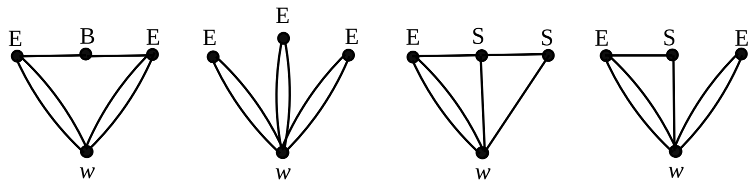


Figure 8.3 Subgraphs of rank 3 and size 6 in H^*.

To every graph G in $\mathcal{O}$ we assign a word in the alphabet $\{B, S\}$ corresponding to the type of vertices in the spine of the dual tree, read from an arbitrarily chosen end of the spine. Given such a word, we can recover the graph up to 2-isomorphism; hence, by Whitney's Theorem, each word determines a unique matroid. Clearly one word and its reversal encode the same matroid, but no other pair of words can give the same matroid. We say that $G \in \mathcal{O}$ is *symmetric* if its word is equal to its reversal. We refer to this class of graphs as $\mathcal{SO}$. For instance, the graph in Figure 8.1 belongs to $\mathcal{SO}$, since its word is BSBBSB.

Now comes the main result of this section, where we prove that cycle matroids of graphs in $\mathcal{SO}$ are T-unique.

Theorem 8.7 *If G is a graph in $\mathcal{SO}$, then the matroid $M(G)$ is T-unique.*

Proof. Let M be a matroid T-equivalent to $M(G)$. By the previous lemmas and their proofs we know that $M \cong M(H)$, where H is a graph in $\mathcal{O}$ with the same number of vertices of type E, S, and B as G. Thus, by Lemma 8.4 it is enough to prove that $\tau(H)$ and $\tau(G)$ are isomorphic.

Let $a_0a_1a_2 \cdots a_2a_1a_0$ be the word in the alphabet $\{B, S\}$ encoding G, and let $b_0b_1b_2 \cdots c_2c_1c_0$ be the one encoding H. The goal is to prove that $b_i = a_i = c_i$ for all i; we do this by induction on i. By definition of the class $\mathcal{O}$, we have $a_0 = b_0 = c_0 = B$. Assume $b_j = a_j = c_j$ for $j \leq i$, and let us prove that $b_{i+1} = a_{i+1} = c_{i+1}$. The basic tool is the following easy to prove claim.

Claim Let G' be a graph in $\mathcal{O}$, let $\{x_0, x_1, x_2, \dots y_2, y_1, y_0\}$ be the (ordered set of) vertices in the spine of $\tau(G')$, and let w be the vertex in $(G')^*$ corresponding to the outer face. Then the only subgraphs of rank r and size $2r$ in $(G')^*$ are those induced by $\{x_0, x_1, \dots, x_p, y_0, y_1, \dots, y_q, w\} \cup V_0 \cup V_1$ for some p, q, where V_0 is the set of

E-vertices that are adjacent to some vertex in $\{x_0, x_1, \ldots, x_p, y_0, y_1, \ldots, y_q\}$, and V_1 is any set of E-vertices disjoint from V_0.

Let r_i be the rank of the subgraph in G^* induced by w and the vertices corresponding to the labels $\{a_0, a_1, \ldots, a_i\}$ in either extreme of the spine. By inductive hypothesis, the vertices in H^* corresponding to the outer face and the labels $\{b_0, b_1, \ldots, b_i\}$ or $\{c_0, c_1, \ldots, c_i\}$ also span a rank-r_i subgraph. By the previous claim and the fact that $e(G) = e(H)$, the graphs G^* and H^* have the same number of edge-sets with rank r_i and size $2r_i$. Again by the claim above, if $a_{i+1} = \mathrm{S}$ then the number of subgraphs in G^* of rank $r_i + 1$ and size $2(r_i + 1)$ increases by two, corresponding to those induced by w and the labels $\{a_0, \ldots, a_i, a_{i+1}\}$ in either extreme of the spine; however, if $a_{i+1} = \mathrm{B}$ then the number of such subgraphs does not change. But the number of rank-$(r_i + 1)$ edge-sets of size $2(r_i + 1)$ is determined by the Tutte polynomial, and this implies that $b_{i+1} = a_{i+1} = c_{i+1}$, as was to be proved. □

The above result implies the existence of exponentially many T-unique graphic matroids with the same rank and size.

Theorem 8.8 *The number of connected T-unique matroids of rank* r *and size* $2r - 1$ *is at least* α^r *for some* $\alpha > 1$.

Proof. We count how many graphs are there in $\mathcal{SO}$ such that the dual tree has $n + 6$ vertices; this is the same as counting matroids of rank $n + 7$. There are 6 vertices in the dual tree that are fixed (three at each end). The remaining n can be split as $n = 2e + 2b + 2s$ with $e = b$. Since the word codifying the tree is symmetric, it is enough to choose b positions among $n/2 - e$. This leads to the following formula for the number of graphs in $\mathcal{SO}$ of rank $n + 7$, for even n.

$$s_n = \sum_{i=0}^{\lfloor \frac{n}{4} \rfloor} \binom{\frac{n}{2} - i}{i}$$

It is straightforward to check that $s_n + s_{n+2} = s_{n+4}$. Therefore, s_n is the Fibonacci number $F_{n/2}$, and thus grows as $\phi^{n/2}$, where ϕ is the golden ratio. □

To find a large family of T-unique matroids we started with a T-closed class and successively added restrictions until we were able to prove T-uniqueness. One might wonder whether we could have stopped before, that is, whether all series-parallel networks, or all outerplanar graphs, are T-unique. Giménez [31] showed that this is not the case by finding several pairs of maximal outerplanar graphs having the same Tutte polynomial. One such pair is shown in Figure 8.4. Note that if we add two leaves to the first graph, one adjacent to each of the vertices labelled v, v' in this figure, by Theorem 8.7 the associated cycle matroid would be T-unique. A question that remains open is whether the family of the cycle matroids of (not necessarily maximal) outerplanar graphs is T-closed. That is equivalent to finding whether the outerplanar character can be read from the Tutte polynomial, at least for series-parallel networks. It would not be surprising that the class is not T-closed but the counterexamples to this are too large to be found by hand.

8.2 Generalized Catalan matroids

In this section we study the T-uniqueness of a very different family of matroids. Generalized Catalan matroids are a class of transversal matroids that are closed under taking minors and

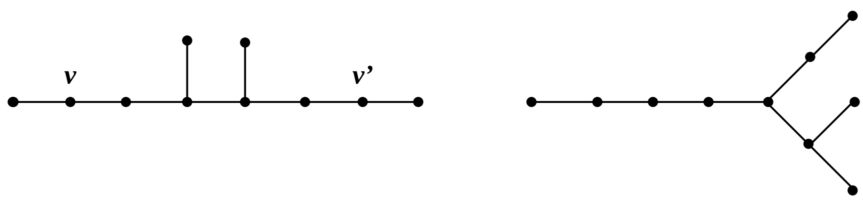


Figure 8.4 The dual trees of two T-equivalent non-2-isomorphic outerplanar graphs

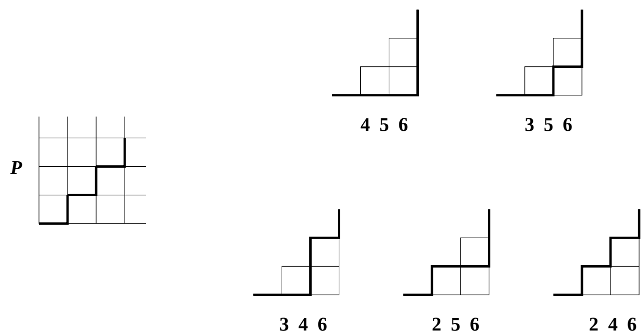


Figure 8.5 A lattice path P and the bases of the matroid $M[P]$ viewed as the sets of N steps in a path not going above P.

duals, and have a particularly simple characterization by excluded minors. This characterization implies a very restricted structure on cyclic flats, and this structure is the key point in proving T-uniqueness. As transversal matroids in general [47], generalized Catalan matroids are not representable over small fields. By a simple counting argument we show that for each q there are exponentially many T-unique matroids not representable over any field GF(q') for $q' \leq q$.

8.2.1 Definition and basic properties

Generalized Catalan matroids can be seen as a subfamily of lattice path matroids, which were recently introduced in [14]. As mentioned above, they are transversal matroids, but here we define them in terms of lattice paths. All paths in this section use steps $E = (1, 0)$ and $N = (0, 1)$. If P is a path, by P^n we denote the concatenation of n copies of P.

Let P be a path from $(0, 0)$ to (m, r) that uses steps E and N. The *generalized Catalan matroid* $M[P]$ is a rank-r matroid on the ground set $[r + m] = \{1, 2, \ldots, r + m\}$; an r-subset of $[r + m]$ is a basis of $M[P]$ if it corresponds to the N steps of a path from $(0, 0)$ to (m, r) that does not go above P (see Figure 8.5 for an example). We denote by $\mathcal{GC}$ the set of generalized Catalan matroids. The name "generalized Catalan matroid" arises from the fact that if we take as P the path $(EN)^r$, then the number of bases of $M[P]$ is the Catalan number $C_r = \frac{1}{r+1}\binom{2r}{r}$. All uniform matroids are generalized Catalan matroids; in particular, $U_{s,n} \cong M[N^s E^{n-s}]$.

Generalized Catalan matroids have been studied previously under different names and points of view, but it is not hard to check that all definitions actually give the same matroid. They first appeared in a paper by Welsh [57]. Later Oxley, Prendergast, and Row [45] gave the excluded minor characterization. Recently Ardila [3] rediscovered them in yet another context.

Deletion, contraction, and duality have easy interpretations in terms of the path P. The following lemma is easy to prove [13, 14].

Lemma 8.9 *Let P be a path from $(0,0)$ to (m,r), and let i, $1 \leq i \leq m+r$, be an element of the ground set of $M[P]$. Then,*

(i) $M[P]^* \cong M[P^*]$, *where P^* is the path obtained from P by interchanging E and N steps;*

(ii) $M[P]\backslash i \cong M[P\backslash i]$, *where $P\backslash i$ is the path obtained from P by deleting the first E step strictly after the $(i-1)$-st step, or the last step if all steps from the i-th are N;*

(iii) $M[P]/i \cong M[P/i]$, *where P/i is the path obtained from P by deleting the last N step strictly before the $(i+1)$-st step, or the first step if all steps until the i-th are E.*

It is immediate from the previous lemma that the class $\mathcal{GC}$ is a minor-closed class of matroids. Its set of excluded minors can be easily described. Define the matroid D_i as the truncation to rank i of the direct sum of two i-circuits; in other words, D_i consists of two i-circuits placed as freely as possible in rank i. Hence in D_i all i-subsets are bases, except for the two i-circuits. This observation implies that $D_i \notin \mathcal{GC}$. The following theorem is from [45].

Theorem 8.10 *The excluded minors for $\mathcal{GC}$ are D_i for $i \geq 2$.*

Recall that a nonempty flat F is a *cyclic flat* if every element of F is in at least one circuit of $M|F$. Equivalently, F is a cyclic flat if $M|F$ has no isthmuses. The following theorem gives a useful characterization of generalized Catalan matroids in terms of cyclic flats.

Theorem 8.11 *A matroid is a generalized Catalan matroid if and only if its cyclic flats form a chain under inclusion. Furthermore, after deleting loops all cyclic flats are connected. Also, the path that defines the matroid can be recovered from the knowledge of the ranks and sizes of the cyclic flats, together with the rank of the matroid.*

Proof. The first assertion is proved in [45]. The second follows easily from the first, since the connected components of a cyclic flat are also cyclic. For the third, let $F_1 \subset F_2 \subset \cdots \subset F_s$ be the chain of cyclic flats of a generalized Catalan matroid M; let r_i and t_i denote the rank and size of F_i, respectively. It is easy to prove that $M \cong M[P]$ with

$$P = N^{r_1} E^{t_1 - r_1} N^{r_2 - r_1} E^{t_2 - t_1 - (r_2 - r_1)} \cdots N^{r_n - r_{n-1}} E^{t_n - t_{n-1} - (r_n - r_{n-1})} N^{r - r_n},$$

where r is the rank of M. □

Before turning to T-uniqueness, we mention here that formulas and algorithms for computing Tutte polynomials of generalized Catalan matroids can be found in [14].

8.2.2 T-uniqueness and non-representability

We now prove that generalized Catalan matroids are T-unique.

Theorem 8.12 *The matroid $M[P]$ is T-unique.*

Proof. Let M be a matroid T-equivalent to $M[P]$. Let S and $S(M)$ be the ground sets of $M[P]$ and M, respectively. We denote by r and t the rank and size of $M[P]$. As in the proof of Theorem 8.11, we denote by $F_1, F_2, \ldots, F_s$ the cyclic flats of $M[P]$, with $F_i \subset F_{i+1}$, $r_i = r(F_i)$ and $t_i = |F_i|$. To prove that $M[P]$ is T-unique we show that the cyclic flats of M form a chain $G_1 \subset G_2 \subset \cdots \subset G_s$ with $r(G_i) = r_i$, $|G_i| = t_i$. By Theorem 8.11, this is enough to prove that $M \cong M[P]$. We prove the existence and properties of G_i by induction on i.

Among the cyclic flats with smallest rank in M, pick one that has the maximum number of elements; the rank and size of this flat are T-invariants, and in fact are r_1 and t_1, respectively. Let G_1 be a cyclic flat of M with $r(G_1) = r_1$ and $|G_1| = t_1$. Since G_1 is a minimal cyclic flat, it consists of a circuit plus some other points added freely (in fact, it is isomorphic to U_{r_1,t_1}); hence, G_1 contains $\binom{t_1}{r_1+1}$ circuits of size r_1+1. In $M[P]$, all (r_1+1)-circuits belong to F_1; since the number of (r_1+1)-circuits in G_1 is the same as in F_1, there can be no other (r_1+1)-circuits in M, and hence there is no other cyclic flat of rank r_1. This is the first step of the induction.

Assume we have a chain of cyclic flats $G_1 \subset \cdots \subset G_i$ with $r(G_j) = r_j$ and $|G_j| = t_j$ for all $j \leq i$. This implies that the G_j are the only cyclic flats in M of rank at most r_i, and that all circuits of rank at most r_i are subsets of G_i.

If $i = s$, then $r - r_i = t - t_i$; thus, all points of $M[P]$ outside F_s are isthmuses, and the same holds for all points of M outside G_s. Hence $M[P]$ is the direct sum of $M[P]|F_s$ with $r - r_s$ isthmuses, and M is the direct sum of $M|G_s$ with $r - r_s$ isthmuses. Since the cyclic flats of $M|G_s$ form a chain, by Theorem 8.11 it follows that $M|G_s \cong M[P]|F_s$. Therefore, M and $M[P]$ are isomorphic.

Suppose now that $i < s$ and $r_{i+1} = r_i + k$. We have to prove that M does not contain any cyclic flat of rank $r_i + k'$ for any k' with $1 \leq k' < k$. If such a cyclic flat existed, M would have a circuit of rank between $r_i + 1$ and $r_i + k - 1$ (if all circuits had rank at most r_i, then the flat would be contained in G_i). We prove by induction on k' that M has no circuit of rank $r_i + k'$ for k' with $1 \leq k' < k$.

We start with a property of $M[P]$. Let A be a subset of F_i with nullity one (that is, containing only one circuit), and add to A any $r_i - r(A) + k'$ points of $S - F_i$ for some k' with $1 \leq k' < k$. Then the resulting set A' also has nullity one. This is because if A' had nullity two or more, then it would contain a second circuit C not entirely contained in F_i. Then the closure of C would be a cyclic flat of rank at most $r_{i+1} - 1$; but the cyclic flats of $M[P]$ that have rank less than r_{i+1} are included in F_i, and this is a contradiction. We also have that all subsets of $M[P]$ with rank $r_i + k'$ and nullity one are of the form described here, since necessarily the circuit of the set must belong to F_i. We next prove that the same holds for M.

> **Claim 1** For k' with $1 \leq k' < k$, any subset B of G_i of nullity one together with $r_i - r(B) + k'$ points outside G_i has nullity one and rank $r_i + k'$, and these account for all sets of nullity one and rank $r_i + k'$.

Proof. We prove both assertions simultaneously by induction on k'.

Suppose $B \subseteq G_i$ has nullity one, and consider the union of B and $r_i - r(B) + 1$ points of $S(M) - G_i$. If this set B' had nullity bigger than one, then it would contain a circuit C not entirely contained in G_i. Since the total number of points is $r_i + 2$, the rank of C cannot be $r_i + 1$. But it cannot be less than $r_i + 1$, since all circuits of rank r_i or less are included in G_i. Hence B' has nullity one.

For the converse, let B' be a set with nullity one and rank $r_i + 1$. Since the cyclic flats of the matroid $M|G_i$ form a chain and $r(F_j) = r(G_j)$ and $|G_j| = |F_j|$ for $j \leq i$, by Theorem 8.11 we get $M|G_i \cong M[P]|F_i$. Hence all subsets of G_i of nullity one are in bijection with the subsets of F_i of nullity one. By the results of the last paragraph, this bijection can be extended to a mapping from the sets of nullity one and rank $r_i + 1$ in M onto the sets with these same rank and nullity in $M[P]$. Since the number of sets with rank $r_i + 1$ and nullity one is a T-invariant, this last mapping is in fact a bijection. This concludes the proof of the base case of the induction.

Now let k' be an integer with $1 < k' < k$; consider a set $B \subseteq G_i$ with nullity one and adjoin to it $r_i - r(B) + k'$ points of $S(M) - G_i$ with $k' < k$. If the resulting set B' had nullity two or more, then it would contain a circuit C containing some points of $S(M) - G_i$. By deleting enough points of $B' - C$, we can get down to a set with nullity one and rank $r_i + \widetilde{k}$, for $\widetilde{k}$ with $1 \leq \widetilde{k} < k'$. This is a contradiction since by induction hypothesis the circuit of a set with nullity one and rank at most $r_i + k' - 1$ is contained in F_i.

The converse follows along the same lines as the case $k' = 1$. □

Since circuits have nullity one, the previous claim implies that there cannot be a circuit of rank $r_i + k'$ in M for k' with $1 \leq k' < k$. Hence, M contains no cyclic flat of rank $r_i + k'$.

Now we have to show that there is a unique cyclic flat with rank $r_{i+1} = r_i + k$ and that this flat has size t_{i+1} and contains G_i.

In $M[P]$, the maximum size of a rank-($r_i + k - 1$) flat is $t_i + k - 1$; by T-equivalence the same holds in M. By Theorem 1.16, both $M[P]$ and M have a unique flat of rank r_{i+1} with strictly more than $t_i + k$ elements (in fact, with t_{i+1} elements). Let G_{i+1} be the unique flat of M with rank r_{i+1} and size t_{i+1}. If G_{i+1} were not cyclic, then we could delete one of its isthmuses and get a flat with rank $r_i + k - 1$ and size greater than $t_i + k$, thus contradicting the observation above. To conclude the induction step we show that $G_i \subset G_{i+1}$. We need the following claim.

Claim 2 The only sets in $M[P]$ that have rank $r_i + k$ and size $t_i + k$ are formed by either

(i) joining k elements of $S - F_i$ to F_i, or
(ii) removing any $t_{i+1} - t_i - k$ elements from F_{i+1}.

Proof. From the fact that the maximum size of a rank-($r_i + k - 1$) flat is $t_i + k - 1$ it follows that the sets described in (i)-(ii) have rank $r_i + k$ and size $t_i + k$. Let $A \subset S$ be such that $r(A) = r_i + k$ and $|A| = t_i + k$; we show that A is of the form of either (i) or (ii). We distinguish two cases.

(a) $M[P]|A$ has at least one isthmus. The elements of A that are not isthmuses are contained in at least one circuit of $M[P]|A$; this circuit has rank at most $r_i + k - 1$, hence at most r_i. Since a circuit of rank at most r_i is contained in F_i, all the elements of A that are not isthmuses belong to F_i. Let A' denote the subset of the elements of A that are not isthmuses of $M[P]|A$. We have that $A' \subseteq F_i$ and that $n(A') = n(F_i)$. From this it follows that $A' = F_i$ and hence A consists of F_i together with k arbitrary elements of $S - F_i$.

(b) $M[P]|A$ has no isthmuses. Then every element of A belongs to a circuit of rank at most $r_i + k$, and hence is contained in F_{i+1}.

□

Consider now the subsets of $S(M)$ that have rank $r_i + k$ and size $t_i + k$. The sets formed by either

(1) joining k elements of $S(M) - G_i$ to G_i, or

(2) removing $t_{i+1} - t_i - k$ elements from G_{i+1}

have rank $r_i + k$ and size $t_i + k$ by the same argument as in the proof of Claim 2. Observe that the number of sets described in (i) (respectively, (ii)) equals the number of sets in (1) (resp., (2)). Note also that since $F_i \subset F_{i+1}$, the intersection of (i) and (ii) is the largest possible. The fact that the number of sets with rank $r_i + k$ and size $t_i + k$ is a T-invariant implies that the intersection of (1) and (2) is also the largest possible. Therefore, $G_i \subset G_{i+1}$, as was to be proved. □

Corollary 8.13 *There exist at least $\binom{n}{r}$ non-isomorphic T-unique matroids with rank r and n elements. Of these, $\binom{n-2}{r-1}$ are connected.*

Proof. There are $\binom{n}{r}$ possible upper paths P that give a generalized Catalan matroid with n elements and rank r; different paths give nonisomorphic matroids by Theorem 8.11. Since a generalized Catalan matroid is not connected only because of loops and isthmuses, the second statement follows by considering paths that start at $(0, 1)$ and end at $(n - r - 1, r)$. □

Observe that for the family $\mathcal{SO}$ in the previous section, the size of the matroid was determined by its rank, whereas generalized Catalan matroids provide many T-unique matroids for all ranks and sizes.

We study now the representability of generalized Catalan matroids. Recall that a matroid M is representable over the field F if M is isomorphic to a matroid having as elements the set of columns of a matrix over F and with the independence relation being linear independence.

The following lemma gives a trivial sufficient condition for a matroid to not be representable over GF(q).

Lemma 8.14 *The matroids $U_{2,q+2}$ and $U_{q,q+2}$ are among the excluded minors for* GF(q)*-representability.*

Recall that $U_{2,q+2}$ and $U_{q,q+2}$ are isomorphic to $M[N^2E^q]$ and $M[N^qE^2]$.

Graphic matroids are representable over all fields [43]; hence by Theorem 8.8, there are exponentially many T-unique matroids that are representable over all fields. Now we show that for each q there are many T-unique matroids that are not representable over GF(q).

Theorem 8.15 *The number of T-unique nonisomorphic non-*GF(q)*-representable matroids with rank r and size $m + r$ is at least*

$$\binom{m+r}{r} - rmp_1(q) - rp_2(q) - mp_3(q) - p_4(q),$$

where $p_1(q), p_2(q), p_3(q), p_4(q)$ are expressions in q that grow like $\mathcal{O}(4^q)$.

Proof. Consider all possible paths Q from $(0,0)$ to (m,r). If Q does not contain any of the pairs of steps $\{(A,Y),(A,Z),(B,Y),(B,Z)\}$ in Figure 8.6, then $M[P]$ is not representable over $\mathrm{GF}(q)$ because it contains a minor isomorphic to either $U_{2,q+2}$ or $U_{q,q+2}$. The result follows by counting the number of paths that contain each of these pairs; note that some paths that contain one of the pairs of steps might also give rise to non-$\mathrm{GF}(q)$-representable matroids, since they could contain another of the excluded minors for $\mathrm{GF}(q)$-representability.

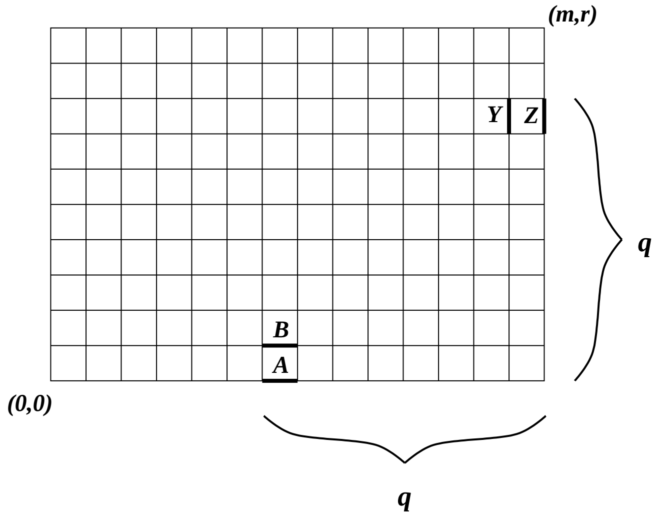


Figure 8.6 A path containing none of A,B,Y,Z gives a non-$GF(q)$-representable matroid.

For each pair we count the number of paths that contain it.

$$\begin{array}{lcl} (A,Y) & \longrightarrow & \binom{2q-3}{q-1}(r-q) \\ (A,Z) & \longrightarrow & \binom{2q-2}{q-1} \\ (B,Y) & \longrightarrow & (m-q)\binom{2q-4}{q-2}(r-q) \\ (B,Z) & \longrightarrow & (m-q)\binom{2q-3}{q-1} \end{array}$$

The desired number of paths is then $\binom{m+r}{r}$ minus the four quantities above. □

Corollary 8.16 *For q a prime power, and for r,m large enough, almost all generalized Catalan matroids with rank r and nullity m are not representable over* $\mathrm{GF}(q)$.

As a complement to the results in this chapter, it would be interesting to find an exponentially large family of T-unique matroids that are not representable over any field. One trivial example would consist of taking the direct sum of a generalized Catalan matroid and a fixed nonrepresentable T-unique matroid. Of course, it would be far more interesting to find a family consisting of connected matroids.

Conclusions

This thesis is focused on T-uniqueness, that is, the study of graphs and matroids determined by their Tutte polynomials. We now review and discuss the results we have obtained and suggest lines of future research.

In Chapter 2 we proved that the Tutte polynomial of a connected matroid is irreducible in the (x, y)-plane. A related question is to study the algebraic behaviour of the Tutte polynomial when evaluated at various curves of the plane. For instance, if we evaluate the Tutte polynomial in the line $y = 0$, after a change of variables we get the characteristic polynomial of the matroid (which generalizes the chromatic polynomial of a graph). Similarly, the Tutte polynomial in the line $x = 0$ gives the flow polynomial in the graphic case. Factorizations of characteristic polynomials have received much attention (see [36]). In particular, there are several instances of 2-connected graphs whose chromatic and flow polynomials are not irreducible. For example,

$$\begin{aligned}\chi(K_5; x) &= x(x-1)(x-2)(x-3)(x-4),\\ \phi(K_5; x) &= (x-1)(x^2-4x+5)(x^3-5x^2+11x-9).\end{aligned}$$

Thus, by restricting the Tutte polynomial to curves of the plane we may get a factorization that does not arise from the connected components of the matroid. The question is to study the irreducibility of the Tutte polynomial along other curves of the plane. In particular, the curve $xy = 1$ would give us information on the Jones polynomial of an alternating link. The hyperbolas $(x-1)(y-1) = \alpha$ are also curves of interest since they play a role in complexity.

In addition to its own importance as an algebraic property, having the factorization of the Tutte polynomial implies that a matroid is T-unique if and only if each of its connected components is T-unique. Note that the operation of taking duals trivially preserves T-uniqueness. It would be interesting to have other constructions that produce T-unique matroids from given ones.

Chapters 3, 4, and 5 are devoted to T-unique graphs. In the first two we showed that several well-known families of graphs are T-unique; these results are inspired in previous work on chromatic uniqueness, but the richness of the Tutte polynomial allows us to treat effectively graphs whose chromatic uniqueness is not settled yet. Although the proofs in these chapters rely on similar ideas, each of the families considered requires its own specific proof. For the graphs in Chapter 4 we also had to prove some results independent from the Tutte polynomial. The spirit of Chapter 5 is to develop a technique with a wider range of application; this technique allows us to prove the T-uniqueness of some kinds of line graphs. Although limited in scope, this method is more general than those used in Chapters 3 and 4.

At this stage, the main problem concerning T-uniqueness for graphs is whether almost all graphs are T-unique. (Actually, this is not only a problem on Tutte polynomials, but can also be posed for other polynomials, such as the chromatic, the characteristic or the matching polynomials). All the instances of T-unique graphs considered in this thesis have a high degree of symmetry, in the sense that many of them are regular and their edges and vertices can be classified into a few types. That is what makes it possible to build constructive proofs of their T-uniqueness.

In order to study the conjecture, one should be able to prove T-uniqueness without the need to reconstruct the graph. Or, in order to prove that it is false, be able to produce a sufficiently large set of non-T-unique graphs; one way to achieve this is to find a method to produce a T-equivalent companion for a significant number of graphs. However, there has been not much progress in this direction, neither for the Tutte nor for any of the polynomials mentioned above. In fact, there is no nontrivial polynomial that is known to determine almost all graphs up to isomorphism. By a "nontrivial" polynomial we mean a polynomial that does not encode explicitly the structure of the graph and that is defined in a somewhat natural way. One possible candidate is the U polynomial of Noble and Welsh [42] that extends by far the Tutte polynomial and that, in fact, is equivalent to the polychromate of Brylawski [22, 51]; for instance, it is not known whether U distinguishes among all trees with a given number of vertices, whereas they all share the same Tutte polynomial.

In Chapter 6 we have introduced chordal matroids as a generalization of chordal graphs and we have proved a characterization theorem that gives a sufficient condition for a chordal matroid to be T-unique. Thus, as in Chapter 5 we give a general framework for proving T-uniqueness. What makes the application of the theorem less straightforward is that it involves finding a bijection with certain properties. We use this tool to prove the T-uniqueness of truncations of complete graphs and matroids of complete bipartite graphs. In Chapter 7 we studied matroids that have a prominent role in matroid structure theory: wheels, whirls, and spikes. We generalized the usual definitions by adding a number of points to each line, and prove their T-uniqueness and find some T-closed classes. From the T-uniqueness proofs we deduce characterizations of these matroids in terms of flat statistics and other T-invariants. A question of interest would be to know if these characterizations are minimal, like those of Kung [37], or if they are not, to find minimal characterizations of these and other matroids.

Since the number of matroids is much larger than the number of possible Tutte polynomials, it does not make sense to ask whether almost all matroids are T-unique. Knowing that almost no matroid is T-unique, we can ask how many T-unique matroids can be found with the same rank and size. The upper bound on this number is given by the number of possible Tutte polynomials. From [24, Exercise 6.9] it follows that there are at most $\mathcal{O}(2^{m^3})$ distinct Tutte polynomials for matroids with m elements. The question would be then how close we can get to this bound. The examples in Chapter 8 give families of size $\mathcal{O}(2^m)$. One could slightly increase this number by using direct sums, but the most interesting examples are with connected matroids. One result that would complete what we have done in Chapter 8 is to have an exponentially large family of T-unique matroids that are not representable over any field.

One general observation that arises from our work in T-uniqueness is that, even if the Tutte polynomial contains a wealth of invariants, not many of them are really useful in proving T-uniqueness. For instance, the number of colourings of a graph is a T-invariant, but since it is difficult to compute even for small graphs, one can practically make no use of it. Roughly speaking, from our experience it follows that the invariants one would like to have are not T-invariants. One invariant which we do not know if it can be deduced from the Tutte polynomial is vertex-connectivity for graphs (note that matroid connectivity is not). In particular, we know of no pair of T-equivalent graphs with one 3-connected and the other not 3-connected. It is likely though that with a large enough database such an example could be found.

Appendix

We list several properties and invariants that cannot be deduced from the Tutte polynomial and give some interesting examples of T-equivalent graphs. The pairs of T-equivalent matroids in Figures A.1 and A.3 are well-known (see [24]); the other counterexamples were found by computer search.

Theorem A.1 *The following invariants are not T-invariants.*

(1) *Whether a matroid is binary, graphic, cographic, or the cycle matroid of a planar graph.*
(2) *The matroid connectivity.*
(3) *The branch-width.*
(4) *Whether the matroid has an element in free position.*
(5) *The Tutte polynomial of the simplification.*
(6) *The number of 4-circuits.*
(7) *The number of flats of each rank and size.*
(8) *The degree sequence, if the matroid is graphic.*

Proof. Figure A.1 shows a pair of T-equivalent matroids; the left side one is not binary (it has a $U_{2,4}$ minor) and the right side one is the cycle matroid of the graph K_4 with a double edge. This proves assertion (1). This pair of matroids is also a counterexample for assertions (5)–(7); note that the number of 4-circuits is a T-invariant if the matroid is simple. The simple graphs in Figure A.2 are another counterexample to some of the statements in assertion (1).

The matroids in Figure A.3 prove (2)–(4). The first matroid has Tutte and vertical connectivity equal to two, and for the second one both connectivities are three. J. Bonin observed that by adding two elements freely to each matroid, their branch-widths become 3 and 4, respectively. Observe that the Tutte polynomial of the free extension of a matroid can be computed from the Tutte polynomial of the original matroid [20], but the fact that there is an element in free position is not a T-invariant.

The two graphs in Figure A.4 have different degree sequence, but they are T-equivalent; this proves (8). □

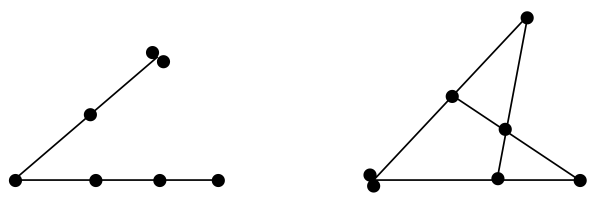


Figure A.1 A nonbinary and a graphic T-equivalent matroids.

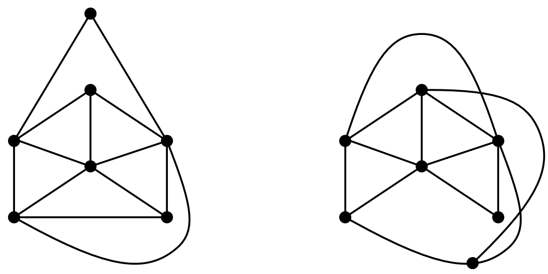


Figure A.2 A planar and a nonplanar T-equivalent graphs.

The graph in Figure A.5 is nonplanar and has symmetric Tutte polynomial (thus, its dual is a cographic but not graphic matroid with the same Tutte polynomial). Furthermore, this graph is T-unique. Hence, it is a simple graph that is T-unique as a graph but not as a matroid.

The graphs in Figure A.6 are 3-connected and T-equivalent. One is the dual of the other, hence their Tutte polynomial is symmetric. They are the smallest pair of T-equivalent 3-connected planar graphs.

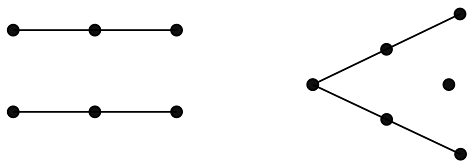


Figure A.3 The smallest T-equivalent matroids.

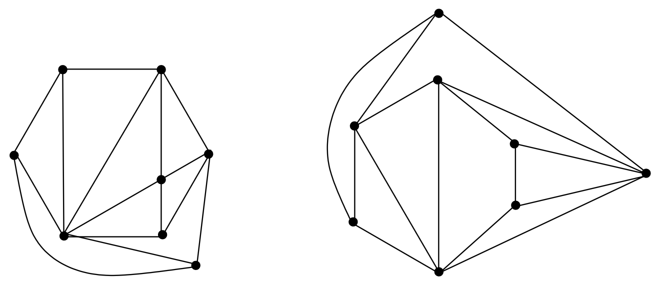


Figure A.4 T-equivalent graphs with different degree sequence.

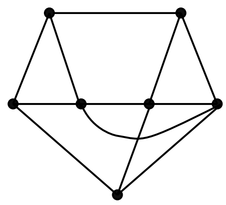


Figure A.5 A nonplanar graph with symmetric Tutte polynomial.

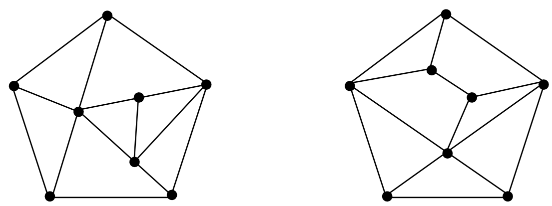


Figure A.6 A 3-connected graph T-equivalent to its dual.

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