On graphs determined by their Tutte polynomial

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Abstract

We say that a graph G is T-unique if any other graph having the same Tutte polynomial as G is necessarily isomorphic to G. In this paper we show that several well-known families of graphs are T-unique: wheels, squares of cycles, complete multipartite graphs, ladders, Möbius ladders, and hypercubes. In order to prove these results, we show that several parameters of a graph, like the number of cycles of length 3, 4 and 5, and the edge-connectivity are determined by its Tutte polynomial.

1 Introduction

The Tutte polynomial $T(G;x,y)$ of a graph G is a powerful invariant with many applications not only in graph theory but also in other fields such as knot theory and statistical physics. One important feature of the Tutte polynomial is that by evaluating $T(G;x,y)$ at special points and lines in the plane one obtains several parameters of G. For example, $T(G;1,1)$ is the number of spanning trees of G, $T(G;2,0)$ is the number of acyclic orientations and $(-1)^{n-k(G)}x^{k(G)}T(G;1-x,0)$ is the chromatic polynomial $p(G,x)$, where $k(G)$ denotes the number of connected components of G and $n=|V(G)|$ (see [3, 10, 21] for useful surveys on the Tutte polynomial).

A natural question is then to what extent the Tutte polynomial determines a given graph. It is easy to construct non-isomorphic graphs with the same Tutte polynomial, for instance any two trees with the same number of vertices, but this is more difficult if one demands the graphs to have higher connectivity. The first examples for 3, 4 and 5-connected graphs were given by Tutte [20]. Later on Brylawski [9] gave examples for any given connectivity, and more recently simpler examples have been constructed in [4].

These considerations motivate the following definitions. From now on we consider graphs without isolated vertices. We say that two graphs G_1 and G_2 are *T-equivalent* if $T(G_1;x,y)=T(G_2;x,y)$. If a graph G is such that for all graphs H T-equivalent to G we have that $H\cong G$, we say that G is *T-unique*. Hence a T-unique graph is completely determined by its Tutte polynomial. Our aim in this paper is to study which information about G is contained in $T(G;x,y)$ and use it to prove that some families of graphs are T-unique.

The problem of finding graphs determined by a polynomial invariant has been studied in other cases. For instance, a great deal of work has been done in the case of the chromatic polynomial. Recall that it is defined as

$$p(G,x) = \text{the number of } x\text{-colorings of } G,$$

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where an x-coloring consists of assigning to each vertex in $V(G)$ a color from the set $\{1, \ldots, x\}$ such that no pair of adjacent vertices have the same color. It is not difficult to see that $p(G, x)$ is in fact a polynomial. The coefficients and structure of this polynomial have been extensively studied and it is known that $p(G, x)$ contains significant information about G. In particular, one can deduce from $p(G, x)$ the number of vertices and the number of edges of G, the girth $g(G)$ and the number of cycles of length $g(G)$.

In some cases these parameters are enough to determine G completely, in which case then G is said to be χ-unique. Some families of graphs are known to be χ-unique: complete graphs K_n, cycles C_n, and bipartite graphs $K_{p,q}$ for $p, q \geq 2$ (see [11], [12], [13] and [18] for more on χ-unique graphs). Given the relationship mentioned above between the chromatic and the Tutte polynomials, it is easy to check that all these graphs are also determined by $T(G; x, y)$ and provide our first examples of T-unique graphs. But the converse relation does not hold, that is, there are T-unique graphs that are not χ-unique. The two graphs in Fig. 1 have the same chromatic polynomial [12], but we prove in Theorem 3.1 that the first of them, the wheel W_5 with five spokes, is T-unique.

This example shows, as expected, that $T(G; x, y)$ is a stronger invariant than $p(G, x)$. In our study of T-unique graphs we will focus on those that are not known to be χ-unique.

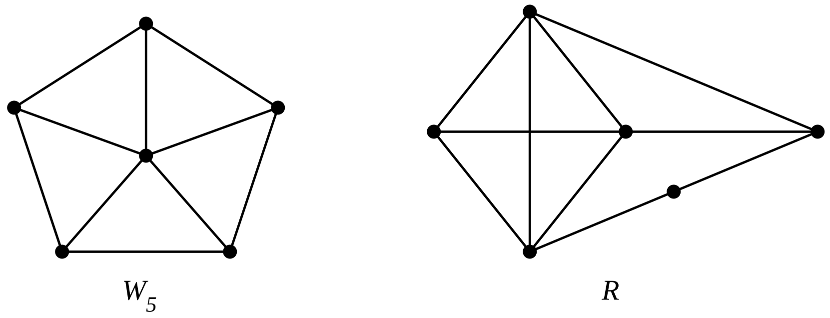


Figure 1: Two graphs with the same chromatic polynomial but different Tutte polynomial.

The definition of the Tutte polynomial can be naturally extended to the class of all matroids (see next section) and one can define T-uniqueness for matroids as well: a matroid M is said to be T-unique if it is isomorphic to any other matroid with the same Tutte polynomial. Bonin and Miller [6] have shown that several well-known families of matroids are T-unique: uniform matroids, projective and affine geometries of rank four or more, the matroids $M(K_n)$ of complete graphs, and certain Dowling geometries. Additional examples are presented in [1] and [19]. This topic is discussed further in the last section.

The organization of the paper is as follows. Section 2 is devoted to the definition and basic properties of the Tutte polynomial; in Theorem 2.4 we prove that certain parameters of a graph G are determined by $T(G; x, y)$, such as the number of cycles of length 3, 4 and 5, and the edge-connectivity. This is our main tool in Sections 3, 4, 5 and 6 where we prove that some well-known families of graphs are T-unique: wheels, squares of cycles, complete multipartite graphs, ladders and Möbius ladders, and hypercubes. We conclude with some remarks on open and related problems.

2 Preliminaries

Let $G = (V, E)$ be a graph with vertex set V and edge set E (loops and multiple edges are allowed). For every subset $A \subseteq E$, its *rank* is $r(A) = n - k(G|A)$, where $n = |V|$ and $k(G|A)$ is the number of connected components of the spanning subgraph (V, A). The *nullity* of A is $n(A) = |A| - r(A)$. The *rank-size generating polynomial* is defined as:

$$F(G; x, y) = \sum_{A \subseteq E} x^{r(A)} y^{|A|}.$$

Notice that the coefficient of $x^i y^j$ in $F(G; x, y)$ counts the number of spanning subgraphs in G with rank i and j edges. The *Tutte polynomial* is given by

$$T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)}.$$

It should be clear that both $F(G; x, y)$ and $T(G; x, y)$ contain exactly the same information about G. However, the Tutte polynomial has several interesting properties not shared by F. In particular, it satisfies the fundamental contraction-deletion rule

$$T(G; x, y) = T(G - e; x, y) + T(G/e; x, y),$$

provided that $e \in E(G)$ is neither a bridge nor a loop, where $G - e$ and G/e denote, respectively, the result of deleting and contracting the edge e in G.

Let $T(G; x, y) = \sum b_{ij} x^i y^j$. For the proof of the next three results see [10].

Lemma 2.1 *If G has neither bridges nor loops then*

$$r(G) = \max\{i \colon b_{i0} \neq 0\}, \qquad n(G) = \max\{j \colon b_{0j} \neq 0\}.$$

The next theorem tells us that the Tutte polynomial includes information about the 2-connectivity of a graph.

Theorem 2.2 *If $r(G)$ and $n(G)$ are both positive, then the number of blocks of G (that is, 2-connected components) is $min\{i : b_{i0} \neq 0\}$. Otherwise, G has $|E(G)|$ blocks. In particular, if G is 2-connected and H is T-equivalent to G, then H is also 2-connected.*

An alternative way of characterizing 2-connectivity is the following: G is 2-connected if, and only if, $T(G; x, y)$ is irreducible in $\mathbb{Z}[x, y]$. This result, conjectured by Brylawski [7], has been proved recently in [16].

The Tutte polynomial of a matroid M is defined in an analogous way, where $r(A)$ is the rank function of the matroid. Of course, if M is graphical then both definitions agree. If $M^*(G)$ denotes the dual matroid of the graph G then, from the basic properties of $M^*(G)$, one can prove the following result.

Proposition 2.3 *If $M^*(G)$ is the dual matroid of G, then $T(M^*(G); x, y) = T(G; y, x)$.*

This is the only fact about matroids that we use in this paper.

From here on we use the following notation. A cycle of length n will be called C_n or an n-cycle; for $n = 3, 4, 5$ we also refer to them as triangles, squares and pentagons. A *chord* of a cycle C is an edge joining two non-consecutive vertices of C. If $P(x, y)$ is a polynomial, we denote by $[x^i y^j]P(x, y)$ the coefficient of $x^i y^j$ in $P(x, y)$.

To prove that a graph is T-unique, we first show that some parameters of this graph are determined by its Tutte polynomial. Theorem 2.4 provides a list of such parameters. Some of them are also determined by the chromatic polynomial, but in the proof we only use facts about $T(G;x,y)$ and $F(G;x,y)$. Although most of the statements in the next theorem can be deduced from results contained in [8] and [14], we prefer to prove them here for completeness.

Theorem 2.4 *Let $G=(V,E)$ be a 2-connected graph. Then the following parameters of G are determined by its Tutte polynomial:*

1. *The number of vertices and the number of edges.*
2. *For every k, the number of edges with multiplicity k. In particular, whether G is simple or not.*
3. *The number of cycles of shortest length $g(G)$.*
4. *The edge-connectivity $\lambda(G)$. In particular, a lower bound for the minimum degree $\delta(G)$.*
5. *If G is simple, the number of cliques of each size. In particular, the clique-number $\omega(G)$.*
6. *If G is simple, the number of cycles of length three, four and five. For the cycles of length four, it is also possible to know how many of them have exactly one chord.*

Proof. 1) For a 2-connected graph, $|V(G)| = r(G)+1$, so $|V(G)|$ is one plus the degree of x in $T(G;x,y)$. The number of edges $|E(G)|$ is $n(G)+r(G)$, that is, the sum of the degrees of x and y in $T(G;x,y)$.

2) A set $\{e_1,\ldots,e_k\} \subseteq E(G)$ of k parallel edges contributes to $F(G;x,y)$ with a term xy^k and, conversely, every term xy^k in $F(G;x,y)$ comes from such a set. Let

$$p = \max\{j : [xy^j]F(G;x,y) \neq 0\},$$

and let P_i, $2 \le i \le p$, be the number of subsets of i parallel edges that are not included in any set of $i+1$ parallel edges. We prove by induction on $j = p-i$ that P_i is determined by $F(G;x,y)$, hence by $T(G;x,y)$.

For $j=0$, we have $i=p$ and $P_p = [xy^p]F(G;x,y)$. Now assume that we know $P_p,\ldots,P_{p-j}$ and let us calculate P_{p-j-1}. The coefficient $[xy^{p-j-1}]F(G;x,y)$ counts the number of sets of $p-j-1$ parallel edges. Since they may be included in larger sets of parallel edges, we have

$$P_{p-j-1} = [xy^j]F(G;x,y) - \sum_{i=p-j}^{p} P_i \binom{i}{p-j-1}.$$

With respect to the second assertion, the graph G is simple if $p=1$.

3) Considering which subsets $A \subseteq E(G)$ contribute to the coefficients of $x^{k-1}y^k$ in $F(G;x,y)$, it is clear that $g(G) = \min\{k : [x^{k-1}y^k]F(G;x,y) \neq 0\}$ is the girth of G, and that $[x^{g(G)-1}y^{g(G)}]F(G;x,y)$ is the number of cycles of shortest length.

4) The edge-connectivity $\lambda(G)$ is the minimum size of an edge cut-set of G. But edge cut-sets in G correspond to circuits in $M^*(G)$, the dual matroid of G (see [17]). Therefore $\lambda(G)$ equals the length of a minimum circuit in $M^*(G)$. The proof of statement 3 applies also to a matroid substituting minimum circuits by shortest cycles, so we can calculate $\lambda(G)$ from $T(M^*(G);x,y)$. But as $T(M^*(G);x,y) = T(G;y,x)$, $\lambda(G)$ is determined by the Tutte polynomial of G. The well-known relation $\lambda(G) \le \delta(G)$ ensures that every vertex in G has degree at least $\lambda(G)$.

5) If G is a simple graph, a subset $A \subseteq E(G)$ is a clique of order n (that is, $A \cong K_n$) if and only if A contributes to the coefficient of $x^{n-1}y^{\binom{n}{2}}$ in $F(G;x,y)$. Then the number of cliques of order n is $[x^{n-1}y^{\binom{n}{2}}]F(G;x,y)$ and the clique number is $\omega(G) = \max\{n : [x^{n-1}y^{\binom{n}{2}}]F(G;x,y) \neq 0\}$.

6) Cycles of length three are cliques, so by the previous statement we know how many of them there are, call this quantity c_3. All cycles of length four are counted by the coefficient of x^3y^4 in $F(G;x,y)$, but cycles of length three plus one edge are also counted by this coefficient. So c_4, the number of 4-cycles, equals $[x^3y^4]F(G;x,y) - c_3(|E(G)| - 3)$. Note that cliques K_4 are 4-cycles with two chords, and that subsets of $E(G)$ formed by one 4-cycle and one of its chords are counted precisely by $[x^3y^5]F(G;x,y)$. By the previous part, we know the number k_4 of cliques K_4. Let c_4^- be the number of 4-cycles without chords, and let c_4^+ be the number of 4-cycles with exactly one chord. Then the following relations hold:

$$\begin{aligned} c_4 &= 3k_4 + c_4^+ + c_4^-; \\ [x^3y^5]F(G;x,y) &= 6k_4 + c_4^+. \end{aligned}$$

From this we can readily compute c_4^- and c_4^+.

Finally, consider which subsets of $E(G)$ contribute to $[x^4y^5]F(G;x,y)$. There are three classes:

- Cycles of length five.
- Cycles of length four plus one edge that is not a chord of the cycle.
- Cycles of length three plus two edges such that they do not form a 4-cycle with a chord.

The cardinalities of the last two classes are, respectively, $c_4(|E(G)| - 4) - [x^3y^5]F(G;x,y)$ and $c_3\binom{|E(G)|-3}{2} - 2[x^3y^5]F(G;x,y)$, so we can deduce the number of cycles of length five in G. □

3 Wheels and squares of cycles

The *wheel* W_n is the graph obtained from a cycle C_n by adding a new vertex adjacent to all the vertices in the cycle (that is, $W_n = C_n + K_1$). The wheels with an even number of spokes W_{2n} are χ-unique, and also W_3 and W_9 are. But W_5 and W_7 are not, and the question of whether W_{2n+1} is χ-unique is still open (see [12]).

The following theorem shows that all wheels are T-unique; our proof does not depend on the parity of the number of vertices.

Theorem 3.1 *For every $n \geq 3$, the wheel W_n is T-unique.*

Proof. As W_3 and W_4 are χ-unique graphs, we begin our proof by taking a graph H T-equivalent to W_n for some $n \geq 5$. Applying Theorems 2.2 and 2.4 we obtain the following information: H is a two-connected simple graph, $|V(H)| = n + 1$, $|E(H)| = 2n$, H has n triangles, and n cycles of length four, all of which have exactly one chord.

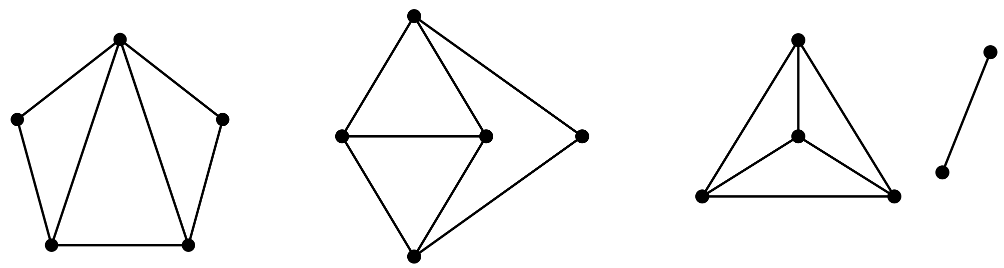
Claim 1 There is no edge in $E(H)$ belonging to three or more triangles.

Proof. Let us show that it is possible to obtain from $T(H;x,y) = T(W_n;x,y)$ the number of subsets of $E(H)$ consisting of three triangles meeting at a single edge, or, which is the same, subgraphs $K_{2,3}$ with an extra edge joining the two vertices in the partite set of cardinatlity two: call these subsets $K_{2,3}^+$. All of them contribute to the coefficient of x^4y^7 in $F(H;x,y)$. We study first which other subsets of $E(H)$ contribute to this coefficient. There are three possibilities, see Fig. 2:

1. Cycles of length five with two chords with a common vertex (that is, a triangulated pentagon).

2. Subgraphs $K_{2,3}$ with an extra edge joining two vertices in the partite set of order three.

3. Complete subgraphs K_4 plus one edge.



(1) (2) (3)

Figure 2: All possible subgraphs in H with rank 4 and size 7.

We already know that the situation described in 3 is impossible because there are no K_4 in H. Using the fact that each cycle of length four must have one and only one chord it is easy to prove that situation 2 is also impossible. Observe that this implies that a cycle of length five cannot have three or more chords.

Therefore, the only contributions to $[x^4y^7]F(G;x,y)$ are $K_{2,3}^+$ and triangulated pentagons, so

$$[x^4y^7]F(G;x,y) = t + c \tag{1}$$

where c is the number of cycles of length five as in 1 and t is the number of $K_{2,3}^+$.

To see that $t = 0$ we need another equation involving t and c. In order to obtain this second equation, we analyze the coefficient of x^4y^6 in $F(H;x,y)$, that is, subgraphs in H with rank 4 and size 6. The possibilities are the following, see Fig. 3:

1. Two edge-disjoint triangles.

2. A cycle of length four with its chord and any other edge.

3. A cycle of length five with one chord.

4. A subgraph $K_{2,3}$.

The contributions of 1 and 2 are, respectively, $\binom{n}{2} - n$ and $n(2n-5)$. Note that each cycle of length five with one chord must be triangulated, for if not there would be a cycle of length four without chords. Since the maximum number of chords in a cycle of length five is two, we deduce that the number of cycles of length five with one chord equals twice the number of triangulated pentagons. On the other hand, in $K_{2,3}$ there are three cycles of length four without chords; as we have seen previously the chord must join the two vertices in the 2-stable set. This means that in H the number or $K_{2,3}$ equals the number of $K_{2,3}^+$. Hence

$$[x^4y^6]F(G;x,y) = t + 2c + \binom{n}{2} - n + n(2n-5) \tag{2}$$

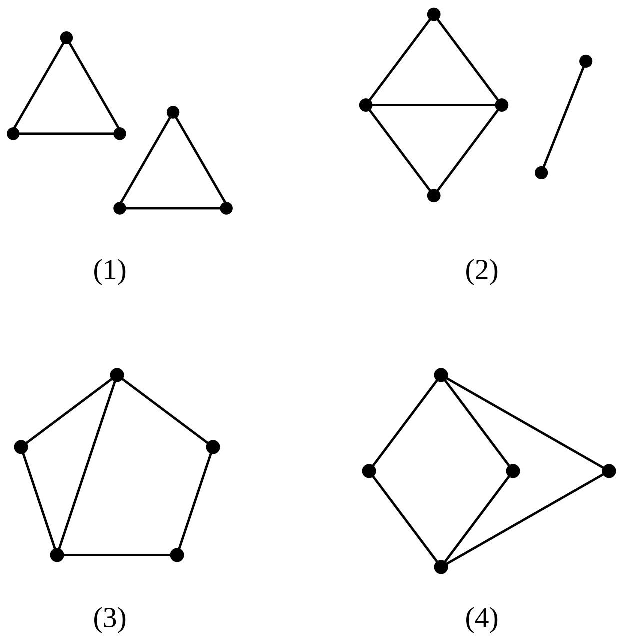


Figure 3: All possible subgraphs in H with rank 4 and size 6.

Equations (1) and (2) imply that it is possible to obtain t and c from the coefficients of $T(H;x,y) = T(W_n;x,y)$. As in W_n there is no edge belonging to three triangles, we deduce that $t=0$. $\square$

We call an edge $e \in E(H)$ a *diagonal* edge if it is the chord of one cycle of length four; we write $e \in C_4^+$. If e is not the chord of any cycle, we call it a *non-diagonal* edge.

Claim 2 In H there are n diagonal edges and n non-diagonal edges

Proof. Let $\mathcal{U} = \{(e, C_4^+) : e \in C_4^+\}$. Define ρ_i as the number of cycles of length 4 having edge e_i as a chord. By Claim 1 we know that $0 \leq \rho_i \leq 1$. The cardinality of $\mathcal{U}$ equals n because there are exactly n cycles of length four. But also $|\mathcal{U}| = \rho_1 + \cdots + \rho_{2n}$, so there must be n edges with $\rho_i = 1$ and n edges with $\rho_i = 0$. $\square$

From now on we assume that the diagonal edges are $e_1, \ldots, e_n$; that is, $\rho_1 = \cdots = \rho_n = 1$ and $\rho_{n+1} = \cdots = \rho_{2n} = 0$.

Claim 3 A diagonal edge belongs to exactly two triangles and a non-diagonal edge belongs to exactly one.

Proof. Consider the set $\mathcal{T} = \{(e,t) : e \text{ is an edge of triangle } t\}$. Define τ_i as the number of triangles that contain e_i. From the claims above we know that $\tau_i = 2$ for $1 \leq i \leq n$ and that $\tau_i \leq 1$ for $n+1 \leq i \leq 2n$. Summing over all triangles in H, we deduce that $|\mathcal{T}| = 3n$, and summing over all edges, we obtain $|\mathcal{T}| = \tau_1 + \cdots + \tau_{2n} = 2n + \tau_{n+1} + \cdots + \tau_{2n} \leq 3n$. So $\tau_i = 1$ for $n+1 \leq i \leq 2n$ and the claim is proved. $\square$

Claim 4 If a vertex $v \in V(H)$ is incident with a non-diagonal edge, then there is a second non-diagonal edge incident with v.

Proof. Let f be the non-diagonal edge incident with v and let t_1 be the unique triangle that contains f. This triangle t_1 contains another edge incident with v, call it f_1. If f_1 is a non-diagonal edge, we have finished. If not, f_1 is a diagonal edge and there exists a triangle t_2 different from t_1 and containing f_1. Let f_2 be the second edge of t_2 incident with v. Applying repeatedly the previous argument, we construct a succession of diagonal edges $f, f_1, f_2, \ldots, f_k$ incident with v. Note that edges are not repeated in this list, because there is no edge belonging to three triangles. As the number of edges is finite, we must necessarily end in a non-diagonal edge incident with v, necessarily different from f_1. □

Claim 5 There is no subgraph in H isomorphic to the wheel W_p for $p < n$.

Proof. W_p has rank p and size $2p$; the proof consists of showing that the coefficient of $x^p y^{2p}$ in $F(H; x, y)$ is zero for all $p < n$. This is equivalent to proving that in W_n there is no subgraph G of rank p and size $2p$. If a subgraph of W_n does not contain the central vertex, then it is a forest or a cycle of length n and its size is at most its rank plus one, so it cannot be a wheel. Therefore, G must contain the central vertex. The connected component in G that contains this vertex has rank k and size at most $2k - 1$, and the rest of the components are forests. So again it is impossible to reach the required relation between rank and size. □

Finally, let

$$\mathcal{V} = \{(v, e) \in V(G) \times \{e_{n+1}, \ldots, e_{2n}\} : v \text{ is incident with } e\},$$

where $\{e_{n+1}, \ldots, e_{2n}\}$ is the set of non-diagonal edges. Clearly $|\mathcal{V}| = 2n$. We define δ'_i as the non-diagonal degree of v_i, that is, the number of non-diagonal edges incident with v_i. By claim 4, either δ'_i is zero or is greater than one. As $|\mathcal{V}| = \delta'_1 + \cdots + \delta'_{n+1} = 2n$, there is at least one vertex with non-diagonal degree equal to zero; assume it is v_1. That means v_1 is incident only with diagonal edges.

We can now reconstruct the wheel W_n from all the information we have. Let f_1 be an edge incident with v_1; f_1 belongs to a triangle t_1 which contains a second edge, f_2, incident with v_1. As f_2 is diagonal, there exists a new triangle t_2 containing f_2 and an edge f_3 incident with v_1. Applying successively the same argument, we obtain a list of diagonal edges $f_1, f_2, \ldots, f_k$ incident with v_1 and such that f_i and f_{i+1} belong to the same triangle. Observe that by Claim 1, f_1 is the only edge that can be repeated in this list. Therefore, the process must end when $f_k = f_1$. If $k < n$, then we obtain a wheel W_k in H, which is impossible by Claim 5. So $k = n$ and $H \cong W_n$. □

The technique introduced in the above proof can be applied to a different family of graphs. The *square of cycle* C_n^2 is the graph obtained from the cycle C_n by adding all the edges between vertices at distance two.

Theorem 3.2 *For every $n \geq 3$, the graph C_n^2 is T-unique.*

Proof. For $n = 3, 4, 5$, the graph C_n^2 is isomorphic to K_n, which we know is T-unique. For $n = 6, 7, 8$, the table of χ-unique graphs in [12] ensures that C_6^2, C_7^2 and C_8^2 are also T-unique. So we restrict to the general case $n \geq 9$.

Let H be a graph T-equivalent to C_n^2 for some $n \geq 9$. Applying theorems 2.2 and 2.4 to H, we obtain the following information: H is a two-connected simple graph, $|V(H)| = n$, $|E(H)| = 2n$,

H has n triangles, H has neither K_4 nor any cycle of length four without chords, and H has n cycles of length four with exactly one chord. (Note that if $n = 6$, H has twelve 4-cycles with one chord, and if $n = 7, 8$, H has some 4-cycles without chords). From statement 5 in Theorem 2.4, we know that the minimum degree in H is at least four. Since the sum of all degrees must be $4n$, and as there are exactly n vertices, we conclude that H is 4-regular.

Claim 1 There is no edge in $E(H)$ belonging to three or more triangles.

Proof. The proof is exactly the same as in Claim 1 in Theorem 3.1; the only difference is that now H has rank equal to $n - 1$ instead of n.

Observe that in this case both t and c are zero; this implies that in H there is no subgraph with rank 4 and size 7. □

As before, we call an edge $e \in E(H)$ a diagonal edge if it is the chord of a cycle of length four. If e is not the chord of any cycle, it is called a non-diagonal edge. The proofs of Claims 2 and 3 in Theorem 3.1 remain valid and we can state the following.

Claim 2 In H there are n diagonal edges and n non-diagonal edges. A diagonal edge belongs to exactly two triangles and a non-diagonal edge belongs to exactly one.

Claim 3 There is no subgraph in H isomorphic to W_p for all p.

Proof. Every wheel W_p has a vertex of degree p. If $p \geq 5$, it is clear that H cannot contain W_p. The only cases remaining are W_3 and W_4 (note that W_2 is not simple). W_3 is the same graph as K_4, and we already know that there are no K_4 in H. In order to see that H does not contain any W_4 we must study the coefficient of x^4y^8 in $F(H; x, y)$. This coefficient counts the number of subgraphs in H with rank 4 and size 8. Observe that deleting one edge in these subgraphs, we obtain size 7 subgraphs with rank either 3 or 4. It is easy to see that no simple graph can have rank 3 and size 7 and, as observed in the proof of Claim 1, H cannot contain a subgraph of rank 4 and size 7. So $[x^4y^8]F(H; x, y)$ is zero and there is no wheel W_4 in H. □

Claim 4 Every vertex $v \in V(H)$ is incident with two diagonal and two non-diagonal edges.

Proof. Let δ'_i be the non-diagonal degree of vertex v_i, $1 \leq i \leq n$. First, note that all the facts used in the proof of Claim 4 in Theorem 3.1 remain true, so we can use a similar argument to prove that δ'_i is either zero or greater than one. Now, if there were any vertex v_i with $\delta'_i = 0$, we could build a wheel W_p in H, just like in the final step of Theorem 3.1; therefore, $\delta'_i \geq 2$ for $1 \leq i \leq n$. The sum of all non-diagonal degrees must equal $2n$, since there are n non-diagonal edges, each of them containing two vertices. This implies that $\delta'_i = 2$ for all vertices v_i. The result on the diagonal edges follows from 4-regularity. □

We know now the local structure around every vertex $v \in V(H)$. It is easy to prove that the four edges incident with v must be joined as depicted in Fig. 4, where d_1, d_2 are diagonal edges and f_1, f_2 are non-diagonal edges.

The last step in the proof consists of determining the global structure of H using this local information.

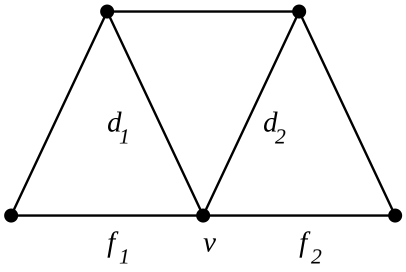


Figure 4: Local structure around every vertex in H.

Claim 5 The subgraph H_1 induced by the diagonal edges in H is a cycle.

Proof. We already know that H_1 is 2-regular, so each of its connected components is a cycle. Suppose now that H_1 is disconnected. Then there is a non-diagonal edge e joining two vertices $v_1 \in V(\Gamma_1)$ and $v_2 \in V(\Gamma_2)$ in two different components Γ_1 and Γ_2. Let x_1 and y_1 (respectively, x_2 and y_2) be the neighbors of v_1 (resp. v_2) in Γ_1 (resp. Γ_2). As $x_1v_1, y_1v_1, x_2v_2, y_2v_2$ are all diagonal edges, the local structure of H implies that $x_1y_1, x_2y_2 \in E(H)$, and also that either x_1v_2 or y_1v_2 belongs to $E(H)$; assume $x_1v_2 \in E(H)$. Observe that the vertices v_1, v_2 and x_1 form a triangle supported by the non-diagonal edge v_1v_2. Now v_1 must be adjacent to either x_2 or y_2, but in both cases we obtain a second triangle containing v_1v_2, which is impossible. Therefore, H_1 is a two-regular connected graph, hence a cycle. □

Finally, note that any pair of adjacent diagonal edges in H_1 is joined by a non-diagonal edge. This implies that $H \cong C_n^2$. □

All graphs C_n^2 for $n \leq 9$ are known to be χ-unique; for greater values of n the question remains open. In [12] they are conjectured to be χ-unique.

4 Complete multipartite graphs

The chromatic uniqueness of complete multipartite graphs is far from being settled. As was mentioned before, $K_{p,q}$ is χ-unique if $p \geq q \geq 2$ but for instance, $K_{1,p,q}$ is not χ-unique. On the other hand, $K(p_1, \ldots, p_r)$ is χ-unique if $|p_i - p_j| \leq 1$ for all $i, j = 1, 2, \ldots, r$, because in this case it is a Turán graph [12, Th. 19].

In this section we show that complete multipartite graphs are T-unique.

Theorem 4.1 *For every set of positive integers $p_1, p_2, \ldots, p_r$ the complete multipartite graph $K_{p_1,p_2,\ldots,p_r}$ is T-unique, with the only exception of $K_{1,p}$.*

Proof. As we said before, complete graphs K_n and bipartite graphs $K_{p,q}$ with $p, q \geq 2$ are χ-unique, hence T-unique. So let H be a graph T-equivalent to $K_{p_1,p_2,\ldots,p_r}$ for some $r \geq 3$. Then we know that H is 2-connected, has $n = \sum p_i$ vertices, has $\sum_{i<j} p_ip_j$ edges, and that it is r-partite since its chromatic number is r.

Let K_{r+1}^- denote the complete graph K_{r+1} with one edge deleted, and let us consider the set

$$\mathcal{C}(H) = \{(A, B) : A \subseteq B \subseteq E(H),\ A = K_r,\ B = K_{r+1}^-\}.$$

If $\alpha(H)$ is the number of subgraphs of H equal to K_{r+1}^-, then

$$|\mathcal{C}(H)| = 2\alpha(H),$$

since every K_{r+1}^- contains exactly two subgraphs equal to K_r. Since $K_{p_1,p_2,\ldots,p_r}$ and H are T-equivalent, they contain the same number $p_1 \cdots p_r$ of r-cliques. Let $W_1(H), \ldots, W_{p_1\cdots p_r}(H)$ be the r-cliques of H, and let $c_i(H)$ be the number of subgraphs of H equal to K_{r+1}^- that contain W_i. Then

$$|\mathcal{C}(H)| = c_1(H) + \cdots + c_{p_1\cdots p_r}(H) \leq p_1 \cdots p_r(n-r), \tag{3}$$

the last inequality because every r-clique in H can be extended in at most $n-r$ (the number of remaining vertices) ways to a K_{r+1}^- in H.

If we define $\mathcal{C}$, α, $W_1, \ldots, W_{p_1\cdots p_r}$, and $c_1, \ldots, c_{p_1\cdots p_r}$ analogously for the graph $K_{p_1,\ldots,p_r}$, then we have

$$|\mathcal{C}| = 2\alpha = c_1 + \cdots + c_{p_1\cdots p_r} = p_1 \cdots p_r(n-r),$$

since now every r-clique in $K_{p_1,\ldots,p_r}$ is contained in *exactly* $n-r$ subgraphs equal to K_{r+1}^-. But $\alpha(H) = \alpha$, since these quantities correspond precisely to edge-sets, respectively in H and in $K_{p_1,\ldots,p_r}$, of rank r and cardinality $\binom{r+1}{2} - 1$, and are determined by the Tutte polynomial. We deduce that the inequality in (3) is in fact an equality and that $c_i(H) = p_1 \cdots p_r(n-r)$ for every i.

We can now prove that H is a *complete* r-partite graph. Let $u, v \in V(H)$ be in different parts of the r-partition of H and let $\{w_1, \ldots, w_r\}$ be the clique $W_1(H)$ in H, where u and w_1, and also v and w_2, are in the same part. Since $c_1(H) = p_1 \cdots p_r(n-r)$, necessarily $\{u, w_2, \ldots, w_r\}$ is a clique in H; by the same reason $\{u, v, w_3, \ldots, w_r\}$ must be a clique, hence u and v are adjacent and H is complete.

We are not finished yet since, in general, we only have $H = K_{q_1,\ldots,q_r}$ for some q_i. By counting the number of cliques of every possible size from 1 to r in both graphs (all of them determined by the Tutte polynomial) we obtain:

$$\begin{aligned}
\sum_i p_i &= \sum_i q_i \\
\sum_{i<j} p_i p_j &= \sum_{i<j} q_i q_j \\
\cdots &= \cdots \\
p_1 \cdots p_r &= q_1 \cdots q_r.
\end{aligned}$$

But it is well-known that two sets of r numbers having the same elementary symmetric functions must be equal, and this concludes the proof. $\square$

5 Ladders and Möbius ladders

The graph $L_n = C_n \times K_2$ is called a *ladder* (or a *prism*). The *Möbius ladders* M_n are constructed from an even cycle C_{2n} by joining every pair of opposite vertices. These graphs are only known to be χ-unique for $n = 3, 4, 5$, and the ladders L_n are conjectured to be χ-unique for all n.

Theorem 5.1 *For every $n \geq 3$, the ladders L_n are T-unique.*

Proof. In [12] the ladders L_3, L_4 and L_5 are shown to be χ-unique, hence T-unique. Let now H be a graph T-equivalent to L_n for some $n \geq 6$. From Theorems 2.2 and 2.4 we deduce the following information: H is a 2-connected simple graph with $2n$ vertices and $3n$ edges, in H there is no cycle of length three or five, and there are n cycles of length four, all of them without chords. Since the Tutte polynomial specializes to the chromatic polynomial, it is possible to determine the chromatic number $\chi(H)$ from $T(H; x, y)$: $\chi(H) = 2$ if n is even, and $\chi(H) = 3$ if

n is odd. We also know from Theorem 2.4 that all the vertices in H have degree al least three, and since the sum of the degrees of all the vertices equals $6n$ (twice the number of edges), we conclude that H is 3-regular.

Some other parameters, such as cycles of length six or bipartite subgraphs $K_{2,3}$ are not usually determined by the Tutte polynomial. But if we use the above information about H, it is possible to make some improvement in our treatment of the rank-size generating polynomial $F(H, x, y)$. We say that two squares are *adjacent* if they have one common edge, or, which is the same, if their union is a cycle of length six with a chord. The two following claims tell us how many pairs of adjacent squares there are.

Claim 1 If $T(H, x, y) = T(L_n, x, y)$, then

1. In H there is no subgraph isomorphic to $K_{2,3}$ (Fig. 5.1).
2. In H there is no cycle of length six with two chords (Fig. 5.2).
3. In H there are n cycles of length six with exactly one chord (Fig. 5.3)
4. In H there is no subgraph isomorphic to the graph in Fig. 5.4. Therefore, if a square is adjacent to two other squares, the only possible configuration is that of Fig. 5.5.

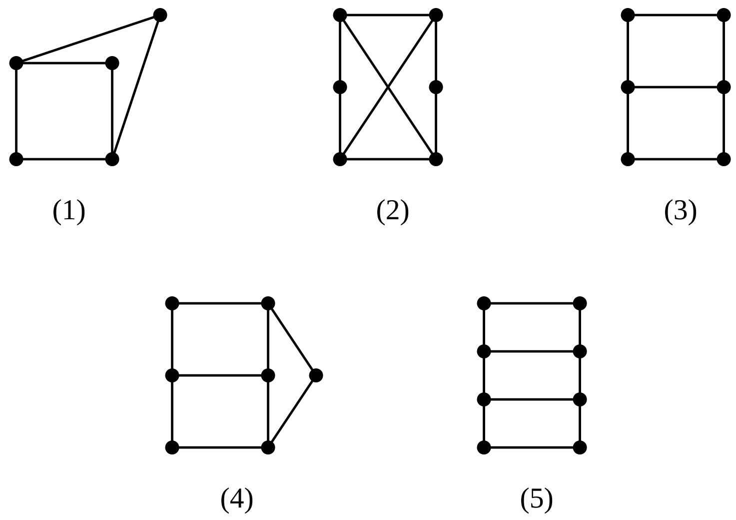


Figure 5: Illustrating Claim 1.

Proof. All of them follow by studying which subsets $A \subseteq E(H)$ are counted by suitable coefficients of $F(H, x, y)$, and using the properties of H proved previously. Note that the only chords allowed in a cycle of length six are those joining two vertices at distance three, otherwise H would contain a triangle. $\square$

Claim 2 Every square in H is adjacent to two other squares.

Proof. Let c_i be the number of squares adjacent to the i-th square, $1 \leq i \leq n$. Observe first that in Fig. 5.5 there is no place for a third square adjacent to the central one, so $c_i \leq 2$ for all

$1 \le i \le n$. Let $\mathcal{C}_4$ be the set of all squares in H, and let $\mathcal{H}$ be the set of all cycles of length six with one chord; now define $\mathcal{S} = \{(c,h) \in \mathcal{C}_4 \times \mathcal{H} : c \subseteq h\}$. The cardinality of $\mathcal{S}$ is clearly $2n$, since there are n elements in $\mathcal{H}$ and each of them contains two squares. But we also have that $|\mathcal{S}| = c_1 + \cdots + c_n \le 2n$. So equality must hold and $c_i = 2$ for every i. □

Finally, given a square s_1 there is a second square s_2 adjacent to s_1, and a third one s_3 adjacent to s_2. We add in this way repeatedly a new square until we return to s_1. The union of these squares $s_1, \ldots, s_k$ is a connected component in H, because all the vertices already have three neighbors. Since H is 2-connected, $k = n$ and H is isomorphic to either L_n or M_n. But we can distinguish L_n from M_n using the chromatic number ($\chi(M_n) = 3$ if n is even and $\chi(M_n) = 2$ if n is odd), so $H \cong L_n$. □

The same proof gives the following.

Theorem 5.2 *The Möbius ladders M_n are Tutte-unique for all $n \ge 2$.*

Proof. Let H' be a graph T-equivalent to M_n for some n. If $n \ge 5$, all the facts proved for H in the above theorem are still valid for H'; when we arrive at the end of the proof, the inversion of the chromatic numbers leads us to $H' \cong M_n$.

For the particular cases remaining, we have that $M_2 \cong K_4$ and $M_3 \cong K_{3,3}$ are both covered by Theorem 4.1, and M_4 can be treated directly without difficulty. □

6 Hypercubes

The *n-cube* Q^n is defined as the product of n copies of K_2. Cubes have been extensively studied and have been characterized in several ways. We need the following characterization due to Mulder (see [2]).

Lemma 6.1 *A connected n-regular graph is isomorphic to the n-cube if, and only if, it has 2^n vertices, and every pair of vertices at distance 2 have precisely two common neighbors.*

We can now prove that cubes are T-unique.

Theorem 6.2 *The n-cube is T-unique for every $n \ge 2$.*

Proof. Let H be T-equivalent to the n-cube. Then H is bipartite and 2-connected, has 2^n vertices, and $n2^{n-1}$ edges. Since the n-cube has edge-connectivity equal to n, so does H. Hence H has minimum degree at least n, and by counting incidences between vertices and edges, we deduce that H is id fact n-regular.

In view of the previous lemma, we only need to check that every pair of vertices at distance 2 in H have precisely two common neighbors. Observe that two vertices at distance 2 must have a common neighbor, and that two common neighbors define a cycle C_4 (all C_4's are chordless since H is bipartite); conversely, every C_4 gives raise to two pairs of vertices with two commons neighbors. Since Q^n and H are T-equivalent, they have the same number $2^{n-2}\binom{n}{2}$ of C_4.

For every pair of vertices x and y at distance 2 in H, let $n(x,y)$ be number of common neighbors they have. Then $n(x,y) \le 2$, since if x and y have a, b, c as common neighbors, then $\{x, y, a, b, c\}$ induces a subgraph in H isomorphic to $K_{2,3}$, which has size 5 and rank 4; but such a subgraph does not exist in Q^n, hence neither in H.

Now count in two ways the pairs $(\{x,y\}, z)$, where z is a common neighbor of x and y. Since H is n-regular, we have

$$2^n \binom{n}{2} = \sum_{d(x,y)=2} n(x,y).$$

Counting C_4's, there are at least $2 \cdot 2^{n-2}\binom{n}{2}$ different pairs $\{x, y\}$ at distance 2 with $n(x, y) = 2$. Since the sum is equal to $2^n\binom{n}{2}$, these are all the pairs and the claim is proved. $\square$

We remark that n-cubes are conjectured to be χ-unique, but this has been proved only for $n = 2, 3$.

7 Concluding remarks

We have shown that several families of graphs are uniquely determined by their Tutte polynomial. A natural line of work is to continue the search for T-unique graphs, and we expect to present new results in this line in a forthcoming paper. One difficulty with this question is that each graph, or rather each family of graphs, seems to require a different proof adapted to the particular properties of the given graph or family. It would be interesting to be able to determine "large" families of T-unique graphs, for instance to show that for every n there is an exponential number of T-unique graphs on n vertices. Observe that Theorem 4.1 gives $p(n)$ non-isomorphic T-unique graphs of order n, where $p(n)$ is the number of partitions of the integer n, and it is well-known that $p(n)$ grows like $Ae^{B\sqrt{n}}$ for some positive constants A and B [15].

In this direction we call the attention of the reader to the following conjecture made in [4]:

Conjecture: Almost every graph is T-unique,

in the sense that the probability that a graph is T-unique among all graphs with n vertices tends to one as $n \to \infty$. On the other hand, the situation for matroids is exactly the opposite, almost every matroid is *not* T-unique. This follows from the following counting argument that appears as Exercise 6.9 in [10]: there are at most $2^{(m+1)^3/4}$ distinct Tutte polynomials corresponding to matroids on m elements, but the number of non-isomorphic matroids on m elements is at least $2^{2^{m-O(\log m)}}$.

A related problem is that of finding large sets of non-isomorphic graphs or matroids all with the same Tutte polynomial. As we mentioned in the introduction the problem becomes interesting when one requires a certain degree of connectivity. In [4] the authors show how to construct a family of $2^{n/10}$ highly connected non-isomorphic graphs on n vertices, all with the same Tutte polynomial (hence, all with the same number of vertices). With respect to matroids, Bonin [5] develops a connection between the theory of inequivalent representations of matroids over finite fields and T-equivalent representable matroids. This connection is used to produce many sequences of families of 3-connected T-equivalent representable matroids where the sizes of the families grow exponentially as a function of the rank.

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